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On Some Classes of Enriched Cyclic Contractive Self-Mappings and Their Boundedness and Convergence Properties

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Abstract

This paper focuses on dealing with several types of enriched cyclic contractions defined in the union of a set of non-empty closed subsets of normed or metric spaces. In general, any finite number $p \geq 2$ of subsets is permitted in the cyclic arrangement. The types of examined single-valued enriched cyclic contractions are, in general, less stringent from the point of view of constraints on the self-mappings compared to p -cyclic contractions while the essential properties of these last ones are kept. The convergence of distances is investigated as well as that of sequences generated by the considered enriched cyclic mappings. It is proved that, both in normed spaces and in simple metric spaces, the distances of sequences of points in adjacent subsets converge to the distance between such subsets under weak extra conditions compared to the cyclic contractive case, which is simply that the contractive constant be less than one. It is also proved that if the metric space is a uniformly convex Banach space and one of the involved subsets is convex then all the sequences between adjacent subsets converge to a unique set of best proximity points, one of them per subset which conform a limit cycle, although the sets of best proximity points are not all necessarily singletons in all the subsets.

Keywords: enriched contractions; cyclic mappings; enriched cyclic contractions; best proximity points

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1. Introduction

Contractive cyclic self-mappings have been widely studied in recent years in metric and in Banach spaces. See, for instance [1–7] and some of the references therein. If a pair of subsets has a proximal normal structure and a relative non-expansive mapping T on $A \cup B$ satisfies $T(A) \subset B$ and $T(B) \subset A$, where A and B are weakly compact and convex subsets of the Banach space $(X, \|\cdot\|)$, then the existence of some proximal point $z \in A \cup B$ such that $\|z - Tz\| = \text{dist}(A, B)$ is proved [7]. Also, cyclic ϕ_A -contractions have been considered in [8] on partially ordered and orbitally complete metric spaces while the existence of best proximity points for cyclic mappings in multiplicative metric spaces has been focused on in [9], and also in [10–12] for b-metric and b-metric-like spaces. On the other hand, certain links between cyclic contractions and Hardy–Rogers contractions were addressed in [13], while some proximity concerns were discussed in [14] within the framework of Busemann convex metric spaces. In [15], and, respectively, References [16,17], a special type of semi-cyclic mapping, under a special contractive constraint, as well

as best proximity results related to S -cyclic mappings, were obtained. The concepts of approximative compactness and bounded compactness which guarantee the non-emptiness of the sets of best proximity points of non-empty subsets A and B of a metric space have been discussed, for instance, in [1,18,19].

It can be mentioned that cyclic contractions involving a set of subsets of a metric space can be an appropriate framework to investigate the stabilization under eventual switching actions between different model parameterizations and/or dynamics characterizations in those continuous-time and discrete-time dynamic systems which can exhibit different modes of operation [20–24]. The usefulness of this framework is supported by the fact that each of the corresponding system configurations could be described by one of the subsets of the cyclic disposal and could also be adapted to the case of positive dynamic systems [25–28] under similar circumstances. Furthermore, in [28], cyclic contractions are eventually addressed as part of mixed iterative processes in conjunction with strict contractions, defined by alternative self-mappings that operate within the individual subsets of the cyclic disposal.

On the other hand, the so-called enriched-type contractions are defined by some extra parameters apart from the contractive constants and, in general, the typical properties of contractions such as boundedness and convergence of distances and existence of fixed points are achievable under less stringent conditions. The existing background literature on those classes of contractions is abundant. In particular, in [29], some general results which extend the Banach contraction mapping principle from metric spaces are obtained while other corresponding results for enriched mappings defined on Banach spaces are obtained as well. In [30], it is shown that any of the enriched contractions within a large class of defined contractive mappings have a unique fixed point and that this fixed point can be approximated by means of an appropriate Krasnoselskij iterative scheme. Such a class includes, amongst other contractive type mappings, Picard–Banach contractions and some non-expansive mappings. In [31], Chatterjea-enriched contractions were introduced and fixed point theorems were proved. Also, a convergence theorem was given for the Krasnoselskij iteration which approximates the fixed points of Chatterjea-enriched mappings in Banach spaces. In [32], enriched Ćirić–Reich–Rus contractions in Banach spaces and in convex metric spaces are focused on and fixed point theorems are derived. On the other hand, enriched Ćirić quasi-contractions in Banach spaces and in convex metric spaces were discussed in [33] and some related fixed point theorems are proved. Also, an interpolative enriched cyclic Kannan-type contraction has been defined in [34] in a Banach space, while the existence and uniqueness of the fixed point of such a mapping were proved. In [35], a new approach of enriched contractions and enriched non-expansive mappings were given which permits in a novel way the use of Mann iteration related to Krasnoselskij iteration. On the other hand, in [36,37], fixed point theorems for enriched Kannan mappings in $CAT(0)$ spaces and, respectively, enriched rational-type contractions in both quasi-Banach spaces and also in generalized convex b -metric spaces have been addressed. In [38], a convergence theorem for cyclic contractions based on the Krasnoselskij iteration was proved while also determining “a priori” and “a posteriori” error estimates. In [39], a generalized cyclic contraction has been defined in Banach spaces and a related fixed point theorem was proved. Furthermore, a novel iterated function system of the generalized enriched cyclic contractions was presented. Finally, monotone-enriched non-expansive mappings for the approximation of fixed points in ordered $CAT(0)$ spaces have been studied in [40] while, in [41], the equivalence of some iterative schemes in Banach spaces, related to the average operator, was demonstrated for enriched contractions and for enriched asymptotically non-expansive maps.

In order to contextualize this research, it can be pointed out that the enriched contractive conditions often relax those required in classical contractions. Therefore, it is of

interest to study enriched contractions associated with cyclic mappings. A central problem to be addressed in the current research is the study of the properties of boundedness and convergence of sequences. Related problems are essential and common in certain semi-analytical and numerical-based iterative procedures used to solve non-linear differential equations [42,43], and some of the references therein. Methods which have been used for that purpose are, for instance, the Variational Iteration Method, the Homotopy Perturbation Method and the Point Solution Method. In this context, the so-called Variational Iteration Method is applied to solve non-linear problems whose main objective is to prove the convergence properties of the iterations. Its application fields are multiple as, for instance, non-linear wave equations, non-linear fractional calculus or non-linear oscillations among others. It can be pointed out that this method constructs approximate solutions by means of appropriate corrections functionals involving the use of Lagrange multipliers and it requires a moderate computational effort. On the other hand, the so-called Homotopy Perturbation Method is a semi-analytical technique which is used as well to solve non-linear differential equations. It combines the known concept of homotopy with perturbation methodology with the purpose of finding approximate series solutions and it is computationally implemented in practice with recursion methods. Also, the classical Point Solution Method consists of finding a particular solution from the general one by replacing the arbitrary constants of the general solution by those which fix the solution to the data of a particular point, usually, the initial condition. The numerical implementation of this methodology usually implies also the involvement of approximative iterative methods.

This paper deals with several types of enriched cyclic contractions which are defined in the union of a finite set of non-empty closed subsets of normed spaces or metric spaces with cardinality at least two. Those enriched cyclic contractions are, in general, subject to weaker conditions on the self-mappings compared to those involving cyclic contractions. The convergence of distances is investigated as well as that of sequences generated by the considered enriched cyclic mappings. It is proved that, both in normed spaces and in simple metric spaces, the distances of sequences of points in adjacent subsets converge to the distance between such subsets. Also, if the metric space is a uniformly convex Banach space and one of the involved subsets is convex then all the sequences between adjacent subsets converge to a unique set of best proximity points, one of them within each subset. This set of convergence points constitutes a limit cycle to which all the sequences converge from any initial conditions. However, such best proximity points are not necessarily unique in all the subsets of the cyclic disposal. It should be noticed that, in fact, the subsets of the cyclic disposal are not required to be either boundedly compact or approximatively compact.

The rest of the paper is organized as follows. Section 2 provides some definitions on enriched cyclic contractions whose relevance for results is formally discussed later on. In particular, the various enriched cyclic contractions, which are defined as operating on p non-empty and closed subsets, are the so-called:

- (a) the $\{\theta_n\}_{n=0}^{\infty}$ -enriched $p(\geq 2)$ -cyclic contraction; (b) its weak version (both in normed spaces); and (c) the modified $\{\theta_n\}_{n=0}^{\infty}$ -enriched $p(\geq 2)$ -cyclic contraction, and (d) the $\{\theta_n\}_{n=0}^{\infty}$ - $p(\geq 2)$ -cyclic contraction in the wide sense (both in metric spaces).

Then, the boundedness of the sequences generated for any initial condition in the union of the subsets of the cyclic disposal and the related convergence of distances to the distance in-between adjacent subsets are proved in the same section under certain not very strong constraints on the parameters which define the enriched cyclic contractions. Section 3 considers that the normed or metric spaces are, in addition, uniformly convex Banach spaces and that one of the subsets of the cyclic disposal is convex and closed while the others are just closed. It has been proved that all the sequences converge to a fixed set of best proximity points, one of them within each of the subsets, for any given initial

conditions in the union of the subsets irrespective of the best proximity sets being all unique or not. In this sense, all the sequences under arbitrary initial points in the union of all the subsets converge to a limit cycle formed by one best proximity point per subset. Finally, conclusions end the paper.

2. Convergence of Distances in Some Classes of Enriched Cyclic Contractions and Cyclic Contractions in the Wide Sense

Let $(X, \|\cdot\|)$ be a normed space and let $d : X \times X \rightarrow \mathbf{R}_{0+}$ be the norm-induced metric $d(x, y) = \|x - y\|$ such the $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$, that is, $T(A_i) \subset A_{i+1}$; $\forall i \in \bar{p}$, where $A_i \subset X$ are non-empty closed sets with

$$\text{dist}(A_i, A_{i+1}) = d(A_i, A_{i+1}) = \inf_{x \in A_i, y \in A_{i+1}} \|x - y\| = D; \forall i \in \bar{p}$$

where $\bar{p} = \{1, 2, \dots, p\}$ with $A_{p+1} = A_1$, being a θ -enriched $p(\geq 2)$ -cyclic contraction which satisfies the following condition for some $\theta \in \mathbf{R}$:

$$\|Tx - Ty - \theta(x - y)\| \leq K\|x - y\| + (1 - K)D \quad (1)$$

for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$ and some real constant $K \in [0, 1)$. Note that the above condition implies the weaker one:

$$\|Tx - Ty\| \leq (K + |\theta|)\|x - y\| + (1 - K)D \quad (2)$$

but the converse is not true, in general.

Definition 1. Let $(X, \|\cdot\|)$ be a normed space. A $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$, where A_i are non-empty closed subsets of X with $\text{dist}(A_i, A_{i+1}) = \inf_{x \in A_i, y \in A_{i+1}} \|x - y\| = D; \forall i \in \bar{p}$, is said to be a $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction if it satisfies the following condition:

$$\|T^{n+1}x - T^{n+1}y - \theta_n(T^n x - T^n y)\| \leq K\|T^n x - T^n y\| + (1 - K)D \quad (3)$$

for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$, any $n \in \mathbf{Z}_{0+} = \{z \in \mathbf{Z} : z \geq 0\}$, some $\{\theta_n\}_{n=0}^\infty \subset \mathbf{R}$ and some real constant $K \in [0, 1)$.

Definition 2. Let $(X, \|\cdot\|)$ be a normed space. A $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$ is said to be a weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction if the condition (3) of Definition 1 is replaced with the weaker one:

$$\|T^{n+1}x - T^{n+1}y\| \leq (K + |\theta_n|)\|T^n x - T^n y\| + (1 - K)D \quad (4)$$

for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$.

Note that if the self-mapping T on $\cup_{i \in \bar{p}} A_i$ is an enriched $p(\geq 2)$ -cyclic contraction then it is also a weak $\{\theta_n\}_{n=0}^\infty$ enriched $p(\geq 2)$ -cyclic contraction since $\|T^{n+1}x - T^{n+1}y - \theta_n(T^n x - T^n y)\| \geq \|T^{n+1}x - T^{n+1}y\| - |\theta_n|\|T^n x - T^n y\|$.

Lemma 1. Consider the normed space $(X, \|\cdot\|)$ and let $d : X \times X \rightarrow \mathbf{R}_{0+}$ be the norm-induced metric $d(x, y) = \|x - y\|$ such the $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$ satisfies the weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction condition (4) for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$ and any $n \in \mathbf{Z}_{0+}$, where $A_i; i \in \bar{p}$ are non-empty closed subsets of X . Then, the following properties hold:

- (i) Assume that for some finite $N \in \mathbf{Z}_{0+}$ $\{|\theta_n|\}_{n=N}^\infty$ is sufficiently small. Then, $\{d(T^n x, T^n y)\}_{n=0}^\infty$ is bounded for any finite $d(x, y)$. Also, $\{T^n x\}_{n=0}^\infty$ is bounded for any given $x \in \cup_{i \in \bar{p}} A_i$.
- (ii) $\sup_{n \in \mathbf{Z}_{0+}} d(T^n x, T^n y) \leq M$ if $\max_{1 \leq j \leq N} d(T^j x, T^j y) \leq M$ for some $M = M(x, y, N) (> D) \in \mathbf{R}^+$ and that $\{|\theta_n|\}_{n=N}^\infty$ is sufficiently small such that $\sup_{n \geq N (\in \mathbf{Z}_{0+})} |\theta_n| \leq (1 - K)(1 - D/M)$. Also, $\{T^n x\}_{n=0}^\infty$ is bounded for any given $x \in \cup_{i \in \bar{p}} A_i$.
- (iii) Assume that $D > 0$ and that Property (ii) holds, and define $K_1 \in [K, 1)$ by $K_1 = K + \sup_{n \geq N (\in \mathbf{Z}_{0+})} |\theta_n| \leq 1 - (1 - K)D/M$. Then, $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \leq \frac{1-K}{1-K_1} D$.
- (iv) Assume that $|\theta_n| = \beta_n \alpha^{n-N}$; $n (\geq N) \in \mathbf{Z}_{0+}$ for some real sequence $\{\beta_n\}_{n=N}^\infty \subset [0, \bar{\beta})$ and some $\alpha \in [0, 1)$. Then, there exists the limit $\lim_{n \rightarrow \infty} d(T^n x, T^n y) = D$.
- (v) If the self-mapping T on $\cup_{i \in \bar{p}} A_i$ is a $\{\theta_n\}_{n=0}^\infty$ -enriched $p (\geq 2)$ -cyclic contraction (Definition 1) then Properties [(i)–(iv)] hold.

Proof. Consider the norm-induced metric $d(x, y) = \|x - y\|$. Then,

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\leq (K + |\theta_n|)d(T^n x, T^n y) + (1 - K)D \\ &= Kd(T^n x, T^n y) + [|\theta_n|d(T^n x, T^n y) + (1 - K)D] \end{aligned} \quad (5)$$

Let $N \in \mathbf{Z}_{0+}$ be arbitrarily fixed and let $M = M(N) > D (\in \mathbf{R}_+)$ be such that $\max_{1 \leq j \leq N} d(T^j x, T^j y) \leq M$. Then,

$$\begin{aligned} d(T^{N+1}x, T^{N+1}y) &\leq Kd(T^N x, T^N y) + [|\theta_N|d(T^N x, T^N y) + (1 - K)D] \\ &\leq (K + |\theta_N|)M + (1 - K)D \leq M \end{aligned} \quad (6)$$

if $1 - K - |\theta_N| \geq \frac{1-K}{M} D$, that is, if $|\theta_N| \leq (1 - K)(1 - D/M)$. It follows by complete induction that $\sup_{n \in \mathbf{Z}_{0+}} d(T^n x, T^n y) \leq M$ provided that $\sup_{n \geq N (\in \mathbf{Z}_{0+})} |\theta_n| \leq (1 - K)(1 - D/M)$.

Now, it is proved that $\{T^{pn}x\}_{n=0}^\infty$ is bounded for any $x \in \cup_{i \in \bar{p}} A_i$. Proceed by contradiction by assuming that $\{T^{pn}x\}_{n=0}^\infty$ is unbounded for some $x \in \cup_{i \in \bar{p}} A_i$ while $d(T^{pn}x, T^{pn}y)$ has been proved to be bounded for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ and any $i \in \bar{p}$. Take $x \in A_i$ for some $i \in \bar{p}$ such that, for any given $C \in \mathbf{R}_+$, there exists some $N = N(C, x) \in \mathbf{Z}_{0+}$ such that, since $\{T^{pn}x\}_{n=0}^\infty$ is unbounded, $d(T^p x, T^{pN-1}x) \leq C < d(T^p x, T^{pN+1}x)$ with $\{T^{pN \pm 1}x\}_{n=0}^\infty \subset A_{i+1}$. The constant C might be chosen to be dependent on x , if suited. Assume that, in accordance with the smallness constraint of Property (ii) $\theta = \sup_{n \geq N (\in \mathbf{Z}_{0+})} |\theta_n| \leq (1 - K)(1 - D/M)$ and define $K_1 = K + \sup_{n \geq N (\in \mathbf{Z}_{0+})} |\theta_n|$. Then, one has

$$\begin{aligned} C &< d(T^p x, T^{pN+1}x) \leq K_1^p d(x, T^{p(N-1)+1}x) + (1 - K_1^p)D \\ &\leq K_1^p (d(x, T^{pN-1}x) + d(T^{pN-1}x, T^{p(N-1)+1}x)) + (1 - K_1^p)D \\ &\leq K_1^p (C + d(T^{pN-1}x, T^{pN-p+1}x)) + (1 - K_1^p)D \\ &\leq K_1^p (C + K_1^{p(N-1)} d(Tx, T^{p-1}x) + (1 - K_1^{p(N-1)})D) + (1 - K_1^p)D \end{aligned}$$

so that the subsequent inequalities hold:

$$\begin{aligned} (1 - K_1^p)C &\leq K_1^{pN} d(Tx, T^{p-1}x) + (K_1^p - K_1^{pN} + 1 - K_1^p)D \\ &\leq K_1^{pN} d(Tx, T^{p-1}x) + (1 - K_1^{pN})D \end{aligned}$$

then $C \leq (1 - K_1^p)^{-1} [K_1^{pN} (d(Tx, T^{p-1}x) - D) + D]$. However, this constraint contradicts the choice $C > (1 - K_1^p)^{-1} [K_1^p (d(Tx, T^{p-1}x) - D) + D]$. As a result, $\{T^{np}x\}_{n=0}^\infty$ is bounded for any $x \in \cup_{i \in \bar{p}} A_i$ and, from the contraction condition, the sequences $\{T^{np+j}x\}_{n=0}^\infty$ for $j \in \bar{p} - 1$ cannot be unbounded either. This also implies that $\{T^n x\}_{n=0}^\infty$ is bounded for any $x \in \cup_{i \in \bar{p}} A_i$ under the smallness constraint of the enrichment condition of Property (ii). The parallel proof under the more general smallness constraint for sufficiently large iteration steps is similar by taking the initialization of sequences after the finite step where the corresponding parallel smallness condition holds. Properties [(i) and (ii)] have been proved. Now, to prove Property (ii) note that

$$K_1 \leq 1 - D/M + KD/M = 1 - (1 - K)D/M = \frac{M - (1 - K)D}{M} < 1$$

since $D > 0$. Thus, we have

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\leq K_1 d(T^n x, T^n y) + (1 - K)D = K_1 d(T^{n-1}x, T^{n-1}y) + (1 - K_1)D + (K_1 - K)D \\ &\leq K_1 [K_1 d(T^{n-2}x, T^{n-2}y) + (1 - K_1)D + (K_1 - K)D] + (1 - K_1)D + (K_1 - K)D \\ &\leq K_1^2 d(T^{n-2}x, T^{n-2}y) + (1 - K_1^2)D + (1 + K_1)(K_1 - K)D \\ &\leq K_1^2 [K_1 d(T^{n-3}x, T^{n-3}y) + (1 - K_1)D + (K_1 - K)D] + (1 - K_1^2)D + (1 + K_1)(K_1 - K)D \\ &\leq K_1^3 d(T^{n-3}x, T^{n-3}y) + (1 - K_1^3)D + (1 + K_1 + K_1^2)(K_1 - K)D \\ &\leq K_1^{n+1-N} d(T^N x, T^N y) + (1 - K_1^{n+1-N})D + (K_1 - K)D \left(\sum_{j=0}^{n-N} K_1^j \right) \end{aligned} \quad (7)$$

$$\begin{aligned} &\leq K_1^{n+1-N} d(T^N x, T^N y) + (1 - K_1^{n+1-N})D + \frac{(1 - K_1^{n+1-N})(K_1 - K)}{1 - K_1} D \\ &\leq K_1^{n+1-N} \left(d(T^N x, T^N y) - \frac{1 - K}{1 - K_1} D \right) + \frac{1 - K}{1 - K_1} D \\ &\leq \left(1 - \frac{(1 - K)D}{M} \right)^{n+1-N} \left(M - \frac{1 - K}{1 - K_1} D \right) + \frac{1 - K}{1 - K_1} D; \quad n(\geq N) \in \mathbb{Z}_0^+ \end{aligned} \quad (8)$$

and

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \leq \frac{1 - K}{1 - K_1} D.$$

Property (iii) has been proved. To prove Property (iv), note that if $K_2 = K + |\theta_N| = K + \bar{\beta}\alpha^N$, then Property (iv) follows, since

$$D \leq \sup_{N+m \geq n \geq N} d(T^n x, T^n y) \leq \frac{1 - K}{1 - K - \bar{\beta}\alpha^{N+m}} D;$$

$$\lim_{n \rightarrow \infty} d(T^n x, T^n y) = \limsup_{n \rightarrow \infty} d(T^n x, T^n y) = \limsup_{m \rightarrow \infty} \sup_{N+m \geq n \geq N} d(T^n x, T^n y) = \frac{1 - K}{1 - K} D = D.$$

Property (v) is obvious, since if T is a $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction (then it is also a weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction. $\square$

Example 1. Assume that T is a linear operator on a normed space $(X, \|\cdot\|)$ and let $d : X \times X \rightarrow \mathbb{R}_{0+}$ be the norm-induced metric $d(x, y) = \|x - y\|$ for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ and any $i \in \bar{p}$, where A_i for $i \in \bar{p}$ are non-empty closed subsets of X which intersect. Assume that the restriction of T to $\cup_{i \in \bar{p}} A_i$ is, in addition, a θ -enriched $p(\geq 2)$ -cyclic contraction, that is,

$$\|(T - \theta I)x - (T - \theta I)y\| \leq K\|x - y\| \quad (9)$$

for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$, and for some real constant $K \in [0, 1)$. Take $y = Tx \in A_{i+1}$ for each $x \in A_i$; $\forall i \in \bar{p}$. Then, if I is the identity operator on X , we have the following:

$$\begin{aligned} \|(T - \theta I)x - (T - \theta I)Tx\| &= \|Tx - \theta x - T^2x + \theta Tx\| = \|T^2x + \theta x - Tx - \theta Tx\| \\ &= \|(T - I)(T - \theta I)x\| \leq K\|(I - T)x\|; \forall x \in \cup_{i \in \bar{p}} A_i \end{aligned} \quad (10)$$

which holds for any $x \in X$, then also for any $x \in \cup_{i \in \bar{p}} A_i$, if $\|(T - I)(T - \theta I)\| \leq K\|(I - T)\|$, which is guaranteed if $\|T - \theta I\| \leq K$.

Proposition 1. In Example 1, the following properties hold:

- (i) The linear operator T is one-to-one and it has a closed range if and only if $\mu(T)$, the minimum modulus of T , is positive, that is, if and only if $\mu(T) = \inf_{\|x\|=1} \|Tx\| = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|} > 0$.
Furthermore, if T is invertible then $\mu(T) = \|T^{-1}\|^{-1}$.
- (ii) The linear operator $(T - \theta I)$ is one-to-one and of closed range if and only if $\mu(T - \theta I) > 0$.
If it is furthermore invertible, then $\mu(T - \theta I) = \|(T - \theta I)^{-1}\|^{-1}$ and, if T is invertible with $|\theta| < \frac{1}{\|T^{-1}\|} = \frac{1}{\mu^{-1}(T)} = \mu(T)$ then $\mu\|(T - \theta I)^{-1}\| \leq \frac{1}{\mu(T) - |\theta|}$.
- (iii) If Property (ii) holds, then $|\theta| < \mu(T) \leq K + |\theta|$.
- (iv) Assume that, for some $K_1 \in (0, 1)$, $\|T^n\| \leq K_1^n / (K\|I - T\|)$ for $n \in \mathbf{Z}_{0+}$. Then, $\|(T - \theta I)T^n x - (T - \theta I)T^{n+1}x\| \rightarrow 0$ as $n \rightarrow \infty$ for any finite $x \in \cup_{i \in \bar{p}} A_i$.

Proof. Property (i) follows from [44] (Lemma 2.5.2). See also [45]. Property (ii) follows from Property (i) and Banach's perturbation lemma [46–48], since

$$\mu^{-1}(T - \theta I) = \|(T - \theta I)^{-1}\| \leq \frac{\|T\|^{-1}}{1 - |\theta|\|T^{-1}\|} = \frac{\mu^{-1}(T)}{1 - |\theta|\mu^{-1}(T)} = \frac{1}{\mu(T) - |\theta|} \quad (11)$$

Property (iii) follows from Property (ii) since $(T - \theta I)$ is invertible and T is an invertible θ -enriched $p(\geq 2)$ -cyclic contraction with contractive constant K and $|\theta| < \mu(T)$ so that $\mu(T) - |\theta| \leq |\mu(T) - \theta| \leq \mu(T - \theta I) \leq \|T - \theta I\| \leq K$.

$$\text{Since } \mu(T - \theta I) = \inf_{\|x\|=1} \|(T - \theta I)x\| \leq \sup_{\|x\|=1} \|(T - \theta I)x\| = \|T - \theta I\|.$$

Property (iv) follows, since if $K\|I - T\| \leq K_1 (\in (0, 1))$, $\|T\| \leq K_1^2 / (K\|I - T\|)$, ..., $\|T^n\| \leq K_1^n / (K\|I - T\|)$ for $n \in \mathbf{Z}_{0+}$. Equivalently, $\|I - T\| \leq K^{-1} \inf_{n \in \mathbf{Z}_{0+}} K_1^n / \|T^n\|$, which is

also equivalent to $\|I - T\| \leq K^{-1} \liminf_{n \rightarrow \infty} K_1^n / \|T^n\|$ since T is bounded from the condition $K\|I - T\| \leq K_1$ and the fact that $K_1^n \rightarrow 0$ as $n \rightarrow \infty$. Then, by using recursion in (10), one gets the following:

$$\begin{aligned} \|(T - \theta I)T^n x - (T - \theta I)T^{n+1}x\| &= \|(T - I)(T - \theta I)T^n x\| \\ &\leq K\|(I - T)T^n x\| \leq K_1^n \|x\|; \forall x \in \cup_{i \in \bar{p}} A_i \end{aligned} \quad (12)$$

and the property follows. $\square$

Retake a modified Definition 1 for a metric space, which is not necessarily a normed space, by taking into account the signs or null value of the elements of the sequence $\{\theta_n\}_{n=0}^{\infty}$ and their eventual fixed upper bounds as follows.

Definition 3. Let (X, d) be a metric space. A $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$, where A_i are non-empty closed subsets of X , is said to be a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction if it satisfies the following condition:

$$\left| d(T^{n+1}x, T^{n+1}y) + \theta_n d(T^n x, T^n y) \right| \leq K d(T^n x, T^n y) + (1 - K)D \quad (13)$$

for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}$, some sequence $\{\theta_n\}_{n=0}^\infty \subset \mathbf{R}$ and some real constant $K \in [0, 1)$.

Note the following for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, any $i \in \bar{p}; \forall n \in \mathbf{Z}_{0+}$:

(a) If $0 \leq \theta_n < K$ then

$$d(T^{n+1}x, T^{n+1}y) \leq (K - \theta_n)d(T^n x, T^n y) + (1 - K)D \quad (14)$$

(b) If $-\frac{d(T^{n+1}x, T^{n+1}y)}{d(T^n x, T^n y)} \leq \theta_n < 0$ then we reach a similar contractive result as the above one, since $\theta_n < 0$, so that

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\leq (K + |\theta_n|)d(T^n x, T^n y) + (1 - K)D \\ &= (K - \theta_n)d(T^n x, T^n y) + (1 - K)D \end{aligned} \quad (15)$$

Note that by defining $K_n = K - \theta_n$ with $-d(T^{n+1}x, T^{n+1}y)/d(T^n x, T^n y) \leq \theta_n < K$, we get the following:

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\leq K_n d(T^n x, T^n y) + (1 - K_n)D - \theta_n D \\ &\leq \left(\prod_{j=0}^n K_j \right) d(T^j x, T^j y) + D \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] (1 - K + \theta_i) \right) \\ &= \left(\prod_{j=0}^n K_j \right) d(T^j x, T^j y) + \left(1 - \left(\prod_{j=0}^n K_j \right) \right) D - D \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] \theta_i \right); \forall n \in \mathbf{Z}_{0+} \end{aligned} \quad (16)$$

Lemma 2. Let (X, d) be a metric space and assume that a $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$, where A_i are non-empty closed subsets of X , is a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction which satisfies the condition (13) such that there exists $\lim_{n \rightarrow \infty} \left(\prod_{j=0}^n K_j \right) = 0$, where $K_n = K - \theta_n$; $n \in \mathbf{Z}_{0+}$, and some $K \in [0, 1)$ and the sequence $\{\theta_n\}_{n=0}^\infty$ satisfies the condition $-d(T^{n+1}x, T^{n+1}y)/d(T^n x, T^n y) \leq \theta_n < K$; $n \in \mathbf{Z}_{0+}$, for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$. Then, one has the following properties:

(i)

$$\begin{aligned} D &\leq \limsup_{n \rightarrow \infty} d(T^{n+1}x, T^{n+1}y) \leq \left(1 + \limsup_{n \rightarrow \infty} \left(-\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] \theta_i \right) \right) D \\ &= \left(1 - \liminf_{n \rightarrow \infty} \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] \theta_i \right) \right) D \end{aligned} \quad (17)$$

and if, in addition, the sequence $\{\theta_n\}_{n=0}^\infty$ is such that it exists $\lim_{n \rightarrow \infty} \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] \theta_i \right) = 0$ then $\lim_{n \rightarrow \infty} d(T^{n+1}x, T^{n+1}y) = D$.

(ii) The sequence $\{T^n x\}_{n=0}^\infty$ is bounded for any given $x \in \cup_{i \in \bar{p}} A_i$.

Proof. The proof of Property (i) is direct from (16). To prove Property (ii), we proceed by contradiction arguments. Assume that, for some $x \in \cup_{i \in \bar{p}} A_i$, $\{T^{n_p} x\}_{n=0}^\infty$ is unbounded so that there is some strictly increasing sequence $\{T^{p_{m_k}} x\}_{k=0}^\infty$ for some strictly increasing sequence of non-negative integers $\{n_k\}_{k=0}^\infty$ so that $\{|T^{p_{m_k}} x|\}_{k=0}^\infty \rightarrow +\infty$. Then, we have from (16) that

$$\begin{aligned}
d(T^{pn_k}x, x) &< d(T^{pn_k+j_k+1}x, x) \leq d(T^{pn_k-\downarrow_k+1}(T^{pn_k+j_k+\downarrow_k}x), T^{pn_k-\downarrow_k+1}x) + d(T^{pn_k-\downarrow_k+1}x, x) \\
&\leq \left(\prod_{j=pn_k-\downarrow_k}^{pn_k+j_k} K_j\right) d(T^{pn_k-\downarrow_k}(T^{pn_k+j_k+\downarrow_k}x), T^{pn_k-\downarrow_k}x) + d(T^{pn_k-\downarrow_k+1}x, x) \\
&+ \left(1 - \left(\prod_{j=pn_k-\downarrow_k}^{pn_k+j_k} K_j\right)\right) D - D\left(\sum_{i=pn_k}^{pn_k+j_k} \left[\prod_{j=i}^{pn_k+j_k} K_{j+1}\right] \theta_i\right) + d(T^{pn_k-\downarrow_k+1}x, x)
\end{aligned} \tag{18}$$

for some strictly increasing positive integer sequences $\{j_k\}_{k=0}^\infty$ and $\{\downarrow_k\}_{k=0}^\infty$ such that $\{T^{pn_k+j_k+\downarrow_k+1}x\}_{k=0}^\infty \subset \{T^{pn_k+j_k+1}x\}_{k=0}^\infty \subset \{T^{pn_k+1}x\}_{k=0}^\infty$ then,

$$\begin{aligned}
d(T^{pn_k}x, x) - d(T^{pn_k-\downarrow_k+1}x, x) &< \left(\prod_{j=pn_k-\downarrow_k}^{pn_k+j_k} K_j\right) d(T^{pn_k-\downarrow_k}(T^{pn_k+j_k+\downarrow_k}x), T^{pn_k-\downarrow_k}x) \\
&+ \left(1 - \left(\prod_{j=pn_k-\downarrow_k}^{pn_k+j_k} K_j\right)\right) D - D\left(\sum_{i=pn_k}^{pn_k+j_k} \left[\prod_{j=i}^{pn_k+j_k} K_{j+1}\right] \theta_i\right)
\end{aligned} \tag{19}$$

and, by taking $k \rightarrow \infty$, $\downarrow_k \rightarrow \infty$ with $(n_k - \downarrow_k) \rightarrow \infty$, it follows, under the conditions of Property (i), that $\left(\prod_{j=pn_k}^{pn_k+j_k} K_j\right) \rightarrow 0$ and $\left(\sum_{i=pn_k}^{pn_k+j_k} \left[\prod_{j=i}^{pn_k+j_k} K_{j+1}\right] \theta_i\right) \rightarrow 0$, and from the previous claimed assumption that $\{T^{pn_k}x\}_{k=0}^\infty \rightarrow +\infty$ to establish a contradiction, one gets $0 < (d(T^{pn_k}x, x) - d(T^{pn_k-\downarrow_k+1}x, x)) \rightarrow +\infty$. Then, the contradiction $+\infty < D < +\infty$ follows by taking limits in (19) and Property (i) if $\{T^n x\}_{n=0}^\infty$ is unbounded. As a result, $\{T^n x\}_{n=0}^\infty$ is bounded for any given $x \in \cup_{i \in \overline{p}} A_i$ and Property (ii) is proved. $\square$

Remark 1. Note in Lemma 2 that $\lim_{n \rightarrow \infty} \left(\prod_{j=0}^n K_j\right) = 0$ if $\limsup_{N \rightarrow \infty, (n-N) \rightarrow \infty} \{K_j\}_{j=N}^{N+n} \subset [0, 1)$.

Remark 2. The condition $\lim_{n \rightarrow \infty} \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_i\right) = 0$ in Lemma 2 is achievable with changes of sign in the sequence $\{\theta_n\}_{n=0}^\infty$ under “ad hoc” extra conditions for the limits of some of its convergent subsequences. For instance, such a condition holds if

$$\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{2i} \rightarrow \alpha, \quad \sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{2i+1} \rightarrow -\alpha \quad \text{as } n \rightarrow \infty \text{ for some arbitrary } |\alpha| < \infty;$$

or, if

$$\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{4i} \rightarrow \alpha, \quad \sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{2i(i \neq 4)} \rightarrow \beta, \quad \sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{4i+1} \rightarrow -\alpha,$$

$$\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1}\right] \theta_{2i+1(i \neq 4)} \rightarrow -\beta \quad \text{as } n \rightarrow \infty \text{ for some arbitrary } |\alpha| < \infty, |\beta| < \infty.$$

Definition 4. Let (X, d) be a metric space and let A_i be a set of non-empty closed subsets of X . Then, a $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \overline{p}} A_i$ is said to be a $\{\theta_n\}_{n=0}^\infty$ $p(\geq 2)$ -cyclic contraction in the wide sense if it satisfies the following conditions for some $K \in \mathbf{R}$:

(C.1)

$$d(T^{n+1}x, T^{n+1}y) \leq (K + \theta_n)d(T^n x, T^n y) + (1 - K - \theta_n)D \tag{20}$$

for any given $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ and any $i \in \overline{p}$.

(C.2) $\{\theta_n\}_{n=0}^\infty \subset (-K, \overline{K})$ for some finite $\overline{K} \in \mathbf{R}_+$.

(C.3) There is a strictly increasing sequence $\{N_k\}_{k=0}^\infty \subset \mathbf{Z}_{0+}$ for some finite N_0 such that $\sup_{k \in \mathbf{Z}_{0+}} (N_{k+1} - N_k) < +\infty$ and, for some real constant $\hat{K} \in [0, 1)$, $\hat{K}_k = \prod_{j=N_k}^{N_{k+1}-1} [K + \theta_j] \leq \hat{K}$.

Remark 3. Note that, compared to Definition 3, related to a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction, now it is not required via Definition 4 that $d(T^{n+1}x, T^{n+1}y) \geq |\theta_n|d(T^n x, T^n y)$ if $\theta_n < 0$ for any $n \in \mathbf{Z}_{0+}$ while it is not required that $K < 1$, either. However, it is required for the cyclic mapping to be globally contractive over a set of steps, which is addressed through the condition C.3, but it is not necessarily required for the mapping to be step by step contractive.

Lemma 3. Let (X, d) be a metric space and consider a $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$, where A_i are non-empty closed subsets of X , which is a $\{\theta_n\}_{n=0}^\infty$ - $p(\geq 2)$ -cyclic contraction in the wide sense. Then, $\lim_{n \rightarrow \infty} d(T^{n+1}x, T^{n+1}y) = D$. Also, the sequence $\{T^n x\}_{n=0}^\infty$ is bounded for any given $x \in \cup_{i \in \bar{p}} A_i$.

Proof. Note from Definition 3 that for any $k \in \mathbf{Z}_{0+}$ for any given $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ and for any $i \in \bar{p}$:

$$\begin{aligned} D &\leq d(T^{N_{k+1}}x, T^{N_{k+1}}y) \leq \hat{K}_k d(T^{N_k}x, T^{N_k}y) + (1 - \hat{K}_k)D \\ D &\leq d(T^{N_{k+2}}x, T^{N_{k+2}}y) \leq \hat{K}_{k+1}(\hat{K}_k d(T^{N_k}x, T^{N_k}y) + (1 - \hat{K}_k)D) + (1 - \hat{K}_{k+1})D \\ &= \hat{K}_{k+1}\hat{K}_k d(T^{N_k}x, T^{N_k}y) + \hat{K}_{k+1}(1 - \hat{K}_k)D + (1 - \hat{K}_{k+1})D \\ &= \hat{K}_{k+1}\hat{K}_k d(T^{N_k}x, T^{N_k}y) + (1 - \hat{K}_{k+1}\hat{K}_k)D \\ &= \hat{K}_{k+1}\hat{K}_k(d(T^{N_k}x, T^{N_k}y) - D) + D \end{aligned}$$

$$D \leq d(T^{N_n}x, T^{N_n}y) \leq \left(\prod_{k=0}^{n-1} [\hat{K}_k]\right) (d(T^{N_0}x, T^{N_0}y) - D) + D; \quad \forall n \in \mathbf{Z}_{0+} \quad (21)$$

From Definition 4 (C.3), $\left(\prod_{k=0}^{n-1} [\hat{K}_k]\right) \rightarrow 0$ as $n \rightarrow \infty$. Then, $d(T^{N_n}x, T^{N_n}y) \rightarrow D$ as $n \rightarrow \infty$. On the other hand, this also implies from (20) and (21) that,

$$\begin{aligned} \limsup_{n \rightarrow \infty} (d(T^{N_n+1}x, T^{N_n+1}y) - (K + \theta_n)d(T^{N_n}x, T^{N_n}y) - (1 - K - \theta_n)D) \\ = \limsup_{n \rightarrow \infty} (d(T^{N_n+1}x, T^{N_n+1}y) - (K + \theta_n)D - (1 - K - \theta_n)D) \\ \Rightarrow \left[0 \leq \limsup_{n \rightarrow \infty} (d(T^{N_n+1}x, T^{N_n+1}y) - D) \leq 0\right] \end{aligned} \quad (22)$$

so that it exists the limit $\lim_{n \rightarrow \infty} d(T^{N_n+1}x, T^{N_n+1}y) = D$. Proceed by complete induction by assuming that

$\lim_{n \rightarrow \infty} d(T^{N_n+j-1}x, T^{N_n+j-1}y) = D$ for all $i \in \mathbf{Z}_+$ and a given $j \in \mathbf{Z}_+$ > 2 . Then, from (22), one gets

$$\limsup_{n \rightarrow \infty} (d(T^{N_n+j}x, T^{N_n+j}y) - (K + \theta_n)d(T^{N_n+j-1}x, T^{N_n+j-1}y) - (1 - K - \theta_n)D) \quad (23)$$

which implies that $0 \leq \limsup_{n \rightarrow \infty} (d(T^{N_n+j}x, T^{N_n+j}y) - D)$ so that there exists the limit $\lim_{n \rightarrow \infty} d(T^{N_n+j}x, T^{N_n+j}y) = D$ for any $j \in \mathbf{Z}_+$ as claimed. On the other hand, the boundedness of the sequence $\{T^n x\}_{n=0}^\infty$ for any given $x \in \cup_{i \in \bar{p}} A_i$ is proved under similar arguments as those used in the proof of Lemma 2(ii). $\square$

Note that the condition C.3 holds, in particular, if $N_k = kN$ for all $k \in \mathbf{Z}_{0+}$ and some $N \in \mathbf{Z}_+$ and, if for some real constant $\hat{K} \in [0, 1)$, $\hat{K}_k = \prod_{j=kN}^{(k+1)N-1} [K + \theta_j] \leq \hat{K}$. Note also that, in this case, if $N = 1$ and $\{\theta_n\}_{n=0}^\infty \equiv 0$ then the self-mapping T on $\cup_{i \in \bar{p}} A_i$ is a $p(\geq 2)$ -cyclic contraction since $\hat{K}_k = K; \forall k \in \mathbf{Z}_{0+}$.

Remark 4. It is now seen that $\{\theta_n\}_{n=0}^\infty$ - $p(\geq 2)$ -cyclic contractions in the wide sense are non-expansive while modified and weak enriched cyclic contractions can be locally expansive.

If T on $\cup_{i \in \bar{p}} A_i$ is a $\{\theta_n\}_{n=0}^\infty$ - $p(\geq 2)$ -cyclic contraction in the wide sense then it is always non-expansive as it is seen as follows. Assume that this is not the case so that it is expansive, then for some $i \in \bar{p}$ and some $n \in \mathbf{Z}_{0+}$, there is a pair $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ such that $d(T^{n+1}x, T^{n+1}y) > d(T^n x, T^n y)$. Then, one has from (17) that

$$D \leq d(T^n x, T^n y) < d(T^{n+1}x, T^{n+1}y) \leq (K + \theta_n)d(T^n x, T^n y) + (1 - K - \theta_n)D$$

which implies that $(1 - K - \theta_n)d(T^n x, T^n y) < (1 - K - \theta_n)D$ so that $d(T^n x, T^n y) < D$, a contradiction.

Now, if T on $\cup_{i \in \bar{p}} A_i$ is a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction which could be potentially expansive, then we have from (13) and (14) that, for some $i \in \bar{p}$ and some $n \in \mathbf{Z}_{0+}$, there is a pair $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ such that

$$d(T^n x, T^n y) < d(T^{n+1}x, T^{n+1}y) \leq (K - \theta_n)d(T^n x, T^n y) + (1 - K)D$$

$$[(1 - K + \theta_n)D \leq (1 - K + \theta_n)d(T^n x, T^n y) < (1 - K)D] \Rightarrow [\theta_n < 0]$$

if $D > 0$. However, it is admitted also for $\{\theta_n\}_{n=0}^\infty$ to take non-negative values in Definition 3 fulfilling $0 \leq \theta_n < K$, see (14). As a conclusion, a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction is non-expansive although it could be locally expansive if $\theta_n < 0$.

On the other hand, it follows from (4) in Definition 2 that if T is a weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction, then it can be locally expansive as it might be deduced under a similar reasoning.

3. Best Proximity Points and Convergence of Sequences in Uniformly Convex Banach Spaces

The above section has addressed the convergence of distances to the distances between adjacent closed subsets in normed or metric spaces as well as the boundedness of sequences generated by several types on enriched cyclic contractions. The number of adjacent subsets in the cyclic disposal can be equal or greater than two and it is not assumed that such subsets are bounded. Some further results are formulated in this section if the considered metric spaces are uniformly convex Banach spaces under possible convexity of some of the subsets in the cyclic disposal.

Two essential concepts which are referred to in the sequel are that of best proximity point and that of uniformly convex Banach space that we now describe succinctly and with some intuitive subtract to fix the main following ideas and obtained results.

If two non-empty subsets of a metric space have disjoint closures, then best proximity points are the pairs of points, one located in the closure of each subset, that are as close to each other as possible. Those points are located, in particular, in the respective mentioned subsets if such subsets are closed. If the subsets are closed and, furthermore, intersect, then the best proximity points are at zero distance, so that they are coincident and they can be fixed points of certain mappings defined on such subsets like, for instance, contractive self-mappings or contractive cyclic self-mappings.

On the other hand, a real normed space $(X, \|\cdot\|)$ is said to be strictly convex if, for all $x, y \in X$, with $\|x\| = 1, \|y\| = 1$ and $x \neq y$, $\|x + y\|/2 < 1$. The space $(X, \|\cdot\|)$ is said to be uniformly convex, or uniformly rotund, if for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\|/2 \leq 1 - \delta$ holds. In addition, if $(X, \|\cdot\|)$ is a Banach space then it is said to be a uniformly convex Banach space. It holds that every uniformly convex Banach space is strictly convex and reflexive, but the converses are not true, in general. Intuitively, in a uniformly convex Banach space, every sequence of

points that are uniformly close to the boundary of the unit ball has a midpoint that is close to the interior of the ball.

The above explanations immediately lead to the conclusion that best proximity points may exist either in simple metric spaces or in more sophisticated spaces like normed spaces or uniformly convex Banach spaces.

The essential part of the next result is concerned with weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contractions in the case when the metric space is a uniformly convex Banach space and, at least one of the subsets of the cyclic disposal is closed and convex while the other ones are just supposed to be closed.

Lemma 4. Let A_i for $i \in \bar{p}$ be p non-empty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, such that A_j is convex for some $j \in \bar{p}$. Then, the following properties hold:

- (i) Let $\{x_n\}_{n=0}^\infty$ and $\{z_n\}_{n=0}^\infty$ be sequences in A_j and $\{y_n\}_{n=0}^\infty$ be a sequence in A_{j+1} such that $\|y_n - z_n\| \rightarrow D$ as $n \rightarrow \infty$ and that, for each positive real constant ε , there exists $N_0 \in \mathbb{Z}_{0+}$ such that $\|x_m - y_n\| \leq D + \varepsilon; \forall n, m (> n \geq N_0) \in \mathbb{Z}_{0+}$. Then, there exists $N_1 \in \mathbb{Z}_{0+}$ such that $\|x_m - z_n\| \leq D + \varepsilon; \forall n, m (> n \geq N_1) \in \mathbb{Z}_{0+}$.
- (ii) Let $\{x_n\}_{n=0}^\infty$ and $\{z_n\}_{n=0}^\infty$ be sequences in A_j and $\{y_n\}_{n=0}^\infty$ be a sequence in A_{j+1} such that $\|y_n - z_n\| \rightarrow D$ and $\|y_n - x_n\| \rightarrow D$ as $n \rightarrow \infty$. Then, $\|z_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) Let the self-mapping T on $\cup_{i \in \bar{p}} A_i$ be a weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction which satisfies (4) under the condition of Lemma 1(iv). Let $\{x_{np}\}_{n=0}^\infty$ and $\{z_{np}\}_{n=0}^\infty$ be sequences in A_j and $\{y_{np}\}_{n=0}^\infty$ be a sequence in A_{j+1} such that $x_{np} = T^{np}x$, $z_{(n+m)p} = T^{(n+m)p}z$, $y_{np+1} = Tx_{np}$ (or $y_{np+1} = Tz_{(n+m)p}$) for any given $x, z \in A_j$ and any $n, m \in \mathbb{Z}_{0+}$. Then, $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbb{Z}_{0+}$.
- (iv) Under the conditions of Property (iii), $\left\{ \|T^j x_{np} - T^j z_{(n+m)p}\| \right\}_{n=0}^\infty \rightarrow 0; \forall j \in \overline{p-1} \cup \{0\}, \forall m \in \mathbb{Z}_{0+}$.
- (v) Properties (iii) and (iv) also hold if $x, z \in \cup_{i \in \bar{p}} A_i$.
- (vi) If the self-mapping T on $\cup_{i \in \bar{p}} A_i$ is a $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction (Definition 1) then Properties [(iv) and (v)] hold as well.

Proof. Property (i) and Property (ii) follow, respectively, as direct extensions of Lemma 3.7 and Lemma 3.8 of [1], since A_j is convex and closed and A_{j+1} is closed and $(X, \|\cdot\|)$ is a uniformly convex Banach space. To prove Property (iii), note that, from Lemma 1(iv), $\|y_{np+1} - z_{(n+m)p}\| \rightarrow D$ and $\|y_{np+1} - x_{np}\| \rightarrow D$ as $n \rightarrow \infty$ for any $x, z \in A_j$ and any $m \in \mathbb{Z}_{0+}$. This implies, from Property (ii), that $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbb{Z}_{0+}$ since A_j is convex and A_{j+1} is closed. Property (iii) has been proved.

Now, note from (4) that, since $D \leq \|T^n x - T^n y\|$ for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ and any $i \in \bar{p}$, we have

$$\begin{aligned} \|T^{n+1}x - T^{n+1}y\| &\leq (K + 1 + |\theta_n| - K) \|T^n x - T^n y\| \leq \left(1 + \sup_{n \in \mathbb{Z}_{0+}} |\theta_n|\right) \|T^n x - T^n y\| \\ &\leq (1 + \bar{\theta}) \|T^n x - T^n y\|; \forall n \in \mathbb{Z}_{0+}. \end{aligned}$$

Then, all the composed self-mappings $T^n : \cup_{i \in \bar{p}} A_i \rightarrow \cup_{i \in \bar{p}} A_i$ for $n \in \mathbb{Z}_{0+}$ are Lipschitz-continuous, with Lipschitz constant $(1 + \bar{\theta})$, then they are continuous as well. As a result, by taking $n = 1$, the weak $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction $T : \cup_{i \in \bar{p}} A_i \rightarrow \cup_{i \in \bar{p}} A_i$ is an everywhere continuous mapping in its definition domain. Now, since $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbb{Z}_{0+}$ from Property (iii) and since $T^j : \cup_{i \in \bar{p}} A_i \rightarrow \cup_{i \in \bar{p}} A_i$ are everywhere continuous for $j \in \bar{p}$, then $\left\{ \|T^j x_{np} - T^j z_{(n+m)p}\| \right\}_{n=0}^\infty \rightarrow 0; \forall j \in \overline{p-1} \cup \{0\}, \forall m \in \mathbb{Z}_{0+}$, which proves Property

(iv). Property (v) follows directly from Properties (iii) and (iv) since, if $x, z \in \cup_{i \in \bar{p}} A_i$ then, it is obvious that $x \in A_k$ and $z \in A_{\downarrow}$ for some $k, \downarrow \in \bar{p}$, since all the subsets A_i for $i \in \bar{p}$ are closed, so that, the conclusions from those proofs are identical by replacing $x \rightarrow \bar{x} = T^{j-k}x$ if $k(\in \mathbf{Z}_{0+}) \leq j$ and $x \rightarrow \bar{x} = T^{j+p-k}x$ if $p \geq k(\in \mathbf{Z}_+) > j$ $z \rightarrow \bar{z} = T^{j-\downarrow}x$; and if $k(\in \mathbf{Z}_{0+}) \leq \downarrow$ and $x \rightarrow \bar{x} = T^{j+p-\downarrow}x$ if $p \geq k(\in \mathbf{Z}_+) > \downarrow$.

Property (v) follows directly from Properties (iii) and (iv) and Lemma 1(v). $\square$

In the same way, we have the following result if the mapping T on $\cup_{i \in \bar{p}} A_i$ is a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction in a uniformly convex Banach space:

Lemma 5. Let A_i for $i \in \bar{p}$ be p non-empty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, such that A_j is convex for some $j \in \bar{p}$ and let the self-mapping T on $\cup_{i \in \bar{p}} A_i$ be a modified $\{\theta_n\}_{n=0}^\infty$ -enriched $p(\geq 2)$ -cyclic contraction which satisfies (13) subject to the following:

- (1) There exists $\lim_{n \rightarrow \infty} \left(\prod_{j=0}^n K_j \right) = 0$, where $K_n = K - \theta_n$; $n \in \mathbf{Z}_{0+}$, and some $K \in [0, 1)$ and the sequence $\{\theta_n\}_{n=0}^\infty$ satisfies the condition $-d(T^{n+1}x, T^{n+1}y)/d(T^n x, T^n y) \leq \theta_n < K$; $n \in \mathbf{Z}_{0+}$, for any $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$; and
- (2) The sequence $\{\theta_n\}_{n=0}^\infty$ is such that it exists $\lim_{n \rightarrow \infty} \left(\sum_{i=0}^n \left[\prod_{j=i}^n K_{j+1} \right] \theta_i \right) = 0$.

Then, the following properties hold:

- (i) Let $\{x_{np}\}_{n=0}^\infty$ and $\{z_{np}\}_{n=0}^\infty$ be sequences in A_j and $\{y_{np}\}_{n=0}^\infty$ be a sequence in A_{j+1} such that $x_{np} = T^{np}x$, $z_{(n+m)p} = T^{(n+m)p}z$, $y_{np+1} = Tx_{np}$ (or $y_{np+1} = Tz_{(n+m)p}$) for any given $x, z \in A_j$ and any $n, m \in \mathbf{Z}_{0+}$. Then, $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$.
- (ii) Under the conditions of Property (i), $\left\{ \|T^j x_{np} - T^j z_{(n+m)p}\| \right\}_{n=0}^\infty \rightarrow 0$; $\forall j \in \overline{p-1} \cup \{0\}$, $\forall m \in \mathbf{Z}_{0+}$.
- (iii) Properties (i) and (ii) also hold for any $x, z \in \cup_{i \in \bar{p}} A_i$.

Proof. Firstly, it follows from Lemma 2(ii) that $\|y_{np+1} - z_{(n+m)p}\| \rightarrow D$ and $\|y_{np+1} - x_{np}\| \rightarrow D$ as $n \rightarrow \infty$ for any $x, z \in A_j$ and any $m \in \mathbf{Z}_{0+}$. This implies, from Lemma 4(ii), that $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$ since A_j is convex and A_{j+1} is closed and Property (i) follows. From (14) and (15), it follows that the self-mapping T on $\cup_{i \in \bar{p}} A_i$ is Lipschitz-continuous, and then everywhere continuous, then, taking into account Property (i) $\left\{ \|T^j x_{np} - T^j z_{(n+m)p}\| \right\}_{n=0}^\infty \rightarrow 0$; $\forall j \in \overline{p-1} \cup \{0\}$, $\forall m \in \mathbf{Z}_{0+}$, which proves Property (ii). Finally, the proof of Property (iii) is similar to that of Lemma 4(v). $\square$

A similar result can be directly established for a $\{\theta_n\}_{n=0}^\infty$ - $p(\geq 2)$ -cyclic contraction in the wide sense, which is now given without proof, since it is closed to that of Lemma 5.

Lemma 6. Let A_i for $i \in \bar{p}$ be p non-empty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, such that A_j is convex for some $j \in \bar{p}$ and let the self-mapping T on $\cup_{i \in \bar{p}} A_i$ be a $p(\geq 2)$ -cyclic self-mapping T on $\cup_{i \in \bar{p}} A_i$ which is a $\{\theta_n\}_{n=0}^\infty$ - $p(\geq 2)$ -cyclic contraction in the wide sense, that is, it satisfies the conditions (C.1)–(C.3) of Definition 4 for the norm-induced metric.

Then, the following properties hold:

- (i) Let $\{x_{np}\}_{n=0}^\infty$ and $\{z_{np}\}_{n=0}^\infty$ be sequences in A_j and $\{y_{np}\}_{n=0}^\infty$ be a sequence in A_{j+1} such that $x_{np} = T^{np}x$, $z_{(n+m)p} = T^{(n+m)p}z$, $y_{np+1} = Tx_{np}$ (or $y_{np+1} = Tz_{(n+m)p}$) for any given $x, z \in A_j$ and any $n, m \in \mathbf{Z}_{0+}$. Then, $\|z_{(n+m)p} - x_{np}\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$.
- (ii) Under the conditions of Property (i), $\left\{ \|T^j x_{np} - T^j z_{(n+m)p}\| \right\}_{n=0}^\infty \rightarrow 0$; $\forall j \in \overline{p-1} \cup \{0\}$, $\forall m \in \mathbf{Z}_{0+}$.

(iii) Properties (i) and (ii) also hold for any $x, z \in \cup_{i \in \bar{p}} A_i$.

The result $\left\{ \left\| T^j x_{np} - T^j z_{(n+m)p} \right\| \right\}_{n=0}^{\infty} \rightarrow 0; \forall j \in \overline{p-1} \cup \{0\}, \forall m \in \mathbf{Z}_{0+}$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$ initialized in each of the subsets of the cyclical disposal, for any sequences $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ initialized within any of the subsets, obtained in Lemma 4(iii), Lemma 5(i), and Lemma 6(ii) concludes the following. If the metric space is a uniformly convex Banach space (then, complete) and its p subsets in the cyclic disposal are all non-empty and closed while (at least) one of them is convex, such sequences are Cauchy sequences and then convergent. The convergence points of such sequences belong to the sets of best proximity points of adjacent subsets (which have to be then non-empty) from the associated property $\left\| z_{(n+m)p} - T x_{np} \right\| \rightarrow 0$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$ for any initial points in any of the subsets so for sequences $\{T x_{np}\}_{n=0}^{\infty} \subset A_{i+1}$; and $\{x_{np}\}_{n=0}^{\infty} \subset A_i$ and $\{z_{(n+m)p}\}_{n=0}^{\infty} \subset A_i$ for any $i \in \bar{p}$ (See Lemma 5(i) More formally, one establishes the following main result of this article:

Theorem 1. Let A_i for $i \in \bar{p}$ be p non-empty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, such that A_j is convex for some $j \in \bar{p}$ and let the self-mapping T on $\cup_{i \in \bar{p}} A_i$ be any of the enriched $p(\geq 2)$ -cyclic contractions of Definitions 1–3 such that:

- (a) If T satisfies Definition 2, or Definition 1, then it is also subject to the conditions of Lemma 1(iv).
- (b) If T satisfies Definition 3 then it is also subject to the conditions (1) and (2) of Lemma 5.

Then the p sets of best proximity points $A_{i0} \subset A_i$, subset of A_i , are all non-empty for $i \in \bar{p}$ and all sequences $\{T^{np+j}x\}_{n=0}^{\infty} \left(\subset A_{i+\uparrow} \right) \rightarrow z_{i+\uparrow} \left(= T^{\uparrow} z_i \in A_{i+\uparrow,0} \subset A_{i+\uparrow} \right)$ for any $x \in A_i$ and any $i, \uparrow \in \bar{p}$.

The above result also holds if the mapping T satisfies Definition 4.

Proof. In any of the considered cases, it has been concluded that (a) $\left\{ \left\| T^j x_{np} - T^j z_{(n+m)p} \right\| \right\}_{n=0}^{\infty} \rightarrow 0; \forall j \in \overline{p-1} \cup \{0\}, \forall m \in \mathbf{Z}_{0+}$ as $n \rightarrow \infty$ for any $m \in \mathbf{Z}_{0+}$ and any $x, z \in \cup_{i \in \bar{p}} A_i$; (b) $T : \cup_{i \in \bar{p}} A_i \rightarrow \cup_{i \in \bar{p}} A_i$ and its composed self-mappings are continuous in $\cup_{i \in \bar{p}} A_i$; and (c) the sequences $T^{np}x$ are Cauchy, then convergent, for any $x \in \cup_{i \in \bar{p}} A_i$. Thus, if $x \in A_i$ for any $i \in \bar{p}$ then $\{T^{np}x\}_{n=0}^{\infty} \rightarrow z_i (\in A_i)$ since A_i is closed and $\{d(T^{np}x, T^{np+1}x)\}_{n=0}^{\infty} \rightarrow D$. Then, the best proximity set $A_{i0} = \{z \in A_i : d(z, y) = D, \text{ some } y \in A_{i+1}\}$ in A_i to A_{i+1} is non-empty. Furthermore, $d(z_i, Tz_i) = D$ since, otherwise, if this is not the case, so that $d(z_i, Tz_i) > D$, then $\{d(T^{np}x, T^{np+1}x)\}_{n=0}^{\infty} \rightarrow D$ would fail. As a result, $Tz_i = z_{i+1} \in A_{i+1,0} (\neq \emptyset) \subset A_{i+1}$ so that $A_{\uparrow 0} \neq \emptyset$ for all $\uparrow \in \bar{p}$. Thus, $\{T^{np+j}x\}_{n=0}^{\infty} \left(\subset A_{i+\uparrow} \right) \rightarrow z_{i+\uparrow} \left(= T^{\uparrow} z_i \in A_{i+\uparrow,0} \subset A_{i+\uparrow} \right)$ for any $x \in A_i$ and any $i, \uparrow \in \bar{p}$ as claimed. $\square$

Remark 5. Note that Theorem 1 does not prove the existence of unique best proximity points at each of the subsets A_i for $i \in \bar{p}$ but it guarantees that there exists at least one of such points per subset and that the various enriched cyclic maps under study generate limit cycles for any sequences under any initial conditions such that the cycles consists of a best proximity point per subset. The key ideas which support this result are that the generated sequences through the self-mapping from any initial conditions converge to a best proximity point of the convex and closed subset to its adjacent subset and, since the self-mapping is continuous single-valued, this limit best proximity point is mapped to a single point at its adjacent closed subset (which has to be also a best proximity point since distances converge to D) and so on for the subsequent subsets of the disposal. The sets of these best proximity points form a limit cycle to which all the sequences converge for any initial condition located in any of the subsets.

Corollary 1. *If in Theorem 1, all the subsets A_i are, in addition, convex for $i \in \bar{p}$, then all the best proximity points are unique.*

Proof. Theorem 1 above and Theorem 3.10 of [1] proves the uniqueness of the best proximity points for $p = 2$. $\square$

Corollary 2. *If in Theorem 1, $\cap_{i \in \bar{p}} A_i \neq \emptyset$, then the set of best proximity points are identical, and coincident with a unique fixed point z of T on $\cup_{i \in \bar{p}} A_i$, and all the sequences $\{T^n x\}_{n=0}^{\infty} \rightarrow z$ for any $x \in \cup_{i \in \bar{p}} A_i$.*

Proof. It is direct from Theorem 1 since $D = 0$, since $\cap_{i \in \bar{p}} A_i \neq \emptyset$, all the best proximity points are confluent in a unique fixed point $z \in \cap_{i \in \bar{p}} A_i$ and $\{T^n x\}_{n=0}^{\infty} \rightarrow z; \forall x \in \cup_{i \in \bar{p}} A_i$. $\square$

4. Conclusions

This paper has presented several types of enriched cyclic contractions defined in the union of a finite set of at least two non-empty closed subsets of normed spaces or metric. The discussed single-valued enriched cyclic contractions are, in general, subject to weaker constraints on the self-mappings than in the case of p -cyclic contractions. However, the essential properties of these last ones are kept in the case of the enriched cyclic contractions. The convergence of distances has been addressed as well as the convergence of sequences generated by the considered enriched cyclic mappings. It is proved that, in both normed spaces and in simple metric spaces, the distances of sequences of points in adjacent subsets converge to the distance between such subsets under weak extra conditions compared to the cyclic contractive case, which is simply that the contractive constant be less than one. It is also proved that if the metric space, or if the normed space, is additionally a uniformly convex Banach space while one of the involved subsets in the cyclic disposal is, furthermore, convex, then all the sequences between adjacent subsets converge to a unique set of best proximity points. Such points are allocated one per subset and they conform a limit cycle to which all the sequences converge for any initial conditions in the union of the subsets. However, the sets of existing best proximity points are not necessarily guaranteed to be singletons in all the subsets.

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