mathematics

MDPI

Article

Enhanced Oscillation Criteria of Solutions for Half-Linear Dynamic Equations on Arbitrary Time Scales

Taher S. Hassan [1,2,3,4,*], Loredana Florentina Iambor [5,*], Mouataz Billah Mesmouli [1], Eltigani I. Hassan [6], Khaled Aldwoah [7] and Akbar Ali [1]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; m.mesmouli@uoh.edu.sa (M.B.M.)
[2] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
[3] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[4] Jadara University Research Center, Jadara University, Irbid 21110, Jordan
[5] Department of Mathematics and Computer Science, University of Oradea, Univeritatii nr. 1, 410087 Oradea, Romania
[6] Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 13318, Saudi Arabia; eiabdalla@imamu.edu.sa
[7] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia; aldwoah@iu.edu.sa
* Correspondence: tshassan@yahoo.com (T.S.H.); iambor.loredana@uoradea.ro (L.F.I.)

Abstract

This paper presents some oscillation criteria for second-order half-linear dynamic equations defined on unbounded above arbitrary time scales. These criteria offer sufficient conditions for all solutions of the equations to display oscillatory behavior. We investigate both delay and advanced cases of these equations, and our approach encompasses a broader class of dynamic equations than previously considered in the literature. The results of this study not only generalize well-known oscillation criteria used in differential equations but also significantly broaden their applicability to arbitrary time scales. Additionally, we provide illustrative examples to demonstrate the relevance and accuracy of our findings.

Keywords: oscillation; second order; half-linear; dynamic equations; time scales

MSC: 34K11; 39A10; 39A99

check for updates

Academic Editors: Andrey Amosov and Wen Yuan

Received: 10 July 2025
Revised: 10 August 2025
Accepted: 22 August 2025
Published: 26 August 2025

Citation: Hassan, T.S.; Iambor, L.F.; Mesmouli, M.B.; Hassan, E.I.; Aldwoah, K.; Ali, A. Enhanced Oscillation Criteria of Solutions for Half-Linear Dynamic Equations on Arbitrary Time Scales. *Mathematics* **2025**, *13*, 2743. https://doi.org/10.3390/math13172743

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Oscillation is a significant concern for applied researchers, as it originates from mechanical vibrations and has been extensively explored in science and engineering. Oscillation models may incorporate delay or advanced terms to reflect how solutions depend on past or future time frames. While some research has been conducted on oscillation in delay equations, see [1–10], studies on advanced oscillation are relatively scarce, see [11–15].

A time scale $\mathbb{T}$ is any closed real subset. Define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ as

$$\sigma(\nu) = \inf\{s \in \mathbb{T} : s > \nu\},$$

and it is stated that $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $\nu \in \mathbb{T}$ given that

$$f^{\Delta}(\nu) := \lim_{s \to \nu} \frac{f(\nu) - f(s)}{\nu - s},$$

exists when $\sigma(\nu) = \nu$ and when f is continuous at ν and $\sigma(\nu) > \nu$,

$$f^{\Delta}(\nu) := \frac{f(\sigma(\nu)) - f(\nu)}{\sigma(\nu) - \nu}.$$

The classical theories of differential and difference equations are notably represented when this time scale is equal to the reals or integers. There are numerous additional time scales that are intriguing, which leads to the emergence of many applications. Beyond merely unifying the corresponding theories for differential and difference equations, this novel theory of these so-called "dynamic equations" also encompasses "in-between" cases. In other words, we permitted the consideration of q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^{\gamma} : \gamma \in \mathbb{N}_0 \text{ for } q > 1\}$. These equations possess significant practical implications in quantum theory (refer to [16]). Additionally, we permitted the consideration of different time scales, including $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ represents the set of harmonic numbers. For an introduction to the calculus of time scales, see Hilger [17] and Bohner and Peterson [18]. For further insights into the calculus of time scales, refer to sources [18–20].

This study is devoted to analysing advanced and delayed oscillatory behavior in the context of half-linear dynamic equations on time scales. Particular attention is given to the half-linear case, which extends the classical linear framework by symmetrically incorporating nonlinearities. Such equations represent a natural generalization of the Laplace equation and have been shown to arise in a variety of applied disciplines, including the theory of non-Newtonian fluids, the modelling of polytropic gas flow through porous media, and various problems in mathematical biology (see, e.g., [21–32]). Therefore, we are concerned with half-linear second-order dynamic equations of the form

$$\left[r(\nu)\phi\left(\chi^{\Delta}(\nu)\right)\right]^{\Delta} + p(\nu)\phi(\chi(g(\nu))) = 0, \tag{1}$$

on an arbitrary time scale $\mathbb{T}$ such that $\sup \mathbb{T} = \infty$, where $\nu \in [\nu_0, \infty)_{\mathbb{T}} := [\nu_0, \infty) \cap \mathbb{T}$, $\nu_0 \geq 0$, $\nu_0 \in \mathbb{T}$, $\phi(u) := |u|^{\beta-1}u$, $\beta > 0$, p is a positive *rd*-continuous function on $\mathbb{T}$, $g : \mathbb{T} \to \mathbb{T}$ is an *rd*-continuous function satisfying $\lim_{\nu\to\infty} g(\nu) = \infty$, and r is a positive rd-continuous function on $\mathbb{T}$ such that

$$\mathcal{R}(\nu) := \int_{\nu_0}^{\nu} \frac{\Delta\tau}{r^{1/\beta}(\tau)} \to \infty \text{ as } \nu \to \infty. \tag{2}$$

By a solution of Equation (1) we mean a nontrivial real-valued function $\chi \in \mathrm{C}^1_{\mathrm{rd}}[\nu_\chi, \infty)_{\mathbb{T}}$ for some $\nu_\chi \geq \nu_0$ with $\nu_0 \in \mathbb{T}$ such that $\chi^{\Delta}, r(\nu)\phi(\chi^{\Delta}(\nu)) \in \mathrm{C}^1_{\mathrm{rd}}[\nu_\chi, \infty)_{\mathbb{T}}$ and $\chi(\nu)$ satisfies (1) on $[\nu_\chi, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. We consider only those solutions $\chi(\nu)$ of (1) which hold $\sup\{|\chi(\nu)| : \nu \geq \nu_0\} > 0$ for all $\nu \in [\nu_0, \infty)_{\mathbb{T}}$. We shall not investigate solutions that vanish in the neighborhood of infinity. A solution $\chi(\nu)$ of (1) is considered oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is categorised as nonoscillatory. Equation (1) is classified as oscillatory if all its solutions exhibit oscillation. Below are the oscillation results for the differential equations related to the results of (1) over time scales. Additionally, this section provides a comprehensive summary of the significant contributions made by this paper. Since Sturm's considerable contribution to the literature, oscillation theory has heavily relied on Euler differential equations and their many generalizations. One of the most well-known and widely used is the Euler equation

$$\chi''(\nu) + \frac{\lambda}{\nu^2}\chi(\nu) = 0, \tag{3}$$

which is oscillatory if and only if

$$\lambda > \frac{1}{4}.$$

One of the most important criteria for oscillation in second-order differential equations is the Kneser-type criterion (see [33]), which employs Sturmian comparison methods, and the oscillatory behavior of the Euler Equation (3) demonstrates that the all solutions of linear differential equation

$$\chi''(\nu) + p(\nu)\chi(\nu) = 0, \tag{4}$$

oscillate if

$$\liminf_{\nu\to\infty} \nu^2 p(\nu) > \frac{1}{4}. \tag{5}$$

Since then, many works have appeared that deduce Kneser-type oscillation criteria for a variety of differential equations. Representative contributions in this area can be found in [34–36] as follows:

(I) All solutions of linear differential equation

$$\left[r(\nu)\chi'(\nu)\right]' + p(\nu)\chi(\nu) = 0 \tag{6}$$

oscillate if

$$\liminf_{\nu\to\infty} r(\nu)\mathcal{R}^2(\nu)p(\nu) > \frac{1}{4}. \tag{7}$$

(II) All solutions of half-linear differential equation

$$\left[\phi(\chi'(\nu))\right]' + p(\nu)\phi(\chi(\nu)) = 0 \tag{8}$$

oscillate if

$$\liminf_{\nu\to\infty} \nu^{\beta+1} p(\nu) > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \tag{9}$$

(III) All solutions of half-linear differential equation

$$\left[r(\nu)\phi(\chi'(\nu))\right]' + p(\nu)\phi(\chi(\nu)) = 0 \tag{10}$$

oscillate if

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)p(\nu) > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \tag{11}$$

(IV) All solutions of half-linear delay differential equation

$$\left[r(\nu)\phi(\chi'(\nu))\right]' + p(\nu)\phi(\chi(g(\nu))) = 0, \tag{12}$$

where $g(\nu) \le \nu$, oscillate if

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu) > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \tag{13}$$

Chatzarakis et al. [37] established Kneser-type oscillation criteria for half-linear advanced differential Equation (12), where $g(\nu) \ge \nu$, and proved the following results.

Theorem 1 (see [37]). *Let $g(\nu) \ge \nu$ on $[\nu_0, \infty)$. All solutions of (12) oscillate if one of the following criteria holds:*

(i)

$$\liminf_{\nu\to\infty} \frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\nu)} = \infty, \tag{14}$$

and

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)p(\nu) > 0; \tag{15}$$

(ii)

$$\lambda := \liminf_{\nu\to\infty} \frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\nu)} < \infty, \tag{16}$$

and

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)p(\nu) > \max\{c(\psi) : 0 < \psi < 1\}, \tag{17}$$

where

$$c(\psi) := \beta\psi^{\beta}(1-\psi)\lambda^{-\beta\psi}.$$

Recently, Hassan et al. [38] found some interesting oscillation criteria of Kneser-type for the dynamic equation

$$\left[r(\nu)\phi\left(\chi^{\Delta}(\nu)\right)\right]^{\Delta} + p(\nu)\phi(\chi(g(\nu))) = 0 \tag{18}$$

as in the following theorem.

Theorem 2 (see [38]). *If* $\varrho := \liminf_{\nu\to\infty} \frac{\mathcal{R}(\nu)}{\mathcal{R}^{\sigma}(\nu)} > 0$ *and*

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\Omega(\nu))p(\nu) > \frac{1}{\varrho^{\beta(\beta+1)}}\left(\frac{\beta}{\beta+1}\right)^{\beta+1}, \tag{19}$$

where

$$\Omega(\nu) := \begin{cases} g(\nu), & g(\nu) \le \nu \\ \\ \nu, & g(\nu) \ge \nu, \end{cases}$$

then all solutions of (18) *oscillate.*

In light of the context provided and influenced by the significant contributions of [33–38], this paper aims to establish novel oscillation criteria of Kneser-type for the dynamic Equation (1). These criteria not only generalize existing results within the framework of time scales but also offer new insights into the qualitative behavior of solutions. The developed theory is supported by rigorous mathematical analysis and aims to enhance the existing literature on oscillation theory for dynamic equations. We encourage readers to refer to papers [39–47] and the references cited therein for further information.

2. Main Results

In this section, we aim to prove the main results of this paper. We start by presenting the following preliminary lemma.

Lemma 1 (see [48]). *Let* (1) *have an eventually positive solution* $\chi(\nu)$. *Then*

$$\chi^{\Delta}(\nu) > 0 \quad \text{and} \quad \left[r(\nu)\phi(\chi^{\Delta}(\nu))\right]^{\Delta} < 0 \tag{20}$$

eventually.

Theorem 3. *Let $g(\nu) \geq \sigma(\nu)$ on $[\nu_0, \infty)_{\mathbb{T}}$. If*

$$\liminf_{\nu\to\infty} \frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\sigma(\nu))} = \infty, \tag{21}$$

and

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\nu))p(\nu) > 0, \tag{22}$$

where

$$\beta_1 := \begin{cases} 1, & 0 < \beta \leq 1, \\ \beta, & \beta \geq 1, \end{cases}$$

and

$$\beta_2 := \begin{cases} \beta, & 0 < \beta \leq 1, \\ 1, & \beta \geq 1, \end{cases}$$

then all solutions of (1) *oscillate.*

Proof. Let (1) have a nonoscillatory solution χ on $[\nu_0, \infty)_{\mathbb{T}}$. Without loss of generality, we suppose that $\chi(\nu) > 0$ on $[\nu_0, \infty)_{\mathbb{T}}$. According to Lemma 1, $\chi^{\Delta}(\nu) > 0$ and $\left[r(\nu)\phi(\chi^{\Delta}(\nu))\right]^{\Delta} < 0$ eventually. By using the fact that $\left[r(\nu)\phi(\chi^{\Delta}(\nu))\right]^{\Delta} < 0$ eventually, we get

$$\begin{aligned} \chi(\nu) &> \phi^{-1}\left[r(\nu)\phi(\chi^{\Delta}(\nu))\right] \int_{\nu_0}^{\nu} \frac{\Delta\tau}{r^{1/\beta}(\tau)} \\ &= \phi^{-1}\left[r(\nu)\phi(\chi^{\Delta}(\nu))\right] \mathcal{R}(\nu), \end{aligned}$$

that is,

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) < \chi(\nu) \leq \chi(\sigma(\nu)). \tag{23}$$

Define

$$\kappa_* := \frac{1}{\beta} \liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\nu))p(\nu). \tag{24}$$

Therefore,

$$\frac{1}{\beta} r^{1/\beta}(\nu)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\nu))p(\nu) \geq \kappa \in (0, \kappa_*), \tag{25}$$

eventually. Using the fact that $\chi^{\Delta}(\nu) > 0$ eventually, we get

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \int_{\nu}^{\infty} p(\tau)\phi(\chi(g(\tau)))\,\Delta\tau \geq \phi(\chi(g(\nu))) \int_{\nu}^{\infty} p(\tau)\Delta\tau. \tag{26}$$

Substituting (25) into (26), we have

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \beta\kappa\phi(\chi(g(\nu))) \int_{\nu}^{\infty} \frac{\Delta\tau}{r^{1/\beta}(\tau)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\tau))}.$$

Using the Pötzsche chain rule, we obtain

$$\begin{aligned} \left(\frac{-1}{\mathcal{R}^{\beta}(\tau)}\right)^{\Delta} &= \frac{\left(\mathcal{R}^{\beta}(\tau)\right)^{\Delta}}{\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \\ &= \frac{\beta \int_0^1 [(1-h)\mathcal{R}(\tau) + h\mathcal{R}^{\sigma}(\tau)]^{\beta-1}\mathrm{d}h}{r^{1/\beta}(\tau)\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \end{aligned}$$

$$\leq \begin{cases} \dfrac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}, & 0<\beta\leq 1, \\ \dfrac{\beta}{r^{1/\beta}(\tau)\mathcal{R}^{\beta}(\tau)\mathcal{R}(\sigma(\tau))}, & \beta\geq 1, \end{cases}$$
$$= \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\tau))}.$$

which implies that

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) \geq \sqrt[\beta]{\kappa}\chi(g(\nu)) \geq \sqrt[\beta]{\kappa}\chi(\sigma(\nu)), \tag{27}$$

which implies $\sqrt[\beta]{\kappa} < 1$ and so $\left(\dfrac{\chi(\nu)}{\mathcal{R}^{\sqrt[\beta]{\kappa}}(\nu)}\right)^{\Delta} \geq 0$. By virtue of (21),

$$\frac{\chi(g(\nu))}{\chi(\sigma(\nu))} \geq \left(\frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\sigma(\nu))}\right)^{\sqrt[\beta]{\kappa}} > \frac{1}{\sqrt[\beta]{\kappa}}.$$

Hence, (27) becomes

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) > \chi(\sigma(\nu)),$$

which is a contradiction with (23). This completes the proof. □

Example 1. *Consider the second-order half-linear advanced dynamic equation*

$$\left(\rho\nu^{\gamma}\phi\left(\chi^{\Delta}(\nu)\right)\right)^{\Delta} + \frac{1}{\nu^{\beta-\gamma+1}}\phi(\chi(\nu^{\varsigma})) = 0, \quad \nu\in[\nu_0,\infty), \tag{28}$$

where $\rho,\gamma>0$ and $\varsigma>1$ are constants such that $\beta>\gamma$. Therefore,

$$\int_{\nu_0}^{\infty}\frac{\Delta\tau}{r^{1/\beta}(\tau)} = \frac{1}{\sqrt[\beta]{\rho}}\int_{\nu_0}^{\infty}\frac{1}{\sqrt[\beta]{\tau^{\gamma}}}\mathrm{d}\tau = \infty,$$

and

$$\mathcal{R}(\nu) = \int_{\nu_0}^{\nu}\frac{\Delta\tau}{r^{1/\beta}(\tau)} = \frac{1}{\sqrt[\beta]{\rho}}\int_{\nu_0}^{\nu}\frac{1}{\sqrt[\beta]{\tau^{\gamma}}}\mathrm{d}\tau = \frac{\beta}{\sqrt[\beta]{\rho}(\beta-\gamma)}\left(\sqrt[\beta]{\nu^{\beta-\gamma}} - \sqrt[\beta]{\nu_0^{\beta-\gamma}}\right),$$

which implies that

$$\begin{aligned}
&\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta_1}(\nu)\mathcal{R}^{\beta_2}(\sigma(\nu))p(\nu) \\
&= \sqrt[\beta]{\rho}\left(\frac{\beta}{\sqrt[\beta]{\rho}(\beta-\gamma)}\right)^{\beta_1+\beta_2}\liminf_{\nu\to\infty}\frac{\sqrt[\beta]{\nu^{\gamma}}\left(\sqrt[\beta]{\nu^{\beta-\gamma}} - \sqrt[\beta]{\nu_0^{\beta-\gamma}}\right)^{\beta_1+\beta_2}}{\nu^{\beta-\gamma+1}} \\
&= \frac{1}{\rho}\left(\frac{\beta}{\beta-\gamma}\right)^{\beta+1}\liminf_{\nu\to\infty}\left(1-\frac{\sqrt[\beta]{\nu_0^{\beta-\gamma}}}{\sqrt[\beta]{\nu^{\beta-\gamma}}}\right)^{\beta+1} > 0,
\end{aligned}$$

and

$$\liminf_{\nu\to\infty}\frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\sigma(\nu))} = \liminf_{\nu\to\infty}\frac{\sqrt[\beta]{\nu^{\varsigma(\beta-\gamma)}} - \sqrt[\beta]{\nu_0^{\beta-\gamma}}}{\sqrt[\beta]{\nu^{\beta-\gamma}} - \sqrt[\beta]{\nu_0^{\beta-\gamma}}} = \infty.$$

Then, by Theorem 3, all solutions of (28) oscillate.

Thanks to Theorem 3, in the following theorem we assume $\liminf_{\nu\to\infty} \frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\sigma(\nu))} < \infty$.

Theorem 4. *Let* $0 < \beta \leq 1$ *and* $g(\nu) \geq \sigma(\nu)$ *on* $[\nu_0, \infty)_{\mathbb{T}}$. *If*

$$\omega_* := \liminf_{\nu\to\infty} \frac{\mathcal{R}(g(\nu))}{\mathcal{R}(\sigma(\nu))} < \infty \tag{29}$$

and

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\sigma(\nu))p(\nu) > \max\{c(\psi) : 0 < \psi < 1\}, \tag{30}$$

where

$$c(\psi) := \beta\psi^{\beta}(1-\psi)\omega_*^{-\beta\psi},$$

then all solutions of (1) *oscillate.*

Proof. Let (1) have a nonoscillatory solution χ on $[\nu_0, \infty)_{\mathbb{T}}$. Without loss of generality, we suppose that $\chi(\nu) > 0$ on $[\nu_0, \infty)_{\mathbb{T}}$. According to Lemma 1, $\chi^{\Delta}(\nu) > 0$, $\left(\frac{\chi(\nu)}{\mathcal{R}(\nu)}\right)^{\Delta} < 0$, and $\left[r(\nu)\phi(\chi^{\Delta}(\nu))\right]^{\Delta} < 0$ eventually. As shown in the proof of Theorem 3, we have

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) < \chi(\nu). \tag{31}$$

By defining

$$\gamma_* := \frac{1}{\beta}\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\sigma(\nu))p(\nu),$$

we have

$$\frac{1}{\beta}r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\sigma(\nu))p(\nu) \geq \gamma \in (0, \gamma_*), \tag{32}$$

eventually. Since $\chi^{\Delta}(\nu) > 0$ eventually, from Equation (1) we have

$$\begin{aligned} r(\nu)\phi(\chi^{\Delta}(\nu)) &\geq \int_{\nu}^{\infty} p(\tau)\phi(\chi(g(\tau)))\Delta\tau \geq \int_{\nu}^{\infty} p(\tau)\phi(\chi(\sigma(\tau)))\Delta\tau \\ &\geq \phi(\chi(\nu))\int_{\nu}^{\infty} p(\tau)\Delta\tau. \end{aligned} \tag{33}$$

Substituting (32) into (33), we get

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \beta\gamma\phi(\chi(\nu))\int_{\nu}^{\infty} \frac{\Delta\tau}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}.$$

Using the Pötzsche chain rule, we obtain

$$\begin{aligned} \left(\frac{-1}{\mathcal{R}^{\beta}(\tau)}\right)^{\Delta} &= \frac{\left(\mathcal{R}^{\beta}(\tau)\right)^{\Delta}}{\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \\ &= \frac{\beta\int_0^1[(1-h)\mathcal{R}(\tau) + h\mathcal{R}^{\sigma}(\tau)]^{\beta-1}\mathrm{d}h}{r^{1/\beta}(\tau)\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \\ &\leq \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}. \end{aligned}$$

which implies that

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \gamma\phi(\chi(\nu))\frac{1}{\mathcal{R}^{\beta}(\nu)}.$$

Define for $n \in \mathbb{N}_0$,

$$\zeta_{n+1} := \begin{cases} \sqrt[\beta]{\gamma_*}, & n = 0, \\ \dfrac{\zeta_1 \omega_*^{\zeta_n}}{\sqrt[\beta]{1-\zeta_n}}, & n \in \mathbb{N}. \end{cases}$$

Hence,

$$\mathcal{R}(\nu) r^{1/\beta}(\nu) \chi^{\Delta}(\nu) \geq \sqrt[\beta]{\gamma}\chi(\nu) = \varepsilon_1 \zeta_1 \chi(\nu), \tag{34}$$

eventually, where $\varepsilon_1 := \dfrac{\sqrt[\beta]{\gamma}}{\zeta_1}$ is arbitrary but fixed from $(0,1)$ tending to 1 as $\gamma \to \gamma_*$. From (31) and (34), we have $\varepsilon_1 \zeta_1 < 1$, which implies that

$$\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\Delta} = \frac{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\chi^{\Delta}(\nu) - \left(\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\right)^{\Delta}\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))}. \tag{35}$$

Again, by using the Pötzsche chain rule we obtain

$$\begin{aligned} \left(\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\right)^{\Delta} &= \varepsilon_1\zeta_1 \int_0^1 \left[\mathcal{R}(\nu) + h\mu(\nu)\mathcal{R}^{\Delta}(\nu)\right]^{\varepsilon_1\zeta_1 - 1} \mathrm{d}h \, \mathcal{R}^{\Delta}(\nu) \\ &= \varepsilon_1\zeta_1 \int_0^1 [(1-h)\mathcal{R}(\nu) + h\mathcal{R}(\sigma(\nu))]^{\varepsilon_1\zeta_1 - 1} \mathrm{d}h \, \frac{1}{r^{1/\beta}(\nu)} \\ &\leq \varepsilon_1\zeta_1 \frac{\mathcal{R}^{\varepsilon_1\zeta_1 - 1}(\nu)}{r^{1/\beta}(\nu)}. \end{aligned} \tag{36}$$

Using (36) in (35), we have

$$\begin{aligned} \left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\Delta} &\geq \frac{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\chi^{\Delta}(\nu) - \varepsilon_1\zeta_1\mathcal{R}^{\varepsilon_1\zeta_1 - 1}(\nu) r^{-1/\beta}(\nu)\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))} \\ &= \frac{\mathcal{R}(\nu) r^{1/\beta}(\nu)\chi^{\Delta}(\nu) - \varepsilon_1\zeta_1\chi(\nu)}{r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))} \geq 0. \end{aligned}$$

This together with (1) and (32) shows that

$$\begin{aligned} r(\nu)\phi(\chi^{\Delta}(\nu)) &\geq \int_{\nu}^{\infty} p(\tau)\phi(\chi(g(\tau)))\Delta\tau \\ &= \int_{\nu}^{\infty} p(\tau)\left(\frac{\chi(g(\tau))}{\mathcal{R}^{\varepsilon_1\zeta_1}(g(\tau))}\right)^{\beta} \mathcal{R}^{\beta\varepsilon_1\zeta_1}(g(\tau))\Delta\tau \\ &\geq \left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\beta} \int_{\nu}^{\infty} \mathcal{R}^{\beta\varepsilon_1\zeta_1}(g(\tau)) p(\tau)\Delta\tau \\ &\geq \gamma\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\beta} \int_{\nu}^{\infty} \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))} \left(\frac{\mathcal{R}(g(\tau))}{\mathcal{R}(\sigma(\tau))}\right)^{\beta\varepsilon_1\zeta_1} \Delta\tau. \end{aligned}$$

It follows from the definition of ω_* that

$$\frac{\mathcal{R}(g(\tau))}{\mathcal{R}(\sigma(\tau))} \geq \omega \in (1, \omega_*),$$

eventually. Therefore,

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \gamma\omega^{\beta\varepsilon_1\zeta_1}\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\beta} \int_{\nu}^{\infty} \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\Delta\tau$$

$$\geq \frac{\gamma\omega^{\beta\varepsilon_1\zeta_1}}{1-\varepsilon_1\zeta_1}\left(\frac{\chi(\nu)}{\mathcal{R}(\nu)}\right)^{\beta}$$

since

$$\begin{aligned}\left(\frac{-1}{\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)}\right)^{\Delta} &= \frac{\left(\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\right)^{\Delta}}{\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\\ &= \frac{\beta(1-\varepsilon_1\zeta_1)\int_0^1[(1-h)\mathcal{R}(\tau)+h\mathcal{R}^{\sigma}(\tau)]^{\beta(1-\varepsilon_1\zeta_1)-1}\mathrm{d}h}{r^{1/\beta}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\\ &\leq \frac{\beta(1-\varepsilon_1\zeta_1)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)-1}(\tau)}{r^{1/\beta}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\\ &= \frac{\beta(1-\varepsilon_1\zeta_1)}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) &\geq \sqrt[\beta]{\frac{\gamma}{1-\varepsilon_1\zeta_1}}\omega^{\varepsilon_1\zeta_1}\chi(\nu)\\ &= \varepsilon_2\zeta_2\chi(\nu),\end{aligned} \tag{37}$$

eventually, where

$$\varepsilon_2 := \frac{\sqrt[\beta]{\gamma}}{\zeta_1}\left(\frac{\omega^{\varepsilon_1}}{\omega_*}\right)^{\zeta_1}\sqrt[\beta]{\frac{1-\zeta_1}{1-\varepsilon_1\zeta_1}} = \varepsilon_1\left(\frac{\omega^{\varepsilon_1}}{\omega_*}\right)^{\zeta_1}\sqrt[\beta]{\frac{1-\zeta_1}{1-\varepsilon_1\zeta_1}}$$

is arbitrary but fixed from $(0,1)$ tending to 1 as $\gamma\to\gamma_*$. Inductively, one can show that for any $n\in\mathbb{N}_0$,

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu)\geq\varepsilon_{n+1}\zeta_{n+1}\chi(\nu), \tag{38}$$

eventually, where for $n\in\mathbb{N}_0$,

$$\varepsilon_{n+1} := \begin{cases}\dfrac{\sqrt[\beta]{\gamma}}{\zeta_1}, & n=0\\[2ex] \varepsilon_n\left(\dfrac{\omega^{\varepsilon_n}}{\omega_*}\right)^{\zeta_n}\sqrt[\beta]{\dfrac{1-\zeta_n}{1-\varepsilon_n\zeta_n}}, & n\in\mathbb{N},\end{cases} \tag{39}$$

is arbitrary but fixed from $(0,1)$ tending to 1 as $\gamma\to\gamma_*$. We note that for $n\in\mathbb{N}$,

$$\zeta_{n+1}=\lambda_n\zeta_n \text{ and } \lambda_n>1, \tag{40}$$

where

$$\lambda_{n+1} := \begin{cases}\dfrac{\omega_*^{\zeta_1}}{\sqrt[\beta]{1-\zeta_1}}, & n=0,\\[2ex] \omega_*^{\zeta_n(\lambda_n-1)}\sqrt[\beta]{\dfrac{1-\zeta_n}{1-\lambda_n\zeta_n}}, & n\in\mathbb{N}.\end{cases} \tag{41}$$

If we pick, for any $n\in\mathbb{N}$, $\varepsilon_{n+1}>\dfrac{1}{\lambda_n}$, then from (38) and (40), we have

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu)\geq\varepsilon_{n+1}\zeta_{n+1}\chi(\nu)=\varepsilon_{n+1}\lambda_n\zeta_n\chi(\nu)>\zeta_n\chi(\nu). \tag{42}$$

From (31) and (42), we get

$$\zeta_n < 1 \text{ for any } n \in \mathbb{N}_0.$$

Thus, the sequence $\{\zeta_n\}$ is bounded above and nondecreasing, and so $\{\zeta_n\}$ converges. Say,

$$\lim_{\nu\to\infty} \zeta_n = \psi,$$

where ψ is the smaller positive root of the equation

$$\psi^{\beta}(1-\psi)\omega_*^{-\beta\psi} = \gamma_*.$$

This is a contradiction with (30), due to

$$\gamma_* > \max\left\{\psi^{\beta}(1-\psi)\omega_*^{-\beta\psi} : 0 < \psi < 1\right\}.$$

This completes the proof. □

Example 2. *All solutions of second-order Euler dynamic equation*

$$\left[r(\nu)\phi\left(\chi^{\Delta}(\nu)\right)\right]^{\Delta} + \frac{\mu}{r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\sigma(\nu))}\phi(\chi(\sigma(\nu))) = 0,\ 0 < \beta \le 1,$$

oscillate if $\mu > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}$ *by using Theorem 4. It is well known that this condition is ideal for the second-order Euler differential equation*

$$\left[r(\nu)\phi(\chi'(\nu))\right]' + \frac{\mu}{r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)}\phi(\chi(\nu)) = 0, \tag{43}$$

and equation (43) has a nonoscillatory solution $\chi(\nu) = \mathcal{R}^{\frac{\beta}{\beta+1}}(\nu)$ *if* $\mu = \left(\frac{\beta}{\beta+1}\right)^{\beta+1}$. *That is to say, the constant* $\left(\frac{\beta}{\beta+1}\right)^{\beta+1}$ *provides the lower bound of the oscillation for (43).*

Theorem 5. *Let* $0 < \beta \le 1$ *and* $g(\nu) \le \sigma(\nu)$ *on* $[\nu_0, \infty)_{\mathbb{T}}$. *If*

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu) > \left(\frac{\beta}{1+\beta}\right)^{1+\beta}, \tag{44}$$

then all solutions of (1) oscillate.

Proof. Let (1) have a nonoscillatory solution χ on $[\nu_0, \infty)_{\mathbb{T}}$. Without loss of generality, we suppose that $\chi(\nu) > 0$ and $\chi(g(\nu)) > 0$ on $[\nu_0, \infty)_{\mathbb{T}}$. As shown in the proof of Theorem 4, we have

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) < \chi(\nu), \tag{45}$$

$$\chi^{\Delta}(\nu) > 0,\ \left(\frac{\chi(\nu)}{\mathcal{R}(\nu)}\right)^{\Delta} < 0,\ \left[r(\nu)\phi(\chi^{\Delta}(\nu))\right]^{\Delta} < 0,$$

and

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \ge \int_{\nu}^{\infty} p(\tau)\phi(\chi(g(\tau)))\,\Delta\tau,$$

eventually. Define

$$\delta_* := \frac{1}{\beta}\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu). \tag{46}$$

In view of (46), there is $\delta \in (0, \delta_*)$ such that

$$\frac{1}{\beta} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu) \geq \delta,$$

eventually. Since $\chi^{\Delta}(\nu) > 0$ and $\left(\frac{\chi(\nu)}{\mathcal{R}(\nu)}\right)^{\Delta} < 0$ eventually, from Equation (1) we have that for a large ν,

$$\begin{aligned} r(\nu)\phi(\chi^{\Delta}(\nu)) &\geq \int_{\nu}^{\infty} p(\tau)\phi(\chi(g(\tau)))\Delta\tau \\ &\geq \delta \int_{\nu}^{\infty} \frac{\beta\phi(\chi(g(\tau)))}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(g(\tau))}\Delta\tau \\ &\geq \delta \int_{\nu}^{\infty} \frac{\beta\phi(\chi(\sigma(\tau)))}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}\Delta\tau \\ &\geq \delta\phi(\chi(\nu)) \int_{\nu}^{\infty} \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}\Delta\tau. \end{aligned} \tag{47}$$

By the Pötzsche chain rule, we have

$$\begin{aligned} \left(\frac{-1}{\mathcal{R}^{\beta}(\tau)}\right)^{\Delta} &= \frac{\left(\mathcal{R}^{\beta}(\tau)\right)^{\Delta}}{\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \\ &= \frac{\beta \int_0^1 [(1-h)\mathcal{R}(\tau) + h\mathcal{R}^{\sigma}(\tau)]^{\beta-1}\mathrm{d}h}{r^{1/\beta}(\tau)\mathcal{R}^{\beta}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))} \\ &\leq \frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta}(\sigma(\tau))}. \end{aligned}$$

Hence,

$$r(\nu)\phi(\chi^{\Delta}(\nu)) \geq \delta\phi(\chi(\nu)) \int_{\nu}^{\infty} \left(\frac{-1}{\mathcal{R}^{\beta}(\tau)}\right)^{\Delta} \Delta\tau = \delta\frac{\phi(\chi(\nu))}{\mathcal{R}^{\beta}(\nu)},$$

which implies that

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) \geq \sqrt[\beta]{\delta}\chi(\nu) = \varepsilon_1\zeta_1\chi(\nu), \tag{48}$$

eventually, where ε_1 is defined by (39). From (45) and (48), we have $\varepsilon_1\zeta_1 < 1$, which implies that

$$\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\Delta} = \frac{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\chi^{\Delta}(\nu) - \left(\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\right)^{\Delta}\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))}. \tag{49}$$

Again, using the Pötzsche chain rule we obtain

$$\begin{aligned} \left(\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\right)^{\Delta} &= \frac{\varepsilon_1\zeta_1}{r^{1/\beta}(\nu)} \int_0^1 [(1-h)\mathcal{R}(\nu) + h\mathcal{R}(\sigma(\nu))]^{\varepsilon_1\zeta_1-1}\mathrm{d}h \\ &\leq \frac{\varepsilon_1\zeta_1}{r^{1/\beta}(\nu)}\mathcal{R}^{\varepsilon_1\zeta_1-1}(\nu). \end{aligned} \tag{50}$$

Combining (49) and (50), we have

$$\begin{aligned} \left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\Delta} &\geq \frac{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\chi^{\Delta}(\nu) - \varepsilon_1\zeta_1\mathcal{R}^{\varepsilon_1\zeta_1-1}(\nu)r^{-1/\beta}(\nu)\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))} \\ &= \frac{\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) - \varepsilon_1\zeta_1\chi(\nu)}{r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\varepsilon_1\zeta_1}(\sigma(\nu))} \geq 0. \end{aligned} \tag{51}$$

Using (51) in (47), we get

$$\begin{aligned} r(\nu)\phi(\chi^{\Delta}(\nu)) &\geq \delta\int_{\nu}^{\infty}\frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\left(\left[\frac{\chi(\tau)}{\mathcal{R}^{\varepsilon_1\asymp_1}(\tau)}\right]^{\sigma}\right)^{\beta}\Delta\tau \\ &\geq \delta\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\beta}\int_{\nu}^{\infty}\frac{\beta}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}\Delta\tau. \end{aligned}$$

By the Pötzsche chain rule, we have

$$\begin{aligned} \left(\frac{-1}{\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)}\right)^{\Delta} &= \frac{\left(\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\right)^{\Delta}}{\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))} \\ &= \frac{\beta(1-\varepsilon_1\zeta_1)\int_0^1[(1-h)\mathcal{R}(\tau)+h\mathcal{R}^{\sigma}(\tau)]^{\beta(1-\varepsilon_1\zeta_1)-1}\mathrm{d}h}{r^{1/\beta}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))} \\ &\leq \frac{\beta(1-\varepsilon_1\zeta_1)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)-1}(\tau)}{r^{1/\beta}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))} \\ &= \frac{\beta(1-\varepsilon_1\zeta_1)}{r^{1/\beta}(\tau)\mathcal{R}(\tau)\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\sigma(\tau))}. \end{aligned}$$

Hence,

$$\begin{aligned} r(\nu)\phi(\chi^{\Delta}(\nu)) &\geq \frac{\delta}{1-\varepsilon_1\zeta_1}\left(\frac{\chi(\nu)}{\mathcal{R}^{\varepsilon_1\zeta_1}(\nu)}\right)^{\beta}\int_{\nu}^{\infty}\left(\frac{-1}{\mathcal{R}^{\beta(1-\varepsilon_1\zeta_1)}(\tau)}\right)^{\Delta}\Delta\tau \\ &= \frac{\delta}{1-\varepsilon_1\zeta_1}\left(\frac{\chi(\nu)}{\mathcal{R}(\nu)}\right)^{\beta}, \end{aligned}$$

which implies

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) \geq \sqrt[\beta]{\frac{\delta}{1-\varepsilon_1\zeta_1}}\chi(\nu) = \varepsilon_2\zeta_2\chi(\nu),$$

eventually, where

$$\varepsilon_2 := \frac{\sqrt[\beta]{\delta}}{\zeta_1}\sqrt[\beta]{\frac{1-\zeta_1}{1-\varepsilon_1\zeta_1}} = \varepsilon_1\sqrt[\beta]{\frac{1-\zeta_1}{1-\varepsilon_1\zeta_1}}$$

is arbitrary but fixed from $(0,1)$ tending to 1 as $\delta \to \delta_*$. Inductively, one can show that for any $n \in \mathbb{N}_0$,

$$\mathcal{R}(\nu)r^{1/\beta}(\nu)\chi^{\Delta}(\nu) \geq \varepsilon_{n+1}\zeta_{n+1}\chi(\nu), \tag{52}$$

eventually, where ε_{n+1} for $n \in \mathbb{N}_0$, and

$$\varepsilon_{n+1} := \begin{cases} \dfrac{\sqrt[\beta]{\delta}}{\zeta_1}, & n = 0 \\ \varepsilon_1\sqrt[\beta]{\dfrac{1-\zeta_n}{1-\varepsilon_n\zeta_n}}, & n \in \mathbb{N}, \end{cases} \tag{53}$$

is arbitrary but fixed from $(0,1)$ tending to 1 as $\delta \to \delta_*$. We note that for $n \in \mathbb{N}$,

$$\zeta_{n+1} = \lambda_n\zeta_n \text{ and } \lambda_n > 1, \tag{54}$$

where

$$\lambda_{n+1} := \begin{cases} \dfrac{1}{\sqrt[\beta]{1-\zeta_1}}, & n = 0, \\ \sqrt[\beta]{\dfrac{1-\zeta_n}{1-\lambda_n\zeta_n}}, & n \in \mathbb{N}. \end{cases} \tag{55}$$

If we pick, for any $n \in \mathbb{N}$, $\varepsilon_{n+1} > \dfrac{1}{\lambda_n}$, then from (52) and (54), we have

$$\mathcal{R}(\nu) r^{1/\beta}(\nu) \chi^{\Delta}(\nu) \geq \varepsilon_{n+1}\zeta_{n+1}\chi(\nu) = \varepsilon_{n+1}\lambda_n\zeta_n\chi(\nu) > \zeta_n\chi(\nu). \tag{56}$$

From (45) and (56), we get

$$\zeta_n < 1 \text{ for any } n \in \mathbb{N}_0.$$

Thus, the sequence $\{\zeta_n\}$ is bounded above and increasing, and so $\{\zeta_n\}$ converges. Say,

$$\lim_{\nu\to\infty} \zeta_n = \psi,$$

where ψ is the smaller positive root of the equation

$$\psi^{\beta}(1-\psi) = \delta_*.$$

This is a contradiction with (44), due to

$$\delta_* > \max\left\{\psi^{\beta}(1-\psi) : 0 < \psi < 1\right\} = \frac{\beta^{\beta}}{(1+\beta)^{1+\beta}}.$$

This completes the proof. □

Example 3. *Consider the second-order functional dynamic equation*

$$\left[\sqrt[12]{\nu}\sqrt[3]{\chi^{\Delta}(\nu)}\right]^{\Delta} + \frac{3}{8}\sqrt[3]{\frac{3}{2}}\frac{\mu}{\nu\sqrt[4]{g(\nu)}}\sqrt[3]{\chi(g(\nu))} = 0, \tag{57}$$

where $\mu > 0$ is a constant and $g(\nu) \leq \sigma(\nu)$ on $[\nu_0, \infty)_{\mathbb{T}}$. We have

$$\int_{\nu_0}^{\infty} \frac{\Delta\tau}{r^{1/\beta}(\tau)} = \int_{\nu_0}^{\infty} \frac{\Delta\tau}{\sqrt[4]{\tau}} = \infty,$$

by [18], Example 5.60. Also, by the Pötzsche chain rule, we obtain

$$\mathcal{R}(\nu) = \int_{\nu_0}^{\nu} \frac{\Delta\tau}{r^{1/\beta}(\tau)} = \int_{\nu_0}^{\nu} \frac{\Delta\tau}{\sqrt[4]{\tau}} \geq \frac{4}{3}\int_{\nu_0}^{\nu} \left(\sqrt[4]{\tau^3}\right)^{\Delta}\Delta\tau = \frac{4}{3}\left(\sqrt[4]{\nu^3} - \sqrt[4]{\nu_0^3}\right),$$

and consequently,

$$\begin{aligned} & \liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu) \\ = \; & \mu\sqrt[3]{\frac{1}{4}} \liminf_{\nu\to\infty} \frac{\sqrt[4]{\nu}\left(\sqrt[4]{\nu^3} - \sqrt[4]{\nu_0^3}\right)\sqrt[3]{\sqrt[4]{g^3(\nu)} - \sqrt[4]{\nu_0^3}}}{\nu\sqrt[4]{g(\nu)}} \\ = \; & \mu\sqrt[3]{\frac{1}{4}} \liminf_{\nu\to\infty} \left(1 - \sqrt[4]{\left(\frac{\nu_0}{\nu}\right)^3}\right)\sqrt[3]{1 - \sqrt[4]{\left(\frac{\nu_0}{g(\nu)}\right)^3}} \end{aligned}$$

$$= \mu\sqrt[3]{\frac{1}{4}}.$$

Applying Theorem 5 leads to the oscillation of all solutions of (57) for $\mu > \frac{1}{4}$.

3. Discussion and Conclusions

(I) In this paper, the findings presented apply across all time scales without any restrictive conditions, including $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, and $\mathbb{T}=q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$. Our results extend previous contributions to second-order half-linear differential equations; see the following details:

(1) Criteria (30) and (44) reduce to (5) in the case where $\mathbb{T} = \mathbb{R}$, $\beta = 1$, $r(\nu) = 1$, and $g(\nu) = \nu$;
(2) Criteria (30) and (44) become (7) in the case when $\mathbb{T} = \mathbb{R}$, $\beta = 1$, and $g(\nu) = \nu$;
(3) Criteria (30) and (44) reduce to (9) assuming that $\mathbb{T} = \mathbb{R}$, $r(\nu) = 1$, and $g(\nu) = \nu$;
(4) Criteria (30) and (44) reduce to (11) under the assumptions that $\mathbb{T} = \mathbb{R}$ and $g(\nu) = \nu$;
(5) Criterion (44) becomes (13) assuming that $\mathbb{T} = \mathbb{R}$ and $g(\nu) \le \nu$;
(6) Criteria (21) and (22) reduce to (14) and (15), respectively, in the case where $\mathbb{T} = \mathbb{R}$ and $g(\nu) \ge \nu$;
(7) Criteria (29) and (30) become (16) and (17), respectively, in the case when $\mathbb{T} = \mathbb{R}$ and $g(\nu) \ge \nu$.

(II) We present sharp Kneser-type oscillation criteria for half-linear second-order dynamic equations, considering both cases $g(\nu) \ge \sigma(\nu)$ and $g(\nu) \le \sigma(\nu)$. Our results improve upon previously established Kneser-type criteria, as detailed below:

(i) If $g(\nu) \ge \sigma(\nu)$, then criterion (19) reduces to

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)p(\nu) > \frac{1}{\varrho^{\beta(\beta+1)}}\left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \tag{58}$$

By dint of

$$r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\sigma(\nu))p(\nu) \ge r^{1/\beta}(\nu)\mathcal{R}^{\beta+1}(\nu)p(\nu),$$

we obtain

$$\begin{aligned}\max\left\{\beta\psi^{\beta}(1-\psi)\omega_*^{-\beta\psi} : 0 < \psi < 1\right\} &\le \max\left\{\beta\psi^{\beta}(1-\psi) : 0 < \psi < 1\right\}\\ &= \left(\frac{\beta}{\beta+1}\right)^{\beta+1}\\ &\le \frac{1}{\varrho^{\beta(\beta+1)}}\left(\frac{\beta}{\beta+1}\right)^{\beta+1}.\end{aligned}$$

Theorem 4 improves Theorem 2 (criterion (30) improves (58)).

(ii) If $g(\nu) \le \sigma(\nu)$, then criterion (19) becomes

$$\liminf_{\nu\to\infty} r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\Omega^*(\nu))p(\nu) > \frac{1}{\varrho^{\beta(\beta+1)}}\left(\frac{\beta}{\beta+1}\right)^{\beta+1}, \tag{59}$$

where

$$\Omega^*(\nu) := \begin{cases} g(\nu), & g(\nu) \le \nu \\ \nu, & \nu \le g(\nu) \le \sigma(\nu). \end{cases}$$

By virtue of

$$r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(g(\nu))p(\nu) \geq r^{1/\beta}(\nu)\mathcal{R}(\nu)\mathcal{R}^{\beta}(\Omega^{*}(\nu))p(\nu),$$

we obtain

$$\left(\frac{\beta}{\beta+1}\right)^{\beta+1} \leq \frac{1}{\varrho^{\beta(\beta+1)}}\left(\frac{\beta}{\beta+1}\right)^{\beta+1}.$$

Theorem 5 improves Theorem 2 (criterion (44) improves (59)).

(III) It would be interesting to establish Kneser-type oscillation criteria for a second-order dynamic Equation (1), provided that

$$\int_{\nu_0}^{\infty} \frac{\Delta\tau}{r^{1/\beta}(\tau)} < \infty.$$

Author Contributions: Conceptualization, T.S.H., L.F.I., M.B.M., E.I.H. and K.A.; Methodology, T.S.H. and L.F.I.; Software, E.I.H.; Validation, T.S.H. and K.A.; Formal analysis, T.S.H., M.B.M. and K.A.; Investigation, L.F.I., M.B.M. and E.I.H.; Resources, T.S.H. and E.I.H.; Data curation, A.A.; Writing—original draft, T.S.H., M.B.M., E.I.H. and K.A.; Writing—review & editing, T.S.H., L.F.I., M.B.M., E.I.H., K.A. and A.A.; Supervision, T.S.H. and L.F.I.; Funding acquisition, L.F.I. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Oradea.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Abbas, S.; Negi, S.S.; Grace, S.R.; Agarwal, R.P.; Wang, C. Survey on qualitative theory of dynamic equations on time scale. *Mem. Differ. Equ. Math. Phys.* **2021**, *84*, 1–67.
2. Ishibashi, K. Hille-Nehari type non-oscillation criteria for half-linear dynamic equations with mixed derivatives on a time scale. *Electron. J. Differ. Equ.* **2021**, 1–15. [CrossRef]
3. Sun, S.; Han, Z.; Zhao, P.; Zhang, C. Oscillation for a class of second-order Emden-Fowler delay dynamic equations on time scales. *Adv. Differ. Equ-Ny.* **2010**, *2010*, 642356. [CrossRef]
4. Baculikova, B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. *Appl. Math. Lett.* **2019**, *91*, 68–75. [CrossRef]
5. Zhang, Q.; Gao, L.; Wang, L. Oscillation of second-order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl.* **2011**, *61*, 2342–2348. [CrossRef]
6. Bazighifan, O.; El-Nabulsi, E.M. Different techniques for studying oscillatory behavior of solution of differential equations. *Rocky Mt. J. Math.* **2021**, *51*, 77–86. [CrossRef]
7. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation criteria for second order sublinear dynamic equations with damping term. *J. Differ. Equ. Appl.* **2011**, *17*, 505–523.
8. Zhu, Y.R.; Mao, Z.X.; Liu, S.P.; Tian, J.F. Oscillation criteria of second-order dynamic equations on time scales. *Mathematics* **2021**, *9*, 1867. [CrossRef]
9. Grace, S.R.; Bohner, M.; Agarwal, R.P. On the oscillation of second-order half-linear dynamic equations. *J. Differ. Equ. Appl.* **2009**, *15*, 451–460. [CrossRef]
10. Erbe, L.; Hassan, T.S. New oscillation criteria for second order sublinear dynamic equations. *Dynam. Systems Appl.* **2013**, *22*, 49–63.
11. Jadlovská, I. Iterative oscillation results for second-order differential equations with advanced argument. *Electron. J. Differ. Eq.* **2017**, *2017*, 1–11.
12. Bohner, M.; Vidhyaa, K.S.; Thandapani, E. Oscillation of noncanonical second-order advanced differential equations via canonical transform. *Constr. Math. Anal.* **2022**, *5*, 7–13. [CrossRef]
13. Hassan, T.S.; El-Nabulsi, R.A.; Iqbal, N.; Abdel Menaem, A. New criteria for oscillation of advanced noncanonical nonlinear dynamic equations. *Mathematics* **2024**, *12*, 824. [CrossRef]

14. Chatzarakis, G.E.; Džurina, J.; Jadlovská, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [CrossRef]
15. Chatzarakis, G.E.; Moaaz, O.; Li, T.; Qaraad, B. Some oscillation theorems for nonlinear second-order differential equations with an advanced argument. *Adv. Differ. Equ.* **2020**, *2020*, 160. [CrossRef]
16. Kac, V.; Chueng, P. *Quantum Calculus*; Universitext; Springer: New York, NY, USA, 2002.
17. Hilger, S. Analysis on measure chains—A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [CrossRef]
18. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
19. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
20. Agarwal, R.P.; Hazarika, B.; Tikare, S. *Dynamic Equations on Time Scales and Applications*; CRC Press: Boca Raton, FL, USA, 2024.
21. Frassu, S.; Viglialoro, G. Boundedness in a chemotaxis system with consumed chemoattractant and produced chemorepellent. *Nonlinear Anal.* **2021**, *213*, 112505. [CrossRef]
22. Li, T.; Viglialoro, G. Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. *Differ. Integral Equ.* **2021**, *34*, 315–336. [CrossRef]
23. Li, T.; Acosta-Soba, D.; Columbu, A.; Viglialoro, G. Dissipative Gradient Nonlinearities Prevent δ-Formations in Local and Nonlocal Attraction–Repulsion Chemotaxis Models. *Stud. Appl. Math.* **2025**, *154*, e70018. [CrossRef]
24. Hassan, T.S.; Cesarano, C.; El-Nabulsi, R.A.; Anukool, W. Improved Hille-Type Oscillation Criteria for Second-Order Quasilinear Dynamic Equations. *Mathematics* **2022**, *10*, 3675. [CrossRef]
25. Wu, X. Some Oscillation Criteria for a Class of Higher Order Nonlinear Dynamic Equations with a Delay Argument on Time Scales. *Acta Math. Sci.* **2021**, *41*, 1474–1492. [CrossRef]
26. Bohner, M.; Grace, S.R.; Jadlovská, I. Sharp results for oscillation of second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2023**, *4*, 1–23. [CrossRef]
27. Li, T.; Pintus, N.; Viglialoro, G. Properties of solutions to porous medium problems with different sources and boundary conditions. *Z. Angew. Math. Phys.* **2019**, *70*, 86. [CrossRef]
28. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 761–778. [CrossRef]
29. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation criteria for second-order dynamic equations on time scales. *Appl. Math. Lett.* **2014**, *31*, 34–40. [CrossRef]
30. Baculikova, B. Oscillation and asymptotic properties of second order half-linear differential equations with mixed deviating arguments. *Mathematics* **2021**, *9*, 2552. [CrossRef]
31. Demidenko, G.V.; Matveeva, I.I. Asymptotic stability of solutions to a class of second-order delay differential equations. *Mathematics* **2021**, *9*, 1847. [CrossRef]
32. Saker, S.H.; Mohammed, N.; Rezk, H.M.; Saied, A.I.; Aldwoah, K.; Alahmade, A. Unified Framework for Continuous and Discrete Relations of Gehring and Muckenhoupt Weights on Time Scales. *Axioms* **2024**, *13*, 754. [CrossRef]
33. Kneser, A. Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen. *Math. Ann.* **1893**, *42*, 409–435. [CrossRef]
34. Došly, O; Řehák, P. *Half-Linear Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2005.
35. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Second Order Linear, Half-Linear Superlinear and Sublinear Dynamic Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2002.
36. Jadlovská, I.; Džurina, J. Kneser-type oscillation criteria for second-order half-linear delay differential equations. *Appl. Math. Comput.* **2020**, *380*, 125289. [CrossRef]
37. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I. A sharp oscillation criterion for second-order half-linear advanced differential equations. *Acta Math. Hungar.* **2021**, *163*, 552–562. [CrossRef]
38. Hassan, T.S.; Abdel Menaem, A.; Jawarneh, Y.; Iqbal, N.; Ali, A. Oscillation criterion of Kneser type for half-linear second-order dynamic equations with deviating arguments. *AIMS Math.* **2024**, *9*, 19446–19458. [CrossRef]
39. Grace, S.R.; Chhatria, G.N. Improved oscillation criteria for second order quasilinear dynamic equations of noncanonical type. *Rend. Circ. Mat. Palermo, II. Ser.* **2024**, *73*, 127–140. [CrossRef]
40. Hassan, T.S.; Agarwal, R.P.; Mohammed, W.W. Oscillation criteria for third-order functional half-linear dynamic equations. *Adv. Difference Equ.* **2017**, *2017*, 111. [CrossRef]
41. Džurina, J. Oscillation of second order advanced differential equations. *Electron. J. Qual. Theo.* **2018**, 1–9. [CrossRef]
42. Hassan, T.S.; Sun, Y.; Abdel Menaem, A. Improved oscillation results for functional nonlinear dynamic equations of second order. *Mathematics* **2020**, *8*, 1897. [CrossRef]
43. Affan, S.E.; Elabbasy, E.M.; El-Matary, B.M.; Hassan, T.S.; Hassan, A.M. Second-order advanced dynamic equations on time scales: Oscillation analysis via monotonicity properties. *AIMS Math.* **2025**, *10*, 4473–4491 [CrossRef]

44. Hassan, T.S.; El-Nabulsi, R.A.; Abdel Menaem, A. Amended criteria of oscillation for nonlinear functional dynamic equations of second-order. *Mathematics* **2021**, *9*, 1191. [CrossRef]
45. Džurina, J.; Jadlovská, I. A sharp oscillation result for second-order half-linear noncanonical delay differential equations. *Electron. J. Qual. Theo.* **2020**, *46*, 1–14. [CrossRef]
46. Grace, S.R.; Hassan, T.S. Oscillation criteria for higher order nonlinear dynamic equations. *Math. Nachrichten* **2014**, *287*, 1659–1673. [CrossRef]
47. Jadlovská, I. Oscillation criteria of Kneser-type for second-order half-linear advanced differential equations. *Appl. Math. Lett.* **2020**, *106*, 106354. [CrossRef]
48. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation criteria for nonlinear damped dynamic equations on time scales. *Appl. Math. Comput.* **2008**, *203*, 343–357. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.