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Luxemburg Norm Characterizations of BLO Spaces in General Metric Measure Frameworks

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Abstract

This study provides new equivalent descriptions of the Bounded Lower Oscillation (BLO) space through Luxemburg-type L^φ integrability conditions, where φ is a nonnegative function with either convexity or concavity. The framework accommodates various representative forms of φ , such as the power function $\varphi(t) = t^p$, exponential-type functions $\varphi(t) = e^{pt} - 1$, and logarithmic functions $\varphi(t) = \log_+^k t$, with parameters $p \in (0, \infty)$ and $k \in \mathbb{N}$. These results unify and extend existing characterizations of BLO by encompassing a broad class of generating functions.

Keywords: BLO space; Luxemburg; space of homogeneous type; non-atomic measure; Calderón-Zygmund decomposition

MSC: 42B35; 42B25; 46E30

1. Introduction

The classical space $\text{BLO}(\mathbb{R}^n)$, originally formulated by Coifman and Rochberg in [1], comprises all locally integrable functions h such that the following is written:

$$\|h\|_{\text{BLO}(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q h(x) - c_{h,Q} dx < \infty, \quad (1)$$

where the supremum runs over all cubes $Q \subset \mathbb{R}^n$ and the following:

$$c_{h,Q} = \text{ess inf}_{y \in Q} h(y). \quad (2)$$

A significant characterization of $\text{BLO}(\mathbb{R}^n)$ was established by Bennett in 1982 [2], using the natural maximal operator in connection with the $\text{BMO}(\mathbb{R}^n)$ space. Liu and Yang [3] extended Bennett's result [2] to Gauss measure metric space. For a more comprehensive account of the research on the space $\text{BLO}(\mathbb{R}^n)$, we refer the reader to Leckband [4], Jiang [5], Tang [6], Yang [7,8] and the references therein [9–13].

The problem of characterizing BMO under weaker integrability conditions can be traced back to Strömberg [14]. Afterwards, the studies of [15–17] further demonstrate the weak conditions of defining BMO. More recently, assuming concavity of the function φ , Canto, Pérez and Rela in [18] (Theorem 1.1) further researched the findings of [15] by establishing minimal integrability conditions in terms of Luxemburg-type norms, specifically expressed as follows:



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$$\|h\|_{\text{BMO}(\mathbb{R}^n)} \approx \sup_{Q \subset \mathbb{R}^n} \inf_{c \in \mathbb{R}} \left\{ \gamma > 0 : \frac{1}{|Q|} \int_Q \varphi \left(\frac{|h-c|}{\gamma} \right) dx \leq 1 \right\}.$$

Throughout these discussions, setting $\varphi(t) = t^p$ for some $p \in (0, 1)$ yields the equivalence $\text{BMO}(\mathbb{R}^n) = \text{BMO}^p(\mathbb{R}^n)$.

Although the space BMO has been extensively investigated under minimal integrability assumptions, analogous results for BLO remain largely undeveloped. Motivated by this gap, the present work is devoted to establishing Luxemburg-type characterizations of BLO spaces under both convexity and concavity conditions. The Luxemburg norm, originating from the modular approach to Orlicz spaces, provides a classical yet versatile framework for studying generalized integrability conditions. For a recent application of the Luxemburg norm in approximation theory, we refer the reader to Costarelli and Vinti [19].

Our aim is to identify the minimal integrability requirements for characterizing BLO via Luxemburg-type formulations. To this end, we begin by recalling several fundamental notions.

For a cube R , if $\varphi: [0, \infty] \rightarrow [0, \infty]$ is a function and $\varphi \circ h$ is a locally integrable function on R , then the Luxemburg-type norm is given by the following:

$$\|h\|_{\varphi, R} := \inf \left\{ \gamma > 0 : \frac{1}{|R|} \int_R \varphi \left(\frac{|h(x)|}{\gamma} \right) dx \leq 1 \right\}.$$

Moreover, we define the following

$$L^\varphi(R) := \{ \varphi \circ h \text{ locally integrable} : \|h\|_{\varphi, R} < \infty \}.$$

Definition 1. Let $\varphi: [0, \infty] \rightarrow [0, \infty]$. The Luxemburg-type BLO space, denoted by $\text{BLO}_\varphi(\mathbb{R}^n)$, consists of the collection of all measurable functions $h: \mathbb{R}^n \rightarrow [-\infty, +\infty]$ such that the composition $\varphi \circ h$ is locally integrable over $\mathbb{R}^n$, and satisfies the following:

$$\|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} := \sup_{R \subset \mathbb{R}^n} \|h - c_{h,R}\|_{\varphi, R} < \infty,$$

where $c_{h,R}$ is defined as in (2). If $\varphi(t) = t$, then the Luxemburg-type space $\text{BLO}_\varphi(\mathbb{R}^n)$ coincides with the classical space $\text{BLO}(\mathbb{R}^n)$.

We now recall the definitions of convex and concave functions. Let $\gamma \in (0, 1)$ and $0 \leq t_1, t_2 < \infty$. If the following is true:

$$\varphi(\gamma t_1 + (1 - \gamma)t_2) \leq \gamma \varphi(t_1) + (1 - \gamma)\varphi(t_2),$$

then $\varphi: [0, \infty] \rightarrow [0, \infty]$ is the said convex function. If the following is true:

$$\varphi(\gamma t_1 + (1 - \gamma)t_2) \geq \gamma \varphi(t_1) + (1 - \gamma)\varphi(t_2),$$

then $\varphi: [0, \infty] \rightarrow [0, \infty]$ is said concave function.

Remark 1. Recall [20] (Remark 1.4), if $\varphi: [0, \infty) \rightarrow [0, \infty)$ is convex satisfying $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$, then we find that the inverse function φ^{-1} is well defined on (N_0, ∞) , where N_0 is given by (5). In addition, if φ is concave and satisfies $\lim_{t \rightarrow \infty} \varphi(t) = \infty$, then φ is necessarily increasing on $[0, \infty)$. Furthermore, when φ is increasing and concave

with $\varphi(0) = 0$, it follows from [15] (Remark 1.3) that for any $t_1, t_2 \in \mathbb{R}$, the following is calculated:

$$\varphi(|t_1 + t_2|) \leq \varphi(|t_1|) + \varphi(|t_2|). \quad (3)$$

Definition 2. Let $\varphi: [0, \infty] \rightarrow [0, \infty]$ be a given function, and $(\mathbb{X}, d, \eta)$ denote a space endowed with a quasi-metric d and a doubling measure η . Let $c_{h,B}$ be defined as in (2).

(i) The space $\text{BLO}(\mathbb{X})$ consists of all functions h satisfying the following:

$$\|h\|_{\text{BLO}(\mathbb{X})} := \sup_{Q \subset \mathbb{X}} \frac{1}{\eta(B)} \int_B h(x) - c_{h,B} d\eta(x) < \infty.$$

(ii) The space $\text{BLO}_\varphi(\mathbb{X})$ includes all functions h for which $\varphi \circ h$ is locally integrable function on $(\mathbb{X}, d, \eta)$ and the following:

$$\|h\|_{\text{BLO}_\varphi(\mathbb{X})} := \sup_{B \subset \mathbb{X}} \inf \left\{ \gamma > 0 : \frac{1}{\eta(B)} \int_B \varphi \left(\frac{h(x) - c_{h,B}}{\gamma} \right) d\eta(x) \leq 1 \right\} < \infty.$$

Definition 3. Let $\varphi: [0, \infty] \rightarrow [0, \infty]$ be a function, $c_{h,Q}$ be defined as in (2) and ω be non-atomic measure.

(i) The space $\text{BLO}(\omega)$ consists of all functions h for which the following is calculated:

$$\|h\|_{\text{BLO}(\omega)} := \sup_{Q \in \mathbb{R}^n} \frac{1}{\omega(Q)} \int_Q h(x) - c_{h,Q} d\omega < \infty.$$

(ii) The space $\text{BLO}_\varphi(\omega)$ includes all functions h for which $\varphi \circ h$ is locally integrable function on $(\mathbb{R}^n, \omega)$ and the following:

$$\|h\|_{\text{BLO}_\varphi(\omega)} := \sup_{Q \in \mathbb{R}^n} \inf \left\{ \gamma > 0 : \frac{1}{\omega(Q)} \int_Q \varphi \left(\frac{h(x) - c_{h,Q}}{\gamma} \right) d\omega \leq 1 \right\} < \infty.$$

Definition 4. For any cube $Q(x, L)$ centered at x with side length L , assume the existence of constants $\alpha, C > 0$ for which the following is calculated:

$$\omega(Q(x, L)) \leq CL^\alpha$$

holds for $x \in \text{supp}(\omega)$. Then we assert that the measure ω is non-atomic.

The structure of the paper is organized as follows. Section 2 is devoted to Luxemburg norm characterizations of $\text{BLO}(\mathbb{R}^n)$ in Theorems 1 and 2, where we establish the fundamental equivalence results in the Euclidean setting. Section 3 addresses Luxemburg norm characterizations of $\text{BLO}(\mathbb{X})$ in Theorem 3, extending the previous analysis to the space of homogeneous type. Section 4 presents Luxemburg norm characterization of $\text{BLO}(\omega)$ in Theorem 4, in which we further investigate the weighted framework under non-atomic measures. Finally, Section 5 provides a conclusion, where we summarize the main contributions.

In this paper, the following terminology and notation will be employed: Define $\mathbb{N} = \{1, 2, \dots\}$. For subset $E \subset \mathbb{R}^n$, denote its complement by $E^c = \mathbb{R}^n \setminus E$. The notation $f_1 \lesssim f_2$ indicates the existence of a constant $C > 0$ such that $f_1 \leq Cf_2$; similarly, $f_1 \gtrsim f_2$ means $f_1 \geq Cf_2$. We write $f_1 \approx f_2$ when both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ hold.

2. Luxemburg Norm Characterizations of $\text{BLO}(\mathbb{R}^n)$

In this section, we present Theorems 1 and 2, which establishes the Luxemburg norm characterizations of $\text{BLO}(\mathbb{R}^n)$. To proceed, we begin by recalling the John–Nirenberg-type inequality for the $\text{BLO}(\mathbb{R}^n)$ space, as stated in [9] (Lemma 2.1).

Lemma 1 ([9]). *If $f \in \text{BLO}(\mathbb{R}^n)$, then, there exist constants $c_*, C_{JN} > 0$ for which it holds that the following is true:*

$$|\{x \in R : |h(x) - c_{h,R}| > s\}| \leq C_{JN} |R| e^{-\frac{c_* s}{\|h\|_{\text{BLO}(\mathbb{R}^n)}}}$$

for all $h \in \text{BLO}(\mathbb{R}^n)$, $s \in (0, \infty)$ and $R \subset \mathbb{R}^n$, where $c_{h,R}$ is as in (2).

Lemma 2 ([20]). *Let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be convex satisfying $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Then, for any $t_1 \in [0, \infty)$ and $t_2 \in (N_0, \infty)$, calculate the following:*

$$t_1 \leq \frac{t_2}{\varphi(t_2)} \varphi(t_1) + t_2, \quad (4)$$

where N_0 is defined in (5).

Theorem 1. *Assume that $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a function with $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Then the following hold.*

(i) *Suppose that φ is convex and φ^{-1} denotes the inverse of φ on (N_0, ∞) with the following:*

$$N_0 := \inf\{t \in [0, \infty) : \varphi(t) > 0\}. \quad (5)$$

If there exists $\gamma \in (0, \infty)$ satisfying the following:

$$\int_0^\infty e^{-\gamma \varphi^{-1}(t)} dt < \infty, \quad (6)$$

then the spaces $\text{BLO}(\mathbb{R}^n)$ and $\text{BLO}_\varphi(\mathbb{R}^n)$ can be identified with equivalent norms.

(ii) *If φ is concave, then $\text{BLO}(\mathbb{R}^n) = \text{BLO}_\varphi(\mathbb{R}^n)$. Moreover, there exist constants $c_{n,\varphi}, C_{n,\varphi}$ such that the following is calculated:*

$$c_{n,\varphi} \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} \leq \|h\|_{\text{BLO}(\mathbb{R}^n)} \leq C_{n,\varphi} \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)}.$$

Proof. Now, we first verify (i). Given any $h \in \text{BLO}_\varphi(\mathbb{R}^n)$, we will verify the following:

$$\|h\|_{\text{BLO}(\mathbb{R}^n)} \leq C \left(\frac{2}{\varphi(C)} + 1 \right) \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)}, \quad (7)$$

where $N_0 < C < \infty$ and N_0 is defined in (5). Observe that $\varphi(C) > 0$ for such C . We normalize $\|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} = 1$. Denote by $c_{h,Q}$ the constant defined in (2). For any cube $Q \subset \mathbb{R}^n$, we calculate the following:

$$\frac{1}{|Q|} \int_Q \varphi(h(x) - c_{h,Q}) dx \leq 2.$$

Applying Lemma 2, we deduce the following:

$$\begin{aligned}
\frac{1}{|Q|} \int_Q h - c_{h,Q} dx &\leq \frac{1}{|Q|} \int_Q \left(\frac{C}{\varphi(C)} \varphi(h - c_{h,Q}) + C \right) dx \\
&\leq \frac{2C}{\varphi(C)} + C \\
&= C \left(\frac{2}{\varphi(C)} + 1 \right) \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)}.
\end{aligned}$$

The inequality (7) is thus derived by applying the supremum over all cubes Q .

Next, suppose $h \in \text{BLO}(\mathbb{R}^n)$. Our goal is to prove the following:

$$\|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}(\mathbb{R}^n)}. \quad (8)$$

A sufficient condition for (8) is the existence of a constant $C_0 \in (0, \infty)$ satisfying the following:

$$\frac{1}{|Q|} \int_Q \varphi \left(\frac{h(x) - c_{h,Q}}{C_0 \|h\|_{\text{BLO}(\mathbb{R}^n)}} \right) dx \leq 1. \quad (9)$$

To verify (9), assume $\|h\|_{\text{BLO}(\mathbb{R}^n)} = 1$. Set $\tau = \frac{\gamma}{c_*}$, where γ is as in (6) and c_* is from Lemma 1. By Remark 1, φ^{-1} exists on (N_0, ∞) , with N_0 defined in (5). Then, we calculate the following:

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \varphi \left(\frac{h(y) - c_{h,Q}}{\tau} \right) dy \\
&= \frac{1}{|Q|} \int_0^\infty \left| \left\{ y \in Q : \varphi \left(\frac{h(y) - c_{h,Q}}{\tau} \right) > s \right\} \right| ds \\
&= \frac{1}{|Q|} \int_0^\infty \left| \left\{ y \in Q : h(y) - c_{h,Q} > \tau \varphi^{-1}(s) \right\} \right| ds \\
&\leq C_{JN} \int_0^\infty e^{-c_* \tau \varphi^{-1}(s)} ds.
\end{aligned} \quad (10)$$

From condition (6), one has the following:

$$K := C_{JN} \int_0^\infty e^{-\gamma \varphi^{-1}(s)} ds < \infty.$$

Furthermore, since φ is convex with $\varphi(0) = 0$, then, for any $\lambda \in (0, 1)$ and $s \in [0, \infty)$, calculated as follows:

$$\varphi(\lambda s) = \varphi(\lambda s + (1 - \lambda)0) \leq \lambda \varphi(s) + (1 - \lambda)\varphi(0) = \lambda \varphi(s). \quad (11)$$

We define the following:

$$C_0 := \tau \max\{1, K\},$$

Using this, together with the above inequalities (10) and (11), we derive the following:

$$\begin{aligned}
\frac{1}{|Q|} \int_Q \varphi \left(\frac{h(y) - c_{h,Q}}{C_0 \|h\|_{\text{BLO}(\mathbb{R}^n)}} \right) dy &= \frac{1}{|Q|} \int_Q \varphi \left(\frac{1}{\max\{1, K\}} \frac{h(y) - c_{h,Q}}{\tau} \right) dy \\
&\leq \frac{1}{\max\{1, K\}} \frac{1}{|Q|} \int_Q \varphi \left(\frac{h(y) - c_{h,Q}}{\tau} \right) dy \\
&\leq \frac{K}{\max\{1, K\}} \\
&\leq 1,
\end{aligned}$$

which establishes (9). Then (8) holds. Finally, combining (7) and (8) yields that $\text{BLO}_\varphi(\mathbb{R}^n) = \text{BLO}(\mathbb{R}^n)$ with equivalent norms. This implies that (i) holds.

We now proceed to prove (ii). Assume that $\varphi: [0, \infty) \rightarrow [0, \infty)$ is increasing and concave with $\varphi(0) = 0$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. Let $c_{h,Q}$ be defined as in (2). From Remark 1, we find that φ^{-1} is increasing. Given a cube $Q \subset \mathbb{R}^n$, applying Jensen's inequality, we obtain, for constant $c > 0$, the following:

$$\begin{aligned} \frac{1}{|Q|} \int_Q \varphi \left(\varphi^{-1}(c) \frac{h(z) - c_{h,Q}}{\|h\|_{\text{BLO}(\mathbb{R}^n)}} \right) dz &\leq \varphi \left(\frac{1}{|Q|} \int_Q \varphi^{-1}(c) \frac{h(z) - c_{h,Q}}{\|h\|_{\text{BLO}(\mathbb{R}^n)}} dz \right) \\ &\leq \varphi(\varphi^{-1}(c)) \\ &= c. \end{aligned}$$

From this and Definition 1, the following is assumed:

$$\|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}(\mathbb{R}^n)}.$$

Below, we only need to verify the converse inequality, written as follows:

$$\|h\|_{\text{BLO}(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)},$$

which directly follows from the following inequality

$$\frac{1}{|Q|} \int_Q h(z) - c_{h,Q} dz \lesssim \|h\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \quad \text{for any cube } Q \subset \mathbb{R}^n. \quad (12)$$

Suppose that $\|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)} = 1$. Then the Luxemburg norm condition implies that, for any cube Q , the following is calculated:

$$\frac{1}{|Q|} \int_Q \varphi(h(z) - c_{h,Q}) dz \leq 2. \quad (13)$$

To prove (12), we define the following:

$$X := \sup_{Q \text{ cube}} \frac{1}{|Q|} \int_Q h(z) - c_{h,Q} dz.$$

Since the finiteness of X is not guaranteed, for a large number t , we instead consider the following truncated quantity:

$$X_t := \sup_{Q \text{ cube}} \frac{1}{|Q|} \int_Q \min\{h(z) - c_{h,Q}, t\} dz.$$

Then, we have $X_t \leq t$. From (13), we know the following:

$$\varphi(h - c_{h,Q}) \in L^1(Q).$$

therefore, if we take $L > 2$, applying the Calderón–Zygmund decomposition yields non-overlapping dyadic cubes $\{Q_j\} \subset \mathcal{D}(Q)$ and constant $c_{h,Q} > 0$ such that the following properties hold:

- (i) $L < \frac{1}{|Q|} \int_{Q_j} \varphi(h(z) - c_{h,Q}) dz \leq 2^n L$;
- (ii) $\varphi(h(z) - c_{h,Q}) \leq L$ for a.e. $z \in Q \setminus \cup_j Q_j$;
- (iii) $\frac{1}{|Q|} \sum_j |Q_j| \leq \frac{2}{L}$.

We write the following:

$$\begin{aligned} & \frac{1}{|Q|} \int_Q \min\{h(z) - c_{h,Q}, t\} dz \\ & \leq \frac{1}{|Q|} \int_{Q \setminus \cup_j Q_j} h(z) - c_{h,Q} dz + \frac{1}{|Q|} \int_{\cup_j Q_j} \min\{h(z) - c_{h,Q}, t\} dz \\ & =: \text{I} + \text{II}. \end{aligned}$$

Regarding I, by (ii) and the increasing property of φ^{-1} , it follows that for $z \in Q \setminus \cup_j Q_j$, the following is valid:

$$h(z) - c_{h,Q} = \varphi^{-1}(\varphi(h(z) - c_{h,Q})) \leq \varphi^{-1}(L),$$

which implies the following:

$$\text{I} \leq \frac{1}{|Q|} \int_{Q \setminus \cup_j Q_j} \varphi^{-1}(L) dz \leq \varphi^{-1}(L).$$

Now, we consider II. For any $Q_j \subset Q$, according to (3), (i) and (13), we see the following:

$$\begin{aligned} c_{h,Q_j} - c_{h,Q} &= \varphi^{-1} \left(\frac{1}{|Q_j|} \int_{Q_j} \varphi(|c_{h,Q_j} - c_{h,Q}|) dz \right) \\ &\leq \varphi^{-1} \left(\frac{1}{|Q_j|} \int_{Q_j} \varphi(h(z) - c_{h,Q_j}) dz + \frac{1}{|Q_j|} \int_{Q_j} \varphi(h(z) - c_{h,Q}) dz \right) \\ &\leq \varphi^{-1}(2 + 2^n L). \end{aligned}$$

From this and the following

$$\min\{t, h - c_{h,Q}\} \leq \min\{t, h - c_{h,Q_j}\} + c_{h,Q_j} - c_{h,Q},$$

the following is calculable:

$$\begin{aligned} \text{II} &= \frac{1}{|Q|} \int_{\cup_j Q_j} \min\{t, h(z) - c_{h,Q}\} dz \\ &\leq \frac{1}{|Q|} \sum_j \int_{Q_j} \min\{t, h(z) - c_{h,Q_j}\} dz + \frac{1}{|Q|} \sum_j |Q_j| (c_{h,Q_j} - c_{h,Q}) \\ &\leq \frac{X_t}{|Q|} \sum_j |Q_j| + 2 \frac{1}{L} \varphi^{-1}(2 + 2^n L) \\ &\leq \frac{2X_t}{L} + \frac{2}{L} \varphi^{-1}(2 + 2^n L), \end{aligned}$$

where the penultimate step used the definition of X_t and the last step used (iii).

Combining the above bounds for I and II, we have the following:

$$\frac{1}{|Q|} \int_Q \min\{h(x) - c_{h,Q}, t\} dx \leq \varphi^{-1}(L) + \frac{2X_t}{L} + \frac{2}{L} \varphi^{-1}(2 + 2^n L).$$

Taking the supremum over all intervals, we choose $L = 4$ and obtain that there exists a constant $C_n > 0$ such that we calculate the following:

$$X_t \leq C_n.$$

Finally, we let $t \rightarrow \infty$ and obtain (12). So, we conclude that (ii) holds. This finishes the proof of Theorem 1. $\square$

As application of Theorem 1, we give a related result as follows:

Theorem 2. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a function with $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. If $h \in \text{BLO}_\varphi(\mathbb{R}^n)$, then $h \in \text{BLO}(\mathbb{R}^n)$.

Proof. Assume that φ is measurable function with $\varphi(0) = 0$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. By analyzing the growth of φ at infinity, we construct a polygonal function ϕ such that ϕ is concave and $\phi(s) \leq \varphi(s)$ for large values of s .

Specifically, for $s \leq s_0 \in [0, \infty)$, define $\phi(s) = 0$. For $s \geq s_0$, the function ϕ is defined as a polygonal curve composed of linear segments joining points of the form (s_n, n) and $(s_{n+1}, n+1)$, with $\{s_n\}_{n \in \mathbb{N}}$ chosen so that ϕ is continuous, concave, and satisfies $\phi \leq \varphi$. As a consequence of this construction, the following is assumed:

$$\|h\|_{\text{BLO}_\phi(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)}.$$

Then we only need to verify the following:

$$\|h\|_{\text{BLO}(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}_\phi(\mathbb{R}^n)}. \quad (14)$$

To simplify the argument, let $\|h\|_{\text{BLO}_\phi(\mathbb{R}^n)} = 1$. Assume that $c_{h,Q}$ is defined as in (2). For a given cube Q , we following the proof of (12), if we claim the following:

$$c_{h,Q_j} - c_{h,Q} \leq \varphi^{-1}(2^n L + 2), \quad (15)$$

Then we repeat the argument of (12) and obtain that (14) holds. Moreover, we calculate the following:

$$\|h\|_{\text{BLO}(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}_\phi(\mathbb{R}^n)} \lesssim \|h\|_{\text{BLO}_\varphi(\mathbb{R}^n)}.$$

It remains to show (15). Referring the proof of [18] (pp. 10–11) and replacing c_Q with $c_{h,Q}$, we conclude that (15) holds. This ends the proof of Theorem 2. $\square$

3. Luxemburg Norm Characterization of $\text{BLO}(\mathbb{X})$

In preparation for the proof of Theorem 3, this section first recalls some concepts about the space of homogeneous type (see [21]) and two key Lemmas from [18] (Lemmas 3.1 and 3.2).

Assume that $\mathbb{X}$ is a set and d is a quasi-metric, that is, d satisfies quasi-triangular inequality, written as follows:

$$d(t_1, t_2) \leq \zeta(d(t_1, t_3) + d(t_3, t_2)) \quad \text{for all } t_1, t_2, t_3 \in \mathbb{X},$$

where $\zeta \geq 1$ is a finite constant. For a measure η and a ball $B(y, t)$ with center $y \in \mathbb{X}$ and radius $t > 0$, if there exists constant $C_\eta > 0$ independent of y and t such that we calculate the following:

$$\eta(B(y, 2t)) \leq C_\eta \eta(B(y, t)),$$

then we say that η satisfies the doubling condition. Define $C_{\min}$ to be the smallest constant in the doubling condition, and let $D = \log_2 C_{\min}$ denote the doubling dimension of η .

For any balls B_1, B_2 with $B_1 \subset B_2$, we utilize the above doubling condition to derive the following:

$$\frac{\eta(B_2)}{\eta(B_1)} \leq C_{\eta, \zeta} \left(\frac{r_{B_2}}{r_{B_1}} \right)^D, \quad (16)$$

where $C_{\eta, \zeta}$ is a positive constant, r_{B_1} and r_{B_2} mean the radius of B_1 and B_2 , respectively.

Assume that $\mathbb{X}$ is the set, d is a quasi-metric and η is a doubling measure. The space of homogeneous type is a triple $(\mathbb{X}, d, \eta)$. To simplify the concept, for a ball B , we fix $\sigma > \zeta$ and define the following:

$$B^* = \zeta(1 + 4\zeta)B$$

and

$$\tilde{B} = \sigma B.$$

Lemma 3. Assume that $\mathcal{B} \in \mathbb{X}$ is a family of balls whose radius are bounded. Then one can find a subcollection $\mathcal{B}_1 \subset \mathcal{B}$ consisting of mutually disjoint balls satisfying the following:

$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{B \in \mathcal{B}_1} \zeta(4\zeta + 1)B.$$

Lemma 4. Let $\epsilon > 0$ and B be a ball. There is a sufficiently large constant $L > 1$ for which any ball P whose center lies in B and satisfies the following:

$$\eta(P) \leq \frac{\eta(\tilde{B})}{L},$$

it holds that $r_P \leq \epsilon r_B$. Furthermore, choosing ϵ sufficiently small ensures that $P \subset \tilde{B}$.

Theorem 3. Let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be a function satisfying $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. If φ is concave, then $\text{BLO}(\mathbb{X}) = \text{BLO}_\varphi(\mathbb{X})$ with equivalent norms.

Proof. Proceeding as in the proof of Theorem 1, based on Remark 1, we have that φ^{-1} exist. Let $c_{h,B}$ be defined as in (2). For any ball B , by applying Jensen's inequality, we deduce the following:

$$\begin{aligned} & \frac{1}{\eta(B)} \int_B \varphi \left(\varphi^{-1}(1) \frac{h(z) - c_{h,B}}{\|h\|_{\text{BLO}(\mathbb{X})}} \right) d\eta(z) \\ & \leq \varphi \left(\frac{1}{\eta(B)} \int_B \varphi^{-1}(1) \frac{h(z) - c_{h,B}}{\|h\|_{\text{BLO}(\mathbb{X})}} d\eta(z) \right) \\ & \leq \varphi(\varphi^{-1}(1)) \\ & = 1, \end{aligned}$$

which implies the following:

$$\|h\|_{\text{BLO}_\varphi(\mathbb{X})} \leq \frac{1}{\varphi^{-1}(1)} \|h\|_{\text{BLO}(\mathbb{X})}.$$

Therefore, we establish the following reverse inequality:

$$\|h\|_{\text{BLO}(\mathbb{X})} \lesssim \|h\|_{\text{BLO}_\varphi(\mathbb{X})}.$$

To prove this, we need to show the following:

$$\frac{1}{\eta(B)} \int_B h(z) - c_{h,B} d\eta(z) \lesssim \|h\|_{\text{BMO}_\varphi(\mathbb{X})}, \quad (17)$$

for all B balls. Assume that $\|h\|_{\text{BLO}_\varphi(\mathbb{X})} = 1$. Then, we define the following:

$$\frac{1}{\eta(B)} \int_B \varphi(h(z) - c_{h,B}) d\eta(z) \leq 2. \quad (18)$$

To prove (17), we define the following:

$$X := \sup_{B \text{ ball}} \frac{1}{\eta(B)} \int_B h(z) - c_{h,B} d\eta(z).$$

We claim that X is finite. Assume that $L > 1$ is a parameter to be determined. We define this as follows:

$$\Omega_L = \{z \in B : \varphi(h(z) - c_{h,\tilde{B}}) > L\}.$$

For any $x \in \Omega_L$, we use Lebesgue differentiation theorem to derive that there exists $B_x \subset \tilde{B}$ centered at x satisfying the following:

$$\frac{1}{\eta(B)} \int_B \varphi(h(z) - c_{h,\tilde{B}}) d\eta(z) > L. \quad (19)$$

So, we construct a family $\mathcal{B} = \{B_x\}_x$, where B_x satisfies (19) and $r_{B'_x} \leq 2r_{B_x}$ for all other ball $B'_x \subset \tilde{B}$ satisfying (19). Using Lemma 3, we obtain a maximal subfamily $\mathcal{B}_1 = \{B_j\}$.

Further, when L is chosen sufficiently large and $\eta(B_j) \leq \frac{\eta(\tilde{B})}{L}$, we utilize Lemma 4 to derive $B_j^* \subset \tilde{B}$ and the following:

$$\frac{1}{\eta(B_j^*)} \int_{B_j^*} \varphi(h(z) - c_{h,\tilde{B}}) d\eta(z) \leq L.$$

Moreover, we have the following:

$$\begin{aligned} \sum_j \eta(B_j) &\leq \frac{1}{L} \sum_j \int_{B_j} \varphi(h(z) - c_{h,\tilde{B}}) d\eta(z) \\ &\leq \frac{1}{L} \sum_j \int_{\tilde{B}} \varphi(h(z) - c_{h,\tilde{B}}) d\eta(z) \\ &\leq \frac{2}{L} \eta(\tilde{B}) \\ &\leq \frac{C_\eta}{L} \eta(B), \end{aligned}$$

where C_η refers to a constant that depends solely on η .

Now, we summarize the key features of $\{B_j\}$ as follows:

- (i) For $i \neq j$, and $B_i, B_j \subset \tilde{B}$, we have $B_i \cap B_j = \emptyset$;
- (ii) $\Omega_L \subset \bigcup_j B_j^*$;
- (iii) $B_j^* \subset \tilde{B}$ and $\frac{1}{\eta(B_j^*)} \int_{B_j^*} \varphi(h(z) - c_{h,\tilde{B}}) d\eta(z) \leq L$;
- (iv) there exist constants $C_{\eta,1}, C_{\eta,2} > 0$ such that $\sum_j \eta(B_j^*) \leq C_{\eta,1} \sum_j \eta(B_j) \leq \frac{C_{\eta,2}}{L} \eta(B)$.

To verify (17), by $B \subset \tilde{B}$, we write the following:

$$\begin{aligned} \frac{1}{\eta(B)} \int_B h(z) - c_{h,B} d\eta(z) &\leq \frac{1}{\eta(B)} \int_B h(z) - c_{h,\tilde{B}} d\eta(z) \\ &\leq \frac{1}{\eta(B)} \int_{B \setminus \Omega_L} h(z) - c_{h,\tilde{B}} d\eta(z) \\ &\quad + \frac{1}{\eta(B)} \int_{\Omega_L} h(z) - c_{h,\tilde{B}} d\eta(z) \\ &=: \text{I} + \text{II}. \end{aligned}$$

For term I, by the definition of Ω_L and Remark 1, we obtain, for any $z \in B \setminus \Omega_L$, the following:

$$h(z) - c_{h,\tilde{B}} = \varphi^{-1}\left(\varphi\left(h(z) - c_{h,\tilde{B}}\right)\right) \leq \varphi^{-1}(L),$$

which allows us to obtain the following:

$$\text{I} \leq \frac{1}{\eta(B)} \int_{B \setminus \Omega_L} \varphi^{-1}(L) d\eta(z) \leq \varphi^{-1}(L).$$

Now, we consider II. Based to $\tilde{B} \subset \tilde{B}_j^*$, by (3), (iii) and (18), we have the following:

$$\begin{aligned} c_{h,\tilde{B}} - c_{h,\tilde{B}_j^*} &= \varphi^{-1}\left(\frac{1}{\eta(B_j^*)} \int_{B_j^*} \varphi\left(c_{h,\tilde{B}} - c_{h,\tilde{B}_j^*}\right) d\eta(z)\right) \\ &\leq \varphi^{-1}\left(\frac{1}{\eta(B_j^*)} \int_{B_j^*} \varphi\left(h(z) - c_{h,\tilde{B}}\right) d\eta(z)\right. \\ &\quad \left.+ \frac{1}{\eta(B_j^*)} \int_{B_j^*} \varphi\left(h(z) - c_{h,\tilde{B}_j^*}\right) d\eta(z)\right) \\ &\leq \varphi^{-1}(L + 2C_\eta), \end{aligned}$$

where $C_\eta > 0$ refers to a constant that depends solely on η . Combining properties (ii) and (iv), we estimate II as follows:

$$\begin{aligned} \text{II} &\leq \frac{1}{\eta(B)} \sum_j \int_{B_j^*} h(z) - c_{h,\tilde{B}} d\eta(z) \\ &\leq \frac{1}{\eta(B)} \sum_j \int_{B_j^*} h(z) - c_{h,\tilde{B}_j^*} d\eta(z) \\ &\quad + \frac{1}{\eta(B)} \sum_j \eta(B_j^*) \left(c_{h,\tilde{B}} - c_{h,\tilde{B}_j^*}\right) \\ &\leq \frac{X}{\eta(B)} \sum_j \eta(B_j^*) + \frac{C_{\eta,2}}{L} \varphi^{-1}(L + 2C_\eta) \\ &\leq \frac{C_{\eta,2}X}{L} + \frac{C_{\eta,2}}{L} \varphi^{-1}(L + 2C_\eta), \end{aligned}$$

where the third step used the definition of X and $C_{\eta,2}$ is as in (iv).

Finally, utilizing the bounds for I and II together, we obtain the following:

$$\frac{1}{\eta(B)} \int_B h(z) - c_{h,B} d\eta(z) \leq \varphi^{-1}(L) + \frac{C_{\eta,2}X}{L} + \frac{C_{\eta,2}}{L} \varphi^{-1}(L + 2C_\eta).$$

Taking the supremum over all intervals and choosing $L = 2C_{\eta,2}$, we conclude that there exists constant $C_{\mathbb{X}} > 0$ dependent of η such that we obtain the following:

$$X \leq C_{\mathbb{X}}.$$

This allows us to derive (17). We finish the proof of Theorem 3. $\square$

4. Luxemburg Norm Characterization of $\text{BLO}(\omega)$

This section discusses the equivalent characterization $\text{BLO}(\omega)$ with non-atomic measure ω .

Theorem 4. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. For any non-atomic measure ω , if φ is concave, then $\text{BLO}(\omega) = \text{BLO}_{\varphi}(\omega)$ with equivalent norms.

Proof. Firstly, we use Remark 1 to derive that φ^{-1} exists. For any cube Q , let $c_{h,Q}$ be as in (2). Again applying Jensen inequality, we see the following:

$$\begin{aligned} & \frac{1}{\omega(Q)} \int_Q \varphi \left(\varphi^{-1}(c) \frac{h - c_{h,Q}}{\|h\|_{\text{BLO}(\omega)}} \right) d\omega \\ & \leq \varphi \left(\frac{1}{\omega(Q)} \int_Q \varphi^{-1}(c) \frac{h - c_{h,Q}}{\|h\|_{\text{BLO}(\omega)}} d\omega \right) \\ & \leq \varphi \left(\varphi^{-1}(c) \right) \\ & = c \end{aligned}$$

for all $c > 0$ hold. From this, the following holds:

$$\|h\|_{\text{BLO}_{\varphi}(\omega)} \lesssim \|h\|_{\text{BLO}(\omega)}.$$

Therefore, it remains to show the following:

$$\|h\|_{\text{BLO}(\omega)} \lesssim \|h\|_{\text{BLO}_{\varphi}(\omega)}.$$

In fact, for any cube $Q \subset \mathbb{R}^n$, we only need to verify the following:

$$\frac{1}{\omega(Q)} \int_Q h - c_{h,Q} d\omega \lesssim \|h\|_{\text{BLO}_{\varphi}(\omega)}. \quad (20)$$

Now, we prove (20). Fix cube Q and assume the following:

$$\frac{1}{\omega(Q)} \int_Q \varphi(h - c_{h,Q}) d\omega \leq 2. \quad (21)$$

To verify (20), we define the following:

$$X := \sup_{Q \text{ interval}} \frac{1}{\omega(Q)} \int_Q h - c_{h,Q} d\omega.$$

Again, we claim that X is finite, as established in the proof Theorem 1. Fix $L > 2$ to be precised later. Referring the dyadic construction of [18] (pp. 15–17) and [18] (Lemma 3.3), we replace c_Q with $c_{h,Q}$ and obtain the following Calderón–Zygmund decomposition; the family $\{Q_j\}$ of dyadic cubes satisfies the following:

- (i) $\frac{1}{\omega(Q_j)} \int_{Q_j} \varphi(h - c_{h,Q}) d\omega = L;$
- (ii) $\varphi(h(x) - c_{h,Q}) \leq L$ for ω a.e. $x \in Q \setminus \cup_j Q_j;$
- (iii) $\frac{1}{\omega(Q)} \sum_j \omega(Q_j) \leq \frac{C_n}{L}.$

By $Q_j \subset Q$, (21) and (i), we deduce the following:

$$\begin{aligned} c_{h,Q_j} - c_{h,Q} &= \varphi^{-1} \left(\frac{1}{\omega(Q_j)} \int_{Q_j} \varphi(|c_{h,Q_j} - c_{h,Q}|) d\omega \right) \\ &\leq \varphi^{-1} \left(\frac{1}{\omega(Q_j)} \int_{Q_j} \varphi(h - c_{h,Q_j}) d\omega \right. \\ &\quad \left. + \frac{1}{\omega(Q_j)} \int_{Q_j} \varphi(h - c_{h,Q}) d\omega \right) \\ &\leq \varphi^{-1}(2 + L). \end{aligned}$$

From this and (ii)–(iii), the following holds:

$$\begin{aligned} \frac{1}{\omega(Q)} \int_Q h - c_{h,Q} d\omega &\leq \frac{1}{\omega(Q)} \int_{Q \setminus \cup_j Q_j} \varphi^{-1}(\varphi(h - c_{h,Q})) d\omega \\ &\quad + \frac{1}{\omega(Q)} \sum_j \int_{Q_j} h(x) - c_{h,Q_j} d\omega \\ &\quad + \frac{1}{\omega(Q)} \sum_j \int_{Q_j} c_{h,Q_j} - c_{h,Q} d\omega \\ &\leq \varphi^{-1}(L) + \frac{C_n}{L} X + \frac{C_n}{L} \varphi^{-1}(2 + L). \end{aligned}$$

Taking the supremum over all cubes and choosing $L = 2C_n$, we obtain the following:

$$X \lesssim 1,$$

which establishes the validity of (20). We finish the proof of Theorem 4. $\square$

5. Conclusions

This paper investigates minimal integrability conditions for BLO spaces via Luxemburg-type norms, under assumptions of convexity and concavity on the generating function φ . These characterizations extend from Euclidean spaces to more general metric spaces, homogeneous spaces, and further to frameworks with non-atomic measures, thereby unifying and significantly broadening the scope of BLO theory. The results not only unify and generalize classical BLO frameworks but also provide a flexible analytical bridge for addressing oscillation phenomena related to the boundedness of fractional operators. These findings contribute to the theoretical foundation for analyzing fractional integral and maximal operators on general metric measure spaces and underscore the profound interplay between harmonic analysis, fractional calculus, and geometric frameworks.

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