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New Advances in Index Whittaker Transforms

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Abstract

In this paper, we establish new L^p-boundedness results and derive novel Parseval–Goldstein relations for the index Whittaker transform. Additionally, we obtain Abelian theorems for the distributional index Whittaker transform applied to distributions with compact support.

Keywords: index Whittaker transform; L^p-boundedness; Parseval–Goldstein relations; Abelian theorems; distributions of compact support

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1. Introduction and Preliminaries

We begin by establishing key definitions and notations that will be used throughout the manuscript. For an appropriate function f defined on $\mathbb{R}_+ = (0, \infty)$, the index Whittaker transform is an integral operator introduced by Wimp [1] in 1964 and is defined as

$$(W_\mu f)(\tau) = \int_0^\infty f(x) \frac{W_{\mu,i\tau}(x)}{x^2} dx, \ \tau > 0, \ \mu < \frac{1}{2}, \tag{1}$$

where $W_{\mu,i\tau}$ is the Whittaker function ([2], Section 6.9, p. 264).

In [1] the next inversion formula for (1) was formally obtained:

$$f(x) = \frac{1}{\pi^2} \int_0^\infty \tau \sinh(2\pi\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x)(W_\mu f)(\tau) d\tau, \ x > 0. \tag{2}$$

Several researchers have extensively studied index integral transforms involving the Whittaker function as a kernel, contributing significantly to both theoretical and applied aspects. Yakubovich [3] offers a foundational treatment of index transforms, which has laid the

groundwork for many modern developments in the theory of generalized integral transforms. Foundational results by Srivastava and collaborators established key connections between Whittaker, Hankel, and fractional integral transforms [4–6]. Al-Musallam and Tuan further developed finite and infinite Whittaker-type transforms and their applications on the half-line [7,8], while Becker analyzed integral identities involving Whittaker functions with respect to the index parameter [9].

Modern contributions have significantly extended these ideas. Maan and co-authors formulated Abelian theorems and structural properties for the index Whittaker transform in the context of distributions and Lebesgue spaces [10–12]. Further applications involving pseudo-differential operators and related integral identities were developed in [13]. Srivastava [14] presents an in-depth investigation into broad classes of integral transforms, notably introducing the Srivastava generalized Whittaker transform and exploring its relationships with other transforms, including Hardy's generalized Hankel transform. Recently, Maan [15] obtained interesting uncertainty principles in the context of the continuous index Whittaker wavelet transform, further emphasizing the versatility of the index Whittaker transform in harmonic analysis and signal representation.

The convolution and product formula associated with the index Whittaker transform, introduced in [16], provides a harmonic analysis framework that unifies many of these developments.

For a suitable function g on $\mathbb{R}_+$, we denote

$$(W_\mu^* g)(x) = \int_0^\infty g(\tau) \frac{W_{\mu,i\tau}(x)}{x^2} d\tau, \; x > 0, \; \mu < \frac{1}{2}. \tag{3}$$

The differential equation for $W_{\mu,i\tau}(x)$ is ([17], Formula (13.14.1), p. 334)

$$D_x^2 W_{\mu,i\tau}(x) + \left(\frac{-1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4} + \tau^2}{x^2} \right) W_{\mu,i\tau}(x) = 0.$$

Then $\frac{W_{\mu,i\tau}(x)}{x^2}$ satisfies the differential equation:

$$x^2 D_x^2 \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) + 4x D_x \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) + \left(\mu x - \frac{x^2}{4} + \frac{9}{4} + \tau^2 \right) \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) = 0.$$

We consider the differential operator

$$\mathcal{L}_{\mu,x} \equiv x^2 D_x^2 + 4x D_x + \left(\mu x - \frac{x^2}{4} \right),$$

and its adjoint

$$\mathcal{L}_{\mu,x}^* \equiv x^2 D_x^2 + \left(\mu x - \frac{x^2}{4} - 2 \right).$$

Observe that:

$$\mathcal{L}_{\mu,x} \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) = -\left(\frac{9}{4} + \tau^2 \right) \frac{W_{\mu,i\tau}(x)}{x^2}, \tag{4}$$

and for $k \in \mathbb{N}$:

$$\mathcal{L}_{\mu,x}^k \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) = (-1)^k \left(\frac{9}{4} + \tau^2 \right)^k \frac{W_{\mu,i\tau}(x)}{x^2}.$$

As usual, $C_c^k(\mathbb{R}_+)$, with $k \in \mathbb{N}$, denotes the space of functions on $\mathbb{R}_+$ that are k times continuously differentiable and have compact support.
Now, for $f \in C_c^2(\mathbb{R}_+)$:

$$\left(W_\mu\left[\mathcal{L}_{\mu,x}^* f\right]\right)(\tau) = -\left(\frac{9}{4} + \tau^2\right)(W_\mu f)(\tau),\ \tau > 0,$$

and so for $k \in \mathbb{N}$, $f \in C_c^{2k}(\mathbb{R}_+)$:

$$\left(W_\mu\left[\mathcal{L}_{\mu,x}^{*k} f\right]\right)(\tau) = (-1)^k\left(\frac{9}{4} + \tau^2\right)^k(W_\mu f)(\tau),\ \tau > 0. \tag{5}$$

From the proof of Lemma 4.8 in [16], one has

$$|\Psi(a + i\tau, 1 + 2i\tau; x)| \le \Psi(a, 1; x),\ x > 0,\ \tau > 0,\ a > 0, \tag{6}$$

where for the Tricomi function Ψ one has the relation ([17], Equation (13.14.5), p. 334)

$$\Psi(a + i\tau, 1 + 2i\tau; x) = e^{\frac{x}{2}} x^{-\frac{1}{2} - i\tau} W_{\frac{1}{2} - a, i\tau}(x). \tag{7}$$

From (6) and (7) one obtains

$$|W_{\frac{1}{2} - a, i\tau}(x)| \le W_{\frac{1}{2} - a, 0}(x),\ x > 0,\ \tau > 0,\ a > 0,$$

and so

$$|W_{\mu, i\tau}(x)| \le W_{\mu,0}(x),\ x > 0,\ \tau > 0,\ \mu < \frac{1}{2}. \tag{8}$$

From ([17], Equation (13.14.19), p. 335) and ([17], Equation (13.14.21), p. 335), the behaviors of $W_{\mu,0}(x)$ near 0 and ∞ are as follows:

$$W_{\mu,0}(x) = O(x^{\frac{1}{2}} \log x) \quad \text{as} \quad x \to 0^+, \tag{9}$$
$$W_{\mu,0}(x) \sim x^\mu e^{-\frac{x}{2}} \quad \text{as} \quad x \to +\infty. \tag{10}$$

For $\gamma \in \mathbb{R}$, we define the vector space $\mathcal{L}_\gamma$ as the set of all complex-valued measurable functions f on $\mathbb{R}_+$ for which $x^\gamma f(x) \in L^\infty(\mathbb{R}_+)$. Function spaces of this type were thoroughly examined by Maan and Negrín in their recent work [10]. The norm associated with the space $\mathcal{L}_\gamma$ is defined as

$$\|f\|_\gamma = \|x^\gamma f(x)\|_{L^\infty(\mathbb{R}_+)}.$$

Under this norm, consider the linear transformation

$$T_\gamma : \mathcal{L}_\gamma \to L^\infty(\mathbb{R}_+)$$

defined for each $f \in \mathcal{L}_\gamma$ by

$$(T_\gamma f)(x) = x^\gamma f(x), \quad x \in \mathbb{R}_+.$$

This transformation T_γ serves as an isometric isomorphism between $\mathcal{L}_\gamma$ and $L^\infty(\mathbb{R}_+)$. Consequently, since $L^\infty(\mathbb{R}_+)$ is a Banach space, the space $\mathcal{L}_\gamma$ is also complete and hence forms a Banach space.

Parseval-type and Plancherel-type results play a central role in the theory of integral transforms, establishing isometric relationships between functions and their transforms.

These results highlight the preservation of inner products and energy norms under transformation, forming the analytical backbone of many modern applications (see [18–20]).

Abelian theorems for distributional transforms have garnered significant attention across various integral transform frameworks. The foundational contribution by Zemanian [21] addressed these theorems in the context of the Hankel and K-transforms, laying the groundwork for future studies. In the context of the index Whittaker transform, Maan and Prasad developed analogous results within distribution spaces [12].

These studies contribute significantly to the understanding of the asymptotic behavior of integral transforms, particularly near the origin and at infinity, by connecting them with properties of the underlying distributions or generalized functions. Motivated by this foundation, the present work focuses on establishing Abelian theorems for the index Whittaker transforms within the framework of compactly supported distributions.

The space $\mathcal{E}(\mathbb{R}_+)$ denotes the collection of complex-valued functions on $\mathbb{R}_+$ that are infinitely differentiable, equipped with the topology induced by the seminorms $\Gamma_{k,\Sigma}(\phi) = \max_{x\in\Sigma} |D_x^k \phi(x)|$, where $k \in \mathbb{N}_0$ and Σ ranges over compact subsets of $\mathbb{R}_+$. This topology turns $\mathcal{E}(\mathbb{R}_+)$ into a Fréchet space. Its strong dual, denoted by $\mathcal{E}'(\mathbb{R}_+)$, comprises all distributions supported on compact subsets of $\mathbb{R}_+$. For a detailed treatment of distributional support, see ([22], p. 35).

The paper is organized into six sections. In Section 2, we examine the index Whittaker transform W_μ within the weighted Lebesgue space $L^1(\mathbb{R}_+, x^{-2}dx)$ and derive a Parseval-type identity. Section 3 is devoted to the study of W_μ on the generalized Lebesgue spaces $\mathcal{L}_\gamma$, where we establish a novel Parseval–Goldstein type relation. In Section 4, we investigate Abelian theorems for the distributional index Whittaker transform acting on compactly supported distributions in $\mathcal{E}'(\mathbb{R}_+)$. Section 5 concludes the main discussion with a summary of the principal results. Finally, in Section 6, we highlight an open question related to the connection between W_μ and the differential operator $\mathcal{L}^*_{\mu,x}$, and discuss how deriving inversion formulae may aid in solving differential equations involving this operator.

2. The Index Whittaker Transform W_μ Over the Spaces $L^1(\mathbb{R}_+, x^{-2}dx)$

In this section, we study the index Whittaker transform W_μ, defined by (1), within the weighted space $L^1(\mathbb{R}_+, x^{-2}dx)$. A key result of this analysis is the derivation of a Parseval-type identity.

Proposition 1. *Let w be a measurable function on $\mathbb{R}_+$ such that $w(x) > 0$ almost everywhere on $\mathbb{R}_+$. Suppose $0 < q < \infty$ and $\mu < \frac{1}{2}$. If the integral $\int_0^\infty w(\tau)\, d\tau$ is finite, then the index Whittaker transform W_μ defines a bounded linear operator*

$$W_\mu : L^1(\mathbb{R}_+, x^{-2}\, dx) \to L^q(\mathbb{R}_+, w(x)\, dx).$$

Proof. Making use of (8) and taking into account that, from (9) and (10), the function $W_{\mu,i\tau}(x)$ is bounded on $\mathbb{R}_+$, one has

$$\begin{aligned} \left|(W_\mu f)(\tau)\right| &\leq \int_0^\infty |f(x)| \left|\frac{W_{\mu,i\tau}(x)}{x^2}\right| dx \\ &\leq M \int_0^\infty |f(x)| x^{-2} dx, \text{ for each } \tau > 0,\ \mu < \frac{1}{2}, \text{ for some } M > 0, \end{aligned}$$

and thus the result of this proposition holds. □

Example 1. *Set $\mu < \frac{1}{2}$. Suitable examples of weight functions w that satisfy the conditions in Proposition 1 include:*

$$\begin{aligned} &(i) \quad w(x) = (1+x)^r, \quad \text{with } r < -1, \\ &(ii) \quad w(x) = e^{rx}, \quad \text{with } r < 0. \end{aligned}$$

Theorem 1. *Set $\mu < \frac{1}{2}$. If $f \in L^1(\mathbb{R}_+, x^{-2}dx)$, $g \in L^1(\mathbb{R}_+)$, then the following Parseval-type relation holds:*

$$\int_0^\infty (W_\mu f)(x) g(x) dx = \int_0^\infty f(x) \left(W_\mu^* g\right)(x) dx.$$

Proof. By applying Fubini's theorem in the following, we obtain

$$\begin{aligned} \int_0^\infty (W_\mu f)(\tau) g(\tau) d\tau &= \int_0^\infty \int_0^\infty f(x) \frac{W_{\mu,i\tau}(x)}{x^2} dx g(\tau) d\tau \\ &= \int_0^\infty \int_0^\infty g(\tau) \frac{W_{\mu,i\tau}(x)}{x^2} d\tau f(x) dx \\ &= \int_0^\infty f(x) \left(W_\mu^* g\right)(x) dx. \end{aligned}$$

Observe that $W_\mu^* g$ exists for $g \in L^1(\mathbb{R}_+)$.

In fact, from (8)

$$\begin{aligned} &\left|(W_\mu^* g)(x)\right| \\ &\leq \frac{1}{x^2} \int_0^\infty |g(\tau)| \left|W_{\mu,i\tau}(x)\right| d\tau \\ &\leq \frac{W_{\mu,0}(x)}{x^2} \|g\|_{L^1(\mathbb{R}_+)} < \infty, \end{aligned} \tag{11}$$

for each $x \in \mathbb{R}_+$. □

Proposition 2. *Set $\mu < \frac{1}{2}$. Then the operator W_μ^* defines a bounded linear mapping from $L^1(\mathbb{R}_+)$ into the weighted space $L^q(\mathbb{R}_+, w(x)\,dx)$ for any $0 < q < \infty$, provided that the weight function w satisfies $w(x) > 0$ almost everywhere on $\mathbb{R}_+$ and the function $\frac{W_{\mu,0}(x)}{x^2}$ belongs to $L^q(\mathbb{R}_+, w(x)\,dx)$.*

Proof. From (11), since $g \in L^1(\mathbb{R}_+)$ and $0 < q < \infty$, one has

$$\begin{aligned} \left(\int_0^\infty \left|(W_\mu^* g)(x)\right|^q w(x) dx\right)^{\frac{1}{q}} &\leq \left(\int_0^\infty \left(\frac{W_{\mu,0}(x)}{x^2}\right)^q w(x) dx\right)^{\frac{1}{q}} \|g\|_{L^1(\mathbb{R}_+)} \\ &\leq M \|g\|_{L^1(\mathbb{R}_+)}, \quad \text{for some } M > 0. \end{aligned}$$

□

Example 2. *Set $\mu < \frac{1}{2}$. Suitable examples of weight functions w that satisfy the conditions of Proposition 2 include the following:*

$$\begin{aligned} &\text{(i) } w(x) = x^r, \quad \text{where } r > \frac{3}{2}q - 1. \\ &\text{(ii) } w(x) = (1+x)^r, \quad \text{where } 0 < q < \frac{2}{3}. \\ &\text{(iii) } w(x) = e^{rx}, \quad \text{where either } r < \frac{q}{2} \text{ with } 0 < q < \frac{2}{3}, \text{ or } \frac{1}{4-2\mu} < r < \frac{1}{3}. \end{aligned}$$

Now, from Theorem 1 and relation (5) one obtains the next Parseval–Goldstein relation.

Theorem 2. *If $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, $g \in L^1(\mathbb{R}_+)$, $\mu < \frac{1}{2}$, then the following Parseval-type relation holds:*

$$(-1)^k \int_0^\infty (W_\mu f)(x) g(x) \left(\frac{9}{4} + x^2\right)^{2k} dx = \int_0^\infty (\mathcal{L}_{\mu,x}^{*k} f)(x) \left(W_\mu^* g\right)(x) dx. \tag{12}$$

Proof. In fact, from Theorem 1 one has

$$\int_0^\infty \left(W_\mu \left[\mathcal{L}_{\mu,x}^{*k} f\right]\right)(\tau) g(\tau) d\tau = \int_0^\infty \left(\mathcal{L}_{\mu,x}^{*k} f\right)(x) \left(W_\mu^* g\right)(x) dx, \tag{13}$$

which using relation (5) in the left hand side of (13) yields (12). □

Remark 1. *From Equations (9) and (10), it follows that the function $W_{\mu,0}(x)$ remains bounded on the domain $\mathbb{R}_+$ for all $\mu < \frac{1}{2}$. Consequently, we have the inclusion:*

$$L^1(\mathbb{R}_+, x^{-2}dx) \subset L^1\left(\mathbb{R}_+, \frac{W_{\mu,0}(x)}{x^2} dx\right).$$

Furthermore, this inclusion is strict. For instance, the function $f(x) = x^r$ does not belong to $L^1(\mathbb{R}_+, x^{-2}dx)$ for any real value of r. However, if $r > \frac{1}{2}$, then $f(x) = x^r$ lies in the space $L^1\left(\mathbb{R}_+, \frac{W_{\mu,0}(x)}{x^2} dx\right)$.

Therefore, the conclusions of Proposition 1 and Theorem 1 continue to hold if the function space $L^1(\mathbb{R}_+, x^{-2}, dx)$ is replaced by the larger space $L^1\left(\mathbb{R}_+, \frac{W_{\mu,0}(x)}{x^2} dx\right)$, provided that $\mu < \frac{1}{2}$.

3. The Index Whittaker Transform W_μ Over the Spaces $\mathcal{L}_\gamma$

In this section, we study the index Whittaker transform W_μ, defined by (1), on the Lebesgue-type spaces $\mathcal{L}_\gamma$ introduced in Section 1. A notable result of this analysis is the derivation of a Parseval–Goldstein type identity.
Observe that from (8) one has

$$\begin{aligned} |(W_\mu f)(\tau)| &\leq \int_0^\infty |f(x)| x^\gamma x^{-\gamma} \frac{W_{\mu,0}(x)}{x^2} dx \\ &\leq esssup_{x \in \mathbb{R}_+} |f(x) x^\gamma| \left(\int_0^\infty x^{-\gamma} \frac{W_{\mu,0}(x)}{x^2} dx\right), \ \tau > 0. \end{aligned}$$

Thus from (9) and (10), one obtains for $0 < q < \infty$, $\mu < \frac{1}{2}$ and for w being a measurable function on $\mathbb{R}_+$ such that $w > 0$ a.e. on $\mathbb{R}_+$, $f \in \mathcal{L}_\gamma$, $\gamma < -\frac{1}{2}$, then

$$\int_0^\infty |(W_\mu f)(\tau)|^q w(\tau) d\tau \leq M \|f\|_\gamma^q \int_0^\infty w(\tau) d\tau, \text{ for some } M > 0.$$

So one obtains the next result.

Proposition 3. *Let w be a measurable function on $\mathbb{R}_+$ satisfying $w(x) > 0$ almost everywhere on $\mathbb{R}_+$. Assume $0 < q < \infty$ and $\mu < \frac{1}{2}$. If the integral $\int_0^\infty w(\tau)\, d\tau$ is finite, then for $\gamma < -\frac{1}{2}$, the index Whittaker transform W_μ defines a bounded linear map:*

$$W_\mu : \mathcal{L}_\gamma \to L^q(\mathbb{R}_+, w(x)\, dx).$$

Example 3. *Set $\mu < \frac{1}{2}$. Some admissible examples of weight functions w for which Proposition 3 applies include:*

$$\begin{aligned} &(i) \quad w(x) = (1+x)^r, \quad \text{with } r < -1, \\ &(ii) \quad w(x) = e^{rx}, \quad \text{with } r < 0. \end{aligned}$$

Theorem 3. *Set $\mu < \frac{1}{2}$. If $f \in \mathcal{L}_\gamma$, $\gamma < -\frac{1}{2}$, $g \in L^1(\mathbb{R}_+)$, then the following Parseval–Goldstein relation holds:*

$$\int_0^\infty (W_\mu f)(x) g(x) dx = \int_0^\infty f(x) \left(W_\mu^* g\right)(x) dx.$$

Proof. Applying Fubini's theorem in the following, we obtain

$$\begin{aligned} \int_0^\infty (W_\mu f)(\tau) g(\tau) d\tau &= \int_0^\infty \int_0^\infty f(x) \frac{W_{\mu,i\tau}(x)}{x^2} dx g(\tau) d\tau \\ &= \int_0^\infty \int_0^\infty g(\tau) \frac{W_{\mu,i\tau}(x)}{x^2} d\tau f(x) dx \\ &= \int_0^\infty f(x) \left(W_\mu^* g\right)(x) dx. \end{aligned}$$

Observe that from (11) one has that $(W_\mu^* g)(x)$ exists for $g \in L^1(\mathbb{R}_+)$, for each $x \in \mathbb{R}_+$. □

4. Abelian Theorems for the Index Whittaker Transform Over the Space of Distributions $\mathcal{E}'(\mathbb{R}_+)$

In this section, we establish Abelian theorems for the index Whittaker transform over the space of distributions $\mathcal{E}'(\mathbb{R}_+)$, extending its applicability beyond classical function spaces.

Let f be a distribution with compact support on $\mathbb{R}_+$. The index Whittaker transform $W_\mu f$ is defined via its action on the test function $\frac{W_{\mu,i\tau}(x)}{x^2}$ by the expression:

$$(W_\mu f)(\tau) = \left\langle f(x), \frac{W_{\mu,i\tau}(x)}{x^2} \right\rangle, \quad \tau > 0, \ \mu < \frac{1}{2}. \tag{14}$$

Given that $W_{\mu,i\tau}(x)$ is infinitely differentiable in x, it follows that the function $\frac{W_{\mu,i\tau}(x)}{x^2}$ belongs to the space $\mathcal{E}(\mathbb{R}_+)$ of smooth functions with compact support.

We now utilize ([11], Lemma 2.2) to establish the result below.

Lemma 1. *Let $f \in \mathcal{E}'(\mathbb{R}_+)$ and let W_μ be the transform defined in (14), with $\mu < \frac{1}{2}$. Then, there exists a positive constant M (independent of τ) and a non-negative integer δ, both depending on f, such that the following estimate holds for all $\tau > 0$:*

$$|(W_\mu f)(\tau)| \leq M \left(\frac{9}{4} + \tau^2\right)^\delta. \tag{15}$$

Proof. Recall that $\frac{W_{\mu,i\tau}(x)}{x^2}$ is an eigenfunction of the operator $\mathcal{L}_{\mu,x}$, as established in relation (4).

From ([11], Lemma 2.2), the space $\mathcal{E}(\mathbb{R}_+)$ can be equipped with a topology defined by a family of seminorms $\Gamma_{k,K}$. According to ([23], Proposition 2, p. 97), there exists a compact

subset $\Sigma \subset \mathbb{R}_+$, a positive constant N, and a non-negative integer δ (depending on f), such that for all test functions $\psi \in \mathcal{E}(\mathbb{R}_+)$,

$$|\langle f, \psi \rangle| \leq N \max_{0 \leq k \leq \delta} \max_{x \in \Sigma} \left| \mathcal{L}_{\mu,x}^k \psi(x) \right|. \tag{16}$$

Applying (16) with $\psi(x) = \frac{W_{\mu,i\tau}(x)}{x^2}$, and using the eigenfunction relation (4), we compute:

$$\begin{aligned}
|(W_\mu f)(\tau)| &= \left| \left\langle f(x), \frac{W_{\mu,i\tau}(x)}{x^2} \right\rangle \right| \\
&\leq N \max_{0 \leq k \leq \delta} \max_{x \in \Sigma} \left| \mathcal{L}_{\mu,x}^k \left(\frac{W_{\mu,i\tau}(x)}{x^2} \right) \right| \\
&= N \max_{0 \leq k \leq \delta} \max_{x \in \Sigma} \left| \left(\frac{9}{4} + \tau^2 \right)^k \frac{W_{\mu,i\tau}(x)}{x^2} \right| \\
&\leq N \max_{0 \leq k \leq \delta} \max_{x \in \Sigma} \left| \left(\frac{9}{4} + \tau^2 \right)^k \frac{W_{\mu,0}(x)}{x^2} \right| \\
&\leq M \left(\frac{9}{4} + \tau^2 \right)^\delta,
\end{aligned}$$

for some constant $M > 0$, since Σ is compact and $W_{\mu,i\tau}(x)$ is bounded above by $W_{\mu,0}(x)$ on Σ. □

The smallest integer δ for which the inequality in (16) holds is known as the *order* of the distribution f ([24], Théorème XXIV, p. 88).

Theorem 4 (Abelian theorems). *Let $f \in \mathcal{E}'(\mathbb{R}_+)$ be a distribution of order $\delta \in \mathbb{N}_0$, and let W_μ be the operator defined by (16), $\mu < \frac{1}{2}$. Then*
(i) *for any $\eta > 0$, it follows that*

$$\lim_{\tau \to 0^+} \left(\tau^\eta (W_\mu f)(\tau) \right) = 0,$$

(ii) *for any $\eta > 0$, it follows that*

$$\lim_{\tau \to +\infty} \left(\tau^{-2\delta - \eta} (W_\mu f)(\tau) \right) = 0.$$

Proof. By applying Lemma 1, we obtain the following inequality:

$$|(W_\mu f)(\tau)| \leq M \left(\frac{9}{4} + \tau^2 \right)^\delta, \text{ for all } \tau > 0,$$

where $M > 0$ is a constant. Thus, the proof is concluded. □

Remark 2. *In ([11], Theorem 2.4 (ii)) it was proved that for $\mu < \frac{1}{2}$ and any $\eta > 0$, then*

$$\lim_{\tau \to +\infty} \left(\tau^{-2\delta - \mu - \eta} (W_\mu f)(\tau) \right) = 0.$$

Note that for $0 < \mu < \frac{1}{2}$, then (ii) of Theorem 4 above improves ([11], Theorem 2.4 (ii)).

Let f be a compactly supported, locally integrable function on $\mathbb{R}_+$. Then, f naturally defines a regular distribution T_f of order zero in the space $\mathcal{E}'(\mathbb{R}_+)$, given by

$$\left\langle T_f, \psi \right\rangle = \int_0^\infty f(x)\psi(x)\,dx, \quad \text{for all } \psi \in \mathcal{E}(\mathbb{R}_+).$$

To verify this, we estimate

$$\begin{aligned} \left|\left\langle T_f, \psi \right\rangle\right| &= \left|\int_0^\infty f(x)\psi(x)\,dx\right| \leq \sup_{x\in \mathrm{supp}(f)} |\psi(x)| \int_{\mathrm{supp}(f)} |f(x)|\,dx \\ &= \Gamma_{0,\mathrm{supp}(f)}(\psi) \int_{\mathrm{supp}(f)} |f(x)|\,dx, \end{aligned}$$

where $\mathrm{supp}(f)$ denotes the support of f. This confirms that the distribution T_f is of order zero.

Accordingly, we can express the index Whittaker transform of f as

$$\begin{aligned} F_\mu(\tau) = (W_\mu f)(\tau) &= \left\langle T_f(x), \frac{W_{\mu,i\tau}(x)}{x^2} \right\rangle \\ &= \int_0^\infty f(x)\frac{W_{\mu,i\tau}(x)}{x^2}\,dx, \quad \tau > 0,\ \mu < \frac{1}{2}. \end{aligned} \tag{17}$$

Using Theorem 4, we can extend the scope of the index Whittaker transform to the class of regular distributions in $\mathcal{E}'(\mathbb{R}_+)$ and obtain the result stated below.

Proposition 4. *Let f be a compactly supported, locally integrable function on $\mathbb{R}_+$. Then, the function F_μ defined by (17) for $\mu < \frac{1}{2}$ satisfies the following:*

(i) *For any $\eta > 0$, the following limit holds:*

$$\lim_{\tau\to 0^+} \left(\tau^\eta F_\mu(\tau)\right) = 0.$$

(ii) *For any $\eta > 0$, it also holds that:*

$$\lim_{\tau\to +\infty} \left(\tau^{-\eta} F_\mu(\tau)\right) = 0.$$

5. Final Observations and Conclusions

In this work, we have explored new analytical developments for the index Whittaker transform, an integral operator introduced by Wimp [1], with a kernel involving the classical Whittaker function $W_{\mu,i\tau}(x)$. The investigation began with the analysis of W_μ over the weighted space $L^1(\mathbb{R}_+, x^{-2}dx)$, leading to the establishment of a novel Parseval-type identity.

We further extended our analysis to the class of generalized Lebesgue spaces $\mathcal{L}_\gamma$, recently studied in [10], and derived a new Parseval–Goldstein type relation, thus enriching the structural properties of the transform within such function spaces.

The second half of the study focused on the behavior of the index Whittaker transform in the distributional setting. Specifically, we considered compactly supported distributions in $\mathcal{E}'(\mathbb{R}_+)$ and obtained new Abelian theorems that describe the asymptotic behavior of the transform at infinity.

Taken together, these results provide a robust functional analytic framework for the index Whittaker transform in both classical and distributional contexts. Our contributions significantly extend the existing literature by providing L^p-boundedness properties,

integral identities of Parseval–Goldstein type, and improved asymptotic results in the distributional setting.

This study also serves as motivation to investigate other integral transforms over the types of function spaces considered here. The techniques and results developed in this paper can be extended to a variety of kernel-based integral transforms and distributional frameworks, potentially leading to new discoveries in harmonic analysis, special function theory, and the general theory of integral transforms. Thus, the present work opens promising avenues for further exploration in contemporary mathematical analysis.

6. Open Question

Relation (5) establishes an interesting connection between the index Whittaker transform W_μ and the differential operator $\mathcal{L}^*_{\mu,x}$. Using relation (5) and obtaining inversion formulae for this transform would be a useful tool to solve differential equations containing the operator $\mathcal{L}^*_{\mu,x}$.

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