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Spatial Decay Estimates for Solutions of a Class of Evolution Equations Containing a Biharmonic Operator

Jincheng Shi [1] and Yiwu Lin [2,*]

[1] Department of Applied Mathematics, Guangzhou Huashang College, Guangzhou 511300, China; shijc0818@gdhsc.edu.cn
[2] Department of Applied Mathematics, Guangdong University of Finance, Guangzhou 510521, China
* Correspondence: 26-062@gduf.edu.cn

Abstract

This study delves into the spatial characteristics of solutions for a specific class of evolution equations that incorporate biharmonic operators. The process begins with the construction of an energy function. Subsequently, by employing an integro-differential inequality method, it is deduced that this energy function satisfies an integro-differential inequality. Resolving this inequality enables us to establish an estimate for the spatial decay of the solution. Ultimately, the finding affirms that the spatial exponential decay is reminiscent of Saint-Venant-type estimates.

Keywords: decay estimates; evolution equation; Saint-Venant principle; integro-differential inequality

MSC: 35B30; 35K55; 35Q35

1. Introduction

As a highly regarded research area in the fields of mathematics and mechanics, the Saint-Venant principle, which is elaborated upon in references [1,2], points out that, on a cylindrical surface, the solution to the harmonic equation exhibits exponential growth or decay from a finite endpoint to infinity. The biharmonic equation finds extensive applications in mechanics. For instance, in the analysis of isotropic elastic strain within a planar region, the Airy function satisfies the biharmonic equation. Moreover, the steady Stokes equation on a planar domain can be transformed into and described by the biharmonic equation. Payne and Schaefer [3] were pioneers in extending the findings from the harmonic equation to the biharmonic equation, deriving the Phragmén–Lindelöf alternative results for the biharmonic equation in three distinct regions. Other investigations concerning the Saint-Venant principle for the biharmonic equation can be found in references [4–7]. In recent years, many researchers have conducted studies on spatial decay estimates of solutions or the Phragmén–Lindelöf alternative problem, yielding new outcomes that further enrich the traditional connotations of the Saint-Venant principle. References [8–10] explore various heat equations, elucidating the spatial properties of their solutions.

Our problem is considered on the domain Ω_0, which is an unbounded region defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\},$$

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where h is a positive constant. We use the notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}.$$

Within this coordinate framework, the variable z is designated as the longitudinal coordinate that aligns with the x_1 axis.

In this study, a comma-based subscript notation is employed to represent partial differentiation. Specifically, differentiation with respect to the spatial coordinate x_k is indicated by the notation, k. Consequently, $v_{,i}$ denotes $\frac{\partial v}{\partial x_i}$, and $\dot{\theta}$ denotes $\frac{\partial \theta}{\partial t}$. The conventional summation convention is adopted, wherein repeated Latin subscripts i are summed from 1 to 2. Hence, $\alpha_{,i}\alpha_{,i} = \alpha_{,1}^2 + \alpha_{,2}^2$.

In reference [11], the authors investigated the behavior of solutions to a system of equations governing an evolutionary process composed of an elastic membrane and an elastic plate:

$$\begin{aligned} &v_{,tt} + \Delta^2 v + \lambda \Delta \theta_{,t} = 0 \\ &\theta_{,t} - k\Delta\theta + r\theta - \lambda \Delta v = 0. \end{aligned} \tag{1}$$

Here, v represents the vertical deflection of the plate, θ denotes the temperature difference, and λ, k, and r are all positive constants. $\triangle$ represents the Laplace operator, and $\triangle^2$ denotes the biharmonic operator.

Over the past five decades, numerous scholars have delved into the study of the Saint-Venant principle within the realms of applied mathematics and mechanics. Through extensive research endeavors, the classical Saint-Venant principle has undergone significant theoretical expansions. For a systematic exploration of advancements in this field, readers are referred to the seminal works of Horgan [12]. Established theorems of the Saint-Venant type reveal that energy exhibits exponential decay as the axial distance increases from the proximal end towards infinity along semi-infinite strip or cylindrical geometries (cf. [13–22]).

Recent studies have established foundational results for coupled partial differential equation systems through differential inequality techniques. Specifically, ref. [23] analyzed hyperbolic–hyperbolic coupled systems and derived spatial decay estimates using both first-order and second-order differential inequalities. Building on this, ref. [24] investigated structural stability for the same hyperbolic–hyperbolic system through a second-order differential inequality framework. For hyperbolic–parabolic coupled systems, ref. [25] developed a Saint-Venant-type principle for weighted energy functionals via a second-order differential inequality approach. While these works advance our understanding of solution behavior, the present study introduces a fundamentally novel approach for analyzing systems of greater complexity. Unlike the hyperbolic–parabolic systems in [25], whose relatively simple structures permit the construction of second-order differential inequalities through spatially weighted energy expressions, the systems considered here exhibit structural complexity that renders such traditional methods intractable. The spatial weighting techniques employed in prior works [23–25] become prohibitively cumbersome for our systems, preventing the formulation of the required second-order differential inequalities. To overcome this limitation, we introduce a novel integro-differential inequality method—a technique rarely utilized in previous studies—to establish analogous results for unweighted energy functionals. This innovative approach circumvents the need for spatial weighting, thereby significantly simplifying subsequent analyses such as pointwise solution estimates. By developing this method, we not only address the inherent complexity of the systems under investigation but also open new avenues for analyzing coupled PDEs for which traditional differential inequality approaches are insufficient.

The initial-boundary conditions are

$$\theta(x_1,0,t)=v(x_1,0,t)=v_{,2}(x_1,0,t)=0,\quad x_1>0,t>0, \tag{2}$$

$$\theta(x_1,h,t)=v(x_1,h,t)=v_{,2}(x_1,h,t)=0,\quad x_1>0,t>0, \tag{3}$$

$$\theta(0,x_2,t)=g_1(x_2,t),\quad 0\le x_2\le h,t>0, \tag{4}$$

$$v(0,x_2,t)=g_2(x_2,t),\quad 0\le x_2\le h,t>0, \tag{5}$$

$$v_{,1}(0,x_2,t)=g_3(x_2,t),\quad 0\le x_2\le h,t>0, \tag{6}$$

and

$$\theta(x_1,x_2,0)=v(x_1,x_2,0)=v_{,t}(x_1,x_2,0)=0,\quad 0\le x_2\le h,x_1>0. \tag{7}$$

In this paper, we add some a priori asymptotic decay assumptions for solutions at infinity.

$$\begin{aligned}&\theta_{,\alpha}(x_1,x_2,t),v(x_1,x_2,t),\theta(x_1,x_2,t),v_{,\alpha}(x_1,x_2,t),\\&v_{,\alpha\beta}(x_1,x_2,t)=o(z^{-1})\quad (uniformly\quad in\quad x_2\quad and\quad t)\quad as\quad x_1\to\infty.\end{aligned} \tag{8}$$

In this paper, the initial-boundary conditions must satisfy the following compatibility conditions:

$$g_i(0,t)=g_{i,2}(0,t)=g_i(h,t)=g_{i,2}(h,t)=0,$$

and

$$g_1(x_2,0)=g_2(x_2,0)=g_i(h,t)=g_{1,t}(x_2,0)=0.$$

We primarily investigate the spatial properties of solutions under the assumption that the solutions are classical solutions, meaning all partial derivatives with respect to the time and spatial variables of all orders exist and are continuous. By using the semigroup approach, similarly to [26] for the half-space, one may derive the following Sobolev regularity of the solutions:

$$v\in C^2([0,t],H^4(\Omega_0)),$$

and

$$\theta\in C^1([0,t],H^2(\Omega_0)).$$

2. The Function Expression $F(z,t)$

To obtain the desired decay estimates, we introduce a functional framework delineated by $F(z,t)$, which characterizes the behavior of the solution. This auxiliary function is rigorously constructed based on the theoretical propositions presented in the subsequent analysis. In deriving our function expression, we must incorporate $exp(-\omega t)$ into the expressions. If $exp(-\omega t)$ is not incorporated, inconsistencies may arise when the control

energy exceeds zero. This is because introducing the term enables the inclusion of numerous additional positive terms that eliminate the contradiction.

Before constructing the energy function in this paper, we first clarify the following two fundamental properties:

Lemma 1. *Suppose that ϕ and Φ are differentiable functions and satisfy $\phi = o(z^{-1})$ and $\Phi = o(z^{-1})$ as $z \to \infty$. Then, the following properties hold:*

$$\int_z^\infty \int_{L_\xi} \frac{\partial \phi}{\partial z} \mathrm{d}A = -\int_{L_z} \phi \mathrm{d}x_2, \tag{9}$$

$$\int_z^\infty \int_{L_\xi} \frac{\partial \phi}{\partial z} \Phi \mathrm{d}A = -\int_z^\infty \int_{L_\xi} \phi \frac{\partial \Phi}{\partial z} \mathrm{d}A - \int_{L_z} \phi \Phi \mathrm{d}x_2, \tag{10}$$

and

$$\int_z^\infty \int_{L_\xi} \phi \Phi \mathrm{d}A \leq M(t), \tag{11}$$

where $M(t)$ is a positive continuous function.

Proof. We have

$$\int_z^\infty \int_{L_\xi} \frac{\partial \phi}{\partial z} \mathrm{d}A = \int_{L_\xi} \phi \mathrm{d}x_2|_{\xi=\infty} - \int_{L_z} \phi \mathrm{d}x_2. \tag{12}$$

Since $\phi \to 0$ as $z \to \infty$, and L_ξ is a bounded domain, we have

$$\int_{L_\xi} \frac{\partial \phi}{\partial z} \mathrm{d}x_2|_{\xi=\infty} = 0. \tag{13}$$

By inserting (13) into (12), we obtain the desired result (9).

Integrating by parts, we have

$$\begin{aligned} \int_z^\infty \int_{L_\xi} \frac{\partial \phi}{\partial z} \Phi \mathrm{d}A &= -\int_z^\infty \int_{L_\xi} \frac{\partial \Phi}{\partial z} \phi \mathrm{d}A + \int_{L_\xi} \phi \Phi \mathrm{d}x_2|_{\xi=\infty} - \int_{L_z} \phi \Phi \mathrm{d}x_2 \\ &= -\int_z^\infty \int_{L_\xi} \frac{\partial \Phi}{\partial z} \phi \mathrm{d}A - \int_{L_z} \phi \Phi \mathrm{d}x_2. \end{aligned} \tag{14}$$

Equality (14) is the desired result (10).

For $\phi = o(z^{-1})$ as $z \to \infty$, we obtain the following result: there exist constants $N > 1$ and $C > 0$, such that $|\phi| \leq Cz^{-1}$ and $|\Phi| \leq Cz^{-1}$ for $z > N$. Thus, we obtain

$$\begin{aligned} \int_z^\infty \int_{L_\xi} \phi \Phi \mathrm{d}A &\leq \int_z^N \int_{L_\xi} \phi \Phi \mathrm{d}A + \int_N^\infty \int_{L_\xi} \phi \Phi \mathrm{d}A \\ &\leq \delta(t) + C^2 h \left(\frac{1}{N} - \frac{1}{\infty} \right) \\ &= M(t). \end{aligned} \tag{15}$$

Inequality (15) is the desired result (11). □

Proposition 1. *Suppose that (v, θ) constitutes the classical solution to the initial-boundary value problems described by Equations (1)–(8). Subsequently, we define a function*

$$\begin{aligned} F_1(z,t) = {} & \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\ & + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\ & - \lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{16}$$

$F_1(z,t)$ can also be formulated as (or expressed in the form of)

$$\begin{aligned} F_1(z,t) = {} & \frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\right) - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\ & - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta + \lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta, \end{aligned} \tag{17}$$

where ω is an arbitrary positive constant used to adjust the coefficient relationships, defined later.

Proof. We define a function

$$f_1(z,t) = \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta. \tag{18}$$

Using the divergence theorem, we have

$$\begin{aligned} & \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\ & = -\frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}(\exp(-\omega\eta))_{,\eta}v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\ & = \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\eta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A|_{\eta=t}. \end{aligned} \tag{19}$$

Integrating by parts, we have

$$\begin{aligned} & \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\eta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\ & = -\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta \\ & = \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\ & - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta. \end{aligned} \tag{20}$$

Using (1), we have

$$\begin{aligned} & \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\ & = -\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta - \lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\ & = -\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\ & + \lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta. \end{aligned} \tag{21}$$

We define

$$\begin{aligned} F_1(z,t) = {} & \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\ & + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\ & - \lambda\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)v_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{22}$$

Combining (18)–(22), we have

$$\begin{aligned} F_1(z,t) = {} & \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta \\ & + \lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta, \end{aligned} \tag{23}$$

(23) can be rewritten as the desired result (17). □

Proposition 2. *Suppose that* (v,θ) *constitutes the classical solution to the initial-boundary value problems described by Equations* (1)–(8)*. Subsequently, we define a function*

$$\begin{aligned} F_2(z,t) = {} & k\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\ & + \frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A \\ & + r\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \lambda\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{24}$$

$F_2(z,t)$ can also be formulated as (or expressed in the form of)

$$F_2(z,t) = -\frac{k}{2}\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\right) - \lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta. \tag{25}$$

Proof. Differentiating $(1)_2$ with respect to t, we have

$$\theta_{,tt} - k\Delta\theta_{,t} + r\theta_{,t} - \lambda\Delta v_{,t} = 0. \tag{26}$$

We define $f_2(z,t) = k\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta$.
Using the divergence theorem, we have

$$\begin{aligned} & k\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\ & = -k\int_0^t\int_z^\infty\int_{L_{\bar{\xi}}}\exp(-\omega\eta)\theta_{,\alpha\alpha\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta - k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1\eta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta. \end{aligned} \tag{27}$$

Using (26), we have

$$\begin{aligned}
&-k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}\theta_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}(\theta_{,\eta\eta}+r\theta_{,\eta}-\lambda v_{,\alpha\alpha\eta})\mathrm{d}A\mathrm{d}\eta\\
&=-\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta-\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A\mathrm{d}\eta\\
&-r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{28}$$

We define

$$\begin{aligned}
F_2(z,t)&=k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A\\
&+r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{29}$$

Combining (27)–(29), we have

$$\begin{aligned}
F_2(z,t)&=-k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1\eta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&=-\frac{k}{2}\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\right)-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{30}$$

□

Proposition 3. *Suppose that* (v,θ) *constitutes the classical solution to the initial-boundary value problems described by Equations* (1)–(8)*. Subsequently, we define a function*

$$\begin{aligned}
F_3(z,t)&=\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta\\
&+k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{31}$$

$F_3(z,t)$ can also be formulated as (or expressed in the form of)

$$F_3(z,t)=-k\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta-\lambda\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta.\tag{32}$$

Proof. We define

$$f_3(z,t)=\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.\tag{33}$$

Using the divergence theorem and $(1)_1$, we have

$$\begin{aligned}
&\lambda \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\eta} v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\
&= -\lambda \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\alpha\eta} v_{,\eta} \mathrm{d}A\mathrm{d}\eta - \lambda \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,1\eta} v_{,\eta} \mathrm{d}x_2\mathrm{d}\eta \\
&= -\int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} (\theta_{,\eta\eta} - k\theta_{,\alpha\alpha\eta} + r\theta_{,\eta}) \mathrm{d}A\mathrm{d}\eta - \lambda \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,1\eta} v_{,\eta} \mathrm{d}x_2\mathrm{d}\eta \\
&= -\int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta\eta} \mathrm{d}A\mathrm{d}\eta - k \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\
&- \lambda \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\eta} \theta_{,1\eta} \mathrm{d}x_2\mathrm{d}\eta - r \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A\mathrm{d}\eta \\
&- \lambda \frac{\partial}{\partial z} \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{34}$$

If we define

$$\begin{aligned}
F_3(z,t) = \lambda \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\eta} v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta + \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta\eta} \mathrm{d}A\mathrm{d}\eta \\
+ k \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta + r \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A\mathrm{d}\eta,
\end{aligned} \tag{35}$$

we can also define

$$F_3(z,t) = -k \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\eta} \theta_{,1\eta} \mathrm{d}x_2\mathrm{d}\eta - \lambda \frac{\partial}{\partial z} \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta. \tag{36}$$

□

Proposition 4. *Suppose that (v, θ) constitutes the classical solution to the initial-boundary value problems described by Equations (1)–(8). Subsequently, we define a function*

$$\begin{aligned}
F_4(z,t) = &-\lambda \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) \theta_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta - \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta\eta} \mathrm{d}A\mathrm{d}\eta \\
&- \omega \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A\mathrm{d}\eta - \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A|_{\eta=t} \\
&+ \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) v_{,\alpha\beta\beta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{37}$$

$F_4(z,t)$ can also be formulated as (or expressed in the form of)

$$F_4(z,t) = -\int_0^t \int_{L_z} \exp(-\omega\eta) v_{,1\beta\beta} \theta_{,\eta} \mathrm{d}x_2\mathrm{d}\eta - \lambda \int_0^t \int_{L_z} \exp(-\omega\eta) \theta_{,1\eta} \theta_{,\eta} \mathrm{d}x_2\mathrm{d}\eta. \tag{38}$$

Proof. We define

$$f_4(z,t) = -\lambda \int_0^t \int_z^\infty \int_{L_{\tilde{\xi}}} \exp(-\omega\eta) \theta_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta. \tag{39}$$

Using the divergence theorem and (26), we have

$$\begin{aligned}
&-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&=-k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\alpha\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta\beta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1\eta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&=-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta+\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A|_{\eta=t}-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\beta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\beta\beta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1\eta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{40}$$

We define

$$\begin{aligned}
F_4(z,t)=&-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta-\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A|_{\eta=t}\\
&-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\beta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{41}$$

Combining (59) and (41), we have

$$F_4(z,t)=-\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\beta\beta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1\eta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta.\tag{42}$$

□

Proposition 5. *Suppose that* (v,θ) *constitutes the classical solution to the initial-boundary value problems described by Equations* (1)–(8)*. Subsequently, we define a function*

$$\begin{aligned}
F_5(z,t)=&\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,j\eta}v_{,j\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A\mathrm{d}\eta-\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A|_{\eta=t}\\
&+\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\alpha}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{43}$$

$F_5(z,t)$ can also be formulated as (or expressed in the form of)

$$\begin{aligned}
F_5(z,t)=&-\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\right)\\
&-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{44}$$

Proof. We define

$$f_5(z,t)=\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta.\tag{45}$$

Using the divergence theorem, we have

$$\begin{aligned}&\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta\\&=-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta\beta}v_{,jj}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha1}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned}\tag{46}$$

We now deal with the term $-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta\beta}v_{,jj}\mathrm{d}A\mathrm{d}\eta$ in (46). Using the divergence theorem and $(1)_1$, we have

$$\begin{aligned}&\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\&=\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}(v_{,\eta\eta}+\lambda\theta_{,\alpha\alpha\eta})\mathrm{d}A\mathrm{d}\eta\\&=-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\eta}v_{,\eta}\mathrm{d}A\mathrm{d}\eta+\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A\mathrm{d}\eta\\&+\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A|_{\eta=t}-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\alpha}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\&-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta\\&=\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,j\eta}v_{,j\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\eta}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta\\&+\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A\mathrm{d}\eta+\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A|_{\eta=t}\\&-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\alpha}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned}\tag{47}$$

We define

$$\begin{aligned}F_5(z,t)=&\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,j\eta}v_{,j\eta}\mathrm{d}A\mathrm{d}\eta\\&-\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A\mathrm{d}\eta-\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A|_{\eta=t}\\&+\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\alpha}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.\end{aligned}\tag{48}$$

Combining (46) and (47), we have

$$\begin{aligned}F_5(z,t)=&-\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\right)\\&-\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned}\tag{49}$$

We define a new function

$$F(z,t)=k_1(F_1(z,t)+F_2(z,t))+k_2(F_3(z,t)+F_4(z,t))+F_5(z,t),\tag{50}$$

where k_1 and k_2 are positive constants to be determined later.

From (16), (24), (31), (37), and (43), we have

$$\begin{aligned}
F(z,t) = {} & \frac{k_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
& + \frac{k_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\
& + (k_1k - k_2\lambda)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \left(\frac{k_1\omega}{2} + k_1\gamma\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A \\
& + (k_2\lambda - 1)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + k_2k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + (k_2r - k_2\omega)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
& - k_2\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,\eta}\mathrm{d}A|_{\eta=t} - k_2\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\beta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta - \omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
& - \int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}v_{,\eta}\mathrm{d}A|_{\eta=t} + \lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj\alpha}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{51}$$

□

Proposition 6. *Suppose that* (v,θ) *constitutes the classical solution to the initial-boundary value problems given by Equations* (1)–(8). *We choose* $k_1 = \max\left\{2\sqrt{\frac{16}{\lambda^2}+1}, \frac{4(4+\frac{16}{\lambda^2}+\lambda^2+\frac{4k^2}{\lambda^2})}{3k}\right\}$, $k_2 = \frac{4}{\lambda}$, *and* $\omega = \gamma$. *The function* $F(z,t)$, *defined in* (51), *satisfies the following estimates:*

$$\begin{aligned}
F(z,t) \geq {} & \frac{k_1\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
& + \frac{k_1\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\
& + \frac{k_1k}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \left(\frac{k_1\omega}{2} + k_1\gamma\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A \\
& + \frac{k_2\lambda}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta \\
& = M(z,t).
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
-\frac{\partial F(z,t)}{\partial z} \geq {} & \frac{k_1\omega}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta + \frac{k_1}{4}\int_{L_z}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 \\
& + \frac{k_1\omega}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{k_1}{4}\int_{L_z}\exp(-\omega t)v_{,t}^2\mathrm{d}x_2 \\
& + \frac{k_1k}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \left(\frac{k_1\omega}{2} + k_1\gamma\right)\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1}{4}\int_{L_z}\exp(-\omega t)\theta_{,t}^2\mathrm{d}x_2 \\
& + \frac{k_2\lambda}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{53}$$

Proof. Using Schwarz's inequality, we have

$$k_2 k \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \le \frac{k_2 k \epsilon_1}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\eta} v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta + \frac{k_2 k}{2\epsilon_1} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta, \tag{54}$$

$$\begin{aligned} &(k_2 r - k_2\omega) \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A\mathrm{d}\eta \\ &\le \frac{(k_2 r - k_2\omega)\epsilon_2}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta \\ &+ \frac{k_2 r - k_2\omega}{2\epsilon_2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta, \end{aligned} \tag{55}$$

$$k_2 \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta} \theta_{,\eta} \mathrm{d}A|_{\eta=t} \le \frac{k_2\epsilon_3}{2} \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}A|_{\eta=t} + \frac{k_2}{2\epsilon_3} \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta_{,\eta}^2 \mathrm{d}A|_{\eta=t}, \tag{56}$$

$$k_2 \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\beta\beta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \le \frac{k_2\epsilon_4}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\beta\beta} v_{,\alpha jj} \mathrm{d}A\mathrm{d}\eta + \frac{k_2}{2\epsilon_4} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta, \tag{57}$$

$$\omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,jj} v_{,\eta} \mathrm{d}A\mathrm{d}\eta \le \frac{\omega\epsilon_5}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\alpha} v_{,\beta\beta} \mathrm{d}A\mathrm{d}\eta + \frac{\omega}{2\epsilon_5} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta, \tag{58}$$

$$\int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,jj} v_{,\eta} \mathrm{d}A|_{\eta=t} \le \frac{\epsilon_6}{2} \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\alpha} v_{,\beta\beta} \mathrm{d}A|_{\eta=t} + \frac{1}{2\epsilon_6} \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}A|_{\eta=t}, \tag{59}$$

and

$$\lambda \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,jj\alpha} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \le \frac{\lambda\epsilon_7}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\alpha\beta} v_{,\beta jj} \mathrm{d}A\mathrm{d}\eta + \frac{\lambda}{2\epsilon_7} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta_{,\alpha\eta} \theta_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta, \tag{60}$$

where ϵ_1, ϵ_2, ϵ_3, ϵ_4, ϵ_5, ϵ_6, and ϵ_7 are positive constants, determined later.

Inserting (54)–(60) into (51), we have

$$\begin{aligned}
F(z,t) \geq & \left(\frac{k_1\omega}{2}-\frac{\omega\epsilon_5}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\left(\frac{k_1}{2}-\frac{\epsilon_6}{2}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\\
&+\left(\frac{k_1\omega}{2}-\frac{(k_2r-k_2\omega)\epsilon_2}{2}-\frac{\omega}{2\epsilon_5}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{k_1}{2}-\frac{k_2\epsilon_3}{2}-\frac{1}{2\epsilon_6}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A\\
&+\left(k_1k-k_2\lambda-\frac{k_2}{2\epsilon_4}-\frac{\lambda}{2\epsilon_7}-\frac{k_2k}{2\epsilon_1}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha\eta}\theta_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{k_1\omega}{2}+k_1\gamma-\frac{k_2r-k_2\omega}{2\epsilon_2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{k_1}{2}-\frac{k_2}{2\epsilon_3}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta_{,t}^2\mathrm{d}A\\
&+\left(k_2\lambda-1-\frac{k_2k\epsilon_1}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\left(1-\frac{\lambda\epsilon_7}{2}-\frac{k_2\epsilon_4}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,jj\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{61}$$

Now, we proceed to determine the coefficients on the right-hand side of Equation (61). First, let $\frac{k_1w}{2}-\frac{w\epsilon_5}{2}=\frac{k_1w}{4}$, we get $\epsilon_5=\frac{k_1}{2}$.

Similarly, we choose $\epsilon_6=\frac{k_1}{2}$, so the coefficient of the second term on the right-hand side of Equation (61) satisfies $\frac{k_1}{2}-\frac{\epsilon_6}{2}=\frac{k_1}{4}$.

When addressing the coefficient of the third term on the right-hand side of (61), we adopt a strategic selection of $\omega=\gamma$ to streamline the computational process. Subsequently, we posit the equality $\frac{k_1w}{2}-\frac{w}{2\epsilon_5}=\frac{k_1w}{4}$, so we get $k_1=2$.

Next, we turn our attention to the fourth term's coefficient on the right-hand side of Equation (61). We choose $\epsilon_3=\frac{2k_2}{k_1}$. Let $\frac{k_1}{2}-\frac{k_2\epsilon_3}{2}-\frac{1}{2\epsilon_6}=\frac{k_1}{4}$, so we get $k_1^2=4(k_2^2+1)$.

We also find that the coefficient of the seventh term satisfies the equation $\frac{k_1}{2}-\frac{k_2}{2\epsilon_3}=\frac{k_1}{4}$.

We now focus on analyzing the coefficient of the fifth term on the right-hand side of Equation (61). If we choose $\epsilon_1=\frac{\lambda}{2k}$, we have $k_2\lambda-1-\frac{k_2k\epsilon_1}{2}=\frac{k_2\lambda}{2}-1$. If we further choose $k_2=\frac{4}{\lambda}$, we have $k_2\lambda-1-\frac{k_2k\epsilon_1}{2}=\frac{k_2\lambda}{4}$.

If we choose $\epsilon_4=\frac{1}{2k_2}$, $\epsilon_7=\frac{1}{2\lambda}$, we also find that the coefficient of the last term satisfies the equation $1-\frac{\lambda\epsilon_7}{2}-\frac{k_2\epsilon_4}{2}=\frac{1}{2}$.

Finally, we address the coefficient of the eighth term on the right-hand side of (61). If we let the term $k_1k-k_2\lambda-\frac{k_2}{2\epsilon_4}-\frac{\lambda}{2\epsilon_7}-\frac{k_2k}{2\epsilon_1}=\frac{k_1k}{4}$, we obtain $k_1=\frac{4(k_2\lambda+k_2^2+\lambda^2+\frac{k_2k^2}{\lambda})}{3k}$.

In summary, if we choose $k_1=\max\left\{2\sqrt{k_2^2+1},\frac{4(k_2\lambda+k_2^2+\lambda^2+\frac{k_2k^2}{\lambda})}{3k}\right\}$, $k_2=\frac{4}{\lambda}$, $\gamma=\omega$, $\epsilon_1=\frac{\lambda}{2k}$, $\epsilon_3=\frac{2k_2}{k_1}$, $\epsilon_4=\frac{1}{2k_2}$, $\epsilon_5=\frac{k_1}{2}$, $\epsilon_6=\frac{k_1}{2}$, and $\epsilon_7=\frac{1}{2\lambda}$, from (61), we obtain the desired result (52).

Differentiating (51) with respect to z and following the same procedure as in deriving (52), we can also obtain the desired result (53). □

Proposition 7. *Suppose that (v,θ) constitutes the classical solution to the initial-boundary value problems given by Equations (1)–(8). The function $F(z,t)$ satisfies the following estimates:*

$$\int_z^\infty F(z,t)\mathrm{d}\xi\leq k_3\left(-\frac{\partial F(z,t)}{\partial z}\right)+k_4F(z,t).\tag{62}$$

Proof. From (17), (25), (32), (38), and (44), $F(z,t)$ can also be rewritten as

$$\begin{aligned}
F(z,t) = {} & k_1\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{kk_1}{2}\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - k_1\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - k_1\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda k_1\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& - k_2\lambda\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta - kk_2\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& - k_2\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\beta\beta}\theta_{,\eta}\mathrm{d}x_2\mathrm{d}\eta - \frac{\lambda k_2}{2}\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - \frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\frac{\partial}{\partial z}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - \lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{63}$$

Integrating (63), we have

$$\begin{aligned}
\int_z^\infty F(z,t)\mathrm{d}\xi = {} & k_1\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{k_1k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - k_2\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta - \frac{k_2\lambda}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - k_1\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta - k_1\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta \\
& + k_1\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta - k_2k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
& - k_2\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\beta\beta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta - \lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{64}$$

Using Schwarz's inequality and (53), we have

$$\begin{aligned}
& k_1\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{k_1k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - k_2\lambda\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta - \frac{k_2\lambda}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1}^2\mathrm{d}x_2\mathrm{d}\eta \\
& - \int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& \le \frac{k_1+1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{k_1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\beta\beta}v_{,jj}\mathrm{d}x_2\mathrm{d}\eta \\
& \le k_3\left(-\frac{\partial F(z,t)}{\partial z}\right),
\end{aligned} \tag{65}$$

where $k_3 = \left[\frac{2(k_1+1)}{k_1\omega} + \frac{2}{\omega}\right]$.

Using Schwarz's inequality and (52), we have

$$\begin{aligned}
&-k_1\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta-k_1\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta\\
&+k_1\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta-k_2k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta\\
&-k_2\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\beta\beta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta-\lambda\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,jj}\theta_{,1\eta}\mathrm{d}A\mathrm{d}\eta\\
&\le\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+(1+\frac{\lambda}{2})\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{\lambda+k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{2\lambda+k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\beta\beta}^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&\le k_4F(z,t),
\end{aligned}\tag{66}$$

where $k_4=max\left\{1,\frac{1}{k_1\omega+2k_1r},\frac{4\lambda+2k}{k_1k},\frac{2(\lambda+k)}{k_1\omega},\frac{4+2\lambda}{k_1\omega},\frac{2}{k_2\lambda}\right\}$.

By combining (64)–(66), we obtain the desired result (62).

Inequality (62) will play an important role in deducing our result. □

3. Spatial Decay Estimates

In this section, we derive the spatial decay estimates pertaining to the energy $M(z,t)$. Furthermore, we reformulate (62) as

$$\frac{\partial F(z,t)}{\partial z}+\frac{1}{k_3}\int_z^\infty F(\xi,t)\mathrm{d}\xi\le\frac{k_4}{k_3}F(z,t).\tag{67}$$

Next, we define two functions

$$\phi(z,t)=e^{-\frac{k_4}{k_3}z}F(z,t),\tag{68}$$

and

$$G(z,t)=\phi(z,t)+r\int_z^\infty e^{\frac{k_4}{k_3}(\xi-z)}\phi(\xi,t)\mathrm{d}\xi,\tag{69}$$

where r is a positive constant, defined later.

Since it is difficult to solve (67), we use the form of $G(z,t)$ to solve it.

Proposition 8. *The function $G(z,t)$, defined in (69), satisfies the following estimate:*

$$G(z,t)\le G(0,t)e^{-r_1z},\tag{70}$$

where r_1 is a positive constant, defined later.

Proof. Differentiating (69) with respect to z, we have

$$\begin{aligned}
\frac{\partial G(z,t)}{\partial z}&=\frac{\partial\phi(z,t)}{\partial z}-\frac{k_4}{k_3}r\int_z^\infty e^{\frac{k_4}{k_3}(\xi-z)}\phi(\xi,t)\mathrm{d}\xi-r\phi(z,t)\\
&=-\frac{k_4}{k_3}e^{-\frac{k_4}{k_3}z}F(z,t)+e^{-\frac{k_4}{k_3}z}\frac{\partial F(z,t)}{\partial z}\\
&\quad-\frac{k_4}{k_3}r\int_z^\infty e^{-\frac{k_4}{k_3}z}F(\xi,t)\mathrm{d}\xi-re^{-\frac{k_4}{k_3}z}F(z,t).
\end{aligned}\tag{71}$$

We then obtain

$$\begin{aligned}\frac{\partial G(z,t)}{\partial z}+rG(z,t) &= -\frac{k_4}{k_3}e^{-\frac{k_4}{k_3}z}F(z,t)+e^{-\frac{k_4}{k_3}z}\frac{\partial F(z,t)}{\partial z}\\ &\quad -\frac{k_4}{k_3}r\int_z^\infty e^{-\frac{k_4}{k_3}z}F(\xi,t)\mathrm{d}\xi - re^{-\frac{k_4}{k_3}z}F(z,t)\\ &\quad +re^{-\frac{k_4}{k_3}z}F(z,t)+r^2\int_z^\infty e^{-\frac{k_4}{k_3}z}F(\xi,t)\mathrm{d}\xi.\end{aligned} \tag{72}$$

From (67), we have

$$-\frac{k_4}{k_3}e^{-\frac{k_4}{k_3}z}F(z,t)+e^{-\frac{k_4}{k_3}z}\frac{\partial F(z,t)}{\partial z}\leq -\frac{1}{k_3}e^{-\frac{k_4}{k_3}z}\int_z^\infty F(\xi,t)\mathrm{d}\xi. \tag{73}$$

By inserting (73) into (72), we get

$$\frac{\partial G(z,t)}{\partial z}+rG(z,t)\leq\left(r^2-\frac{k_4}{k_3}r-\frac{k_4}{k_3}\right)\int_z^\infty e^{-\frac{k_4}{k_3}z}F(\xi,t)\mathrm{d}\xi. \tag{74}$$

Let $r^2-\frac{k_4}{k_3}r-\frac{k_4}{k_3}=0$. We choose $r_1=\frac{\frac{k_4}{k_3}+\sqrt{(\frac{k_4}{k_3})^2+\frac{4k_4}{k_3}}}{2}>0$. We then obtain the result

$$\frac{\partial G(z,t)}{\partial z}+r_1G(z,t)\leq 0, \tag{75}$$

Integrating (75), we obtain the desired result (70). □

Proposition 9. *The function $F(z,t)$, defined in (51), satisfies the following estimates:*

$$F(z,t)\leq\frac{2r_1k_3-k_4}{r_1k_3-k_4}F(0,t)e^{-\left(r_1-\frac{k_4}{k_3}\right)z}. \tag{76}$$

Proof. By combining (69) and (70), we have

$$\phi(z,t)\leq G(0,t)e^{-r_1z}. \tag{77}$$

According to the definition of $\phi(z,t)$ in (68), we have

$$F(z,t)\leq G(0,t)e^{-\left(r_1-\frac{k_4}{k_3}\right)z}. \tag{78}$$

We now want to bound $G(0,t)$ in terms of $F(0,t)$.
Using (69) and (70), we obtain

$$F(z,t)+r_1\int_z^\infty F(\xi,t)\mathrm{d}\xi\leq G(0,t)e^{-\left(r_1-\frac{k_4}{k_3}\right)z}. \tag{79}$$

We rewrite inequality (79) as

$$-\frac{\partial}{\partial z}\left[e^{-r_1z}\int_z^\infty F(\xi,t)\mathrm{d}\xi\right]\leq G(0,t)e^{-\left(2r_1-\frac{k_4}{k_3}\right)z}. \tag{80}$$

Integrating (80) from 0 to ∞ , we have

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi\leq\frac{G(0,t)}{2r_1-\frac{k_4}{k_3}}. \tag{81}$$

Using the definition of $G(0,t)$ in (69), we have

$$G(0,t) = F(0,t) + r_1 \int_0^\infty F(\xi,t)\mathrm{d}\xi. \tag{82}$$

Inserting (82) into (81), we obtain

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi \leq \frac{F(0,t) + r_1 \int_0^\infty F(\xi,t)\mathrm{d}\xi}{2r_1 - \frac{k_4}{k_3}}. \tag{83}$$

Solving (83), we obtain

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi \leq \frac{F(0,t)}{r_1 - \frac{k_4}{k_3}}. \tag{84}$$

We thus have

$$\begin{aligned} G(0,t) &= F(0,t) + r_1 \int_0^\infty F(0,t)\mathrm{d}\xi \\ &\leq F(0,t) + \frac{r_1 F(0,t)}{r_1 - \frac{k_4}{k_3}} \\ &= \frac{2r_1 - \frac{k_4}{k_3}}{r_1 - \frac{k_4}{k_3}} F(0,t). \end{aligned} \tag{85}$$

Inserting (85) into (78) , we obtain the desired result (76).
By combining (52) and (76), we obtain the following main theorem. □

Theorem 1. *Assuming that (v,θ) constitutes the classical solution to the initial-boundary value problems governed by Equations (1)–(8), we proceed to establish the corresponding decay estimates for the energy $M(z,t)$, defined in (52):*

$$M(z,t) \leq \frac{2r_1k_3 - k_4}{r_1k_3 - k_4} F(0,t) e^{-\left(r_1 - \frac{k_4}{k_3}\right)z}. \tag{86}$$

4. Conclusions

Inequality (86) establishes the spatial decay estimates, demonstrating that the analytical solution exhibits exponential decay as the spatial coordinate recedes to infinity from the inflow boundary. This asymptotic behavior can be seen as a version of the Saint-Venant principle, wherein far-field perturbations decay proportionally to the characteristic decay rate. To the best of our knowledge, such results have not been previously reported for Equation (1).

Next, we extend our analysis to spatial decay estimates for the solution using a weighted energy functional. However, the methodology developed in this work is not applicable in this context, necessitating further investigation.

Additionally, the study of structural stability properties for the heat equation governed by (1) in unbounded spatial domains represents a promising avenue for future research. A thorough theoretical examination of this topic will be presented in a forthcoming publication dedicated to this subject.

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