

## Article

# Development of Viscosity Iterative Techniques for Split Variational-like Inequalities and Fixed Points Related to Pseudo-Contraactions

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## Abstract

This work presents an extragradient-type iterative process combined with the viscosity method to find a common solution to a split generalized variational-like inequality, a variational inequality, and a fixed point problem associated with a family of  $\varepsilon$ -strict pseudo-contractive mappings and a nonexpansive operator in Hilbert spaces. Strong convergence of the proposed algorithm is established, with some remarks derived from the main theorem. Numerical experiments are carried out to verify the applicability of the method and provide comparative observations. The results broaden and unify a range of existing contributions in this field.

**Keywords:** split generalized general variational-like inequality problem;  $\varepsilon$ -pseudo-contraction mapping; nonexpansive mapping; fixed point problem; iterative methods; variational inequality problem; extragradient methods

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## 1. Introduction

Let  $\mathbb{X}_1$  and  $\mathbb{X}_2$  denote real Hilbert spaces equipped with inner product  $\langle \cdot, \cdot \rangle$  and norm  $|\cdot|$ . Let  $\mathbb{D}_1 \subset \mathbb{X}_1$  and  $\mathbb{D}_2 \subset \mathbb{X}_2$  be nonempty, closed, and convex sets. The fixed point problem (abbreviated as FPP) for a mapping  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_1$  seeks a point  $\mathfrak{w} \in \mathbb{D}_1$  satisfying  $T\mathfrak{w} = \mathfrak{w}$ . The collection of fixed points of  $T$  is represented by  $\text{Fix}(T)$ , that is,  $\text{Fix}(T) = \{\mathfrak{w} \in \mathbb{D}_1 : T\mathfrak{w} = \mathfrak{w}\}$ .

Censor et al. [1] formulated the split feasibility problem (SFP) within the framework of finite-dimensional Hilbert spaces, motivated by applications in areas such as phase retrieval and medical imaging. The SFP is defined as follows: Find  $\mathfrak{t}^* \in \mathbb{D}_1$  such that  $A\mathfrak{t}^* \in \mathbb{D}_2$ , where  $A : \mathbb{X}_1 \rightarrow \mathbb{X}_2$  is a bounded linear transformation. Since its introduction, a wide range of iterative approaches and extensions of the SFP have been actively explored in the literature; for a comprehensive discussion, refer to [2–6].

In this paper, we investigate the split generalized general variational-like inequality problem (abbreviated as SGGVLIP), defined as follows: find  $\mathfrak{t}^* \in \mathbb{D}_1$  such that

$$\mathfrak{B}_1(\mathfrak{w}, \mathfrak{t}^*; \mathfrak{t}^*) + \psi_1(\mathfrak{w}, \mathfrak{t}^*) - \psi_1(\mathfrak{t}^*, \mathfrak{t}^*) \geq 0, \quad \forall \mathfrak{w} \in \mathbb{D}_1, \quad (1)$$

and the associated point  $p^* = At^* \in \mathbb{D}_2$  satisfies

$$\mathfrak{B}_2(v, p^*; p^*) + \psi_2(v, p^*) - \psi_2(p^*, p^*) \geq 0, \quad \forall v \in \mathbb{D}_2, \quad (2)$$

where  $\mathfrak{B}_j : \mathbb{D}_j \times \mathbb{D}_j \times \mathbb{D}_j \rightarrow \mathbb{R}$  for  $j = 1, 2$  are nonlinear trifunctions,  $\psi_j : \mathbb{D}_j \times \mathbb{D}_j \rightarrow \mathbb{R}$  for  $j = 1, 2$  are nonlinear bifunctions, and  $A : \mathbb{X}_1 \rightarrow \mathbb{X}_2$  is a bounded linear operator. The set of all such solutions is denoted by  $\text{Sol}(\text{SGGVLP}(1)–(2))$ . This formulation consists of two interconnected GGVLPs, which are linked through the mapping  $p^* = At^*$  induced by the operator  $A$ .

It is worth mentioning that the SGGVLP framework serves as a generalization of the multiple-sets split feasibility problem and encompasses split variational inequality problems as particular instances. This formulation further extends several well-known problems, including the split zero problem, split feasibility problem, and split equilibrium problem (see, for instance, [2,4,7–9]).

If we set  $\mathbb{X}_1 = \mathbb{X}_2$ ,  $\mathbb{D}_1 = \mathbb{D}_2$ ,  $\mathfrak{B}_1 = \mathfrak{B}_2$ , and  $\psi_1 = \psi_2$ , then SGGVLP((1)–(2)) reduces to a generalized general variational-like inequality problem (abbreviated as GGVLP) that involves finding an element  $t^* \in \mathbb{D}_1$  satisfying

$$\mathfrak{B}_1(w, t^*; t^*) + \psi_1(w, t^*) - \psi_1(t^*, t^*) \geq 0, \quad \forall w \in \mathbb{D}_1 \quad (3)$$

which is introduced by Kazmi et al. [10] and studied in [11,12].

If we set  $\psi_1 = 0$ , then GGVLP(3) reduces to a general variational-like inequality problem (abbreviated as GVLIP) that involves finding an element  $t^* \in \mathbb{D}_1$  satisfying

$$\mathfrak{B}_1(w, t^*; t^*) \geq 0, \quad \forall w \in \mathbb{D}_1. \quad (4)$$

which was introduced by Preda et al. [13]. This formulation has notable applications in areas such as mathematical programming and equilibrium theory; see, for example, refs. [14–18].

Further, if we set  $\mathfrak{B}_1(w, t^*; t^*) = \langle ft^*, \eta(w, t^*) \rangle$ , where  $f : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  and  $\eta : \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathbb{X}_1$  and  $\psi_1 = 0$ , then GGVLP(3) is reduced to the variational-like inequality problem (abbreviated as VLIP) that involves finding an element  $t^* \in \mathbb{D}_1$  satisfying

$$\langle ft^*, \eta(w, t^*) \rangle \geq 0, \quad \forall w \in \mathbb{D}_1,$$

which was introduced and studied by Parida et al. [19], and has applications in mathematical programming problems.

Moreover, if  $\eta(w, t^*) = w - t^*$  for all  $w, t^* \in \mathbb{D}_1$ , then the variational-like inequality problem is reduced to the classical variational inequality problem (abbreviated as VIP) that involves finding an element  $t^* \in \mathbb{D}_1$  satisfying

$$\langle ft^*, w - t^* \rangle \geq 0, \quad \forall w \in \mathbb{D}_1, \quad (5)$$

which was originally introduced by Hartman and Stampacchia [20]. The set of solutions of VIP(5) is denoted by  $\text{Sol}(\text{VIP}(5))$ .

If we set  $\mathfrak{B}_1(w, t^*; t^*) = G_1(t^*, w)$ , where  $G_1 : \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathbb{R}$  and  $\psi_1 \equiv 0$ , then GGVLP(3) reduces to an equilibrium problem (abbreviated as EP) that involves finding an element  $t^* \in \mathbb{D}_1$  satisfying

$$G_1(t^*, w) \geq 0, \quad \forall w \in \mathbb{D}_1, \quad (6)$$

which was introduced by Blum and Oettli [21].

The GGVLP is often interpreted as a trifunction equilibrium problem. As a result, the classical equilibrium problem emerges as a particular instance of this more general

formulation. The equilibrium problem has gained considerable attention due to its profound influence on the development of multiple scientific and engineering disciplines. Notably, numerous classical problems have been shown to fit naturally within the equilibrium framework, offering a unified, flexible, and powerful approach for addressing challenges in nonlinear analysis, optimization, economics, finance, game theory, physics, and engineering [22,23].

Korpelevich [24] proposed the extragradient method within the setting of a Hilbert space  $\mathbb{X}_1$  to solve the VIP(5):

$$\left. \begin{aligned} \eta_0 &\in \mathbb{D}_1, \\ \vartheta_n &= P_{\mathbb{D}_1}(\eta_n - \alpha f\eta_n), \\ \eta_{n+1} &= P_{\mathbb{D}_1}(\eta_n - \alpha f\vartheta_n). \end{aligned} \right\} \quad (7)$$

Here,  $\alpha > 0$  is a step-size parameter,  $f$  is assumed to be monotone and Lipschitz continuous, and  $P_{\mathbb{D}_1}$  denotes the metric projection onto the convex set  $\mathbb{D}_1$ . Under appropriate assumptions, the generated sequence is guaranteed to converge to a solution of the VIP(5).

In 2006, Nadezhkina and Takahashi [25] proposed a modified version of the extragradient method given in (7), and they proved a weak convergence theorem to find a common element of the set of solutions to  $\text{Fix}(T)$  for a nonexpansive mapping and VIP(5) in the setting of Hilbert space.

Further, in 2006, Nadezhkina and Takahashi [26] introduced an alternative extragradient method that integrates the hybrid projection approach [27] with the extragradient scheme [24] and proved a strong convergence theorem to find a common element of the set of a solution to  $\text{Fix}(T)$  for a nonexpansive mapping and VIP(5) in the setting of Hilbert space. For further developments and extensions of this scheme, see [28].

It should be noted that, apart from the hybrid extra-gradient iterative method developed for approximating a common solution to GGVLP(3), the VIP(5) and fixed point problem for nonlinear mappings, no strong convergence theorems for extra-gradient iterative methods have been established so far. Furthermore, the convergence analysis of such methods remains largely unexplored. Hence, the primary objective of this work is to introduce a non-hybrid extra-gradient iterative method for solving SGGVLP((1)–(2)), VIP(5), and fixed point problems for nonlinear mappings and to establish strong convergence.

On the other hand, iterative approaches for strict pseudo-contractions are less advanced than those for nonexpansive mappings, despite the early contribution of Browder and Petryshyn in 1967 [29]. This gap may be attributed to the additional term on the right-hand side of (10), which complicates the convergence analysis of algorithms designed to find a fixed point of the strict pseudo-contraction  $S$ .

However, strict pseudo-contractions provide stronger applicability than nonexpansive mappings in solving inverse problems (see [30]). Therefore, it is interesting to develop the iterative methods for finding a common solution to SGGVLP(1)–(2), VIP(5), and FPPs not only for nonexpansive mappings, but also for a finite family of  $\varepsilon$ -strict pseudo-contraction mappings. For further work, see, for example, [31,32] and the references therein.

Therefore, inspired by earlier works (e.g., [5,12,26,32]), in this paper, we propose an extra-gradient iterative method for approximating a common solution to SGGVLP(1)–(2), VIP(5), and FPPs not only for nonexpansive mappings, but also for a finite family of  $\varepsilon$ -strict pseudo-contraction mappings in Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to the common solution to these problems. Furthermore, we derive some consequences from the main result. Finally, we discuss some numerical examples to demonstrate the applicability of the iterative algorithm. We mention the abbreviation and algebraic expression in Appendices A and B for the convenience of readers.

## 2. Preliminaries

In this section, we compile the key concepts and results needed for the presentation of this work. We denote strong and weak convergence by  $\rightarrow$  and  $\rightharpoonup$ , respectively.

For any  $w_1 \in X_1$   $\exists$ , a unique nearest point to  $w_1$  in  $D_1$  is denoted by  $P_{D_1}w_1$  such that

$$\|w_1 - P_{D_1}w_1\| \leq \|w_1 - w_2\|, \quad \forall w_2 \in D_1.$$

The operator  $P_{D_1}$  is called the metric projection of  $X_1$  onto  $D_1$ . This projection is nonexpansive and satisfies

$$\langle w_1 - w_2, P_{D_1}w_1 - P_{D_1}w_2 \rangle \geq \|P_{D_1}w_1 - P_{D_1}w_2\|^2, \quad \forall w_1, w_2 \in X_1.$$

Additionally,  $P_{D_1}w_1$  is characterized by  $P_{D_1}w_1 \in D_1$  and

$$\langle w_1 - P_{D_1}w_1, w_2 - P_{D_1}w_1 \rangle \leq 0, \quad \forall w_2 \in D_1.$$

This implies that

$$\|w_1 - w_2\|^2 \geq \|w_1 - P_{D_1}w_1\|^2 + \|w_2 - P_{D_1}w_1\|^2, \quad \forall w_1 \in X_1, \forall w_2 \in D_1.$$

In a real Hilbert space  $X_1$ , it is known that

$$\|\beta w_1 + (1 - \beta)w_2\|^2 = \beta\|w_1\|^2 + (1 - \beta)\|w_2\|^2 - \beta(1 - \beta)\|w_1 - w_2\|^2, \quad \forall w_1, w_2 \in X_1 \text{ and } \beta \in [0, 1]; \quad (8)$$

and

$$\|w_1 + w_2\|^2 \leq \|w_1\|^2 + 2\langle w_2, w_1 + w_2 \rangle, \quad \forall w_1, w_2 \in X_1. \quad (9)$$

A mapping  $S : D_1 \rightarrow X_1$  is called an  $\varepsilon$ -strict pseudo-contraction if there exists  $\varepsilon \in [0, 1]$  such that

$$\|Sw_1 - Sw_2\|^2 \leq \|w_1 - w_2\|^2 + \varepsilon\|(I - S)w_1 - (I - S)w_2\|^2, \quad \forall w_1, w_2 \in D_1. \quad (10)$$

- When  $\varepsilon = 0$ ,  $S$  becomes a nonexpansive mapping.
- When  $\varepsilon = 1$ ,  $S$  becomes pseudo-contractive.
- For  $\lambda \in (0, 1)$ , if  $S - \lambda I$  is pseudo-contractive, then  $S$  is called strongly pseudo-contractive.

Thus, the class of  $\varepsilon$ -strict pseudo-contractions lies strictly between nonexpansive and pseudo-contractive mappings. It is worth noting that the class of strongly pseudo-contractive mappings is not included within the class of  $\varepsilon$ -strict pseudo-contractions (see [29] for details).

The following properties and equivalent characterizations hold:

- (i) For all  $w_1, w_2 \in D_1$ , the condition (10) is equivalent to

$$\langle Sw_1 - Sw_2, w_1 - w_2 \rangle \leq \|w_1 - w_2\|^2 - \frac{1 - \varepsilon}{2}\|(I - S)w_1 - (I - S)w_2\|^2.$$

- (ii) The mapping  $S$  is pseudo-contractive if and only if

$$\langle Sw_1 - Sw_2, w_1 - w_2 \rangle \leq \|w_1 - w_2\|^2, \quad \forall w_1, w_2 \in D_1.$$

- (iii) The mapping  $S$  is strongly pseudo-contractive if and only if there exists  $\lambda \in (0, 1)$  such that

$$\langle S\mathfrak{w}_1 - S\mathfrak{w}_2, \mathfrak{w}_1 - \mathfrak{w}_2 \rangle \leq (1 - \lambda) \|\mathfrak{w}_1 - \mathfrak{w}_2\|^2, \quad \forall \mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}_1.$$

**Lemma 1** ([33]). Let  $b_n$  be a sequence of non-negative real numbers with the subsequence  $b_{n_i}$  such that  $b_{n_i} < b_{n_{i+1}}$  for all  $i \in \mathbb{N}$ . Then, there exists a non-decreasing sequence  $\{m_j\} \subset \mathbb{N}$  such that  $\lim_{j \rightarrow \infty} m_j = \infty$ , and, for all sufficiently large  $j \in \mathbb{N}$ , the following hold:

$$b_{m_j} \leq b_{m_{j+1}} \quad \text{and} \quad b_j \leq b_{m_j}.$$

Moreover,  $m_j$  is the largest number  $n$  in the set  $\{1, 2, 3, \dots, j\}$  such that  $b_n < b_{n+1}$ .

**Lemma 2** ([31]). Assume that  $D$  is a strongly positive, self-adjoint, and bounded linear operator on a Hilbert space  $\mathbb{X}_1$  with a positive coefficient  $\bar{\gamma} > 0$  and  $0 < \rho \leq \|D\|^{-1}$ . Then, it follows that  $\|I - \rho D\| \leq 1 - \rho \bar{\gamma}$ .

**Assumption 1** ([10]). Let  $\mathfrak{B}_1 : \mathbb{D}_1 \times \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathbb{R}$  and  $\psi_1 : \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathbb{R}$  satisfy the following conditions:

- (1)  $\mathfrak{B}_1(\mathfrak{w}_1, \mathfrak{w}_2; \mathfrak{w}_3) = 0$  if  $\mathfrak{w}_1 = \mathfrak{w}_2$ ,  $\forall \mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \in \mathbb{D}_1$ ;
- (2)  $\mathfrak{B}_1$  is generalized relaxed  $\alpha$ -monotone, i.e., for any  $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}_1$  and  $t \in (0, 1]$ , we get

$$\mathfrak{B}_1(\mathfrak{w}_2, \mathfrak{w}_1; \mathfrak{w}_2) - \mathfrak{B}_1(\mathfrak{w}_2, \mathfrak{w}_1; \mathfrak{w}_1) \geq \alpha(\mathfrak{w}_1, \mathfrak{w}_2),$$

where  $\alpha : \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathbb{R}$  such that

$$\lim_{t \rightarrow 0} \frac{\alpha(\mathfrak{w}_1, t\mathfrak{w}_2 + (1-t)\mathfrak{w}_1)}{t} = 0;$$

- (3)  $\mathfrak{B}_1(\mathfrak{w}_2, \mathfrak{w}_1; \cdot)$  is hemicontinuous, for any fixed  $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}_1$ ;
- (4)  $\mathfrak{B}_1(\cdot, \mathfrak{w}_1; \mathfrak{w}_3)$  is convex and lower semicontinuous, for any fixed  $\mathfrak{w}_1, \mathfrak{w}_3 \in \mathbb{D}_1$ ;
- (5)  $\mathfrak{B}_1(\mathfrak{w}_1, \mathfrak{w}_2; \mathfrak{w}_3) + \mathfrak{B}_1(\mathfrak{w}_2, \mathfrak{w}_1; \mathfrak{w}_3) = 0$ , for any  $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \in \mathbb{D}_1$ ;
- (6)  $\psi_1$  is skew-symmetric, i.e.,

$$\psi_1(\mathfrak{w}_1, \mathfrak{w}_1) - \psi_1(\mathfrak{w}_1, \mathfrak{w}_2) + \psi_1(\mathfrak{w}_2, \mathfrak{w}_2) - \psi_1(\mathfrak{w}_2, \mathfrak{w}_1) \geq 0, \quad \forall \mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}_1;$$

- (7)  $\psi_1(\cdot, \cdot)$  is weakly continuous and  $\psi_1(\cdot, \mathfrak{w}_2)$  convex for any fixed  $\mathfrak{w}_2 \in \mathbb{D}_1$ .

Now, we define  $F_r^{(\mathfrak{B}_1, \psi_1)} : \mathbb{X}_1 \rightarrow \mathbb{D}_1$  by

$$F_r^{(\mathfrak{B}_1, \psi_1)}(\mathfrak{w}) = \{\mathfrak{x} \in \mathbb{D}_1 : \mathfrak{B}_1(\mathfrak{w}, \mathfrak{x}; \mathfrak{x}) + \psi_1(\mathfrak{x}, \mathfrak{w}) - \psi_1(\mathfrak{w}, \mathfrak{w}) + \frac{1}{r} \langle \mathfrak{w} - \mathfrak{x}, \mathfrak{x} - \mathfrak{w} \rangle \geq 0, \quad \forall \mathfrak{w} \in \mathbb{D}_1\}, \quad (11)$$

where  $r$  is a positive real number.

The next lemma represents a particular instance of Lemmas 3.1–3.3, as established in [10] within the setting of a real Hilbert space.

**Lemma 3** ([10]). Let  $\mathfrak{B}_1, \psi_1$  satisfying the Assumption 1 and  $F_r^{(\mathfrak{B}_1, \psi_1)}$  be defined in (11). Then, the following holds:

- (i)  $F_r^{(\mathfrak{B}_1, \psi_1)}(\mathfrak{w})$  is nonempty for each  $\mathfrak{w} \in \mathbb{X}_1$ ;
- (ii)  $F_r^{(\mathfrak{B}_1, \psi_1)}$  is single-valued;

(iii)  $F_r^{(\mathfrak{B}_1, \psi_1)}$  is a firmly nonexpansive mapping, i.e., for all  $\eta_1, \eta_2 \in \mathbb{X}_1$ ,

$$\|F_r^{(\mathfrak{B}_1, \psi_1)}(\eta_1) - F_r^{(\mathfrak{B}_1, \psi_1)}(\eta_2)\|^2 \leq \langle F_r^{(\mathfrak{B}_1, \psi_1)}(\eta_1) - F_r^{(\mathfrak{B}_1, \psi_1)}(\eta_2), \eta_1 - \eta_2 \rangle;$$

(iv)  $\text{Fix}(F_r^{(\mathfrak{B}_1, \psi_1)}) = \text{Sol}(\text{GGVLIP}(1))$ ;

(v)  $\text{Sol}(\text{GGVLIP}(1))$  is closed and convex.

Further, suppose  $\mathfrak{B}_2 : \mathbb{D}_2 \times \mathbb{D}_2 \times \mathbb{D}_2 \rightarrow \mathbb{R}$  and  $\psi_2 : \mathbb{D}_2 \times \mathbb{D}_2 \times \mathbb{D}_2 \rightarrow \mathbb{R}$  satisfying Assumption 1. For  $s > 0$  and  $\eta \in \mathbb{X}_2$ , define the mapping  $F_s^{(\mathfrak{B}_2, \psi_2)} : \mathbb{X}_2 \rightarrow \mathbb{D}_2$  as follows:

$$F_s^{(\mathfrak{B}_2, \psi_2)}(\eta) = \{\mathfrak{x} \in \mathbb{D}_2 : \mathfrak{B}_2(\mathfrak{w}, \mathfrak{x}; \mathfrak{x}) + \psi_2(\mathfrak{x}, \mathfrak{w}) - \psi_2(\mathfrak{x}, \mathfrak{x}) + \frac{1}{s} \langle \mathfrak{w} - \mathfrak{x}, \mathfrak{x} - \eta \rangle \geq 0, \forall \mathfrak{w} \in \mathbb{D}_2\}. \quad (12)$$

It follows that  $F_s^{(\mathfrak{B}_2, \psi_2)}$  is nonempty, single-valued, firmly nonexpansive,  $\text{Fix}(F_s^{(\mathfrak{B}_2, \psi_2)}) = \text{Sol}(\text{GGVLIP}(2))$  and  $\text{Sol}(\text{GGVLIP}(2))$  is closed and convex.

**Lemma 4** ([34]). Let  $S : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  be a  $\varepsilon_i$ -strictly pseudo-contraction mapping. Then,  $\text{Fix}(S)$  is closed convex and it yields that  $P_{\text{Fix}(S)}$  is well defined.

**Lemma 5** ([35]). For any  $\mathfrak{x}, \mathfrak{w}, \eta \in \mathbb{X}_1$ , we have

$$\|\sigma_n \mathfrak{x} + \gamma_n \mathfrak{w} + \mu_n \eta\|^2 = \sigma_n \|\mathfrak{x}\|^2 + \gamma_n \|\mathfrak{w}\|^2 + \mu_n \|\eta\|^2 - \sigma_n \gamma_n \|\mathfrak{x} - \mathfrak{w}\|^2 - \mu_n \gamma_n \|\mathfrak{w} - \eta\|^2 - \sigma_n \mu_n \|\mathfrak{x} - \eta\|^2,$$

where  $\sigma_n, \gamma_n, \mu_n \in [0, 1]$  with  $\sigma_n + \gamma_n + \mu_n = 1$ .

**Lemma 6** ([32]). To each  $i = 1, 2, 3, \dots, \mathbb{N}$ , where  $\mathbb{N}$  be an integer, consider  $S_i : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  be a  $\varepsilon_i$ -strictly pseudo-contraction mapping for some  $0 \leq \varepsilon_i < 1$  and  $\max_{1 \leq i < \mathbb{N}} \varepsilon_i < 1$  with  $\cap_{i=1}^{\mathbb{N}} \text{Fix}(S_i) \neq \emptyset$ .

Let  $\{\varphi_i^{(n)}\}_{i=1}^{\mathbb{N}}$ , for each  $n \geq 0$ , be a positive sequence with  $\sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} = 1$ . Then,  $\sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} S_i : \mathbb{D}_1 \rightarrow \mathbb{X}_1$ , for each  $n \geq 0$ , be  $\varepsilon$ -strictly pseudo-contraction with coefficient  $\varepsilon = \max_{1 \leq i < \mathbb{N}} \varepsilon_i$  and  $\text{Fix}(\sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} S_i) = \cap_{i=1}^{\mathbb{N}} \text{Fix}(S_i)$ .

**Lemma 7** ([36]). Assume that  $\{a_n\}$  is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \vartheta_n, \quad n \geq 0,$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\vartheta_n\}$  is a sequence in  $\mathbb{R}$  such that

- (i)  $\sum_{n=1}^{\infty} \gamma_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \frac{\vartheta_n}{\gamma_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\vartheta_n| < +\infty$ .

Then,  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. Main Outcome

**Theorem 1.** Let  $\mathbb{D}_1$  and  $\mathbb{D}_2$  be nonempty closed convex subsets of Hilbert spaces  $\mathbb{X}_1$  and  $\mathbb{X}_2$ , respectively. For  $j = 1, 2$ , let  $\mathfrak{B}_j : \mathbb{D}_j \times \mathbb{D}_j \times \mathbb{D}_j \rightarrow \mathbb{R}$  be trifunctions, and let  $\psi_j : \mathbb{D}_j \times \mathbb{D}_j \rightarrow \mathbb{R}$  be bifunctions satisfying Assumption 1. Let  $A : \mathbb{X}_1 \rightarrow \mathbb{X}_2$  be a bounded linear operator. Let  $\mathfrak{f} : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  be a  $\sigma$ -inverse strongly monotone mapping, and let  $g : \mathbb{D}_1 \rightarrow \mathbb{D}_1$  be a  $\vartheta$ -contraction mapping. Further, let  $T : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  be a nonexpansive mapping, and for each  $i = 1, 2, 3, \dots, \mathbb{N}$ , let  $S_i : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  be an  $\varepsilon_i$ -strict pseudo-contraction mapping. Let  $\{\varphi_i^{(n)}\}_{i=1}^{\mathbb{N}}$ , for each  $n \geq 0$ , be a finite sequence of positive numbers with  $\sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} = 1$ . Assume that the common solution set

$\Gamma := \bigcap_{i=1}^{\mathbb{N}} \text{Fix}(S_i) \cap \text{Fix}(T) \cap \text{Sol}(\text{SGVLIP}(1) - (2)) \cap \text{Sol}(\text{VIP}(5)) \neq \emptyset$ . Let the sequence  $\{\eta_n\}$  be generated by the following iterative process:

$$\begin{cases} \eta_0 \in \mathbb{D}_1, \\ u_n = F_{r_n}^{(\mathfrak{B}_1, \psi_1)}(\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n), \\ \mathfrak{d}_n = P_{\mathbb{D}_1}(u_n - \alpha_n f(u_n)), \\ \eta_{n+1} = \eta_n g(\eta_n) + (1 - \eta_n)P_{\mathbb{D}_1}\left[\sigma_n \eta_n + \gamma_n T\mathfrak{d}_n + \mu_n \sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} S_i \eta_n\right], \quad n \geq 0, \end{cases} \quad (13)$$

where the control parameters satisfy the following conditions:

- (i)  $\eta_n, \sigma_n, \gamma_n, \mu_n \in (0, 1)$  and  $\sigma_n + \gamma_n + \mu_n = 1$ ;
- (ii)  $\lim_{n \rightarrow \infty} \eta_n = 0$ ,  $\sum_{n=0}^{\infty} \eta_n = \infty$ ;
- (iii)  $\sum_{n=0}^{\infty} \sum_{i=1}^{\mathbb{N}} |\varphi_i^{(n)} - \varphi_i^{(n-1)}| < +\infty$ ;
- (iv)  $0 \leq \varepsilon_i \leq \sigma_n \leq c < 1$ ,  $\lim_{n \rightarrow \infty} \sigma_n = c$ ;
- (v)  $\lambda \in (0, \frac{1}{L})$ , where  $L$  is the spectral radius of  $A^*A$  and  $A^*$  is the adjoint of  $A$ ;
- (vi)  $\alpha_n \in (0, 2\sigma)$ .

Then, the sequence  $\{\eta_n\}$  converges strongly to a point  $\mathfrak{t}^* \in \Gamma$ , where  $\mathfrak{t}^* = P_{\Gamma}g(\mathfrak{t}^*)$ .

**Proof.** We claim that  $\{\eta_n\}$  is bounded. Setting  $s_n = \sigma_n \eta_n + \gamma_n T\mathfrak{d}_n + \mu_n \sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} S_i \eta_n$ . Let  $\mathfrak{t}^* \in \Gamma$ . Using the concept of non-expansive of  $I - \alpha_n f$  and  $T$ , we compute

$$\begin{aligned} \|\mathfrak{d}_n - \mathfrak{t}^*\| &= \|P_{\mathbb{D}_1}(u_n - \alpha_n f u_n) - \mathfrak{t}^*\| \\ &\leq \|(I - \alpha_n f)u_n - (I - \alpha_n f)\mathfrak{t}^*\| \\ &\leq \|u_n - \mathfrak{t}^*\|, \end{aligned} \quad (14)$$

and thus,

$$\|T\mathfrak{d}_n - \mathfrak{t}^*\| \leq \|\mathfrak{d}_n - \mathfrak{t}^*\|. \quad (15)$$

We calculate

$$\begin{aligned} \|u_n - \mathfrak{t}^*\|^2 &= \|F_{r_n}^{(\mathfrak{B}_1, \psi_1)}(\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n) - \mathfrak{t}^*\|^2 \\ &\leq \|\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n - \mathfrak{t}^*\|^2 \\ &\leq \|\eta_n - \mathfrak{t}^*\|^2 + \lambda^2 \|A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \\ &\quad + 2\lambda \langle \eta_n - \mathfrak{t}^*, A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle. \end{aligned} \quad (16)$$

Thus, we have

$$\begin{aligned} \|u_n - \mathfrak{t}^*\|^2 &\leq \|\eta_n - \mathfrak{t}^*\|^2 + \lambda^2 \langle (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n, AA^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\ &\quad + 2\lambda \langle \eta_n - \mathfrak{t}^*, A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle. \end{aligned} \quad (17)$$

Thus,

$$\begin{aligned} \lambda^2 \langle (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n, AA^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\ \leq L\lambda^2 \langle (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n, (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\ = L\lambda^2 \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2. \end{aligned} \quad (18)$$

Assume that  $\Pi := 2\lambda \langle \eta_n - \mathbf{t}^*, A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle$ ; we have

$$\begin{aligned}
 \Pi &= 2\lambda \langle \eta_n - \mathbf{t}^*, A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\
 &= 2\lambda \langle B(\eta_n - \mathbf{t}^*), (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\
 &= 2\lambda \langle B(\eta_n - \mathbf{t}^*) + (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \\
 &\quad - (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n, (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \\
 &= 2\lambda \left\{ \langle F_{r_n}^{(\mathfrak{B}_2, \psi_2)}A\eta_n - A\mathbf{t}^*, (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle \right. \\
 &\quad \left. - \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \right\} \\
 &\leq 2\lambda \left\{ \frac{1}{2} \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 - \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \right\} \\
 &\leq -\lambda \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2.
 \end{aligned} \tag{19}$$

According to (17)–(19), we get

$$\|u_n - \mathbf{t}^*\|^2 \leq \|\eta_n - \mathbf{t}^*\|^2 + \lambda(L\lambda - 1) \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2. \tag{20}$$

As  $\lambda \in (0, \frac{1}{L})$ , we have

$$\|u_n - \mathbf{t}^*\| \leq \|\eta_n - \mathbf{t}^*\|. \tag{21}$$

We compute

$$\begin{aligned}
 \|\eta_{n+1} - \mathbf{t}^*\| &= \|\eta_n g(\eta_n) + (1 - \eta_n)P_{\mathbb{D}_1} s_n - \mathbf{t}^*\| \\
 &\leq \eta_n \|g(\eta_n) - \mathbf{t}^*\| + (1 - \eta_n) \|P_{\mathbb{D}_1} s_n - \mathbf{t}^*\| \\
 &\leq \eta_n \|g(\eta_n) - \mathbf{t}^*\| + (1 - \eta_n) \|s_n - \mathbf{t}^*\|.
 \end{aligned} \tag{22}$$

Now,

$$\begin{aligned}
 \|g(\eta_n) - \mathbf{t}^*\| &\leq \|g(\eta_n) - g(\mathbf{t}^*)\| + \|g(\mathbf{t}^*) - \mathbf{t}^*\| \\
 &\leq \vartheta \|\eta_n - \mathbf{t}^*\| + \|g(\mathbf{t}^*) - \mathbf{t}^*\|.
 \end{aligned} \tag{23}$$

Setting  $G_n = \sum_{i=1}^{\mathbb{N}} \varphi_i^{(n)} S_i$  and using Lemma 6, we observe that the mapping  $G_n : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  is  $\varepsilon$ -strictly pseudo-contraction with  $\varepsilon = \max_{1 \leq i \leq \mathbb{N}} \varepsilon_i$  and  $\text{Fix}(G_n) = \cap_{i=1}^{\mathbb{N}} \text{Fix}(S_i)$ . Thus, according to Lemma 5, we estimate

$$\begin{aligned}
 \|s_n - \mathbf{t}^*\|^2 &= \|\sigma_n \eta_n + \gamma_n T\mathfrak{d}_n + \mu_n G_n \eta_n - \mathbf{t}^*\|^2 \\
 &= \|\sigma_n (\eta_n - \mathbf{t}^*) + \gamma_n (T\mathfrak{d}_n - \mathbf{t}^*) + \mu_n (G_n \eta_n - \mathbf{t}^*)\|^2 \\
 &= \sigma_n \|\eta_n - \mathbf{t}^*\|^2 + \gamma_n \|T\mathfrak{d}_n - \mathbf{t}^*\|^2 + \mu_n \|G_n \eta_n - \mathbf{t}^*\|^2 \\
 &\quad - \sigma_n \gamma_n \|(\eta_n - T\mathfrak{d}_n)\|^2 - \gamma_n \mu_n \|T\mathfrak{d}_n - G_n \eta_n\|^2 - \sigma_n \mu_n \|\eta_n - G_n \eta_n\|^2
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 &\leq \sigma_n \|\eta_n - \mathbf{t}^*\|^2 + \gamma_n \|\mathfrak{d}_n - \mathbf{t}^*\|^2 + \mu_n (\|\eta_n - \mathbf{t}^*\|^2 + \varepsilon \|\eta_n - G_n \eta_n\|^2) \\
 &\quad - \sigma_n \gamma_n \|(\eta_n - T\mathfrak{d}_n)\|^2 - \gamma_n \mu_n \|T\mathfrak{d}_n - G_n \eta_n\|^2 - \sigma_n \mu_n \|\eta_n - G_n \eta_n\|^2 \\
 &= (\sigma_n + \gamma_n + \mu_n) \|\eta_n - \mathbf{t}^*\|^2 - \mu_n (\sigma_n - \varepsilon) \|\eta_n - G_n \eta_n\|^2 \\
 &\quad - \sigma_n \gamma_n \|\eta_n - T\mathfrak{d}_n\|^2 - \gamma_n \mu_n \|T\mathfrak{d}_n - G_n \eta_n\|^2 \\
 &= \|\eta_n - \mathbf{t}^*\|^2 - \mu_n (\sigma_n - \varepsilon) \|\eta_n - G_n \eta_n\|^2 \\
 &\quad - \sigma_n \gamma_n \|\eta_n - T\mathfrak{d}_n\|^2 - \gamma_n \mu_n \|T\mathfrak{d}_n - G_n \eta_n\|^2
 \end{aligned} \tag{25}$$

that implies

$$\|s_n - t^*\| \leq \|\eta_n - t^*\|. \quad (26)$$

Thus, according to (22), (23), and (26), we have

$$\begin{aligned} \|\eta_{n+1} - t^*\| &\leq \eta_n[\vartheta\|\eta_n - t^*\| + \|g(t^*) - t^*\|] + (1 - \eta_n)\|\eta_n - t^*\| \\ &\leq [1 - \eta_n(1 - \vartheta)]\|s_n - t^*\| + \eta_n\|g(t^*) - t^*\|. \end{aligned} \quad (27)$$

By induction, we get

$$\|\eta_{n+1} - t^*\| \leq \max\{\|v_0 - t^*\|, \frac{1}{1 - \vartheta}\|g(t^*) - t^*\|\}, \quad \forall n \geq 0,$$

which shows that  $\{\eta_n\}$  is bounded, and hence,  $\{u_n\}$  and  $\{d_n\}$  are also bounded.

As  $t^* \in \Gamma$ , therefore, we compute

$$\begin{aligned} \|\eta_{n+1} - t^*\|^2 &= \|\eta_n(g(\eta_n) - t^*) + (1 - \eta_n)(P_{\mathbb{D}_1}s_n - t^*)\|^2 \\ &\leq (1 - \eta_n)\|P_{\mathbb{D}_1}s_n - t^*\|^2 + 2\langle\eta_n(g(\eta_n) - t^*), \eta_{n+1} - t^*\rangle \\ &\leq (1 - \eta_n)\|s_n - t^*\|^2 + 2\eta_n\langle g(\eta_n) - t^*, \eta_{n+1} - t^*\rangle. \end{aligned} \quad (28)$$

Moreover, we estimate

$$\begin{aligned} \langle g(\eta_n) - t^*, \eta_{n+1} - t^*\rangle &= \langle g(\eta_n) - t^*, \eta_n - t^*\rangle + \langle g(\eta_n) - t^*, \eta_{n+1} - \eta_n\rangle \\ &\leq \|g(\eta_n) - h(t^*)\|\|\eta_n - t^*\| + \frac{M}{2}\|\eta_{n+1} - \eta_n\| + \langle g(\eta_n) - t^*, \eta_n - t^*\rangle \\ &\leq \vartheta\|\eta_n - t^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\| + \langle g(\eta_n) - t^*, \eta_n - t^*\rangle, \end{aligned} \quad (29)$$

where  $M = \sup_n \|g(\eta_n) - t^*\|$ . Using (25), (28), and (29), we have

$$\begin{aligned} \|\eta_{n+1} - t^*\|^2 &\leq (1 - \eta_n(1 - 2\vartheta))\|\eta_n - t^*\|^2 + \eta_n M\|\eta_{n+1} - \eta_n\| + 2\eta_n\langle g(\eta_n) - t^*, \eta_n - t^*\rangle \\ &\quad - \mu_n(\sigma_n - \varepsilon)(1 - \eta_n)\|\eta_n - G_n\eta_n\|^2 - (1 - \eta_n)\sigma_n\gamma_n\|\eta_n - Td_n\|^2 \\ &\quad - (1 - \eta_n)\gamma_n\mu_n\|Td_n - G_n\eta_n\|^2 \end{aligned} \quad (30)$$

$$\begin{aligned} \|\eta_{n+1} - t^*\|^2 &\leq (1 - \eta_n(1 - 2\vartheta))\|\eta_n - t^*\|^2 + \eta_n M\|\eta_{n+1} - \eta_n\| \\ &\quad + 2\eta_n\langle g(\eta_n) - t^*, \eta_n - t^*\rangle. \end{aligned} \quad (31)$$

Setting  $q_n = \|\eta_n - t^*\|^2$ . Consider the two cases on  $\{q_n\}$  as follows:

**Case 1.** For every  $n \geq m_0$  where  $m_0 \in \mathbb{N}$ , consider that the sequence  $\{q_n\}$  is decreasing; therefore, it must be convergent. By applying the conditions in (30), we get

$$\lim_{n \rightarrow \infty} \|\eta_n - Td_n\| = 0, \quad \lim_{n \rightarrow \infty} \|Td_n - G_n\eta_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\eta_n - G_n\eta_n\| = 0. \quad (32)$$

Notice that  $\{\eta_n\}$  is bounded; therefore,  $\exists$  is a subsequence  $\{v_{n_j}\}$  of  $\{\eta_n\}$  with  $v_{n_j} \rightharpoonup p \in \mathbb{D}_1$  and satisfies

$$\limsup_{n \rightarrow \infty} \langle h(t^*) - t^*, \eta_n - t^*\rangle = \lim_{j \rightarrow \infty} \langle h(t^*) - t^*, v_{n_j} - t^*\rangle. \quad (33)$$

Define  $H_n = \kappa_n v + (1 - \kappa_n)G_nv$ ,  $\forall v \in \mathbb{D}_1$ , and  $\kappa_n \in [\vartheta, 1)$ . By applying Lemma 4,  $H_n : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  is nonexpansive, and we have

$$\begin{aligned}
\|\eta_n - H_n \eta_n\| &= \|\eta_n - (\kappa_n \eta_n + (1 - \kappa_n) G_n \eta_n)\| \\
&= \|(\kappa_n + (1 - \kappa_n) \eta_n) - (\kappa_n \eta_n + (1 - \kappa_n) G_n \eta_n)\| \\
&= (1 - \kappa_n) \|\eta_n - G_n \eta_n\|.
\end{aligned} \tag{34}$$

Thus,

$$\lim_{n \rightarrow \infty} \|\eta_n - H_n \eta_n\| = 0. \tag{35}$$

Applying the given conditions, we may consider that  $\varphi_i^{(n)} \rightarrow \varphi_i$  as  $n \rightarrow \infty$ ,  $\forall i$ . According to Lemma 6, the map  $G : \mathbb{D}_1 \rightarrow \mathbb{X}_1$  with  $Gv = (\sum_{i=1}^N \varphi_i S_i)v$ ,  $\forall v \in \mathbb{D}_1$  is a  $\varepsilon$ -strict pseudo-contraction, and  $\text{Fix}(G) = \cap_{i=1}^N \text{Fix}(S_i)$ . By applying Lemma 6, given the conditions and boundedness of  $\eta_n$ , we get

$$\begin{aligned}
\|\eta_n - G\eta_n\| &\leq \|\eta_n - G_n \eta_n\| + \|G_n \eta_n - G\eta_n\| \\
&\leq \|\eta_n - G_n \eta_n\| + \sum_{i=1}^N |\varphi_i^{(n)} - \varphi_i| \|S_i \eta_n\|,
\end{aligned} \tag{36}$$

and thus,

$$\lim_{n \rightarrow \infty} \|\eta_n - G\eta_n\| = 0. \tag{37}$$

As

$$\|G_n \eta_n - G\eta_n\| \leq \|G_n \eta_n - \eta_n\| + \|\eta_n - G\eta_n\|, \tag{38}$$

which, according to (31) and (37), yields

$$\lim_{n \rightarrow \infty} \|G_n \eta_n - G\eta_n\| = 0. \tag{39}$$

Again, we notice that the map  $H = tv + (1 - t)Gv$ ,  $\forall v \in \mathbb{D}_1$  and  $t \in [\vartheta, 1)$ ,  $\text{Fix}H = \text{Fix}G$ . Thus, we obtain

$$\begin{aligned}
\|\eta_n - H\eta_n\| &\leq \|\eta_n - H_n \eta_n\| + \|H_n \eta_n - H\eta_n\| \\
&\leq \|\eta_n - H_n \eta_n\| + \|\kappa_n \eta_n + (1 - \kappa_n) H_n \eta_n - tv - (1 - t)Gv\| \\
&\leq \|\eta_n - H_n \eta_n\| + |\kappa_n - t| \|\eta_n - G\eta_n\| + (1 - \kappa_n) \|H_n \eta_n - H\eta_n\|.
\end{aligned} \tag{40}$$

By applying (35), (37), and (39), we have

$$\lim_{n \rightarrow \infty} \|\eta_n - H\eta_n\| = 0. \tag{41}$$

As  $\eta_n \in \mathbb{D}_1$ , therefore,

$$\|\eta_{n+1} - \eta_n\| \leq \eta_n \|h\eta_n - \eta_n\| + (1 - \eta_n) [\sigma_n \|T\eta_n - \eta_n\| + \mu_n \|\eta_n - G_n \eta_n\|].$$

Using the given conditions and (31), we get

$$\lim_{n \rightarrow \infty} \|\eta_{n+1} - \eta_n\| = 0. \tag{42}$$

Applying (28) and (29), we estimate

$$\|\eta_{n+1} - \mathfrak{t}^*\|^2 \leq (1 - \eta_n)\|s_n - \mathfrak{t}^*\|^2 + 2\eta_n[\vartheta\|\eta_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\|]. \quad (43)$$

Using (14), (20), and (24), we compute

$$\|s_n - \mathfrak{t}^*\|^2 \leq (1 - \gamma_n)\|\eta_n - \mathfrak{t}^*\|^2 + \gamma_n\|u_n - \mathfrak{t}^*\|^2 \quad (44)$$

$$\leq \|\eta_n - \mathfrak{t}^*\|^2 + \lambda(L\lambda - 1)\gamma_n\|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \quad (45)$$

According to (43) and (45), we get

$$\begin{aligned} \|\eta_{n+1} - \mathfrak{t}^*\|^2 &\leq (1 - \eta_n)\|\eta_n - \mathfrak{t}^*\|^2 + \lambda(L\lambda - 1)(1 - \eta_n)\gamma_n\|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \\ &\quad + 2\vartheta\eta_n\|\eta_n - \mathfrak{t}^*\|^2 + M\eta_n\|\eta_{n+1} - \eta_n\| \\ \lambda(1 - L\lambda)(1 - \eta_n)\gamma_n\|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 &\leq \|\eta_n - \mathfrak{t}^*\|^2 - \|\eta_{n+1} - \mathfrak{t}^*\|^2 \\ &\quad + 2\vartheta\eta_n\|\eta_n - \mathfrak{t}^*\|^2 + M\eta_n\|\eta_{n+1} - \eta_n\| \\ &\leq (\|\eta_n - \mathfrak{t}^*\| + \|\eta_{n+1} - \mathfrak{t}^*\|)\|\eta_n - \eta_{n+1}\| \\ &\quad + 2\vartheta\eta_n\|\eta_n - \mathfrak{t}^*\|^2 + M\eta_n\|\eta_{n+1} - \eta_n\|. \end{aligned} \quad (46)$$

Applying the given condition and (42), we get

$$\lim_{n \rightarrow \infty} \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\| = 0. \quad (47)$$

Next, we compute

$$\begin{aligned} \|u_n - \mathfrak{t}^*\|^2 &= \|F_{r_n}^{(\mathfrak{B}_1, \psi_1)}(\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n) - \mathfrak{t}^*\|^2 \\ &\leq \|F_{r_n}^{(\mathfrak{B}_1, \psi_1)}(\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n) - F_{r_n}^{(\mathfrak{B}_1, \psi_1)}\mathfrak{t}^*\|^2 \\ &\leq \langle u_n - \mathfrak{t}^*, \eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n - \mathfrak{t}^* \rangle \\ &= \frac{1}{2} \left\{ \|u_n - \mathfrak{t}^*\|^2 + \|\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n - \mathfrak{t}^*\|^2 \right. \\ &\quad \left. - \| (u_n - \mathfrak{t}^*) - [\eta_n + \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n - \mathfrak{t}^*] \|^2 \right\} \\ &= \frac{1}{2} \left\{ \|u_n - \mathfrak{t}^*\|^2 + \|\eta_n - \mathfrak{t}^*\|^2 \right. \\ &\quad \left. - \|u_n - \eta_n - \lambda A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \right\} \\ &= \frac{1}{2} \left\{ \|u_n - \mathfrak{t}^*\|^2 + \|\eta_n - \mathfrak{t}^*\|^2 \right. \\ &\quad - [\|u_n - \eta_n\|^2 + \lambda^2 \|A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|^2 \\ &\quad \left. - 2\lambda \langle u_n - \eta_n, A^*(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n \rangle] \right\}. \end{aligned} \quad (48)$$

Thus,

$$\|u_n - \mathfrak{t}^*\|^2 \leq \|\eta_n - \mathfrak{t}^*\|^2 - \|u_n - \eta_n\|^2 + 2\lambda \|A(z_n - v_n)\| \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n\|. \quad (49)$$

Using (44) and (49) in (43), we get

$$\begin{aligned}
 \|\eta_{n+1} - \mathfrak{t}^*\|^2 &\leq (1 - \eta_n)\|s_n - \mathfrak{t}^*\|^2 + 2\eta_n[\vartheta\|\eta_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\|] \\
 &\leq (1 - \eta_n)(1 - \gamma_n)\|\eta_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n\|\eta_n - \mathfrak{t}^*\|^2 \\
 &\quad - \gamma_n(1 - \eta_n)\|u_n - \eta_n\|^2 \\
 &\quad + 2\lambda(1 - \eta_n)\gamma_n\|A(z_n - v_n)\| \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)Av_n\| \\
 &\quad + 2\eta_n[\vartheta\|\eta_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\|] \\
 \implies \gamma_n(1 - \eta_n)\|u_n - \eta_n\|^2 &\leq \|\eta_n - \mathfrak{t}^*\|^2 - \|\eta_{n+1} - \mathfrak{t}^*\|^2 \\
 &\quad + 2\lambda(1 - \eta_n)\gamma_n\|A(z_n - v_n)\| \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)Av_n\| \\
 &\quad + 2\eta_n[\vartheta\|\eta_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\|] \\
 &\leq (\|\eta_n - \mathfrak{t}^*\| + \|\eta_{n+1} - \mathfrak{t}^*\|)\|\eta_n - \eta_{n+1}\| \\
 &\quad + 2\lambda(1 - \eta_n)\gamma_n\|A(z_n - v_n)\| \|(F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)Av_n\| \\
 &\quad + 2\eta_n[\vartheta\|\eta_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\eta_{n+1} - \eta_n\|].
 \end{aligned} \tag{50}$$

Applying given conditions (42) and (47) in (50), we get

$$\lim_{n \rightarrow \infty} \|u_n - \eta_n\| = 0. \tag{51}$$

Further, we estimate

$$\begin{aligned}
 \|\mathfrak{d}_n - \mathfrak{t}^*\|^2 &= \|P_{\mathbb{D}_1}(u_n - \alpha_n \mathfrak{f}u_n) - P_{\mathbb{D}_1}(\mathfrak{t}^* - \alpha_n \mathfrak{f}\mathfrak{t}^*)\|^2 \\
 &\leq \langle \mathfrak{d}_n - \mathfrak{t}^*, (u_n - \alpha_n \mathfrak{f}u_n) - (\mathfrak{t}^* - \alpha_n \mathfrak{f}\mathfrak{t}^*) \rangle \\
 &\leq \frac{1}{2} \{ \|\mathfrak{d}_n - \mathfrak{t}^*\|^2 + \|(u_n - \alpha_n \mathfrak{f}u_n) \\
 &\quad - (\mathfrak{t}^* - \alpha_n \mathfrak{f}\mathfrak{t}^*)\|^2 - \|(\mathfrak{d}_n - u_n) + \alpha_n(\mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^*)\|^2 \} \\
 &\leq \frac{1}{2} \{ \|\mathfrak{d}_n - \mathfrak{t}^*\|^2 + \|u_n - \mathfrak{t}^*\|^2 - \|(\mathfrak{d}_n - u_n) + \alpha_n(\mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^*)\|^2 \} \\
 &\leq \|u_n - \mathfrak{t}^*\|^2 - \|\mathfrak{d}_n - u_n\|^2 - \alpha_n^2 \|\mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^*\|^2 \\
 &\quad + 2\alpha_n \langle \mathfrak{d}_n - u_n, \mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^* \rangle \\
 &\leq \|u_n - \mathfrak{t}^*\|^2 - \|\mathfrak{d}_n - u_n\|^2 + 2\alpha_n \|\mathfrak{d}_n - u_n\| \|\mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^*\| \\
 &\leq \|\eta_n - \mathfrak{t}^*\|^2 - \|\mathfrak{d}_n - u_n\|^2 + 2\alpha_n \|\mathfrak{d}_n - u_n\| \|\mathfrak{f}u_n - \mathfrak{f}\mathfrak{t}^*\|.
 \end{aligned} \tag{52}$$

From (24), we obtain

$$\begin{aligned}
 \|s_n - \mathfrak{t}^*\|^2 &\leq \sigma_n \|\eta_n - \mathfrak{t}^*\|^2 + \gamma_n \|\mathfrak{d}_n - \mathfrak{t}^*\|^2 + \mu_n \|\eta_n - \mathfrak{t}^*\|^2 + \mu_n \varepsilon \|\eta_n - G_n \eta_n\|^2 \\
 &\leq (1 - \gamma_n) \|\eta_n - \mathfrak{t}^*\|^2 + \gamma_n \|\mathfrak{d}_n - \mathfrak{t}^*\|^2 - \mu_n (\sigma_n - \varepsilon) \|\eta_n - G_n \eta_n\|^2.
 \end{aligned} \tag{53}$$

From (43) and (53), we estimate

$$\begin{aligned}
\|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|^2 &\leq (1 - \eta_n)[(1 - \gamma_n)\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \gamma_n\|\mathfrak{d}_n - \mathfrak{t}^*\|^2 - \mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2] \\
&\quad + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|] \\
&\leq (1 - \eta_n)(1 - \gamma_n)\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n[P_{\mathbb{D}_1}(\mathfrak{u}_n - \alpha_n\mathfrak{f}\mathfrak{u}_n) - P_{\mathbb{D}_1}(\mathfrak{t}^* - \alpha_n\mathfrak{f}\mathfrak{t}^*)] \\
&\quad - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|] \\
&\leq (1 - \eta_n)(1 - \gamma_n)\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n[\|\mathfrak{u}_n - \mathfrak{t}^*\|^2 + \alpha_n(\alpha_n - 2\sigma)\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\|^2] \\
&\quad - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|] \\
&\leq (1 - \eta_n)(1 - \gamma_n)\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n[\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \alpha_n(\alpha_n - 2\sigma)\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\|^2] \\
&\quad - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|] \\
&\leq \|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n\alpha_n(\alpha_n - 2\sigma)\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\|^2 \\
&\quad - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|];
\end{aligned} \tag{54}$$

this implies

$$\begin{aligned}
(1 - \eta_n)\gamma_n\alpha_n(2\sigma - \alpha_n)\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\|^2 &\leq \|\mathfrak{h}_n - \mathfrak{t}^*\|^2 - \|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|^2 - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 \\
&\quad + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|] \\
&\leq (\|\mathfrak{h}_n - \mathfrak{t}^*\| + \|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|)\|\mathfrak{h}_n - \mathfrak{h}_{n+1}\| \\
&\quad - (1 - \eta_n)\mu_n(\sigma_n - \varepsilon)\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 \\
&\quad + 2\eta_n[\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \frac{M}{2}\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\|].
\end{aligned} \tag{55}$$

By applying the given conditions, (32) and (42) in (55), we obtain

$$\lim_{n \rightarrow \infty} \|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\| = 0. \tag{56}$$

Using (28), (29), (52), and (53), we compute

$$\begin{aligned}
\|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|^2 &\leq (1 - \eta_n)(1 - \gamma_n)\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + (1 - \eta_n)\gamma_n[\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 - \|\mathfrak{d}_n - \mathfrak{u}_n\|^2] \\
&\quad + 2\alpha_n\|\mathfrak{d}_n - \mathfrak{u}_n\|\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\| + (1 - \eta_n)\mu_n\varepsilon\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 \\
&\quad + 2\eta_n\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \eta_n M\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\| + 2\eta_n\langle g(\mathfrak{h}_n) - \mathfrak{t}^*, \mathfrak{h}_n - \mathfrak{t}^* \rangle \\
\implies (1 - \eta_n)\gamma_n\|\mathfrak{d}_n - \mathfrak{u}_n\|^2 &\leq \|\mathfrak{h}_n - \mathfrak{t}^*\|^2 - \|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|^2 \\
&\quad + 2(1 - \eta_n)\gamma_n\alpha_n\|\mathfrak{d}_n - \mathfrak{u}_n\|\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\| + (1 - \eta_n)\mu_n\varepsilon\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 \\
&\quad + 2\eta_n\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \eta_n M\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\| + 2\eta_n\langle g(\mathfrak{h}_n) - \mathfrak{t}^*, \mathfrak{h}_n - \mathfrak{t}^* \rangle \\
&\leq (\|\mathfrak{h}_n - \mathfrak{t}^*\| + \|\mathfrak{h}_{n+1} - \mathfrak{t}^*\|)\|\mathfrak{h}_n - \mathfrak{h}_{n+1}\| \\
&\quad + 2(1 - \eta_n)\gamma_n\alpha_n\|\mathfrak{d}_n - \mathfrak{u}_n\|\|\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{t}^*\| + (1 - \eta_n)\mu_n\varepsilon\|\mathfrak{h}_n - G_n\mathfrak{h}_n\|^2 \\
&\quad + 2\eta_n\vartheta\|\mathfrak{h}_n - \mathfrak{t}^*\|^2 + \eta_n M\|\mathfrak{h}_{n+1} - \mathfrak{h}_n\| \\
&\quad + 2\eta_n\langle g(\mathfrak{h}_n) - \mathfrak{t}^*, \mathfrak{h}_n - \mathfrak{t}^* \rangle.
\end{aligned} \tag{57}$$

By applying the given conditions, (31), (42) and (56), we get

$$\lim_{n \rightarrow \infty} \|\mathfrak{d}_n - \mathfrak{u}_n\| = 0. \tag{58}$$

Now, we show that  $t^* \in \text{Fix}(H) = \text{Fix}(G) = \text{Fix}(G_n) = \bigcap_{i=1}^{\mathbb{N}} \text{Fix}(S_i)$ . Let  $t^* \notin \text{Fix}(H)$ . Due to  $\eta_{n_j} \rightarrow t^*$  and  $t^* \neq Ht^*$ , according to the Opial condition, we therefore obtain

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|\eta_{n_j} - t^*\| &< \liminf_{j \rightarrow \infty} \|\eta_{n_j} - Ht^*\| \\ &\leq \liminf_{j \rightarrow \infty} [\|\eta_{n_j} - H\eta_{n_j}\| + \|H\eta_{n_j} - Ht^*\|] \\ &\leq \liminf_{j \rightarrow \infty} \|\eta_{n_j} - t^*\|, \end{aligned} \quad (59)$$

which contradicts our supposition. Hence,  $t^* \in \text{Fix}(H) = \text{Fix}(G) = \text{Fix}(G_n) = \bigcap_{i=1}^{\mathbb{N}} \text{Fix}(S_i)$ . According to (31), we observe that  $\{\eta_n\}$  and  $\{\eta_n\}$  have the same asymptotic behaviour; therefore,  $\exists$  is a subsequence  $\{\eta_{n_j}\}$  of  $\{\eta_n\}$  with  $\eta_{n_j} \rightarrow t^*$ . Again, from (31) and the opial condition, we see that  $t^* \in \text{Fix}(T)$ . Next, we prove that  $t^* \in \text{Sol}(\text{SGVLIP}(1) - (2))$ . We set  $\tau_n := \eta_n + \lambda A^*((F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n)$ . Then,  $u_n = F_{r_n}^{(\mathfrak{B}_1, \psi_1)}\tau_n$ . For any  $v \in \mathbb{D}_1$ , we get

$$\begin{aligned} \mathfrak{B}_1(v, u_n; u_n) + \psi_1(v, u_n) - \psi_1(u_n, u_n) + \frac{1}{r_n} \langle v - u_n, u_n - \tau_n \rangle &\geq 0 \\ \implies \mathfrak{B}_1(v, u_{n_j}; u_{n_j}) + \psi_1(v, u_{n_j}) - \psi_1(u_{n_j}, u_{n_j}) + \langle v - u_{n_j}, \frac{u_{n_j} - \tau_{n_j}}{r_{n_j}} \rangle &\geq 0 \end{aligned} \quad (60)$$

Using the concept of the generalized relaxed  $\alpha$ -monotonicity of  $\mathfrak{B}_1$  and (60), we get

$$\begin{aligned} \|v - u_{n_j}\| \left\| \frac{u_{n_j} - \tau_{n_j}}{r_{n_j}} \right\| &\geq \frac{1}{r_{n_j}} \langle v - u_{n_j}, u_{n_j} - \tau_{n_j} \rangle \\ &\geq \mathfrak{B}_1(v, u_{n_j}; u_{n_j}) - \psi_1(v, u_{n_j}) + \psi_1(u_{n_j}, u_{n_j}) \\ &\geq \alpha(u_{n_j}, v) - \mathfrak{B}_1(v, u_{n_j}; v) + \psi_1(u_{n_j}, u_{n_j}) - \psi_1(u_{n_j}, v). \end{aligned} \quad (61)$$

Applying the lower semicontinuity of  $\alpha$  in the first argument, the continuity of  $\mathfrak{B}_1$  in the second argument, and the continuity of  $\psi_1$ , we obtain

$$\alpha(t^*, v) - \mathfrak{B}_1(v, t^*; v) + \psi_1(t^*, t^*) - \psi_1(t^*, v) \leq 0, \quad \forall v \in \mathbb{D}_1.$$

Assume  $\omega_\zeta := (1 - \zeta)t^* + \zeta\omega$ ,  $\forall \zeta \in (0, 1]$ . As  $\omega, t^* \in \mathbb{D}_1$ , then  $\omega_\zeta \in \mathbb{D}_1$ . Therefore,

$$\alpha(t^*, \omega_\zeta) - \mathfrak{B}_1(\omega_\zeta, t^*; \omega_\zeta) + \psi_1(t^*, t^*) - \psi_1(t^*, \omega_\zeta) \leq 0,$$

which yields that

$$\begin{aligned} \alpha(t^*, \omega_\zeta) &\leq \mathfrak{B}_1(\omega_\zeta, t^*; \omega_\zeta) - \psi_1(t^*, t^*) + \psi_1(t^*, \omega_\zeta) \\ &\leq \zeta \mathfrak{B}_1(\omega, t^*; \omega) + (1 - \zeta) \mathfrak{B}_1(\omega, t^*; \omega_\zeta) - \psi_1(t^*, t^*) + \zeta \psi_1(t^*, \omega) + (1 - \zeta) \psi_1(t^*, t^*) \\ &\leq \zeta [\mathfrak{B}_1(\omega, t^*; \omega_\zeta) + \psi_1(t^*, \omega) - \psi_1(t^*, t^*)]. \end{aligned}$$

As  $\mathfrak{B}_1(\omega, t^*; \cdot)$  is hemicontinuous, we get

$$\lim_{\zeta \rightarrow 0} \{\mathfrak{B}_1(\omega, t^*; \omega_\zeta) + \psi_1(t^*, \omega) - \psi_1(t^*, t^*)\} \geq \lim_{\zeta \rightarrow 0} \frac{\alpha(t^*, \omega_\zeta)}{\zeta}$$

which yields that

$$\mathfrak{B}_1(\omega, t^*; t^*) + \psi_1(t^*, \omega) - \psi_1(t^*, t^*) \geq 0.$$

Thus,  $t^* \in \text{Sol}(\text{GGVLIP}(1))$ . Further, we prove that  $At^* \in \text{Sol}(\text{GGVLIP}(2))$ . As  $\|u_n - \eta_n\| \rightarrow 0$ ,  $u_n \rightarrow t^*$ ,  $n \rightarrow \infty$ , and  $\{\eta_n\}$  are bounded,  $\exists$  is a subsequence  $\{\eta_{n_j}\}$  of  $\{\eta_n\}$  with  $\eta_{n_j} \rightarrow t^*$  and  $A\eta_{n_j} \rightarrow At^*$  because  $A$  is a bounded linear operator.

We set  $q_{n_j} = A\eta_{n_j} - F_{r_n}^{(\mathfrak{B}_2, \psi_2)} A\eta_{n_j}$ . Using (47), we get  $\lim_{j \rightarrow \infty} q_{n_j} = 0$  and  $A\eta_{n_j} - q_{n_j} = F_{r_n}^{(\mathfrak{B}_2, \psi_2)} A\eta_{n_j}$ . By applying Lemma 3, we get

$$\begin{aligned} \mathfrak{B}_2(A\eta_{n_j} - q_{n_j}, v) &+ \phi_1(v, \mathfrak{z}_{n_j}) - \phi_1(\mathfrak{z}_{n_j}, \mathfrak{z}_{n_j}) \\ &+ \frac{1}{r_{n_j}} \langle v - (A\eta_{n_j} - q_{n_j}), (A\eta_{n_j} - q_{n_j}) - A\eta_{n_j} \rangle \geq 0, \quad \forall v \in \mathbb{D}_1. \end{aligned} \quad (62)$$

By taking the limit superior in (62) to be  $j \rightarrow \infty$ , using the concept of the upper semicontinuous in the first argument of  $\mathfrak{B}_2$ , and applying the given conditions, we get

$$\mathfrak{B}_2(At^*, v) + \phi_1(v, t^*) - \phi_1(t^*, t^*) \geq 0, \quad \forall v \in \mathbb{D}_1,$$

which implies  $At^* \in \text{Sol}(\text{GGVLIP}(2))$ . Thus,  $t^* \in \text{Sol}(\text{SGGVLIP}(1)-(2))$ .

Next, we show that  $t^* \in \text{Sol}(\text{VIP}(5))$ . As  $\lim_{n \rightarrow \infty} \|u_n - \mathfrak{d}_n\| = 0$ ,  $\exists \{\mathfrak{z}_{n_j}\}$  and  $\{\varrho_{n_j}\}$  are subsequences of  $\{u_n\}$  and  $\{\mathfrak{d}_n\}$  with  $\mathfrak{z}_{n_j} \rightarrow t^*$  and  $\varrho_{n_j} \rightarrow t^*$ .

Let

$$\vartheta(s) = \begin{cases} A(s) + N_{\mathbb{D}_1}(t^*), & \text{if } t^* \in \mathbb{D}_1, \\ \emptyset, & \text{if } t^* \notin \mathbb{D}_1, \end{cases}$$

where  $N_{\mathbb{D}_1}(t^*) := \{t \in \mathbb{X}_1 : \langle t^* - v, t \rangle \geq 0, \forall t \in \mathbb{D}_1\}$  is the normal cone to  $\mathbb{D}_1$  at  $t^* \in \mathbb{X}_1$ . Hence,  $\vartheta$  is maximal monotone, and  $0 \in \vartheta t^* \Leftrightarrow t^* \in \text{Sol}(\text{VIP}(5))$ . Let  $(t^*, w) \in \text{graph}(\vartheta)$ . Then,  $w \in \vartheta t^* = At^* + N_{\mathbb{D}_1}(t^*)$ , and hence,  $w - At^* \in N_{\mathbb{D}_1}(t^*)$ . Thus,  $\langle t^* - t, w - At^* \rangle \geq 0, \forall t \in \mathbb{D}_1$ . Since  $\mathfrak{d}_n = P_{\mathbb{D}_1}(u_n - \alpha_n Au_n)$  and  $t^* \in \mathbb{D}_1$ , then

$$\begin{aligned} \langle (u_n - \alpha_n Au_n) - \mathfrak{d}_n, \mathfrak{d}_n - t^* \rangle &\geq 0 \\ \implies \langle t - \mathfrak{d}_n, \frac{\mathfrak{d}_n - u_n}{\alpha_n} + Au_n \rangle &\geq 0, \quad \forall p \in \mathbb{D}_1. \end{aligned}$$

As  $\langle p - t, w - Ap \rangle \geq 0$ , for all  $p \in \mathbb{D}_1$  and  $\varrho_{n_j} \in \mathbb{D}_1$ , the monotonicity of  $A$ , we obtain

$$\begin{aligned} \langle p - \varrho_{n_j}, w \rangle &\geq \langle p - \varrho_{n_j}, At^* \rangle \\ &\geq \langle p - \varrho_{n_j}, Ap \rangle - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}} + A\mathfrak{z}_{n_j} \rangle \\ &= \langle p - \varrho_{n_j}, Ap - A\mathfrak{z}_{n_j} \rangle + \langle p - \varrho_{n_j}, A\varrho_{n_j} - A\mathfrak{z}_{n_j} \rangle \\ &\quad - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}} \rangle \\ &\geq \langle p - \varrho_{n_j}, A\varrho_{n_j} - A\mathfrak{z}_{n_j} \rangle - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}} \rangle. \end{aligned}$$

By taking  $j \rightarrow \infty$ , and according to the continuity of  $A$ , we get  $\langle p - t^*, w \rangle \geq 0$ . As  $\vartheta$  is maximal monotone,  $t^* \in \vartheta^{-1}(0)$ , and hence,  $t^* \in \text{Sol}(\text{VIP}(5))$ . Hence,  $t^* \in \Gamma$ .

As  $t^* = P_{\Gamma}h(t^*)$ , according to (33), we therefore have

$$\limsup_{n \rightarrow \infty} \langle h(t^*) - t^*, \eta_n - t^* \rangle = \lim_{j \rightarrow \infty} \langle h(t^*) - t^*, v_{n_j} - t^* \rangle \leq 0. \quad (63)$$

By applying the given conditions, (30), (42), (63), and Lemma 7, we obtain  $q_n \rightarrow 0$ , as  $n \rightarrow \infty$ . Hence,  $\{\eta_n\}$  strongly converges to  $t^* = P_{\Gamma}h(t^*)$ .

**Case 2.** Consider  $\{q_{t_j}\}$  to be a subsequence of  $\{q_t\}$  with  $q_{t_j} < q_{t_{j+1}}, \forall j \geq 0$ . Then, followed by Lemma 1, construct a nondecreasing sequence  $\{m_t\} \subset \mathbb{N}$  with  $m_t \rightarrow \infty$ , as  $t \rightarrow$

$\infty$  and  $\max\{q_{m_t}, q_t\} \leq q_{m_{t+1}}, \forall t$ . As  $r_t \in [c, d] \subset (0, \sigma^{-1})$ ,  $t \geq 0$ ,  $\sigma_t, \gamma_t, \mu_t \in (0, 1)$  with given condition and (30), we get

$$\lim_{t \rightarrow \infty} \|v_{m_t} - Ty_{m_t}\| = 0, \quad \lim_{t \rightarrow \infty} \|Ty_{m_t} - G_{m_t}v_{m_t}\| = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|v_{m_t} - G_{m_t}v_{m_t}\| = 0. \quad (64)$$

We apply the same steps as in case 1 to get

$$\limsup_{t \rightarrow \infty} \langle h(t^*) - t^*, v_{m_t} - t^* \rangle \leq 0.$$

As  $\{v_t\}$  is bounded, and  $\lim_{t \rightarrow \infty} \beta_t = 0$ , it is obtained from (32), (35), and (42)

$$\lim_{t \rightarrow \infty} \|v_{m_{t+1}} - v_{m_t}\| = 0.$$

As  $q_{m_t} \leq q_{m_{t+1}}, \forall t$ , it is obtained from (31) that

$$(1 - 2\vartheta)q_{m_{t+1}} \leq M\|v_{m_{t+1}} - v_{m_t}\| + 2\langle h(t^*) - t^*, v_{m_t} - t^* \rangle.$$

Taking  $t \rightarrow \infty$ , we get  $q_{m_{t+1}} \rightarrow 0$ . As  $q_{m_t} \leq q_{m_{t+1}}, \forall t$ , then  $q_t \rightarrow 0$ , as  $t \rightarrow \infty$ . Thus,  $v_t \rightarrow 0$ , as  $t \rightarrow \infty$ . Hence, we have proved that the sequence  $\{v_n\}$  strongly converges to  $t^* = P_\Gamma h(t^*)$ .  $\square$

Following this approach, we present several remarks that stem from the conclusions of Theorem 1. These remarks provide a concise overview of the theoretical results and pave the way for the broader exploration and application of the proposed iterative scheme across various mathematical and computational settings.

**Remark 1.** Let  $T = I$ ,  $I$  be the identity mapping, and let  $\varepsilon_i = 0$ ; that is, let  $S_i$  be a finite family of nonexpansive mappings in Theorem 1. Then,  $\Gamma := \text{Fix}(S_i) \cap \text{Sol}(\text{SGVLIP}(1 - 2)) \cap \text{Sol}(\text{VIP}(5)) \neq \emptyset$ .

**Remark 2.** Let  $B = I$ ,  $I$  be the identity mapping,  $\mathbb{X}_1 = \mathbb{X}_2$ ,  $\mathbb{D}_1 = \mathbb{D}_2$ ,  $\mathfrak{B}_1 = \mathfrak{B}_2$  and  $\psi_1 = \psi_2$  in Theorem 1. Then,  $\Gamma := \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{Fix}(T) \cap \text{Sol}(\text{GVLIP}(1)) \cap \text{Sol}(\text{VIP}(5)) \neq \emptyset$ .

### 3.1. Numerical Example

**Example 1.** Consider  $\mathbb{X}_1 = \mathbb{X}_2 = \mathbb{R}$  and  $\mathbb{D}_1 = \mathbb{D}_2 = [0, +\infty]$ . Define  $\mathfrak{B}_1(\eta_1, \mathfrak{w}_1; \mathfrak{w}_1) = (\eta_1 - \mathfrak{w}_1)(\eta_1 + 2\mathfrak{w}_1)$ ,  $\forall \eta_1, \mathfrak{w}_1 \in \mathbb{D}_1$ ,  $\mathfrak{B}_2(\eta_2, \mathfrak{w}_2; \mathfrak{w}_2) = (\eta_2 - \mathfrak{w}_2)(\eta_2 + 2\mathfrak{w}_2)$ ,  $\forall \eta_2, \mathfrak{w}_2 \in \mathbb{D}_2$  with  $\alpha(\eta, \mathfrak{w}) = (\mathfrak{w} - \eta)^2 \forall \eta, \mathfrak{w} \in \mathbb{D}_1$ , and  $\psi_1(\eta_1, \mathfrak{w}_1) = \eta_1 \mathfrak{w}_1$ ,  $\psi_2(\eta_2, \mathfrak{w}_2) = \eta_2 \mathfrak{w}_2$ ,  $\forall \eta_2, \mathfrak{w}_2 \in \mathbb{D}_2$ . It is obvious that  $\mathfrak{B}_1, \mathfrak{B}_2, \psi_1$  and  $\psi_2$  meet the conditions specified in Assumption 1. Additionally, assume  $g(\eta) = \frac{\eta}{5}$ ,  $\mathfrak{f}\eta = 3\eta$ ,  $\eta \in \mathbb{D}_1$ ;  $A(\eta) = \frac{\eta}{2}$ ,  $\eta \in \mathbb{X}_1$ ;  $T(\eta) = \frac{\eta}{4}$ ,  $\eta \in \mathbb{D}_1$  and  $S_i(\eta) = -(1 + i)\eta$ ,  $\eta \in \mathbb{D}_1$ ,  $i = 1, 2, 3$ . By setting  $r_n = \frac{2}{3}$ ,  $\alpha_n = \{\frac{1}{5}\}$ ,  $\lambda = \frac{1}{6}$ ,  $\eta_n = \{\frac{1}{10n}\}$ ,  $\sigma_n = 0.7 + \frac{0.1}{n^2}$ ,  $\gamma_n = 0.2 - \frac{0.2}{n^2}$ ,  $\mu_n = 0.1 + \frac{0.1}{n^2}$ , and  $\psi_{i=1}^3 = \frac{1}{3}$ , we find that the sequence generated by Algorithm 1 converges to  $t^* = \{0\} \in \Gamma$ , and the algorithm simplifies to

$$\left\{ \begin{array}{l} u_n = F_{r_n}^{(\mathfrak{B}_1, \psi_1)}(\eta_n + \frac{1}{6}A^*((F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I)A\eta_n) \\ \mathfrak{d}_n = P_{\mathbb{D}_1}(u_n - \frac{1}{5}\mathfrak{f}u_n) \\ \eta_{n+1} = \eta_n g(\eta_n) + (1 - \eta_n)P_{\mathbb{D}_1}[\sigma_n \eta_n + \gamma_n T\mathfrak{d}_n + \mu_n(\varphi_1 S_1 \eta_n + \varphi_2 S_2 \eta_n + \varphi_3 S_3 \eta_n)] \end{array} \right\} \quad (65)$$

The computations and graphical analyses for the proposed algorithm were performed using MATLAB R2015(a). Figures 1 and 2 depict the convergence patterns corresponding to different initial points. The stopping criterion employed is  $\|\eta_{n+1} - \eta_n\| < 10^{-10}$ . Several initial values  $\eta_0$  were tested, and the outcomes are summarized in Tables 1 and 2, which also include comparisons with the results reported in [24,25]. The convergence trends

are further illustrated in Figures 1 and 2. From these results, it is evident that, across different initial points, the proposed algorithm generally completes computations faster (often measured in seconds) than the existing methods.

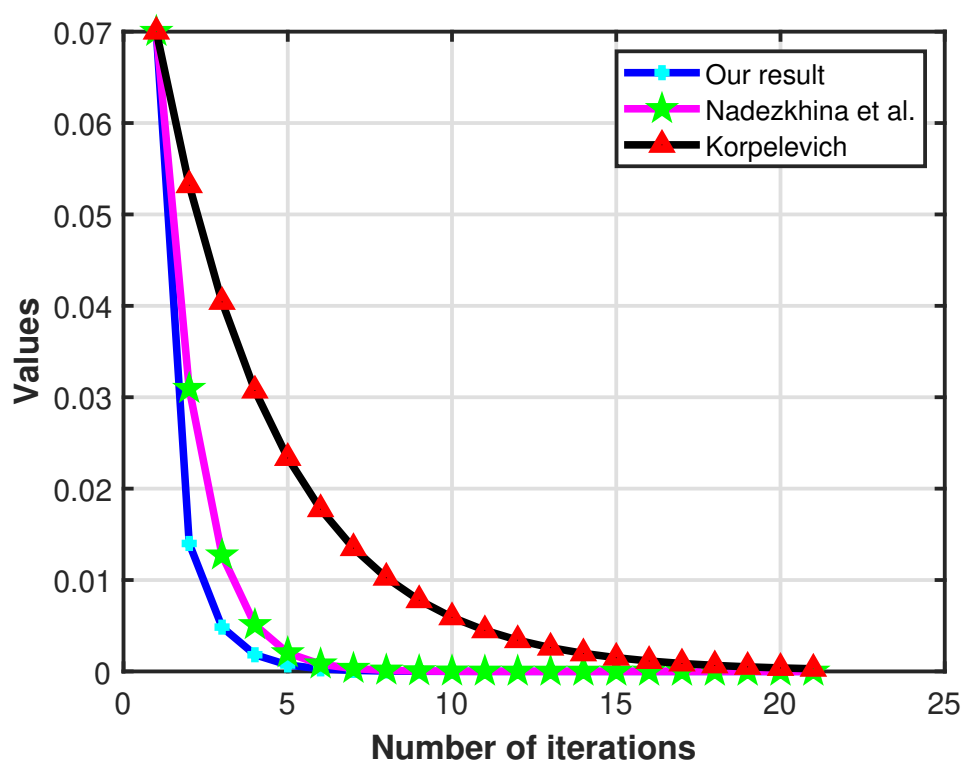


Figure 1. Graph of convergence  $\{\eta_n\}$  at  $\eta_0 = 0.07$  [24,25].

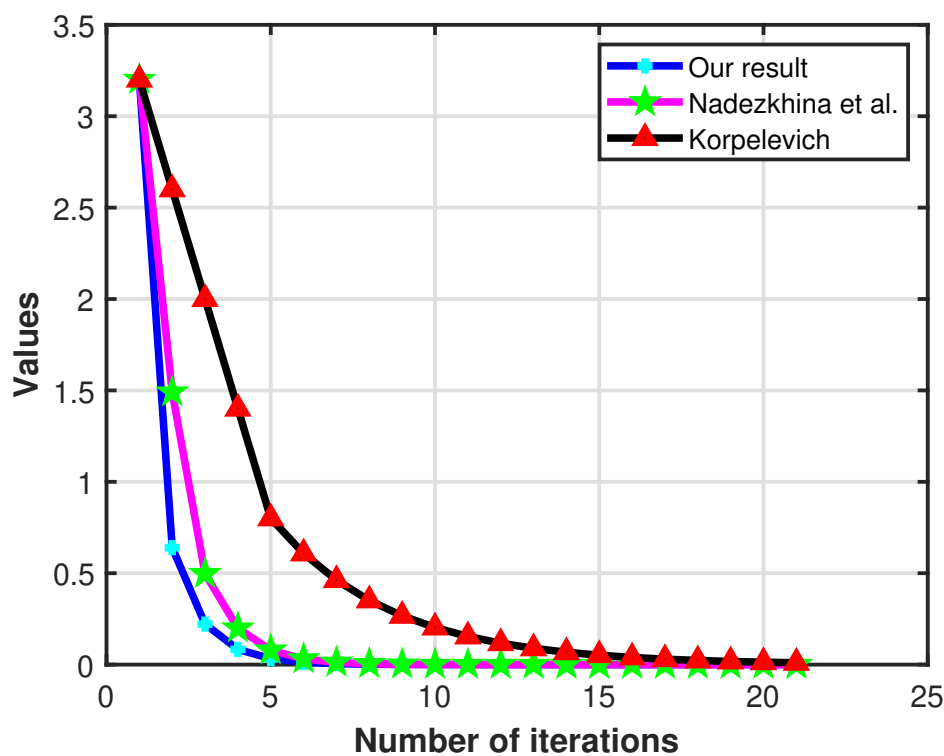


Figure 2. Graph of convergence  $\{\eta_n\}$  at  $\eta_0 = 3.2$  [24,25].

**Table 1.** Comparison of our main results for initial point  $\eta_0 = 0.07$ .

| No. of Iter. | Our Result<br>cpu Time (in s) | Nadezkhina et al. [25]<br>cpu Time (in s) | Korpelevich [24]<br>cpu Time (in s) |
|--------------|-------------------------------|-------------------------------------------|-------------------------------------|
| 1            | 0.014000                      | 0.030940                                  | 0.053200                            |
| 2            | 0.004848                      | 0.012716                                  | 0.040432                            |
| 3            | 0.001825                      | 0.005095                                  | 0.030728                            |
| 4            | 0.000707                      | 0.002015                                  | 0.023354                            |
| 5            | 0.000278                      | 0.000791                                  | 0.017749                            |
| 6            | 0.000110                      | 0.000309                                  | 0.013489                            |
| 7            | 0.000044                      | 0.000120                                  | 0.010252                            |
| 8            | 0.000018                      | 0.000047                                  | 0.007791                            |
| 9            | 0.000007                      | 0.000018                                  | 0.005921                            |
| 10           | 0.000003                      | 0.000007                                  | 0.004500                            |
| 11           | 0.000001                      | 0.000003                                  | 0.003420                            |
| 12           | 0.000000                      | 0.000001                                  | 0.002599                            |

**Table 2.** Comparison of our main results for initial point  $\eta_0 = 3.2$ .

| No. of Iter. | Our Result<br>cpu Time (in s) | Nadezkhina et al. [25]<br>cpu Time (in s) | Korpelevich [24]<br>cpu Time (in s) |
|--------------|-------------------------------|-------------------------------------------|-------------------------------------|
| 1            | 0.640000                      | 1.490000                                  | 2.600000                            |
| 2            | 0.221612                      | 0.497250                                  | 2.000000                            |
| 3            | 0.083414                      | 0.199232                                  | 1.400000                            |
| 4            | 0.032335                      | 0.078796                                  | 0.800000                            |
| 5            | 0.012712                      | 0.030920                                  | 0.608000                            |
| 6            | 0.005037                      | 0.012069                                  | 0.462080                            |
| 7            | 0.002006                      | 0.004693                                  | 0.351181                            |
| 8            | 0.000802                      | 0.001820                                  | 0.266897                            |
| 9            | 0.000321                      | 0.000704                                  | 0.202842                            |
| 10           | 0.000129                      | 0.000272                                  | 0.154160                            |
| 11           | 0.000052                      | 0.000105                                  | 0.117162                            |
| 12           | 0.000021                      | 0.000040                                  | 0.089043                            |
| 13           | 0.000008                      | 0.000016                                  | 0.067673                            |
| ...          | ...                           | ...                                       | ...                                 |
| ...          | ...                           | ...                                       | ...                                 |
| 17           | 0.000000                      | 0.000001                                  | 0.022577                            |

**Example 2.** Consider  $\mathbb{X}_1 = \mathbb{X}_2 = C[0, 1]$  as the space of all continuous real valued functions defined on  $[0, 1]$  with the sup norm as  $\|h\|_\infty = \max_{p \in [0, 1]} |h(p)|$  and  $\mathbb{D}_1 = \mathbb{D}_2 = \{h \in C[0, 1] : h(p) \geq 0, \forall p \in [0, 1]\}$ . The metric projection of  $h \in C[0, 1]$  onto  $\mathbb{D}_1$  is given by  $(P_{\mathbb{D}_1} h)(p) = \max\{h(p), 0\}$ ,  $p \in [0, 1]$ . Define  $\mathfrak{B}_1(h_1, l_1; l_1) = (h_1 - l_1)(h_1 + 2l_1)$ ,  $\forall h_1, l_1 \in \mathbb{D}_1$ ,  $\mathfrak{B}_2(h_2, l_2; l_2) = (h_2 - l_2)(h_2 + 2l_2)$ ,  $\forall h_2, l_2 \in \mathbb{D}_2$  with  $\alpha(h, l) = (h - l)^2 \forall h, l \in \mathbb{D}_1$ , and  $\psi_1(h_1, l_1) = h_1 l_1$ ,  $\psi_2(h_2, l_2) = h_2 l_2$ ,  $\forall h_2, l_2 \in \mathbb{D}_2$ . It is obvious that  $\mathfrak{B}_1, \mathfrak{B}_2, \psi_1$  and  $\psi_2$  meet the conditions specified in Assumption 1. Additionally, assume  $g(h) = \frac{h}{5}$ ,  $f(h) = 3h$ ,  $h \in \mathbb{D}_1$ ;

$A(h) = \frac{h}{2}$ ,  $h \in \mathbb{X}_1$ ;  $T(h) = \frac{h}{4}$ ,  $h \in \mathbb{D}_1$  and  $S_i(h) = -(1+i)h$ ,  $h \in \mathbb{D}_1$ ,  $i = 1, 2, 3$ . By setting  $r_n = \frac{2}{3}$ ,  $\alpha_n = \{\frac{1}{5}\}$ ,  $\lambda = \frac{1}{6}$ ,  $\eta_n = \{\frac{1}{10n}\}$ ,  $\sigma_n = 0.7 + \frac{0.1}{n^2}$ ,  $\gamma_n = 0.2 - \frac{0.2}{n^2}$ ,  $\mu_n = 0.1 + \frac{0.1}{n^2}$ , and  $\psi_{i=1}^3 = \frac{1}{3}$ , we find that the sequence generated by Algorithm 1 converges to  $h^* = \{0\} \in \Gamma$ .

The computations and graphical visualizations for this algorithm were carried out using MATLAB R2015a on a standard HP laptop featuring an Intel Core i7 processor and 8 GB of RAM. The stopping criterion was set to  $\|\eta_{n+1} - \eta_n\| < 10^{-10}$ . We analyzed the convergence for distinct initial points, and the convergence trends are illustrated in Figure 3. Further, we extended the experiment to show how convergence depends on distinct choices of control sequences: vary  $\lambda$  (step-size parameter); vary  $\alpha_n$  (strongly monotone scaling), and vary  $\eta_n$  (weight between contraction mapping and other operators). We show the convergence by taking distinct values of  $\lambda$ ,  $\alpha_n$ , and  $\eta_n$  in Figures 4–6. In the graphs, we assume  $y = \eta$ .

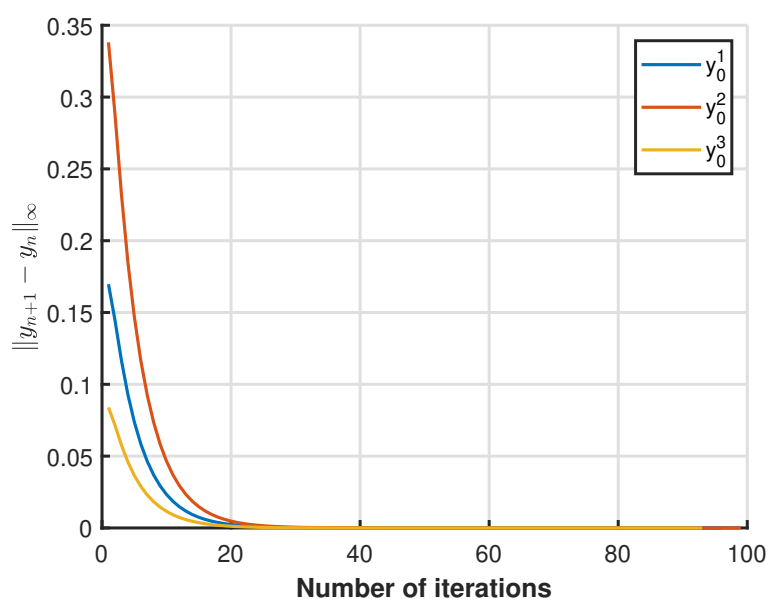


Figure 3. Graph of convergence.

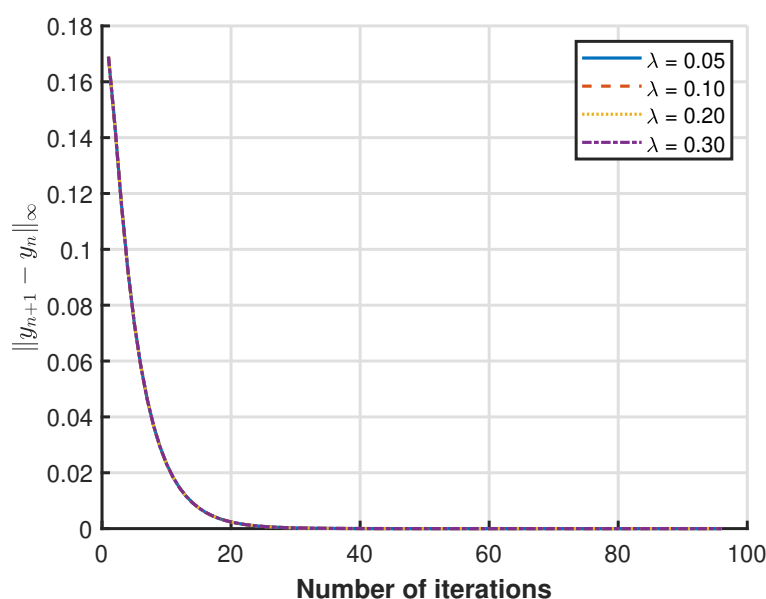
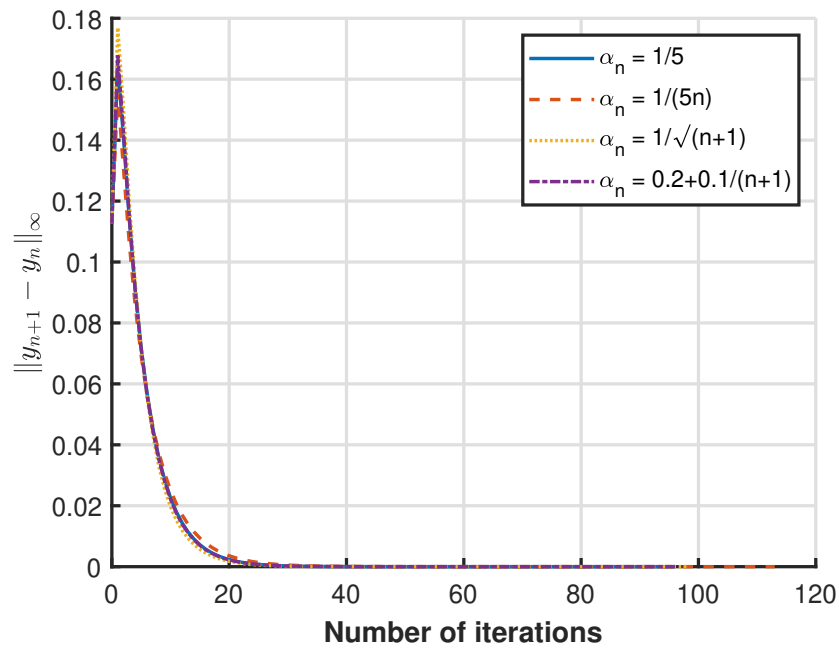
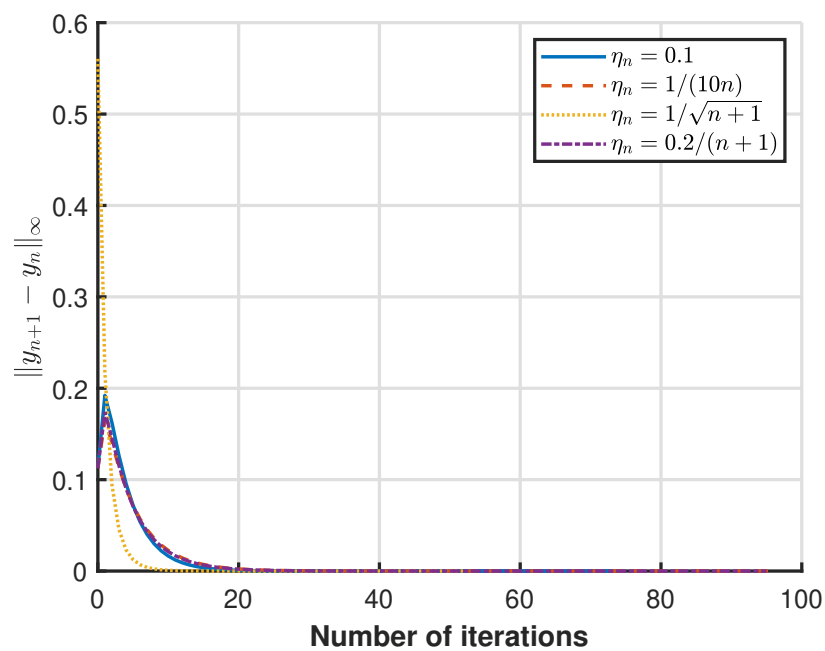


Figure 4. Convergence for distinct values of  $\lambda$ .



**Figure 5.** Convergence for distinct values of  $\{\alpha_n\}$ .



**Figure 6.** Convergence for distinct values of  $\{\eta_n\}$ .

### 3.2. Application: Image Denoising Problem of Iteration (13)

Moreover, to evaluate the effectiveness of our approach, we applied the proposed iterative projection-based scheme to a noisy version of the test image. The original image was first corrupted with Gaussian noise, producing a degraded observation. By applying our iterative method, the noisy input was progressively refined through successive updates, resulting in a denoised image. In standard image denoising, one commonly has  $A = I$ ,  $\mathbb{X}_1$  (signal space) =  $\mathbb{X}_2$  (observation space) =  $\mathbb{R}^{m \times n}$ . Assume the trifunctions  $\mathfrak{B}_j$  and

bifunctions  $\psi_j$  are such that their resolvent-like maps  $F_r^{(\mathfrak{B}_j, \psi_j)}$  coincide with proximal maps of convex functionals  $F_j$ , i.e.,

$$F_r^{(\mathfrak{B}_j, \psi_j)} \equiv \text{prox}_{rF_j}, \quad j = 1, 2,$$

where

$$\text{prox}_{rF}(z) = \arg \min_{w \in \mathbb{D}} \left\{ F(w) + \frac{1}{2r} \|w - z\|^2 \right\}.$$

Then, the iterative process (13) can be written in proximal form as

$$\begin{cases} \eta_0 \in \mathbb{D}_1, \\ \mathfrak{v}_n = \text{prox}_{rF_2}(A\eta_n), \\ \mathfrak{u}_n = \text{prox}_{rF_1}(\eta_n + \lambda A^*(\mathfrak{v}_n - A\eta_n)), \\ \mathfrak{d}_n = P_{\mathbb{D}_1}(\mathfrak{u}_n - \alpha_n \mathfrak{f}(\mathfrak{u}_n)), \\ \eta_{n+1} = \eta_n g(\eta_n) + (1 - \eta_n) P_{\mathbb{D}_1} \left[ \sigma_n \eta_n + \gamma_n T \mathfrak{d}_n + \mu_n \sum_{i=1}^N \varphi_i^{(n)} S_i \eta_n \right], \end{cases} \quad (66)$$

with the same control conditions (i)–(vi).

If  $F_2(z) = \frac{1}{2} \|z - b\|^2$  (standard Gaussian fidelity), then

$$\text{prox}_{rF_2}(z) = \frac{z + rb}{1 + r}.$$

In practice, one often replaces the exact data-prox by an explicit gradient step:

$$\text{prox}_{rF_2}(A\eta_n) \approx A\eta_n - \tau(A\eta_n - b),$$

so that with  $A = I$ , the update in (66) reduces to the familiar proximal-gradient (prox-TV) denoising iteration:

$$\begin{aligned} z_n &= \eta_n - \tau(\eta_n - b), \\ \mathfrak{u}_n &= \text{prox}_{\tau\lambda \text{TV}}(z_n), \\ \eta_{n+1} &= P_{\mathbb{D}_1}(\mathfrak{u}_n). \end{aligned}$$

By setting the gradient step size  $\alpha_n \equiv 0.2$ , the regularization weight (prox parameter)  $\lambda \equiv 0.1$ , and the trivial convex combination weights  $\eta_n \equiv 0$ ,  $\sigma_n \equiv 0$ ,  $\gamma_n \equiv 1$ ,  $\mu_n \equiv 0$ , we represent the image denoising via proximal iteration in Figure 7. In Figure 7, we show that the different blurred images are degraded, along with the peak signal-to-noise ratio (PSNR); this shows the quantitatively measured image-restoring process and also the measured structural similarity index (SSIM). For the convenience of readers, we provide a flowchart of our iterative algorithm in Figure 8.

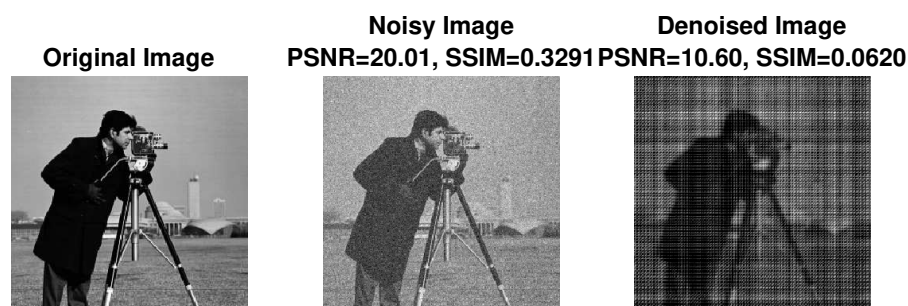


Figure 7. Image denoising via proximal iteration.

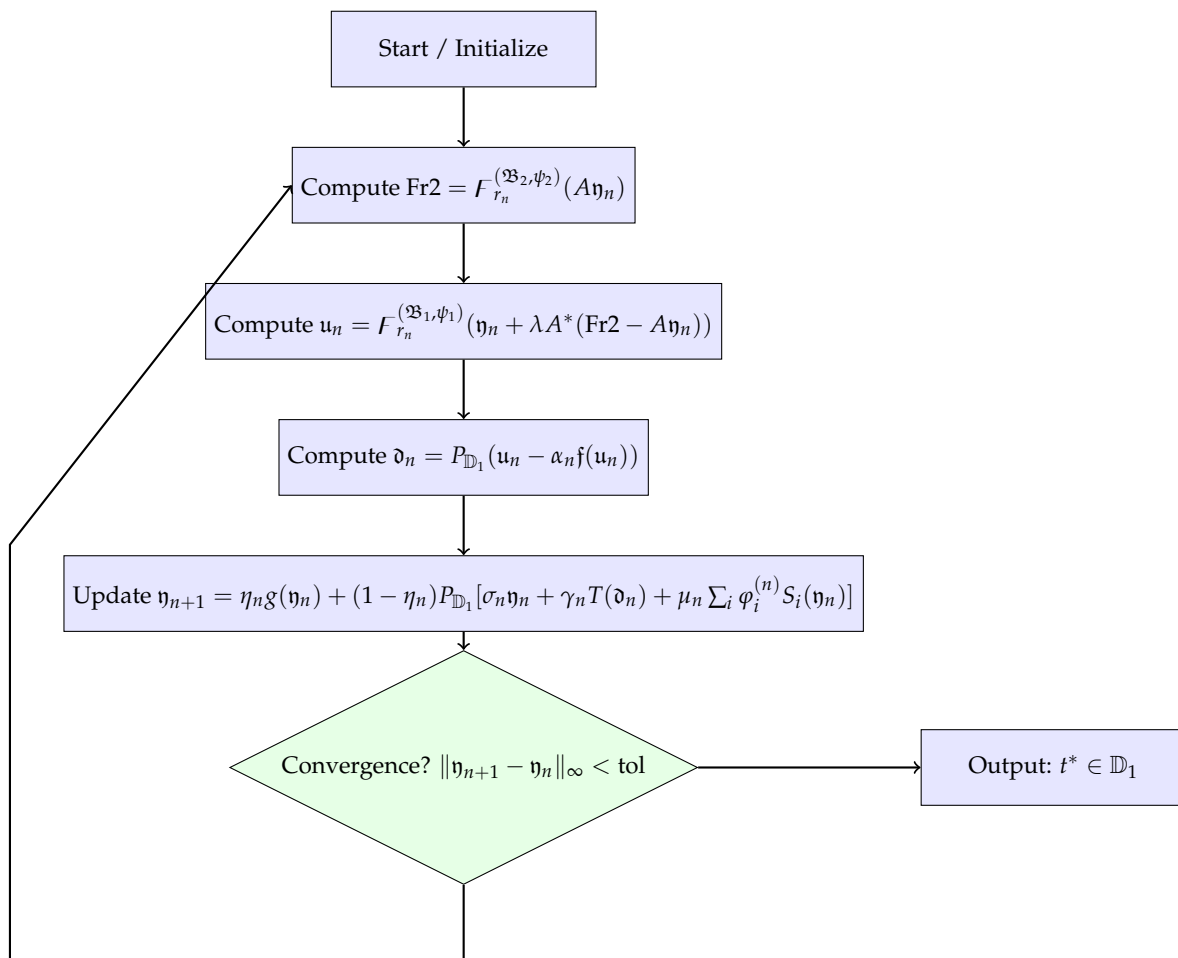


Figure 8. Flowchart of the iterative algorithm.

#### 4. Conclusions

In summary, this work provides important insights into the behavior of the proposed iterative algorithm. The scheme is shown to converge strongly to a common solution for the SGGVLIPs, VIPs, and FPPs associated with a finite set of  $\varepsilon$ -strict pseudocontractive mappings and a nonexpansive operator within a Hilbert space framework. To illustrate its practical relevance, numerical results are presented through detailed tables and graphical representations, including comparisons with existing methods and analyses of convergence trends, which collectively demonstrate the algorithm's effectiveness and computational efficiency.

Although strong convergence is theoretically guaranteed, the algorithm's performance can be influenced by the choice of parameters, and explicit convergence rates are not provided. Practical application may benefit from careful tuning of control sequences to enhance stability and performance. Furthermore, several avenues for future research are identified, including extending the approach to Banach spaces, establishing explicit convergence rates, and exploring inertial or stochastic modifications to broaden its applicability.

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**Conflicts of Interest:** The authors declare that they have no competing interests.

## Appendix A. Acronyms

| Acronym | Meaning                                                       |
|---------|---------------------------------------------------------------|
| FPP     | Fixed Point Problem                                           |
| FPPs    | Fixed Point Problems                                          |
| VIP     | Variational Inequality Problem                                |
| VLIP    | Variational-Like Inequality Problem                           |
| GVLIP   | General Variational-Like Inequality Problem                   |
| GGVLIP  | Generalized General Variational-Like Inequality Problem       |
| SFP     | Split Feasibility Problem                                     |
| GGVLIP  | Split Generalized General Variational-Like Inequality Problem |

## Appendix B. Algebraic Details

| Acronym                                                                                        | Meaning                                                                                                                                                                                                                                                                                                    |
|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $f : \mathbb{D}_1 \rightarrow \mathbb{X}_1$                                                    | $:= \sigma$ -inverse strongly monotone                                                                                                                                                                                                                                                                     |
| $g : \mathbb{D}_1 \rightarrow \mathbb{D}_1$                                                    | $:= \vartheta$ -contraction mapping                                                                                                                                                                                                                                                                        |
| $T : \mathbb{D}_1 \rightarrow \mathbb{X}_1$                                                    | $:=$ nonexpansive mapping                                                                                                                                                                                                                                                                                  |
| $S_i : \mathbb{D}_1 \rightarrow \mathbb{X}_1$                                                  | $:= \varepsilon_i$ -strict pseudo-contractions, $(i = 1, 2, \dots)$                                                                                                                                                                                                                                        |
| $A : \mathbb{X}_1 \rightarrow \mathbb{X}_2$                                                    | $:=$ bounded linear operator                                                                                                                                                                                                                                                                               |
| $\mathfrak{B}_j : \mathbb{D}_j \times \mathbb{D}_j \times \mathbb{D}_j \rightarrow \mathbb{R}$ | $:=$ be trifunctions, $j = 1, 2$                                                                                                                                                                                                                                                                           |
| $\psi_j : \mathbb{D}_j \times \mathbb{D}_j$                                                    | $:=$ be bifunctions, $j = 1, 2$                                                                                                                                                                                                                                                                            |
| $\alpha_n, \lambda, \eta_n, \sigma_n, \gamma_n, \mu_n$                                         | $:=$ are control sequences                                                                                                                                                                                                                                                                                 |
| $F_r^{(\mathfrak{B}_1, \psi_1)}(\eta)$                                                         | $:= \{ \mathfrak{x} \in \mathbb{D}_1 : \mathfrak{B}_1(\mathfrak{w}, \mathfrak{x}; \mathfrak{x}) + \psi_1(\mathfrak{x}, \mathfrak{w}) - \psi_1(\mathfrak{w}, \mathfrak{w}) + \frac{1}{r} \langle \mathfrak{w} - \mathfrak{x}, \mathfrak{x} - \eta \rangle \geq 0, \forall \mathfrak{w} \in \mathbb{D}_1 \}$ |
| $F_s^{(\mathfrak{B}_2, \psi_2)}(\eta)$                                                         | $:= \{ \mathfrak{x} \in \mathbb{D}_2 : \mathfrak{B}_2(\mathfrak{w}, \mathfrak{x}; \mathfrak{x}) + \psi_2(\mathfrak{x}, \mathfrak{w}) - \psi_2(\mathfrak{x}, \mathfrak{x}) + \frac{1}{s} \langle \mathfrak{w} - \mathfrak{x}, \mathfrak{x} - \eta \rangle \geq 0, \forall \mathfrak{w} \in \mathbb{D}_2 \}$ |
| $s_n$                                                                                          | $:= \sigma_n \eta_n + \gamma_n T \mathfrak{d}_n + \mu_n \sum_{i=1}^N \varphi_i^{(n)} S_i \eta_n$                                                                                                                                                                                                           |
| $\Pi$                                                                                          | $:= 2\lambda \langle \eta_n - \mathfrak{t}^*, A^* (F_{r_n}^{(\mathfrak{B}_2, \psi_2)} - I) A \eta_n \rangle$                                                                                                                                                                                               |

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