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Uncertainty Inequalities for the Linear Canonical Dunkl Transform

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Abstract

The aim of this paper is to show some uncertainty inequalities for the linear canonical Dunkl transform (LCDT), including sharp Heisenberg-type, entropic-type, logarithmic-type, Donoho–Stark-type and local-type uncertainty principles.

Keywords: linear canonical Dunkl transform; uncertainty principles

MSC: 47G30; 42B10; 47G10

1. Introduction and Preliminaries

Dunkl operators play a vital role in the analysis of special functions exhibiting reflection symmetries, and recent years have witnessed significant advancements in harmonic analysis linked to root systems. These developments have attracted substantial interest within the field of mathematical physics. To proceed with precision, we first introduce some notation related to the Dunkl transform. For a more comprehensive overview of Dunkl theory, the reader is referred to [1–3].

Let σ_α denote the reflection across a hyperplane in $\mathbb{R}^N$ orthogonal to a nonzero vector $\alpha \in \mathbb{R}^N$. Consider a root system $\mathcal{R}$ such that $\sigma_\alpha(\mathcal{R}) = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. The reflections corresponding to the elements of $\mathcal{R}$ generate a finite subgroup W of the orthogonal group $O(N)$, known as the reflection group. Given a multiplicity function k defined on the root system $\mathcal{R}$, we define the associated weight function ω_k and the index κ by [1]

$$\kappa = \sum_{\alpha\in\mathcal{R}_+} k(\alpha), \qquad \omega_k(t) = \prod_{\alpha\in\mathcal{R}_+} |\langle \alpha, t\rangle|^{2k(\alpha)}, \tag{1}$$

where $\mathcal{R}_+$ denotes a system of positive roots associated with $\mathcal{R}$. In addition, we introduce the Mehta-type constant defined by

$$c_k = \int_{\mathbb{R}^N} e^{-\frac{|t|^2}{2}} \omega_k(t)\, dt, \tag{2}$$

where $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$ denotes the standard Euclidean norm on $\mathbb{R}^N$.

The Dunkl operators $\{T_j\}_{j\in\mathbb{N}}$ are given by the following definition:

$$T_j u(\cdot) := \partial_{x_j} u(\cdot) + \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\langle \alpha, v_j\rangle \frac{u(\cdot) - u(\sigma_\alpha(\cdot))}{\langle \alpha, \cdot\rangle}, \tag{3}$$

where $\{v_j\}$ is an orthonormal basis of $\mathbb{R}^N$.

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The Dunkl kernel $e_k(-i\cdot,\cdot)$ is the unique analytic function on $\mathbb{R}^N$ that solves following the system:

$$\begin{cases} T_j u(t,z) &= z_j u(t,z), \quad j=1,\ldots,N, \\ u(0,z) &= 1, \end{cases} \tag{4}$$

This kernel satisfies the inequality

$$|e_k(it,z)| \leq 1, \qquad \text{for all } z,t \in \mathbb{R}^N. \tag{5}$$

In addition, it admits a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$, and fulfills the identities

$$e_k(\lambda z,t) = e_k(z,\lambda t), \quad e_k(t,z) = e_k(z,t), \quad e_k(z,0) = 1,$$

for all $\lambda \in \mathbb{C}$ and $t,z \in \mathbb{C}^N$.

For $1 \leq p \leq \infty$, and for a fixed multiplicity function k, the space $L_k^p(\mathbb{R}^N)$ consists of measurable functions f on $\mathbb{R}^N$ satisfying the following:

$$\|u\|_p := \left(\int_{\mathbb{R}^N} |u(t)|^p \, d\gamma_k(t) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|u\|_\infty := \text{ess sup}_{t\in\mathbb{R}^N} |u(t)| < \infty, \qquad p = \infty,$$

where $d\gamma_k(t) = {c_k}^{-1}\omega_k(t)dt$.

Moreover, if $u(\cdot) = v(|\cdot|)$, then [4]

$$\int_{\mathbb{R}^N} u(x) d\gamma_k(x) = 2M_k \int_0^\infty v(t) t^{2\kappa+N-1} dt, \tag{6}$$

where

$$M_k := \left(2^{N/2-2+\kappa} \Gamma(N/2+\kappa) \right)^{-1}. \tag{7}$$

For any function $u \in L_k^1(\mathbb{R}^N)$, the Dunkl transform is defined as

$$\mathfrak{D}_k(u)(z) = \int_{\mathbb{R}^N} u(t) e_k(-it,z) d\gamma_k(t), \qquad z \in \mathbb{R}^N. \tag{8}$$

Let $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $SL(2,\mathbb{R})$, with $b \neq 0$. The linear canonical Dunkl transform (LCDT) of a function $u \in L_k^1(\mathbb{R}^N)$ is defined by

$$\mathfrak{D}_k^M(u)(z) = (ib)^{-N/2-\kappa} \int_{\mathbb{R}^N} \mathfrak{B}_k^M(z,t) u(t) d\gamma_k(t), \tag{9}$$

where

$$\mathfrak{B}_k^M(z,t) = e^{\frac{i}{2}\left(\frac{d}{b}|z|^2 + \frac{a}{b}|t|^2\right)} e_k(-iz/b,t), \tag{10}$$

which satisfies the following: For every $z,t \in \mathbb{R}^N$,

$$|\mathfrak{B}_k^M(z,t)| \leq 1. \tag{11}$$

This transformation generalizes the classical linear canonical transform (LCT), which was independently introduced by Collins [5] and Moshinsky–Quesne [6]. It serves as a powerful and versatile tool for exploring complex problems in areas such as signal processing, optics, and quantum physics [7–12]. In recent years, it has attracted considerable interest and has been studied to a broad class of integral transforms [13–24].

We define the differential-difference operator Δ_k^M by the following:

$$\Delta_k^M := \Delta_k - i\frac{d}{b}\left(\sum_{j=1}^{N} m_j T_j - T_j m_j\right) - \frac{d^2}{b^2}|x|^2, \tag{12}$$

where $m_j(u(t)) := t_j u(t)$ and $\Delta_k = \sum_{j=1}^{N} T_j^2$ is the Dunkl–Laplacian operator. Then, we have the following relations:

1. Δ_k^M and Δ_k are connected by

$$e^{-\frac{i}{2}\frac{d}{b}|x|^2} \circ \Delta_k^M \circ e^{\frac{i}{2}\frac{d}{b}|x|^2} = \Delta_k. \tag{13}$$

2. We have for any functions $u,\ v$ in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \Delta_k^M u(x)\overline{v(x)}d\gamma_k(x) = \int_{\mathbb{R}^N} u(x)\overline{\Delta_k^M v(x)}d\gamma_k(x). \tag{14}$$

$$\int_{\mathbb{R}^N} \Delta_k^M f(x)\overline{g(x)}d\gamma_k(x) = \int_{\mathbb{R}^N} f(x)\overline{\Delta_k^M g(x)}d\gamma_k(x). \tag{15}$$

3. The kernel $\mathfrak{B}_k^M(\cdot,t)$ satisfies the following:

$$\begin{cases} \Delta_k^M \mathfrak{B}_k^M(\cdot,t) = -|\frac{t}{b}|^2 \mathfrak{B}_k^M(\cdot,t), \\ \mathfrak{B}_k^M(0,t) = e^{\frac{i}{2}\frac{a}{b}|t|^2}. \end{cases} \tag{16}$$

4. For every $u \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\mathfrak{D}_k^M\Big(|t|^2 u(t)\Big) = -b^2 \Delta_k^M\Big(\mathfrak{D}_k^M(u)\Big), \tag{17}$$

and

$$|z|^2 \mathfrak{D}_k^M(u) = -b^2 \mathfrak{D}_k^M\Big(\Delta_k^{M^{-1}}(u)\Big). \tag{18}$$

We define the chirp multiplication operator L_s, by

$$\mathrm{L}_s u(\cdot) = e^{\frac{is}{2}|\cdot|^2} u(\cdot), \qquad s \in \mathbb{R}, \tag{19}$$

and the dilation operator π_s by

$$\pi_s u(\cdot) = \frac{1}{|s|^{N/2+\kappa}} u(\cdot/s), \qquad s \neq 0. \tag{20}$$

Then on $L_k^1(\mathbb{R}^N)$,

$$\pi_s \circ \mathfrak{D}_k = \mathfrak{D}_k \circ \pi_{\frac{1}{s}}, \tag{21}$$

and

$$e^{i(N/2+\kappa)\frac{\pi}{2}\operatorname{sgn}(b)} \mathfrak{D}_k^M = \mathrm{L}_{\frac{b}{a}} \circ \pi_b \circ \mathfrak{D}_k \circ \mathrm{L}_{\frac{a}{b}}. \tag{22}$$

For all $u \in L_k^1(\mathbb{R}^N)$, its LCDT belongs to $C_0(\mathbb{R}^N)$ and satisfies the following relation:

$$\big\|\mathfrak{D}_k^M(u)\big\|_\infty \leq |b|^{-N/2-\kappa} \big\|u\big\|_1. \tag{23}$$

1. For all $u, v \in L^1_k(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \mathfrak{D}^M_k(u)(x)\overline{v(x)}d\gamma_k(x) = \int_{\mathbb{R}^N} u(x)\overline{\mathfrak{D}^{M^{-1}}_k(v)(x)}d\gamma_k(x). \tag{24}$$

2. For $u \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$, we have $\mathfrak{D}^M_k(u) \in L^2_k(\mathbb{R}^N)$ and

$$\left\|\mathfrak{D}^M_k(u)\right\|_2 = \|u\|_2. \tag{25}$$

3. The LCDT has a unique extension to an isometric isomorphism on $L^2_k(\mathbb{R}^N)$, denoted also by $\mathfrak{D}^M_k$.
4. For all $u, v \in L^2_k(\mathbb{R}^N)$,

$$\left\langle \mathfrak{D}^M_k(u), v \right\rangle_2 = \left\langle u, \mathfrak{D}^{M^{-1}}_k v \right\rangle_2. \tag{26}$$

5. For every $u \in L^1_k(\mathbb{R}^N)$, such that its LCDT belongs to $L^1_k(\mathbb{R}^N)$,

$$\left(\mathfrak{D}^M_k \circ \mathfrak{D}^{M^{-1}}_k\right)(u) = \left(\mathfrak{D}^{M^{-1}}_k \circ \mathfrak{D}^M_k\right)(u) = u, \qquad \text{a.e,} \tag{27}$$

where $M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Since $\mathfrak{D}_k : L^p_k(\mathbb{R}^N) \to L^{p'}_k(\mathbb{R}^N)$ is well defined on $L^p_k(\mathbb{R}^N)$, $p \in [1,2]$, then the LCDT is defined on $L^p_k(\mathbb{R}^N)$ by

$$\mathfrak{D}^M_k = e^{-i(N/2+\kappa)\frac{\pi}{2}\operatorname{sgn}(b)}\left(\mathrm{L}_{\frac{d}{b}} \circ \pi_b \circ \mathfrak{D}_k \circ \mathrm{L}_{\frac{a}{b}}\right).$$

Therefore, we have a Young-type inequality

$$\left\|\mathfrak{D}^M_k(u)\right\|_{p'} \leq |b|^{(\kappa+N/2)(2/p'-1)}\|u\|_p \tag{28}$$

for all $u \in L^p_k(\mathbb{R}^N)$ and $p \in [1,2]$, where p' is the conjugate exponent of p, such that $\frac{1}{p} + \frac{1}{p'} = 1$.

The existing literature presents numerous developments of uncertainty principles for integral operators generalizing the Fourier transform in one dimension, such as the fractional Fourier transform (FRFT) and the LCT [25–33], and in arbitrary dimensions, such as the free metaplectic transform (which can also be considered as a generalization of the LCT) [34–40]. Most of these uncertainty principles have been studied in one or two dimensions, and some of them have considered the general case of the N-dimension, among others only considering real signals or a particular case of the LCT.

In this paper, we will address the N-dimensional case, and prove some uncertainty inequalities for the LCDT, which is a generalization of the LCT, the Dunkl transform, the Fresnel–Dunkl transform, the fractional Dunkl transform, and other [13]. Note that the Fourier transform and the FRFT are two special cases of LCDT (for $k = 0$). Our first result will be the following entropic uncertainty relation: For all $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$,

$$H_k(|f|^2) + H_k(|\mathfrak{D}^M_k(f)|^2) \geq \|f\|_2^2\left((N+2\kappa)\ln|b| - 2\ln(\|f\|_2^2)\right), \tag{29}$$

where

$$H_k(|f|) = -\int_{\mathbb{R}^N} |f|\ln(|f|)d\gamma_k. \tag{30}$$

Then we will provide the following sharp Heisenberg-type uncertainty inequality, that is, for every $f \in L^2_k(\mathbb{R}^N)$,

$$\| |x| f\|_2 \, \| |\xi| \, \mathfrak{D}^M_k(f)\|_2 \geq |b|(N/2+\kappa)\|f\|_2^2, \tag{31}$$

where the equality occurs in (63), if and only if, $f = c\,\phi^k_{0,m}$, $m \in \mathbb{N}$, $c \in \mathbb{C}$. Here $\phi^k_{0,m}$ is the first term of the sequence of Laguerre functions, $\| |x| f\|_2$ is the temporary dispersion, and $\| |\xi| \, \mathfrak{D}^M_k(f)\|_2$ is the frequency dispersion.

Equation (31) is an analogue of the well-known Heisenberg uncertainty inequality for the Dunkl transform [41–45]. For other uncertainty principles in the Dunkl setting, the reader is referred to [46–52].

More generally, for every $s, t > 0$, there is $C > 0$ such that, for every $f \in L^2_k(\mathbb{R}^N)$,

$$\| |x|^s f\|_2^t \| |\xi|^t \mathfrak{D}^M_k(f)\|_2^s \geq C\|f\|_2^{s+t}, \tag{32}$$

where the constant C does not depend on f, but in the general case it is not optimal. For the case $s, t \geq 1$, one can take $C = |b|^{st}(N/2+\kappa)^{st}$. For $p \in (1,2]$, the L^p_k-case of (32) is also studied in Theorem 3.

On the other hand, by showing Nash-type inequalities and Clarkson-type inequalities for the LCDT, we have obtained some uncertainty inequalities involving L^1 and L^p-norms, $p \in (1,2]$. In particular (see also [50,53]), for every $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$, we have

$$\| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+2t}} \| |\xi|^s \mathfrak{D}^M_k(f)\|_2^{\frac{N+2\kappa}{N+2\kappa+2s}} \geq C\|f\|_1^{\frac{N+2\kappa}{N+2\kappa+2s}} \|f\|_2^{\frac{N+2\kappa}{N+2\kappa+2t}}, \tag{33}$$

and

$$\| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+2t}} \| |\xi|^s \mathfrak{D}^M_k(f)\|_2^{\frac{N+2\kappa}{2s}} \geq C\|f\|_2^{\frac{N+2\kappa+2s}{2s} - \frac{2t}{N+2\kappa+2t}}. \tag{34}$$

Other variations of such inequalities have been proven in this paper, for functions in $L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$, where $1 < p < q \leq 2$.

In Section 2.5, we will show a Pitt-type inequality for the LCDT, that is, for all $f \in \mathcal{S}(\mathbb{R}^N)$ we have

$$|b|^{2\alpha} \int_{\mathbb{R}^N} |\xi|^{-2\alpha} \left|\mathfrak{D}^M_k(f)(\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\alpha) \int_{\mathbb{R}^N} |x|^{2\alpha} |f(x)|^2 d\gamma_k(x), \tag{35}$$

where

$$C_k(\alpha) := (1/2)^{2\alpha} \left(\frac{\Gamma((N+2\kappa-2\alpha)/4)}{\Gamma((N+2\kappa+2\alpha)/4)}\right)^2, \qquad 0 \leq \alpha < \kappa + N/2. \tag{36}$$

Using this inequality, we derive the following logarithmic uncertainty inequality for the LCDT: For every $f \in \mathcal{S}(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \log|x| \, |f(x)|^2 d\gamma_k(x) \quad &+ \quad \int_{\mathbb{R}^N} \log|y| |\mathfrak{D}^M_k(f)(y)|^2 d\gamma_k(y) \\ &\geq \quad N\left(\frac{\Gamma'(N/4+\kappa/2)}{\Gamma(N/4+\kappa/2)} + \log(2|b|)\right)\|f\|_2^2. \end{aligned} \tag{37}$$

In Section 2.6, we will prove some Donoho–Stark-type uncertainty principles for functions in $L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$, where $1 \leq p \leq q \leq 2$. In particular, if a nonzero function $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$ is ε_S-concentrated on S for the L^1_k-norm and its LCDT is ε_Σ-concentrated on Σ for the L^2_k-norm, then

$$\gamma_k(S)\,\gamma_k(\Sigma) \geq |b|^{2\kappa+N}(1-\varepsilon_S)^2(1-\varepsilon_\Sigma)^2, \tag{38}$$

where a nonzero function $g \in L_k^p(\mathbb{R}^N)$ is said to beε_S-concentrated on S for the L_k^p-norm, if

$$\|g - \chi_S g\|_p \leq \varepsilon_S \|g\|_p, \qquad \varepsilon_S \in [0,1), \tag{39}$$

where χ is the characteristic function. Notice that, if $\varepsilon_S = 0$ in (39), then g is concentrated on S, and $S = \operatorname{supp} g = \{x \in \mathbb{R}^N : g(x) \neq 0\}$ is the support of g. Moreover, if $\varepsilon_S \in (0,1)$ is close to zero, then S is said to be the essential support of g. Therefore, (38) shows that the supports or the essential supports of a nonzero function f and its LCDT cannot be too small. Consequently, if a function $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ satisfies

$$\gamma_k(\operatorname{supp} f)\, \gamma_k(\operatorname{supp} \mathfrak{D}_k^M(f)) < |b|^{2\kappa+N}, \tag{40}$$

then f is the zero function [54].

In Section 2.7, we will show some local uncertainty inequalities, comparing the concentration of a function f or its LCDT, with the the time or frequency dispersions. Following this, we derive uncertainty inequalities involving the essential supports and time or frequency dispersions. Moreover, we obtain other general forms of Heisenberg's uncertainty inequality.

2. Uncertainty Principles for the LCDT

Our first result will be an entropic-type uncertainty principle for the LCDT.

2.1. Entropic Uncertainty Inequality

A positive measurable function ϱ on $\mathbb{R}^N$ that satisfies

$$\int_{\mathbb{R}^N} \varrho(\xi) d\gamma_k(\xi) = 1. \tag{41}$$

is a probability density function on $\mathbb{R}^N$. Then

$$E_k(\varrho) := -\int_{\mathbb{R}^N} \ln(\varrho(\xi))\varrho(\xi) d\gamma_k(\xi) \tag{42}$$

defines the entropy of ϱ on $\mathbb{R}^N$, in accordance with Shannon [55]. Therefore, if the integral in (42) is well defined, we extend (42) to any positive function ϱ defined on $\mathbb{R}^N$.

For any nonnegative measurable function g on $\mathbb{R}^N$, we define the weighted entropy of g by the following:

$$H_k(g) = -\int_{\mathbb{R}^N} g(y) \ln(g(y)) d\gamma_k(y). \tag{43}$$

Then we have the following entropic uncertainty inequality.

Theorem 1. *For all $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, with $\|f\|_2 = 1$, we have*

$$H_k(|f|^2) + H_k(|\mathfrak{D}_k^M(f)|^2) \geq (N + 2\kappa) \ln(|b|). \tag{44}$$

Proof. For $t > 0$, we have

$$t^2 - t \leq \frac{t^q - t^2}{q-2} \leq \lim_{q \to 2^-} \frac{t^q - t^2}{q-2} = t^2 \ln(t), \qquad q \in [1,2). \tag{45}$$

Let $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ such that $\|f\|_2 = 1$. Then by Hölder's inequality, f is in $L_k^q(\mathbb{R}^N)$ for every $q \in [1,2]$. So, we can define the functions ϕ on $(1,2]$ by

$$\phi(q) = \int_{\mathbb{R}^N} |\mathfrak{D}_k^M(f)(x)|^{\frac{q}{q-1}} d\gamma_k(x) - |b|^{(\kappa+N/2)\frac{q-2}{q-1}} \left(\int_{\mathbb{R}^N} |f(x)|^q d\gamma_k(x) \right)^{\frac{1}{q-1}}.$$

Involving (28), we have for all $q \in (1,2]$, $\phi(q) \leq 0$. Moreover, (25) implies that $\phi(2) = 0$. Then $\frac{d\phi}{dq}(2^-) \geq 0$, as long as the derivative exists.

By (45),

$$\frac{d}{dq}\left(\int_{\mathbb{R}^N} |f|^q d\gamma_k \right)(2^-) = \frac{1}{2} \int_{\mathbb{R}^N} |f|^2 \ln(|f|^2) d\gamma_k. \tag{46}$$

Therefore,

$$\lim_{q \to 2^-} \int_{\mathbb{R}^N} |f(x)|^q d\gamma_k(x) = \int_{\mathbb{R}^N} |f(x)|^2 d\gamma_k(x) = 1. \tag{47}$$

Combining relations (46) and (47), we get

$$\frac{d}{dq}\left(\int_{\mathbb{R}^N} |f(x)|^q d\gamma_k(x) \right)^{\frac{1}{q-1}} (2^-) = -\frac{1}{2} H_k(|f(x)|^2). \tag{48}$$

Then

$$\frac{d}{dq}\left(\int_{\mathbb{R}^N} |\mathfrak{D}_k^M(f)(\xi)|^{\frac{q}{q-1}} d\gamma_k(\xi) \right)(2^-) = -\frac{1}{2} \int_{\mathbb{R}^N} |\mathfrak{D}_k^M(f)(\xi)|^2 \ln(|\mathfrak{D}_k^M(f)(\xi)|^2) d\gamma_k(\xi).$$

By a simple computation we get,

$$\frac{d}{dq}\left(|b|^{(\kappa+N/2)\frac{q-2}{q-1}} \right)(2^-) = (\kappa + N/2) \ln\left(|b|\right). \tag{49}$$

Thus we obtain the desired result. □

Corollary 1. *For all $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,*

$$H_k(|f|^2) + H_k(|\mathfrak{D}_k^M(f)|^2) \geq \|f\|_2^2 \Big((N + 2\kappa) \ln |b| - 2 \ln(\|f\|_2^2) \Big). \tag{50}$$

2.2. Sharp Heisenberg-Type Uncertainty Inequality

Let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) > -1$, and let $L_\ell^{(\lambda)}$ be the Laguerre polynomial given by the following:

$$L_\ell^{(\lambda)}(t) := \frac{(\lambda+1)_\ell}{\ell!} \sum_{j=0}^{\ell} \frac{(-\ell)_j}{(\lambda+1)_j} \frac{t^j}{j!} = \sum_{j=0}^{\ell} \frac{(-1)^j \Gamma(\lambda+1+\ell)}{(\ell-j)! \Gamma(\lambda+1+j)} \frac{t^j}{j!}, \tag{51}$$

where $(\ell)_j := \ell(\ell+1)\cdots(\ell+j-1)$. Then we have the following proposition (see [56]).

Proposition 1. *For a fixed integer $m \in \mathbb{N}$:*

1. *Let*

$$f_{\ell,m}^{(k)}(r) := \left(\frac{2^{1+\lambda_{k,m}} \Gamma(\ell+1)}{2^{\lambda_{k,m}} \Gamma(\lambda_{k,m}+\ell+1)} \right)^{1/2} r^m L_\ell^{(\lambda_{k,m})}\left(r^2\right) \exp\left(-\frac{1}{2} r^2\right), \qquad \ell \in \mathbb{N}, \tag{52}$$

where $\lambda_{k,m} := \frac{2m+2k-1}{2}$. Then the sequence $\left\{ \Phi_{\ell,m}^{(k)}(\cdot) := f_{\ell,m}^{(k)}(|\cdot|) \,:\, \ell \in \mathbb{N} \right\}$ is an orthonormal basis for $L_k^2(\mathbb{R}^N)$.

2. *The sequence of generalized Laguerre functions* $\{\Phi_{\ell,m}^{(k)}\}_{\ell\in\mathbb{N}}$ *is the basis of eigenfunctions of* $\mathfrak{D}_k$ *, for* $L_k^2(\mathbb{R}^N)$*, such that*

$$\mathfrak{D}_k(\Phi_{\ell,m}^{(k)}) = e^{-i\pi(\ell+\frac{m}{2})}\Phi_{\ell,m}^{(k)}. \tag{53}$$

3. *The sequence* $\{\Phi_{\ell,m}^{(k)}\}_{\ell\in\mathbb{N}}$ *forms a complete set of eigenfunctions on* $L_k^2(\mathbb{R}^N)$ *for* $(-\Delta_k)$*, such that*

$$-\Delta_k\Phi_{\ell,m}^{(k)} = (N+2\kappa+4\ell)\Phi_{\ell,m}^{(k)}. \tag{54}$$

For $m\in\mathbb{N}$, we define the sequences $\{\phi_{\ell,m}^k\}_{\ell=0}^\infty$ and $\{\psi_{\ell,m}^k\}_{\ell=0}^\infty$ of modified Laguerre functions as follows:

$$\phi_{\ell,m}^k(x) = \left(\mathrm{L}_{-\frac{a}{b}}\circ\pi_{\sqrt{|b|}}\right)\Phi_{\ell,m}^{(k)}(x) = \frac{e^{-\frac{i}{2}\frac{a}{b}|x|^2}}{|b|^{\frac{N+2\kappa}{4}}}\Phi_{\ell,m}^{(k)}(x/\sqrt{|b|}), \quad x\in\mathbb{R}^N, \tag{55}$$

and

$$\psi_{\ell,m}^k(x) = \left(\mathrm{L}_{\frac{d}{b}}\circ\pi_{\sqrt{|b|}}\right)\Phi_{\ell,m}^{(k)}(x) = \frac{e^{\frac{i}{2}\frac{d}{b}|x|^2}}{|b|^{\frac{N+2\kappa}{4}}}\Phi_{\ell,m}^{(k)}(x/\sqrt{|b|}), \quad x\in\mathbb{R}^N. \tag{56}$$

Proposition 2. *Let* $m\in\mathbb{N}$*. Then,*

1. *The sequences* $\{\phi_{\ell,m}^k\}_{\ell=0}^\infty$ *and* $\{\psi_{\ell,m}^k\}_{\ell=0}^\infty$ *are two orthonormal bases of* $L_k^2(\mathbb{R}^N)$ *such that*

$$\left(e^{i(\kappa+N/2)\frac{\pi}{2}\operatorname{sgn}(b)}\mathfrak{D}_k^M\right)\phi_{\ell,m}^k(x) = e^{-i\pi(\ell+\frac{m}{2})}\psi_{\ell,m}^k(x), \tag{57}$$

and

$$\left(e^{-i(\kappa+N/2)\frac{\pi}{2}\operatorname{sgn}(b)}\mathfrak{D}_k^{M^{-1}}\right)\psi_{\ell,m}^k(x) = e^{-i\pi(\ell+\frac{m}{2})}\phi_{\ell,m}^k(x). \tag{58}$$

2. *Each of the sequences* $\{\phi_{\ell,m}^k\}_{\ell=0}^\infty$ *and* $\{\psi_{\ell,m}^k\}_{\ell=0}^\infty$ *forms a complete set on* $L_k^2(\mathbb{R}^N)$ *of eigenfunctions for the operators*

$$H_k^{M^{-1}} = |x|^2 - |b|^2\Delta_k^{M^{-1}} \quad \text{and} \quad H_k^M = |x|^2 - |b|^2\Delta_k^M \tag{59}$$

such that

$$H_k^{M^{-1}}\phi_{\ell,m}^k = |b|(4\ell+2\kappa+N)\phi_{\ell,m}^k, \tag{60}$$

and

$$H_k^M\psi_{\ell,m}^k = |b|(4\ell+2\kappa+N)\psi_{\ell,m}^k. \tag{61}$$

Proof. The first result is an immediate consequence of the fact that any unitary operator transforms any orthonormal basis to an orthonormal basis. Moreover, as

$$\mathfrak{D}_k\circ\pi_{\sqrt{|b|}} = \pi_{\frac{1}{\sqrt{|b|}}}\circ\mathfrak{D}_k \quad \text{and} \quad \pi_{|b|}\circ\pi_{\frac{1}{\sqrt{|b|}}} = \pi_{\sqrt{|b|}},$$

then

$$\begin{aligned}\left(e^{i(\kappa+N/2)\frac{\pi}{2}\operatorname{sgn}(b)}\mathfrak{D}_k^M\right)(\phi_{\ell,m}^k)(x) &= \left(\mathrm{L}_{\frac{d}{b}}\circ\pi_{|b|}\circ\mathfrak{D}_k\circ\mathrm{L}_{\frac{a}{b}}\right)\circ\left(\mathrm{L}_{-\frac{a}{b}}\circ\pi_{\sqrt{|b|}}\right)\Phi_{\ell,m}^{(k)}(x)\\ &= e^{-i\pi(\ell+\frac{m}{2})}\psi_{\ell,m}^k(x)\end{aligned}$$

and

$$\begin{aligned}\left(e^{-i(\kappa+N/2)\frac{\pi}{2}\operatorname{sgn}(b)}\mathfrak{D}_k^{M^{-1}}\right)\psi_{\ell,m}^k(x) &= \left(\mathrm{L}_{-\frac{a}{b}}\circ\pi_{|b|}\circ\mathfrak{D}_k\circ\mathrm{L}_{-\frac{d}{b}}\right)\circ\left(\mathrm{L}_{\frac{d}{b}}\circ\pi_{\sqrt{|b|}}\right)\Phi_{\ell,m}^{(k)}(x)\\ &= e^{-i\pi(\ell+\frac{m}{2})}\phi_{\ell,m}^k(x).\end{aligned}$$

For the second result, let $u(x)=e^{i\lambda|x|^2}v(tx)$. Then by a simple computation we have

$$\Delta_k^M u(x)=t^2e^{i\lambda|x|^2}\Delta_k v(tx). \tag{62}$$

Thus, using (12), (54), (59) we obtain (60), for $\lambda=-\frac{a}{2b}$, $t=|b|^{-1/2}$, $v=\Phi_{\ell,m}^{(k)}$ and we obtain (61) for $\lambda=\frac{d}{2b}$, $t=|b|^{-1/2}$, $v=\Phi_{\ell,m}^{(k)}$. □

Theorem 2 (Sharp Heisenberg-Type Uncertainty Inequality)**.**
For every $f\in L_k^2(\mathbb{R}^N)$,

$$\big\|\,|x|\,f\big\|_2\,\big\|\,|\xi|\,\mathfrak{D}_k^M(f)\big\|_2\geq|b|(N/2+\kappa)\|f\|_2^2. \tag{63}$$

Furthermore, we have equality in (63), if and only if, $f=c\,\phi_{0,m}^k$, $m\in\mathbb{N}$, $c\in\mathbb{C}$.

Proof. We have

$$f=\sum_{\ell=0}^{\infty}\langle f,\phi_{\ell,m}^k\rangle_2\phi_{\ell,m}^k\ \text{ and }\ \mathfrak{D}_k^M(f)=\sum_{\ell=0}^{\infty}\langle f,\phi_{\ell,m}^k\rangle_2\mathfrak{D}_k^M(\phi_{\ell,m}^k). \tag{64}$$

Then by (59), (61), and (17), we get

$$\begin{aligned}\big\|\,|x|f\big\|_2^2+\big\|\,|\xi|\mathfrak{D}_k^M(f)\big\|_2^2 &= |b|\sum_{\ell=0}^{\infty}(N+2\kappa+4\ell)\Big|\langle f,\phi_{\ell,m}^k\rangle_2\Big|^2\\ &\geq |b|(N+2\kappa)\|f\|_2^2,\end{aligned} \tag{65}$$

with equality in (65), only if $\langle\phi_{\ell,m}^k,f\rangle_2=0$, for $\ell\neq0$.

Now, for $\lambda>0$, let $M_\lambda=\begin{pmatrix}a\lambda^2 & b\\ c & d\lambda^{-2}\end{pmatrix}$, and for $f\in L_k^2(\mathbb{R}^N)$, let $f_\lambda(x)=f(\lambda x)$. Then by replacing f by f_λ in (65), we obtain

$$\lambda^{-2}\big\|\,|x|f\big\|_2^2+\lambda^2\big\|\,|\xi|\mathfrak{D}_k^M(f)\big\|_2^2\geq|b|(N+2\kappa)\|f\|_2^2. \tag{66}$$

Takin the minimum of the left-hand side of (66) over λ, we get the result. □

Corollary 2. *Let $t,s\geq1$. Then for all $f\in L_k^2(\mathbb{R}^N)$,*

$$\big\|\,|x|^sf\big\|_2^t\big\|\,|\xi|^t\mathfrak{D}_k^M(f)\big\|_2^s\geq|b|^{st}(\kappa+N/2)^{st}\|f\|_2^{s+t}. \tag{67}$$

Proof. If $f\in L_k^2(\mathbb{R}^N)$ is a non-vanishing function with finite dispersions

$$\big\|\,|x|^sf\big\|_2,\quad\big\|\,|\xi|^t\mathfrak{D}_k^M(f)\big\|_2<\infty, \tag{68}$$

then

$$\big\|\,|x|^sf\big\|_2^{\frac{2}{s}}\big\|f\big\|_2^{\frac{2}{s'}}=\big\|\,|x|^2|f|^{\frac{2}{s}}\big\|_s^{\frac{2}{s}}\big\|\,|f|^{\frac{2}{s'}}\big\|_{s'}, \tag{69}$$

From Hölder's inequality and (69), it follows that

$$\| |x|f\|_2 \leq \| |x|^s f\|_2^{\frac{1}{s}} \| f\|_2^{\frac{1}{s'}}. \tag{70}$$

Then

$$\| |x|^s f\|_2^t \geq \| |x|f\|_2^{st} \| f\|_2^{(1-s)t}.$$

Likewise,

$$\| |\xi|^t \mathfrak{D}_k^M(f)\|_2^s \geq \| |\xi|\mathfrak{D}_k^M(f)\|_2^{st} \| \mathfrak{D}_k^M(f)\|_2^{(1-t)s}. \tag{71}$$

Then the result follows from (63). □

Since the last corollary's proof uses Hölder's relation, then the result eliminates the special cases in which $0 < s, t < 1$.

2.3. Heisenberg-Type Inequality Involving the L^p-Norm

In this subsection, we will prove an L^p-Heisenberg-type inequality for the LCDT. Our proof is motivated by [57], in which the L^2-case is introduced and studied for Lie groups.

Lemma 1. *Let $q \in [1, \infty)$ and $t \in (0, \infty)$. There is a constant $C > 0$, for which*

$$\left\|e^{-t|\cdot|^2}\right\|_q = Ct^{-\frac{N+2\kappa}{2q}}. \tag{72}$$

Lemma 2. *Let $t > 0$, $1 < p \leq 2$ and $0 < s_1 < \frac{N+2\kappa}{p'}$. Then there exists $C(p,k) > 0$ such that for all $f \in L_k^p(\mathbb{R}^N)$,*

$$\left\|e^{-t|\cdot|^2}\mathfrak{D}_k^M(f)\right\|_{p'} \leq C(p,k)\ t^{-\frac{s_1}{2}} \left\| |x|^{s_1} f\right\|_p. \tag{73}$$

Proof. Assume that $\left\| |x|^{s_1} f\right\|_p < \infty$, and for $r > 0$, let $f_r = \chi_{B_r} f$ and $f^r = f - f_r$. Then, we have

$$|f^r(x)| \leq r^{-s_1} \left| |x|^{s_1} f(x)\right|.$$

Moreover, from (28), we have

$$\begin{aligned}
\left\|e^{-t|\cdot|^2}\mathfrak{D}_k^M\left(\chi_{B_r^c} f\right)\right\|_{p'} &\leq \left\|\mathfrak{D}_k^M(\chi_{B_r^c} f)\right\|_{p'} \left\|e^{-t|\cdot|^2}\right\|_\infty \\
&\leq |b|^{(N/2+\kappa)(2/p'-1)} \left\|\chi_{B_r^c} f\right\|_p \\
&\leq r^{-s_1} |b|^{(N/2+\kappa)(2/p'-1)} \left\| |x|^{s_1} f\right\|_p.
\end{aligned}$$

By (23) we have,

$$\begin{aligned}
\left\|e^{-t|\cdot|^2}\ \mathfrak{D}_k^M(\chi_{B_r} f)\right\|_{p'} &\leq \left\|e^{-t|\cdot|^2}\right\|_{p'} \left\|\mathfrak{D}_k^M(\chi_{B_r} f)\right\|_\infty \\
&\leq |b|^{-\kappa+N/2} \left\|e^{-t|\cdot|^2}\right\|_{p'} \|\chi_{B_r} f\|_1 \\
&\leq |b|^{-\kappa+N/2} \left\|e^{-t|\cdot|^2}\right\|_{p'} \left\| |x|^{-s_1}\chi_{B_r}\right\|_{p'} \left\| |x|^{s_1} f\right\|_p.
\end{aligned}$$

It is straightforward to show that $C > 0$ such that,

$$\left\| |y|^{-s_1}\chi_{B_r}\right\|_{p'} = Cr^{-s_1+\frac{N+2\kappa}{p'}}.$$

Consequently, we obtain

$$\left\|e^{-t|\cdot|^2}\mathfrak{D}_k^M(f)\right\|_{p'} \le \left\|e^{-t|\cdot|^2}\mathfrak{D}_k^M(f\chi_{B_r^c})\right\|_{p'} + \left\|e^{-t|\cdot|^2}\mathfrak{D}_k^M(f\chi_{B_r})\right\|_{p'}$$

$$\le Cr^{-s_1}\left(|b|^{(\kappa+N/2)(2/p'-1)} + r^{\frac{N+2\kappa}{p'}}\left\|e^{-t|\cdot|^2}\right\|_{p'}|b|^{-\kappa+N/2}\right)\||x|^{s_1}f\|_p.$$

For $r = t^{\frac{1}{2}}$, we get the result. □

Theorem 3. *Let $0 < s_1 < \frac{N+2\kappa}{p'}$, $1 < p \le 2$ and let $s_2 > 0$. Then there is a positive constant C such that for every $f \in L_k^p(\mathbb{R}^N)$*

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \le C\left\| |x|^{s_1}f\right\|_p^{\frac{s_2}{s_1+s_2}}\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_{p'}^{\frac{s_1}{s_1+s_2}}. \tag{74}$$

Proof. Assume first that $s_2 \le 2$. Then by (73), we have

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \le Ct^{-\frac{s_1}{2}}\left\| |x|^{s_1}f\right\|_p + \left\|(1-e^{-t|\cdot|^2})\mathfrak{D}_k^M(f)\right\|_{p'}, \qquad t > 0.$$

In addition,

$$\left\|(1-e^{-t|\xi|^2})\mathfrak{D}_k^M(f)\right\|_{p'} \le t^{\frac{s_2}{2}}\left\|(t|\xi|^2)^{-\frac{s_2}{2}}(1-e^{-t|\xi|^2})\right\|_\infty\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_{p'}.$$

If $s_2 \le 2$, then the function $r^{-\frac{s_2}{2}}(1-e^{-r})$ is bounded for all $r \ge 0$. Thus,

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \le C\left(t^{-\frac{s_1}{2}}\left\| |x|^{s_1}f\right\|_p + t^{\frac{s_2}{2}}\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_{p'}\right). \tag{75}$$

The result for $s_2 \le 2$ and $0 < s_1 < \frac{N+2\kappa}{p'}$ follows by minimizing the last inequality over $t > 0$.

On the other hand, let $s_2' \le 2 < s_2$. For $r \ge 0$, we have $r^{s_2'} \le 1 + r^{s_2}$, which for $\varepsilon > 0$ and $r = \frac{\xi}{\varepsilon}$, we obtain the result.

$$\left\| |\xi|^{s_2'}\mathfrak{D}_k^M(f)\right\|_{p'} \le \varepsilon^{s_2'}\left\|\mathfrak{D}_k^M(f)\right\|_{p'} + \varepsilon^{s_2'-s_2}\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_{p'}.$$

For $\varepsilon = \left(\frac{s_2-s_2'}{s_2'}\right)^{\frac{1}{s_2}}\left\|\mathfrak{D}_k^M(f)\right\|_{p'}^{-\frac{1}{s_2}}\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_{p'}^{\frac{1}{s_2}}$, we obtain the inequality. □

Corollary 3. *For $s_1, s_2 > 0$, there is $C > 0$ such that, for every $f \in L_k^2(\mathbb{R}^N)$,*

$$\left\|f\right\|_2 \le C\left\| |x|^{s_1}f\right\|_2^{\frac{s_2}{s_1+s_2}}\left\| |\xi|^{s_2}\mathfrak{D}_k^M(f)\right\|_2^{\frac{s_1}{s_1+s_2}}. \tag{76}$$

Proof. From Equality (25) and Theorem 3, we obtain the result when $0 < s_1 < \kappa + N/2$. If $s_1 \ge \kappa + N/2$, let $s_1' < \kappa + N/2$. For $u \ge 0$, $u^{s_1'} \le 1 + u^{s_1}$ which for $u = |x|/\varepsilon$ gives the inequality

$$(|x|/\varepsilon)^{s_1'} \le 1 + (|x|/\varepsilon)^{s_1}, \quad \text{for all } \varepsilon > 0.$$

It follows that

$$\left\| |x|^{s_1'}f\right\|_2 \le \varepsilon^{s_1'}\left\|f\right\|_2 + \varepsilon^{s_1'-s_1}\left\| |x|^{s_1}f\right\|_2.$$

Minimizing over ε, we get

$$\left\| |x|^{s_1'}f\right\|_2 \le C\left\|f\right\|_2^{\frac{s_1-s_1'}{s_1}}\left\| |x|^{s_1}f\right\|_2^{\frac{s_1'}{s_1}}. \tag{77}$$

Then, by (76) for (s_1' and s_2), and (77), we deduce that

$$\begin{aligned}\|f\|_2 &\leq C\| |x|^{s_1'} f\|_2^{\frac{s_2}{s_1'+s_2}} \| |\xi|^{s_2} \mathfrak{D}_k^M(f)\|_2^{\frac{s_1'}{s_1'+s_2}} \\ &\leq C\|f\|_2^{\frac{s_2(s_1-s_1')}{s_1(s_1'+s_2)}} \| |x|^{s_1} f\|_2^{\frac{s_1's_2}{s_1(s_1'+s_2)}} \| |\xi|^{s_2} \mathfrak{D}_k^M(f)\|_2^{\frac{s_1'}{s_1'+s_2}}.\end{aligned}$$

Hence for $s_1 \geq \kappa + N/2$, we have the following:

$$\|f\|_2^{\frac{s_1'(s_1+s_2)}{s_1(s_1'+s_2)}} \leq C\| |x|^{s_1} f\|_2^{\frac{s_1's_2}{s_1(s_1'+s_2)}} \| |\xi|^{s_2} \mathfrak{D}_k^M(f)\|_2^{\frac{s_1'}{s_1'+s_2}}.$$

Thus, we obtain the result. □

2.4. Heisenberg-Type Inequalities Involving the $L^1 - L^p$-Norms

In this subsection, s and S are two positive, nonzero real numbers.

Theorem 4 (Nash-Type Inequality on $L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$)**.**
Let $1 < p \leq 2$. Then for all $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$,

$$\|\mathfrak{D}_k^M(f)\|_{p'} \leq K_1(s,k,p)\|f\|_1^{\frac{sp'}{N+2\kappa+sp'}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{\frac{N+2\kappa}{N+2\kappa+sp'}}, \tag{78}$$

where

$$K_1(s,k,p) = \left(\frac{2}{N+2\kappa}(sp'/2)^{\frac{N+2\kappa}{N+2\kappa+sp'}} + (2/sp')^{\frac{sp'}{N+2\kappa+sp'}}\right)^{1/p'} M_k^{\frac{s}{N+2\kappa+sp'}} |b|^{-p'\frac{N+2\kappa}{2(N+2\kappa+sp')}}.$$

Proof. For $r > 0$, we have

$$\|\mathfrak{D}_k^M(f)\|_{p'}^{p'} = \|\chi_{B_r}\mathfrak{D}_k^M(f)\|_{p'}^{p'} + \|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_{p'}^{p'}. \tag{79}$$

Firstly,

$$\|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_{p'}^{p'} \leq r^{-sp'}\| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{p'}. \tag{80}$$

By (23), we get

$$\|\chi_{B_r}\mathfrak{D}_k^M(f)\|_{p'}^{p'} \leq \gamma_k(B_r)\|\mathfrak{D}_k^M(f)\|_\infty^{p'} \leq \gamma_k(B_r)|b|^{-(N/2+\kappa)p'}\|f\|_1^{p'}.$$

Since

$$\gamma_k(B_r) := \int_{\mathbb{R}^N} \chi_{B_r}(y)d\gamma_k(y) = \frac{M_k}{N+2\kappa}r^{N+2\kappa}, \tag{81}$$

then

$$\|\chi_{B_r}\mathfrak{D}_k^M(f)\|_{p'}^{p'} \leq \frac{2M_k}{N+2\kappa}r^{N+2\kappa}|b|^{-(N/2+\kappa)p'}\|f\|_1^{p'}. \tag{82}$$

It follows from (79), (80), and (82), that

$$\|\mathfrak{D}_k^M(f)\|_{p'}^{p'} \leq \frac{2M_k}{N+2\kappa}r^{N+2\kappa}|b|^{-(N/2+\kappa)p'}\|f\|_1^{p'} + r^{-sp'}\| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{p'}. \tag{83}$$

For $r = \left(\dfrac{sp'\| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{p'}}{2M_k|b|^{-(N/2+\kappa)p'}\|f\|_1^{p'}}\right)^{\frac{1}{N+2\kappa+sp'}}$, we have the desired result. □

The last theorem implies, in particular that,

$$\|f\|_2 \leq K_1(s,k,2)\|f\|_1^{\frac{2s}{N+2\kappa+2s}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{N+2\kappa}{N+2\kappa+2s}}. \tag{84}$$

Theorem 5 (Clarkson-Type Inequality on $L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$)**.**
Let $1 < p \leq 2$. *Then for all* $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$,

$$\|f\|_1 \leq D_1(s,p,k)\|f\|_p^{\frac{sp'}{N+2\kappa+sp'}} \| |x|^s f\|_1^{\frac{N+2\kappa}{N+2\kappa+sp'}}, \tag{85}$$

where

$$D_1(s,k,p) = \left(\frac{2M_k}{N+2\kappa}\right)^{\frac{s}{N+2\kappa+sp'}} \left(\left(\frac{sp'}{N+2\kappa}\right)^{\frac{N+2\kappa}{N+2\kappa+sp'}} + \left(\frac{N+2\kappa}{sp'}\right)^{\frac{sp'}{N+2\kappa+sp'}} \right). \tag{86}$$

Proof. For $r > 0$, we have

$$\|f\|_1 = \|\chi_{B_r} f\|_1 + \|(1-\chi_{B_r})f\|_1. \tag{87}$$

First,

$$\|(1-\chi_{B_r})f\|_1 \leq r^{-s} \| |x|^s f\|_1. \tag{88}$$

Then

$$\|\chi_{B_r} f\|_1 \leq (\gamma_k(B_r))^{1/p'} \|f\|_p \leq \left(\frac{2M_k}{N+2\kappa}\right)^{1/p'} r^{\frac{N+2\kappa}{p'}} \|f\|_p. \tag{89}$$

Combining the relations (87), (88), and (89), we obtain

$$\|f\|_1 \leq \left(\frac{2M_k}{N+2\kappa}\right)^{1/p'} r^{\frac{N+2\kappa}{p'}} \|f\|_p + r^{-s} \| |x|^s f\|_1. \tag{90}$$

For $r = \left(\dfrac{sp'(N+2\kappa)^{-1/p} \| |x|^s f\|_1}{(2M_k)^{\frac{1}{p'}} \|f\|_p} \right)^{\frac{p'}{N+2\kappa+sp'}}$, we obtain the result. □

The following Heisenberg-type uncertainty inequalities are obtained from the relations (78) and (85).

Corollary 4. *Let* $1 < p \leq 2$. *Then for every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$,

$$\|f\|_1^{\frac{N+2\kappa}{N+2\kappa+sp'}} \|f\|_p^{-\frac{p'tN}{2+N(2\kappa-1+p't)}} \|\mathfrak{D}_k^M(f)\|_{p'} \leq C_1 \| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+p't}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{\frac{N+2\kappa}{N+2\kappa+sp'}}, \tag{91}$$

where

$$C_1 = C_1(s,k,p,t) = D_1(t,k,p)K_1(s,k,p).$$

In addition,

$$\|f\|_p^{-\frac{p't}{N+2\kappa+p't}} \|\mathfrak{D}_k^M(f)\|_{p'}^{\frac{N+2\kappa+sp'}{sp'}} \leq C_2(s,k,p,t) \| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+p't}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{p'}^{\frac{N+2\kappa}{sp'}}, \tag{92}$$

where

$$C_2(s,k,p,t) = D_1(t,k,p)(K_1(s,k,p))^{\frac{N+2\kappa+sp'}{sp'}}.$$

Remark 1. *For $p = 2$, we deduce the following Heisenberg-type uncertainty principles:*

1. *From (91), for every $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$, we have*

$$\|f\|_1^{\frac{N+2\kappa}{N+2\kappa+2s}} \|f\|_2^{\frac{N+2\kappa}{N+2\kappa+2t}} \leq C_1(s,k,2,t) \| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+2t}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{N+2\kappa}{N+2\kappa+2s}}. \tag{93}$$

2. *From (92) we have for all $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$,*

$$\|f\|_2^{\frac{N+2\kappa+2s}{2s} - \frac{2t}{N+2\kappa+2t}} \leq C_2(s,k,2,t) \| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+2t}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{N+2\kappa}{2s}}. \tag{94}$$

3. *From (84) and Theorem 5, for every $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$, we have*

$$\|f\|_1^{1-\frac{4st}{(2s+N+2\kappa)(2t+N+2\kappa)}} \leq C_3 \| |x|^t f\|_1^{\frac{N+2\kappa}{N+2\kappa+2t}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{2t(N+2\kappa)}{(N+2\kappa+2s)(N+2\kappa+2t)}}, \tag{95}$$

where

$$C_3 = C_3(s,k,t) := (K_1(s,k,2))^{\frac{2t}{N+2\kappa+2t}} D_1(t,k,2).$$

Theorem 6 (Nash-Type Relation on $L^p_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$)**.**
Let $1 < p < 2$. Then for every $f \in L^p_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$,

$$\|f\|_2 \leq K_2 \|f\|_p^{\frac{2sp'}{2sp'+(p'-2)(N+2\kappa)}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}}, \tag{96}$$

where

$$K_2 = K_2(s,k,p) = \left(\frac{M_k}{|b|^{N+2\kappa}}\right)^{\frac{s(p'-2)}{2sp'+(p'-2)(N+2\kappa)}} (N/2+\kappa)^{\frac{2s}{2sp'+(p'-2)(N+2\kappa)}}$$
$$\times \left(\frac{2}{N+2\kappa}\left(\frac{sp'}{p'-2}\right)^{\frac{(p'-2)(N+2\kappa)}{(p'-2)(N+2\kappa)+2sp'}} \left(\frac{|b|^{N+2\kappa}}{M_k}\right)^{\frac{2(p'-2)}{(p'-2)(N+2\kappa)+2sp'}} + \left(\frac{p'-2}{sp'}\right)^{\frac{2sp'}{(p'-2)(N+2\kappa)+2sp'}}\right).$$

Proof. For $r > 0$,

$$\|\mathfrak{D}_k^M(f)\|_2^2 = \|\chi_{B_r}\mathfrak{D}_k^M(f)\|_2^2 + \|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_2^2. \tag{97}$$

Firstly,

$$\|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_2^2 \leq r^{-2s} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^2. \tag{98}$$

By Hölder's inequality, (28) and (81), we get

$$\begin{aligned} \left\|\chi_{B_r}\mathfrak{D}_k^M(f)\right\|_2^2 &\leq (\gamma_k(B_r))^{1-2/p'} \|\mathfrak{D}_k^M(f)\|_{p'}^2 \\ &\leq \left(\frac{M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-2/p'} |b|^{(2/p'-1)(N+2\kappa)} \|f\|_p^2. \end{aligned} \tag{99}$$

By (97), (98), and (99), we obtain

$$\|\mathfrak{D}_k^M(f)\|_2^2 \leq r^{-2s} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^2 + \left(\frac{M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-2/p'} |b|^{(2/p'-1)(N+2\kappa)} \|f\|_p^2.$$

By choosing $r = \left(\frac{sp'}{p'-2}\left(\frac{2}{N+2\kappa}\right)^{2/p'} \left(\frac{|b|^{N+2\kappa}}{M_k}\right)^{1-2/p'} \frac{\| |\xi|^s \mathfrak{D}_k^M(f)\|_2^2}{\|f\|_p^2}\right)^{\frac{p'}{2sp'+(p'-2)(N+2\kappa)}}$, and applying Parseval's identity for $\mathfrak{D}_k^M$, we obtain the result. □

Theorem 7 (Clarkson-Type Inequality on $L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$)**.**
Let $1 < p < 2$*. Then for every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\|f\|_p \leq D_2(s,p,k)\|f\|_2^{\frac{2ps}{(2-p)(N+2\kappa)+2ps}} \| |x|^s f\|_p^{\frac{(2-p)(N+2\kappa)}{(2-p)(N+2\kappa)+2ps}}, \tag{100}$$

where

$$D_2(s,p,k) = \left(\frac{N+2\kappa}{2M_k^{1-2/p}}\right)^{\frac{2sp}{2sp(2-p)(N+2\kappa)}} \left(\left(\frac{ps}{2-p}\right)^{\frac{(2-p)(N+2\kappa)}{2sp+(2-p)(N+2\kappa)}} + \left(\frac{2-p}{ps}\right)^{\frac{2ps}{2sp+(2-p)(N+2\kappa)}}\right)^{1/p}.$$

Proof. For $r > 0$, we have

$$\|f\|_p^p = \|\chi_{B_r} f\|_p^p + \|(1-\chi_{B_r})f\|_p^p. \tag{101}$$

Firstly,

$$\|(1-\chi_{B_r})f\|_p^p \leq r^{-ps}\| |x|^s f\|_p^p. \tag{102}$$

Then

$$\|\chi_{B_r} f\|_p^p \leq (\gamma_k(B_r))^{1-p/2}\|f\|_2^p \leq \left(\frac{2M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-p/2}\|f\|_2^p. \tag{103}$$

From (101), (102), and (103), we have

$$\|f\|_p^p \leq r^{-ps}\| |x|^s f\|_p^2 + \left(\frac{2M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-p/2}\|f\|_2^p. \tag{104}$$

For $r = \left(\frac{(\frac{2}{N+2\kappa})^{\frac{p}{2}} spM_k^{p/2-1}\| |x|^s f\|_p^2}{(2-p)\|f\|_2^p}\right)^{\frac{2}{(2-p)(N+2\kappa)+2ps}}$, we obtain the result. □

Corollary 5. *Let* $1 < p < 2$*. Then*

1. *From* (96) *and* (100) *we have for every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\begin{aligned} \|f\|_2^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}} \|f\|_p^{\frac{(p'-2)(N+2\kappa)}{(p'-2)(N+2\kappa)+2sp'}} &\leq M_1(s,k,p,t)\| |x|^t f\|_p^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}} \\ &\times \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{(p'-2)(N+2\kappa)+2sp'}}, \end{aligned} \tag{105}$$

where

$$M_1(s,k,p,t) = K_2(s,k,p)D_2(t,k,p).$$

2. *From* (96) *and* (100) *we have for every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\begin{aligned} \|f\|_2^{1-\frac{4pp'st}{(2sp'+(p'-2)(N+2\kappa))(2pt+(2-p)(N+2\kappa))}} &\leq M_2\| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}} \\ &\times \| |x|^t f\|_p^{\frac{2sp'(2-p)(N+2\kappa)}{(2pt+(2-p)(N+2\kappa))(2sp'+(p'-2)(N+2\kappa))}}, \end{aligned} \tag{106}$$

where

$$M_2 = M_2(s,k,p,t) = K_2(s,k,p)(D_2(t,k,p))^{\frac{2sNp'}{2sNp'+(p'-2)(2+(2k-1)N)}}.$$

3. *From* (96) *and* (100) *we have for every* $f \in L^p_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$,

$$\|f\|_p^{1-\frac{4psp't}{((p'-2)(N+2\kappa)+2sp')(2pt+(2-p)(N+2\kappa))}} \leq M_3 \| |x|^t f\|_p^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}} \times \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{2pt(p'-2)(N+2\kappa)}{(p'-2)(N+2\kappa)+2sp')((2-p)(N+2\kappa)+2tp}}, \tag{107}$$

where

$$M_3 = M_3(s,k,p,t) = D_2(t,k,p)(K_2(s,k,p))^{\frac{2pt}{(2-p)(N+2\kappa)+2pt}}.$$

Theorem 8 (Nash-Type Inequality on $L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$)**.**
Let $1 < p < q \leq 2$. *Then there exists a positive constant* $K_3(s,p,q,k)$ *such that for all* $f \in L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$,

$$\|\mathfrak{D}_k^M(f)\|_{q'} \leq K_3(s,p,q,k)\|f\|_p^{\frac{sp'q'}{(N+2\kappa)(p'-q')+sp'q'}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{\frac{(N+2\kappa)(p'-q')}{(N+2\kappa)(p'-q')+sp'q'}}. \tag{108}$$

Proof. Let $r > 0$ and let $f \in L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$. Then

$$\|\mathfrak{D}_k^M(f)\|_{q'}^{q'} = \|\chi_{B_r}\mathfrak{D}_k^M(f)\|_{q'}^{q'} + \|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_{q'}^{q'}. \tag{109}$$

Firstly,

$$\|(1-\chi_{B_r})\mathfrak{D}_k^M(f)\|_{q'}^{q'} \leq r^{-sq'} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{q'}. \tag{110}$$

From (28) and (81), we have

$$\begin{aligned} \|\chi_{B_r}\mathfrak{D}_k^M(f)\|_{q'}^{q'} &\leq (\gamma_k(B_r))^{1-q'/p'}\|\mathfrak{D}_k^M(f)\|_{p'}^{q'} \\ &\leq \left(\frac{2M_k}{(N+2\kappa)}r^{N+2\kappa}\right)^{1-q'/p'}\|f\|_p^{q'}|b|^{(2q'/p'-q')\kappa+N/2}. \end{aligned} \tag{111}$$

Combining the relations (109), (110), and (111), we obtain

$$\|\mathfrak{D}_k^M(f)\|_{q'}^{q'} \leq \left(\frac{2M_k}{N+2\kappa}r^{N+2\kappa}\right)^{1-q'/p'}|b|^{(2q'/p'-q')\kappa+N/2}\|f\|_p^{q'} + r^{-sq'}\| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{q'}.$$

By choosing

$$r = \left(\frac{sp'q'\left(\frac{2M_k}{N+2\kappa}\right)^{q'/p'}\| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{q'}}{2M_k|b|^{(2q'/p'-q')\kappa+N/2}(p'-q')\|f\|_p^{q'}}\right)^{\frac{p'}{(N+2\kappa)(p'-q')+sp'q'}},$$

we obtain (108). □

Corollary 6. *Let* $1 < p < 2$. *Then for every* $f \in L^p_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$,

$$\|f\|_2 \leq K_3(s,p,2,k)\|f\|_p^{\frac{2sp'}{2sp'+(p'-2)(N+2\kappa)}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}}. \tag{112}$$

Theorem 9 (Clarkson-Type Inequality on $L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$)**.**
Let $1 < p < q \leq 2$. *Then there exists a positive constant* $D_3 = D_3(s,p,q,k)$ *such that, for every* $f \in L^p_k(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$,

$$\|f\|_p \leq D_3\| |x|^s f\|_p^{\frac{(N+2\kappa)(q-p)}{(N+2\kappa)(q-p)+spq}}\|f\|_q^{\frac{spq}{(N+2\kappa)(q-p)+spq}}. \tag{113}$$

Proof. For $r > 0$ and $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$, we have

$$\|f\|_p^p = \|\chi_{B_r} f\|_p^p + \|(1-\chi_{B_r})f\|_p^p. \tag{114}$$

Firstly,

$$\|(1-\chi_{B_r})f\|_p^p \le r^{-ps} \| |x|^s f\|_p^p. \tag{115}$$

By Hölder's inequality, we get

$$\|\chi_{B_r} f\|_p^p \le (\gamma_k(B_r))^{1-p/q} \|f\|_q^p \le \left(\frac{2M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-p/q} \|f\|_q^p. \tag{116}$$

Combining the relations (114), (115), and (116), we get

$$\|f\|_p^p \le \left(\frac{2M_k}{N+2\kappa} r^{N+2\kappa}\right)^{1-p/q} \|f\|_q^p + r^{-ps} \| |x|^s f\|_p^p.$$

By choosing $r = \left(\dfrac{spq \| |x|^s f\|_p^p}{(N+2\kappa)(q-p)\left(\frac{2M_k}{N+2\kappa}\right)^{1-p/q} \|f\|_q^p}\right)^{\frac{q}{(N+2\kappa)(q-p)+spq}}$, we conclude the result.

□

From (108) and (113), we have the following Heisenberg-type uncertainty relations.

Theorem 10. *Let* $1 < p < q \le 2$.

1. *For all* $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$,

$$\begin{aligned} &\|f\|_p^{1-\frac{sp'q'}{(N+2\kappa)(p'-q')+sp'q'}} \|f\|_q^{\frac{-pqt}{(N+2\kappa)(q-p)+pqt}} \|\mathfrak{D}_k^M(f)\|_{q'} \\ \le\ & N_1(s,p,q,k) \| |x|^t f\|_p^{\frac{(N+2\kappa)(q-p)}{(N+2\kappa)(q-p)+pqt}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{\frac{(N+2\kappa)(p'-q')}{(N+2\kappa)(p'-q')+sp'q'}}, \end{aligned} \tag{117}$$

where

$$N_1(s,p,q,k) = D_3(t,p,q,k) K_3(s,p,q,k).$$

2. *For all* $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$,

$$\begin{aligned} &\|\mathfrak{D}_k^M(f)\|_{q'} \le N_2(s,t,p,q,k) \|f\|_q^{\frac{stp'q'pq}{((N+2\kappa)(q-p)+pqt)((N+2\kappa)(p'-q')+sp'q')}} \\ \le\ & \| |x|^t f\|_p^{\frac{sp'q'(q-p)(N+2\kappa)}{((N+2\kappa)(q-p)+pqt)((N+2\kappa)(p'-q')+sp'q')}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_{q'}^{\frac{(N+2\kappa)(p'-q')}{((N+2\kappa)(p'-q')+sp'q')}}, \end{aligned} \tag{118}$$

where

$$N_2(s,t,p,q,k) = K_3(s,p,q,k) (D_3(t,p,q,k))^{\frac{sp'q'}{((N+2\kappa)(p'-q')+sp'q')}}.$$

Corollary 7. *Let* $1 < p < 2$.

1. *For all* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\begin{aligned} \|f\|_p^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}} \|f\|_2^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}} \le\ & N_1(s,p,2,k) \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}} \\ &\times \| |x|^t f\|_p^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}}. \end{aligned} \tag{119}$$

2. *For every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\begin{aligned} \|f\|_2^{1-\frac{4stpp'}{(2pt+(2-p)(N+2\kappa))(2sp'+(p'-2)(N+2\kappa))}} &\leq N_2(s,t,p,2,k) \\ \times \quad \| |x|^t f\|_p^{\frac{2sp'(2-p)(N+2\kappa)}{(2pt+(2-p)(N+2\kappa))(2sp'+(p'-2)(2\kappa+N))}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)}}. \end{aligned} \tag{120}$$

3. *For every* $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$,

$$\|f\|_p \leq N_3 \| |x|^t f\|_p^{\frac{(2-p)(N+2\kappa)}{2pt+(2-p)(N+2\kappa)}} \| |\xi|^s \mathfrak{D}_k^M(f)\|_2^{\frac{(p'-2)(N+2\kappa)}{2sp'+(p'-2)(N+2\kappa)} \frac{2pt}{2pt+(2-p)(N+2\kappa)}}, \tag{121}$$

where

$$N_3 = N_3(s,k,p,t) = (K_3(s,p,2,k))^{\frac{2pt}{2pt+(2-p)(N+2\kappa)}} D_3(t,p,2,k).$$

2.5. Pitt-Type Inequality

Pitt's inequality explains the relation between the variance of a function and its Dunkl transform, which makes it fundamentally important in the Dunkl framework. An optimal version of this inequality has recently been proved in [52]. That is, for every $f \in \mathcal{S}(\mathbb{R}^N) \subseteq L_k^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\xi|^{-2\alpha} \left|\mathfrak{D}_k(f)(\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\alpha) \int_{\mathbb{R}^N} |x|^{2\alpha} |f(x)|^2 d\gamma_k(x), \tag{122}$$

where

$$C_k(\alpha) := (1/2)^{2\alpha} \left(\frac{\Gamma((N+2\kappa-2\alpha)/4)}{\Gamma((N+2\kappa+2\alpha)/4)}\right)^2, \quad 0 \leq \alpha < \kappa + N/2. \tag{123}$$

The following result is an analogue of (122) for the LCDT.

Theorem 11. *For all* $f \in \mathcal{S}(\mathbb{R}^N)$ *we have*

$$|b|^{2\alpha} \int_{\mathbb{R}^N} |\xi|^{-2\alpha} \left|\mathfrak{D}_k^M(f)(\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\alpha) \int_{\mathbb{R}^N} |x|^{2\alpha} |f(x)|^2 d\gamma_k(x), \tag{124}$$

where $C_k(\alpha)$ *is given by* (123) *and* $0 \leq \alpha < \kappa + N/2$.

Proof. From (9) and (10), we have

$$|b|^{N+2\kappa} \int_{\mathbb{R}^N} |\xi|^{-2\alpha} \left|\mathfrak{D}_k^M\left(e^{-\frac{ia}{2b}|y|^2} f\right)(b\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\alpha) \int_{\mathbb{R}^N} |x|^{2\alpha} |f(x)|^2 d\gamma_k(x). \tag{125}$$

Then the result follows by a change of variable. □

The Beckner-type inequality in the Dunkl setting states the following (see [52]): For all $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned} \int_{\mathbb{R}^N} \log|x| \, |f(x)|^2 d\gamma_k(x) + \int_{\mathbb{R}^N} \log|y| \, |\mathfrak{D}_k(f)(y)|^2 d\gamma_k(y) \\ \geq \left(\frac{\Gamma'((N+2\kappa)/4)}{\Gamma((N+2\kappa)/4)} + \log(2)\right) \int_{\mathbb{R}^N} |f(t)|^2 d\gamma_k(t). \end{aligned} \tag{126}$$

The next theorem is an analogue of inequality (126) for the LCDT.

Theorem 12. *For every $f \in \mathcal{S}(\mathbb{R}^N)$, we have*

$$\begin{aligned}\int_{\mathbb{R}^N} \log|x| \left|f(x)\right|^2 d\gamma_k(x) \quad &+ \quad \int_{\mathbb{R}^N} \log|y| \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \\ &\geq \quad N\left(\frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2|b|)\right)\|f\|_2^2. \end{aligned} \tag{127}$$

Proof. For $0 \leq \alpha < \kappa + N/2$, let

$$F(\alpha) = |b|^{2\alpha} \int_{\mathbb{R}^N} |y|^{-2\alpha} \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) - C_k(\alpha) \int_{\mathbb{R}^N} |t|^{2\alpha} \left|f(t)\right|^2 d\gamma_k(t). \tag{128}$$

Then

$$\begin{aligned} F'(\alpha) \quad &= -2|b|^{2\alpha} \int_{\mathbb{R}^N} |y|^{-2\alpha} \log|y| \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \\ &\quad +2|b|^{2\alpha} \log|b| \int_{\mathbb{R}^N} |y|^{-2\alpha} \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \\ &\quad -2C_k(\alpha) \int_{\mathbb{R}^N} \log|t| \, |t|^{2\alpha} \left|f(t)\right|^2 d\gamma_k(t) - C_k'(\alpha) \int_{\mathbb{R}^N} |t|^{2\alpha} \left|f(t)\right|^2 d\gamma_k(t), \end{aligned}$$

where

$$C_k'(\alpha) \quad = -C_k(\alpha) \left(\frac{\Gamma'\left(\frac{N+2\kappa-2\alpha}{4}\right)}{\Gamma\left(\frac{N+2\kappa-2\alpha}{4}\right)} + \frac{\Gamma'\left(\frac{N+2\kappa+2\alpha}{4}\right)}{\Gamma\left(\frac{N+2\kappa+2\alpha}{4}\right)} + 2\log(2)\right).$$

In particular,

$$C_k'(0) = -2\left(\frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2)\right). \tag{129}$$

Using (124), we derive that $F(\alpha) \leq 0$. Moreover, since

$$\begin{aligned} F(0) &= \int_{\mathbb{R}^N} \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) - C_k(0) \int_{\mathbb{R}^N} \left|f(t)\right|^2 d\gamma_k(t) \\ &= \|\mathfrak{D}_k^M(f)\|_2^2 - \|f\|_2^2 = 0, \end{aligned}$$

then we deduce that

$$F'(0^+) := \lim_{\alpha \to 0^+} \frac{F(\alpha)}{\alpha} \leq 0.$$

Equivalently,

$$\begin{aligned} &2\log|b| \int_{\mathbb{R}^N} \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) - 2\int_{\mathbb{R}^N} \log|y| \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \\ &-2C_k(0) \int_{\mathbb{R}^N} \log|t| \left|f(t)\right|^2 d\gamma_k(t) - C_k'(0) \int_{\mathbb{R}^N} \left|f(t)\right|^2 d\gamma_k(t) \leq 0. \end{aligned}$$

It follows that, by (25) and (129),

$$\begin{aligned} &2\log|b| \int_{\mathbb{R}^N} \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) - 2\int_{\mathbb{R}^N} \log|y| \left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \\ &-2\int_{\mathbb{R}^N} \log|t| \left|f(t)\right|^2 d\gamma_k(t) + 2N\left(\frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2)\right)\|f\|_2^2 \leq 0. \end{aligned}$$

Alternatively,

$$\int_{\mathbb{R}^N} \log|t|\,|f(t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^N} \log|y|\,\left|\mathfrak{D}_k^M(f)(y)\right|^2 d\gamma_k(y) \geq \left(\frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2|b|)\right)\|f\|_2^2.$$

As desired. □

Corollary 8. *For all $f \in \mathcal{S}(\mathbb{R}^N)$, we have*

$$\||x|\,f\|_2 \left\||y|\,\mathfrak{D}_k^M(f)\right\|_2 \geq \exp\left(\left(\frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2|b|)\right)\right)\|f\|_2^2. \tag{130}$$

Proof. Applying Jensen's inequality to (127), we obtain

$$\begin{aligned}
&\log\left\{\int_{\mathbb{R}^N} |t|^2 \frac{|f(t)|^2}{\|f\|_2^2} d\gamma_k(t)\right\}^{1/2} \left\{\int_{\mathbb{R}^N} |y|^2 \frac{\left|\mathfrak{D}_k^M(f)(y)\right|^2}{\|f\|_2^2} d\gamma_k(y)\right\}^{1/2} \\
&= \log\left\{\int_{\mathbb{R}^N} |t|^2 \frac{|f(t)|^2}{\|f\|_2^2} d\gamma_k(t)\right\}^{1/2} + \log\left\{\int_{\mathbb{R}^N} |y|^2 \frac{\left|\mathfrak{D}_k^M(f)(y)\right|^2}{\|f\|_2^2} d\gamma_k(y)\right\}^{1/2} \\
&\geq \int_{\mathbb{R}^N} \log|t| \frac{|f(t)|^2}{\|f\|_2^2} d\gamma_k(t) + \int_{\mathbb{R}^N} \log|y| \frac{\left|\mathfrak{D}_k^M(f)(y)\right|^2}{\|f\|_2^2} d\gamma_k(y) \\
&\geq \frac{\Gamma'\left(\frac{N+2\kappa}{4}\right)}{\Gamma\left(\frac{N+2\kappa}{4}\right)} + \log(2|b|),
\end{aligned}$$

which completes the proof. □

Remark 2. *By virtue of the identity [58],*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)} dt, \tag{131}$$

we infer that

$$\exp\left\{\left(\frac{\Gamma'((N+2\kappa)/4)}{\Gamma((N+2\kappa)/4)} + \log(2|b|)\right)\right\} \approx (\kappa + N/2)|b|, \quad \text{for } \kappa + N/2 \gg 1, \tag{132}$$

which is exactly the same constant as in (63).

2.6. Donoho–Stark-Type Uncertainty Inequalities

Let $1 \leq p \leq 2$, $0 \leq \varepsilon_S, \varepsilon_\Sigma < 1$ and let $S, \Sigma \subset \mathbb{R}^N$. Then we say that a nonzero function $g \in L_k^p(\mathbb{R}^N)$, is ε_S-concentrated on S for the L_k^p-norm, if

$$\|g - \chi_S g\|_p \leq \varepsilon_S \|g\|_p, \tag{133}$$

and we say that $\mathfrak{D}_k^M(g)$ is ε_Σ-concentrated on Σ for the $L_k^{p'}$-norm, if

$$\|\mathfrak{D}_k^M(g) - \chi_\Sigma \mathfrak{D}_k^M(g)\|_{p'} \leq \varepsilon_\Sigma \|\mathfrak{D}_k^M(g)\|_{p'}. \tag{134}$$

Notice that if $\varepsilon_S = \varepsilon_\Sigma = 0$ in (133) and (134), then $S = \text{supp}g = \{t \in \mathbb{R}^N : f(t) \neq 0\}$ is the support of f and $\Sigma = \text{supp}\mathfrak{D}_k^M(g) = \{x \in \mathbb{R}^N : \mathfrak{D}_k^M(f)(x) \neq 0\}$ is the support of $\mathfrak{D}_k^M(g)$.

In the remaining of this subsection $\varepsilon_S, \varepsilon_\Sigma$ will be in $(0,1)$ and S, Σ will be two measurable subsets of $\mathbb{R}^N$ with finite measure $0 < \gamma_k(S),\ \gamma_k(\Sigma) < \infty$.

Theorem 13 (Donoho–Stark-Type Inequality on $L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$)**.**
Let $1 < p \leq 2$. If a nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$ is ε_S-concentrated on S for the L_k^1-norm and its LCDT is ε_Σ-concentrated on Σ for the $L_k^{p'}$-norm, then

$$\|\mathfrak{D}_k^M(f)\|_{p'} \leq \frac{(\gamma_k(S))^{1/p'}(\gamma_k(\Sigma))^{1/p'}}{(1-\varepsilon_S)(1-\varepsilon_\Sigma)} \frac{1}{|b|^{\kappa+N/2}} \|f\|_p. \tag{135}$$

Proof. Let $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$, $1 < p \leq 2$. Then

$$\begin{aligned} \|\mathfrak{D}_k^M(f)\|_{p'} &\leq \varepsilon_\Sigma \|\mathfrak{D}_k^M(f)\|_{p'} + \|\chi_\Sigma \mathfrak{D}_k^M(f)\|_{p'} \\ &\leq \varepsilon_\Sigma \|\mathfrak{D}_k^M(f)\|_{p'} + (\gamma_k(\Sigma))^{1/p'} \|\mathfrak{D}_k^M(f)\|_\infty. \end{aligned}$$

Therefore from (23),

$$\|\mathfrak{D}_k^M(f)\|_{p'} \leq \frac{(\gamma_k(\Sigma))^{1/p'}}{(1-\varepsilon_\Sigma)} \frac{1}{|b|^{\kappa+N/2}} \|f\|_1. \tag{136}$$

On the other hand, since f is ε_S-concentrated on S for the L_k^1-norm,

$$\begin{aligned} \|f\|_1 &\leq \varepsilon_S \|f\|_1 + \|\chi_S f\|_1 \\ &\leq \varepsilon_S \|f\|_1 + (\gamma_k(S))^{1/p'} \|f\|_p. \end{aligned}$$

Thus,

$$\|f\|_1 \leq \frac{(\gamma_k(S))^{1/p'}}{1-\varepsilon_S} \|f\|_p. \tag{137}$$

Combining (136) and (137), we obtain the result of this theorem. □

Corollary 9. *If a nonzero function $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ is ε_S-concentrated on S for the L_k^1-norm and its LCDT is ε_Σ-concentrated on Σ for the L_k^2-norm, then*

$$\gamma_k(S)\gamma_k(\Sigma) \geq |b|^{2\kappa+N}(1-\varepsilon_S)^2(1-\varepsilon_\Sigma)^2. \tag{138}$$

Theorem 14 (Donoho–Stark-Type Relation on $L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$)**.**

Let $1 < p < q \leq 2$. If a nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$ is ε_S-concentrated on S for the L_k^p-norm and its LCDT is ε_Σ-concentrated on Σ for the $L_k^{q'}$-norm, then

$$\|\mathfrak{D}_k^M(f)\|_{q'} \leq \frac{(\gamma_k(S))^{\frac{q-p}{pq}}(\gamma_k(\Sigma))^{\frac{p'-q'}{p'q'}}}{(1-\varepsilon_S)(1-\varepsilon_\Sigma)} |b|^{\frac{(N+2\kappa)(2-p')}{2p'}} \|f\|_q. \tag{139}$$

Proof. Let $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$, $1 < p < q \leq 2$. Then we have

$$\begin{aligned} \|\mathfrak{D}_k^M(f)\|_{q'} &\leq \|\chi_\Sigma \mathfrak{D}_k^M(f)\|_{q'} + \varepsilon_\Sigma \|\mathfrak{D}_k^M(f)\|_{q'} \\ &\leq (\gamma_k(\Sigma))^{\frac{p'-q'}{p'q'}} \|\mathfrak{D}_k^M(f)\|_{p'} + \varepsilon_\Sigma \|\mathfrak{D}_k^M(f)\|_{q'}. \end{aligned}$$

Thus by (28),

$$\left\|\mathfrak{D}_k^M(f)\right\|_{q'} \leq \frac{(\gamma_k(\Sigma))^{\frac{p'-q'}{p'q'}}}{(1-\varepsilon_\Sigma)}|b|^{\frac{(N+2\kappa)(2-p')}{2p'}}\left\|f\right\|_p. \tag{140}$$

On the other hand, since f is ε_S-concentrated on S for the L_k^p-norm,

$$\begin{aligned}\left\|f\right\|_p &\leq \varepsilon_S\left\|f\right\|_p + \left\|\chi_S f\right\|_p \\ &\leq \varepsilon_S\left\|f\right\|_p + (\gamma_k(S))^{\frac{q-p}{pq}}\left\|f\right\|_q.\end{aligned}$$

Thus,

$$\left\|f\right\|_p \leq \frac{(\gamma_k(S))^{\frac{q-p}{pq}}}{1-\varepsilon_S}\left\|f\right\|_q. \tag{141}$$

Combining (140) and (141), we obtain the result of this theorem. □

Corollary 10. *If a nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, $1 < p < 2$, is ε_S-concentrated on S for the L_k^p-norm and its LCDT is ε_Σ-concentrated on Σ for the L_k^2-norm, then*

$$(\gamma_k(S))^{\frac{2-p}{2p}}(\gamma_k(\Sigma))^{\frac{p'-2}{2p'}} \geq |b|^{\frac{2p'}{(N+2\kappa)(2-p')}}(1-\varepsilon_S)(1-\varepsilon_\Sigma). \tag{142}$$

Let $1 \leq p \leq 2$ and let $B^p(\Sigma) = \left\{h \in L_k^p(\mathbb{R}^N) \,:\, \chi_\Sigma \mathfrak{D}_k^M(h) = \mathfrak{D}_k^M(h)\right\}$ be the set of functions of $L_k^p(\mathbb{R}^N)$ whose LCDT are supported on Σ. Then we say that f is ε_Σ-bandlimited on Σ for the L_k^p-norm if there is $h \in B^p(\Sigma)$ such that

$$\left\|f-h\right\|_p \leq \varepsilon_\Sigma\left\|f\right\|_p. \tag{143}$$

Theorem 15. *If a nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$, $1 < p \leq q \leq 2$, is ε_S-concentrated on S for the L_k^p-norm and ε_Σ-bandlimited on Σ for the L_k^q-norm, then*

$$\left\|f\right\|_p \leq \frac{(\gamma_k(S))^{\frac{q-p}{pq}}}{1-\varepsilon_S}\left((1+\varepsilon_\Sigma)(\gamma_k(S))^{1/q}(\gamma_k(\Sigma))^{1/q}|b|^{-\frac{N+2\kappa}{p_2}} + \varepsilon_\Sigma\right)\left\|f\right\|_q. \tag{144}$$

Proof. Let $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$, $1 < p < q \leq 2$. Then

$$\begin{aligned}\left\|f\right\|_p &\leq \varepsilon_S\left\|f\right\|_p + \left\|\chi_S f\right\|_p \\ &\leq \varepsilon_S\left\|f\right\|_p + (\gamma_k(S))^{\frac{q-p}{pq}}\left\|\chi_S f\right\|_q.\end{aligned}$$

Thus,

$$\left\|f\right\|_p \leq \frac{1}{1-\varepsilon_S}(\gamma_k(S))^{\frac{q-p}{pq}}\left\|\chi_S f\right\|_q. \tag{145}$$

Since f is ε_Σ-bandlimited for the L_k^q-norm, then there exists $h \in B^q(E)$ such that

$$\left\|f-h\right\|_q \leq \varepsilon_\Sigma\left\|f\right\|_q.$$

It follows that

$$\begin{aligned}\left\|\chi_S f\right\|_{L_k^q(\mathbb{R}^N)} &\leq \left\|\chi_S h\right\|_q + \left\|\chi_S(f-h)\right\|_q \\ &\leq \left\|\chi_S h\right\|_q + \varepsilon_\Sigma\left\|f\right\|_q.\end{aligned} \tag{146}$$

For $h \in B^q(E)$, we have from (27), $h = \left(\mathfrak{D}_k^{M^{-1}}\right)(\chi_\Sigma \mathfrak{D}_k^M(h))$, and by (28) we have

$$|h(\cdot)| \leq \frac{(\gamma_k(\Sigma))^{1/q}}{|b|^{\kappa+N/2}} \left\|\mathfrak{D}_k^M(h)\right\|_{q'} \leq \frac{(\gamma_k(\Sigma))^{1/q}}{|b|^{\frac{N+2\kappa}{q}}} \|h\|_q.$$

Thus,

$$\left\|\chi_S h\right\|_q = \left(\int_S |h(t)|^q d\gamma_k(t)\right)^{1/q} \leq (\gamma_k(S))^{1/q} \frac{(\gamma_k(\Sigma))^{1/q}}{|b|^{\frac{N+2\kappa}{q}}} \|h\|_q.$$

Then by (146) and the fact that $\|h\|_q \leq (1+\varepsilon_\Sigma)\|f\|_q$, we get

$$\left\|\chi_S f\right\|_q \leq \left((1+\varepsilon_\Sigma)(\gamma_k(S))^{1/q}(\gamma_k(\Sigma))^{1/q}|b|^{-\frac{N+2\kappa}{q}} + \varepsilon_\Sigma\right)\|f\|_q. \tag{147}$$

By (145) we conclude. □

Corollary 11. *If a nonzero function $f \in L_k^p(\mathbb{R}^N)$, $1 < p \leq 2$, is ε_S-concentrated on S and ε_Σ-bandlimited on Σ for the $L_k^p(\mathbb{R}^N)$-norm, then*

$$\gamma_k(S)\,\gamma_k(\Sigma) \geq |b|^{N+2\kappa} \frac{(1-\varepsilon_S-\varepsilon_\Sigma)^p}{(1+\varepsilon_\Sigma)^p}. \tag{148}$$

Remark 3. *If we take $\varepsilon_S = \varepsilon_\Sigma = 0$ in the previous inequalities, we derive the following Matolcsi–Szucs-type relations:*

1. *For every nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^1(\mathbb{R}^N)$, $1 < p \leq 2$, we have*

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \leq \frac{1}{|b|^{\kappa+N/2}} (\gamma_k(\mathrm{supp} f))^{1/p'} \left(\gamma_k\left(\mathrm{supp}\mathfrak{D}_k^M(f)\right)\right)^{1/p'} \|f\|_p. \tag{149}$$

2. *For all nonzero function $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, we have*

$$\gamma_k(\mathrm{supp} f)\gamma_k\left(\mathrm{supp}\mathfrak{D}_k^M(f)\right) \geq |b|^{N+2\kappa}. \tag{150}$$

3. *For all nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^q(\mathbb{R}^N)$, $1 < p \leq q \leq 2$, we have*

$$\left\|\mathfrak{D}_k^M(f)\right\|_{q'} \leq (\gamma_k(\mathrm{supp} f))^{\frac{q-p}{pq}} \left(\gamma_k\left(\mathrm{supp}\mathfrak{D}_k^M(f)\right)\right)^{\frac{p'-q'}{p'q'}} |b|^{\frac{(N+2\kappa)(2-p')}{2p'}} \|f\|_q. \tag{151}$$

4. *For all nonzero function $f \in L_k^p(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, $1 < p \leq 2$, we have*

$$(\gamma_k(\mathrm{supp} f))^{\frac{2-p}{2p}} \left(\gamma_k\left(\mathrm{supp}\mathfrak{D}_k^M(f)\right)\right)^{(1/2-1/p')} \geq |b|^{(1/2-1/p')(N+2\kappa)}. \tag{152}$$

2.7. Local Uncertainty Inequalities

In this section we will show another type of uncertainty relations involving general dispersions and the fraction of the norm on sets of finite measure. We start with the following lemma [59].

Lemma 3. *Let $p \in (1,2]$.*

1. *If $0 < \theta < \frac{N+2\kappa}{p'}$, then*

$$\left\| |y|^{-\theta} \chi_{B_s} \right\|_{p'} = \left(\frac{2M_k}{N+2\kappa-\theta p'}\right)^{\frac{1}{p'}} s^{-\theta+\frac{N+2\kappa}{p'}}. \tag{153}$$

2. *If* $\theta > \frac{N+2\kappa}{p'}$, *then*

$$C(\theta,p) := \int_{\mathbb{R}^N} \frac{d\gamma_k(y)}{\left(1+|y|^{p\theta}\right)^{\frac{p'}{p}}} = \frac{2M_k}{p\theta} \frac{\Gamma\left(\frac{N+2\kappa}{\theta p}\right)\Gamma\left(\frac{p'}{p}-\frac{N+2\kappa}{\theta p}\right)}{\Gamma(p'/p)}. \tag{154}$$

Theorem 16. *Let* $\theta > 0$, $p \in (1,2]$ *and let* $S \subset \mathbb{R}^N$ *be a subset of finite measure* $0 < \gamma_k(S) < \infty$. *Then for every* $f \in L^p_k(\mathbb{R}^N)$, *we have*

$$\left\|\chi_S \mathfrak{D}^M_k(f)\right\|_{p'} \leq \begin{cases} C_1\left(\gamma_k(S)\right)^{\frac{\theta}{N+2\kappa}} \left\| |x|^\theta f\right\|_p, & 0 < \theta < \frac{N+2\kappa}{p'}, \\ C_2\left(\gamma_k(S)\right)^{\frac{1}{p'}} \|f\|_p^{1-\frac{N+2\kappa}{\theta p'}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, & \theta > \frac{N+2\kappa}{p'}, \\ C_3\left(\gamma_k(S)\right)^{\frac{1}{2p'}} \|f\|_p^{\frac{1}{2}} \left\| |x|^\theta f\right\|_p^{\frac{1}{2}}, & \theta = \frac{N+2\kappa}{p'}, \end{cases}$$

where

$$\begin{aligned} C_1 = C_1(\theta,p) &= 2|b|^{\frac{(2-p')(N+2\kappa)}{2p'}-\theta}\left(\frac{M_k}{N+2\kappa-\theta p'}\right)^{\frac{\theta}{N+2\kappa}}, \\ C_2 = C_2(\theta,p) &= |b|^{-\kappa+N/2}\left(\frac{N\theta p'}{(\theta p'-(2\kappa-1))N-2}\right)^{\frac{1}{p}}\left(\frac{\theta p'}{N+2\kappa}-1\right)^{\frac{N+2\kappa}{\theta p p'}}\left(C(\theta,p)\right)^{\frac{1}{p'}}, \\ C_3 = C_3(\theta,p) &= 2C_1\left(\frac{N+2\kappa}{2p'},p\right). \end{aligned}$$

Proof. Let $s > 0$, and suppose that

$$\left\| |x|^\theta f\right\|_p < \infty.$$

Define $f_s = f\chi_{B_s}$ and $f^s = f - f_s$. Since

$$|f^s(x)| \leq s^{-\theta} |x|^\theta f(x)|,$$

then by (28) we have

$$\begin{aligned} \left\|\chi_S \mathfrak{D}^M_k(f\chi_{B_s^c})\right\|_{p'} &\leq \left\|\mathfrak{D}^M_k(f\chi_{B_s^c})\right\|_{p'} \\ &\leq |b|^{\frac{(N+2\kappa)(2-p')}{2p'}} \left\|f\chi_{B_s^c}\right\|_p \\ &\leq |b|^{\frac{(N+2\kappa)(2-p')}{2p'}} s^{-\theta} \left\| |x|^\theta f\right\|_p. \end{aligned}$$

On the other hand, by (23) and Hölder's inequality,

$$\begin{aligned} \left\|\chi_S \mathfrak{D}^M_k(f\chi_{B_s})\right\|_{p'} &\leq \left\|\mathfrak{D}^M_k(f\chi_{B_s})\right\|_\infty \left\|\chi_S\right\|_{p'} \\ &\leq \frac{1}{|b|^{N/2+\kappa}} \left(\gamma_k(S)\right)^{\frac{1}{p'}} \left\|f\chi_{B_s}\right\|_1 \\ &\leq \frac{1}{|b|^{N/2+\kappa}} \left(\gamma_k(S)\right)^{\frac{1}{p'}} \left\| |x|^{-\theta}\chi_{B_s}\right\|_{p'} \left\| |x|^\theta f\right\|_p. \end{aligned}$$

Therefore, from Lemma 3,

$$\begin{aligned}&\left\|\chi_S\mathfrak{D}_k^M(f)\right\|_{p'} \le \left\|\chi_S\mathfrak{D}_k^M(f_s)\right\|_{p'} + \left\|\chi_S\mathfrak{D}_k^M(f^s)\right\|_{p'}\\ \le\ & s^{-\theta}\left(|b|^{(1/p'-1/2)(N+2\kappa)} + \frac{1}{|b|^{\kappa+N/2}}\left(\frac{M_k}{N+2\kappa-\theta p'}\right)^{\frac{1}{p'}}(\gamma_k(S))^{\frac{1}{p'}}s^{\frac{N+2\kappa}{p'}}\right)\left\||x|^\theta f\right\|_p.\end{aligned}$$

Choosing

$$s = |b|\left(\frac{N+2\kappa-\theta p'}{2M_k}\right)^{\frac{1}{N+2\kappa}}(\gamma_k(S))^{\frac{-1}{N+2\kappa}},$$

we obtain the first inequality.

The second inequality holds if $\|f\|_p = \infty$ or $\||x|^\theta f\|_p = \infty$. Assume that

$$\|f\|_p + \||x|^\theta f\|_p < \infty.$$

As above using (23), we obtain

$$\begin{aligned}\left\|\chi_S\mathfrak{D}_k^M(f)\right\|_{p'} &\le \left\|\mathfrak{D}_k^M(f)\right\|_\infty\left\|\chi_S\right\|_{p'}\\ &\le \frac{1}{|b|^{N/2+\kappa}}(\gamma_k(S))^{\frac{1}{p'}}\|f\|_1.\end{aligned} \tag{155}$$

Finally, by Lemma 3, we have

$$\begin{aligned}\|f\|_1^p &= \left(\int_{\mathbb{R}^N}(1+|x|^{p\theta})^{\frac{1}{p}}|f(x)|\frac{d\gamma_k(x)}{(1+|x|^{p\theta})^{\frac{1}{p}}}\right)^p\\ &\le \left(\int_{\mathbb{R}^N}\frac{d\gamma_k(x)}{(1+|x|^{p\theta})^{\frac{p'}{p}}}\right)^{\frac{p}{p'}}\left(\|f\|_p^p + \||x|^\theta f\|_p^p\right)\\ &\le (C(\theta,p))^{\frac{p}{p'}}\left(\|f\|_p^p + \||x|^\theta f\|_p^p\right).\end{aligned} \tag{156}$$

Therefore $f \in L_k^1(\mathbb{R}^N)$, and by replacing $f(x)$ by $f_\lambda(x) := f(\lambda x)$, $\lambda > 0$, in (156), we get

$$\|f\|_1^p \le (C(\theta,p))^{\frac{p}{p'}}\left(\lambda^{(p-1)(N+2\kappa)}\|f\|_p^p + \lambda^{(p-1)(N+2\kappa)-\theta p}\||x|^\theta f\|_p^p\right).$$

In particular, the inequality holds at the critical point

$$\lambda = \left(\frac{\theta p'}{N+2\kappa}-1\right)^{\frac{1}{p\theta}}\left(\frac{\||x|^\theta f\|_p}{\|f\|_p}\right)^{\frac{1}{\theta}}$$

which implies that

$$\|f\|_1 \le \mathfrak{K}(\theta,p,k)\|f\|_p^{1-\frac{N+2\kappa}{\theta p'}}\||x|^\theta f\|_p^{\frac{N+2\kappa}{\theta p'}}, \tag{157}$$

where

$$\mathfrak{K}(\theta,p,k) = (C(\theta,p))^{\frac{1}{p'}}\left(\frac{\theta p'}{\theta p'-2\kappa-N}\right)^{\frac{1}{p}}\left(\frac{\theta p'}{N+2\kappa}-1\right)^{\frac{N+2\kappa}{\theta p p'}}, \tag{158}$$

which allows as to conclude.

Finally, since for $\lambda > 0$, we have

$$(|x|/\lambda)^{\frac{N+2\kappa}{2p'}} \le 1 + (|x|\lambda)^{\frac{N+2\kappa}{p'}},$$

then

$$\| |x|^{\frac{N+2\kappa}{2p'}} f\|_p \leq \lambda^{\frac{N+2\kappa}{2p'}} \|f\|_p + \lambda^{-\frac{N+2\kappa}{2p'}} \| |x|^{\frac{N+2\kappa}{p'}} f\|_p.$$

Optimizing in λ, we obtain

$$\| |x|^{\frac{N+2\kappa}{2p'}} f\|_p \leq 2\|f\|_p^{\frac{1}{2}} \| |x|^{\frac{N+2\kappa}{p'}} f\|_p^{\frac{1}{2}}.$$

Thus, using the first case (for $\theta = \frac{N+2\kappa}{2p'}$) we deduce that

$$\begin{aligned} \|\chi_S \mathfrak{D}_k^M(f)\|_{p'} &\leq C_1\left(\frac{N+2\kappa}{2p'}, p\right)(\gamma_k(S))^{\frac{1}{2p'}} \| |x|^{\frac{N+2\kappa}{2p'}} f\|_p \\ &\leq 2C_1\left(\frac{N+2\kappa}{2p'}, p\right)(\gamma_k(S))^{\frac{1}{2p'}} \|f\|_p^{\frac{1}{2}} \| |x|^{\frac{N+2\kappa}{p'}} f\|_p^{\frac{1}{2}}. \end{aligned}$$

The proof is complete. □

Remark 4. *Notice that, in the case of $\theta = \frac{N+2\kappa}{p'}$, one can provide an other inequality that improves that in Theorem 16. In fact, taking*

$$s = (1-\varepsilon)\frac{2\langle k+N\rangle}{p'}, \qquad 0 < \varepsilon < 1,$$

and using the following relation,

$$\| |x|^{\theta-\theta\varepsilon} f\|_r \leq C\|f\|_r^{\varepsilon} \left\| |x|^{\theta} f\right\|_r^{1-\varepsilon}, \tag{159}$$

we get for every $0 < \varepsilon < 1$,

$$\|\chi_S \mathfrak{D}_k^M(f)\|_{p'} \leq C_4(\varepsilon, \theta, p)(\gamma_k(S))^{\frac{1-\varepsilon}{p'}} \|f\|_p^{\varepsilon} \| |x|^{\theta} f\|_p^{1-\varepsilon}. \tag{160}$$

Corollary 12.

1. *If $0 < \theta < \kappa + N/2$, then there is $C > 0$, such that for any ε-bandlimited function f on S,*

$$(\gamma_k(S))^{\frac{\theta}{N+2\kappa}} \| |x|^{\theta} f\|_2^2 \geq C\left(1-\varepsilon^2\right)\|f\|_2^2. \tag{161}$$

2. *If $\theta > \kappa + N/2$, then there is $C > 0$, such that for any ε-bandlimited function f on S,*

$$\gamma_k(S)\|f\|_2^{2-\frac{N+2\kappa}{\theta}} \| |x|^{\theta} f\|_2^{\frac{N+2\kappa}{\theta}} \geq C\left(1-\varepsilon^2\right)\|f\|_2^2. \tag{162}$$

3. *For all $\lambda \in (0,1)$, there there is $C > 0$, such that for any ε-bandlimited function f on S,*

$$(\gamma_k(S))^{1-\lambda}\|f\|_2^{2\lambda} \| |x|^{\theta} f\|_2^{2-2\lambda} \geq C(1-\varepsilon^2)\|f\|_2^2. \tag{163}$$

Proof. Since $f \in L_k^2(\mathbb{R}^N)$ is ε-bandlimited on S, then

$$\|\chi_S \mathfrak{D}_k^M(f)\|_2^2 = \|f\|_2^2 - \|\chi_{S^c} \mathfrak{D}_k^M(f)\|_2^2 \geq (1-\varepsilon^2)\|f\|_2^2.$$

The results follows by Theorem 16 and Inequality (160). □

Corollary 13. *Let $\theta > 0$ and $1 < p \leq 2$. Then for all $f \in L_k^p(\mathbb{R}^N)$,*

$$\left\|\mathfrak{D}_k^M(f)\right\|_{L_k^{\frac{p'(2+N(2\kappa-1))}{2+N(2\kappa-1)-Np'\theta},p'}} \leq C_1 \left\| |x|^\theta f\right\|_p, \qquad 0<\theta<\tfrac{N+2\kappa}{p'},$$

$$\left\|\mathfrak{D}_k^M(f)\right\|_{L_k^{\infty,p'}} \leq C_2 \|f\|_p^{1-\frac{N+2\kappa}{\theta p'}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, \quad \theta>\tfrac{N+2\kappa}{p'},$$

$$\left\|\mathfrak{D}_k^M(f)\right\|_{L_k^{2p',p'}} \leq C_3 \|f\|_p^{\frac{1}{2}} \left\| |x|^\theta f\right\|_p^{\frac{1}{2}}, \qquad \theta=\tfrac{N+2\kappa}{p'},$$

where $L_k^{p,q}(\mathbb{R}^N)$ *is the Lorentz space defined by the following norm:*

$$\|g\|_{L_k^{p,q}} = \sup_{S\subset\mathbb{R}^N, 0<\gamma_k(S)<\infty} (\gamma_k(S))^{\frac{1}{p}-\frac{1}{q}} \|\chi_S g\|_q. \tag{164}$$

Theorem 17. *Let* $\theta,\delta>0$ *and* $1<p\leq 2$. *Then for all* $f\in L_k^p(\mathbb{R}^N)$,

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \leq \begin{cases} \tilde{C}_1 \left\| |x|^\theta f\right\|_p^{\frac{\delta}{\theta+\delta}} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{\frac{\theta}{\theta+\delta}}, & 0<\theta<\frac{N+2\kappa}{p'}, \\ \tilde{C}_2 \left(\|f\|_p^{1-\frac{N+2\kappa}{\theta p'}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}} \right)^{\frac{\delta p'}{N+2\kappa+\delta p'}} \\ \qquad \times \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{\frac{N+2\kappa}{N+2\kappa+\delta p'}}, & \theta>\frac{N+2\kappa}{p'}, \\ \tilde{C}_3 \|f\|_p^{\frac{\delta}{\theta+2\delta}} \left\| |x|^\theta f\right\|_p^{\frac{\delta}{\theta+2\delta}} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{\frac{\theta}{\theta+2\delta}}, & \theta=\frac{N+2\kappa}{p'}, \end{cases}$$

where

$$\begin{aligned} \tilde{C}_1 = \tilde{C}_1(\theta,\delta,p) &= \left((\tfrac{\delta}{\theta})^{\frac{\theta}{\theta+\delta}} + (\tfrac{\theta}{\delta})^{\frac{\delta}{\theta+\delta}} \right)^{\frac{1}{p'}} \left(C_1 \sigma_k^{\frac{\theta}{N+2\kappa}} \right)^{\frac{\delta}{\theta+\delta}}, \\ \tilde{C}_2 = \tilde{C}_2(\theta,\delta,p) &= \left(\left(\tfrac{n\delta p'}{(2\kappa-1)N+2} \right)^{\frac{N+2\kappa}{N+2\kappa+\delta p'}} + (\tfrac{N+2\kappa}{\delta p'})^{\frac{\delta p'}{N+2\kappa+\delta p'}} \right)^{\frac{1}{p'}} \left(\sigma_k C_2^{p'} \right)^{\frac{\delta}{N+2\kappa+\delta p'}}, \\ \tilde{C}_3 = \tilde{C}_3(\theta,\delta,p) &= \left((\tfrac{2\delta}{\theta})^{\frac{\theta}{\theta+2\delta}} + (\tfrac{\theta}{2\delta})^{\frac{2\delta}{\theta+2\delta}} \right)^{\frac{1}{p'}} \left(\sigma_k^{\frac{1}{p'}} C_3 \right)^{\frac{2\delta}{\theta+2\delta}}, \end{aligned}$$

and $\sigma_k := \gamma_k(B_1) = \frac{2M_k}{N+2\kappa}$.

Proof. Let $0<\theta<\frac{N+2\kappa}{p'}$, $\delta>0$ and $r>0$. Then

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq \left\|\chi_{B_r}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} + \left\|\chi_{B_r^c}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'}. \tag{165}$$

From Theorem 16 and Lemma 3, we have

$$\left\|\chi_{B_r}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq C_1^{p'} \sigma_k^{\frac{\theta p'}{N+2\kappa}} r^{\theta p'} \left\| |x|^\theta f\right\|_p^{p'}. \tag{166}$$

Moreover, it is easy to see that

$$\left\|\chi_{B_r^c}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq r^{-\delta p'} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{p'}. \tag{167}$$

Combining the relations (165), (166), and (167), we get

$$\begin{aligned}\left\|\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} &\leq C_1^{p'}\sigma_k^{\frac{\theta p'}{N+2\kappa}} r^{\theta p'} \left\| |x|^\theta f\right\|_p^{p'} \\ &+ r^{-\delta p'} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{p'}.\end{aligned}$$

The first inequality follows by choosing

$$r = \left(\frac{\delta}{\theta C_1^{p'}\sigma_k^{\frac{-n\theta p'}{(2\kappa-1)N+2}}}\right)^{\frac{1}{(\theta+\delta)p'}} \left(\frac{\left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}}{\left\| |x|^\theta f\right\|_p}\right)^{\frac{1}{\theta+\delta}},$$

Now let $r, \delta > 0$ and $\theta > \frac{N+2\kappa}{p'}$. Then from Theorem 16 and Lemma 3, we have

$$\left\|\chi_{B_r}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq \sigma_k C_2^{p'} r^{(N+2\kappa)} \|f\|_p^{p'-\frac{N+2\kappa}{\theta}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta}}. \tag{168}$$

Combining the relations (165), (167), and (168), we get

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq \sigma_k C_2^{p'} r^{(N+2\kappa)} \|f\|_p^{p'-\frac{N+2\kappa}{\theta}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta}} + r^{-\delta p'} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{p'}.$$

Choosing

$$r = \left(\frac{\delta p'}{(N+2\kappa)\sigma_k C_2^{p'}}\right)^{\delta p'+N+2\kappa} \left(\frac{\left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{p'}}{\|f\|_p^{p'-\frac{N+2\kappa}{\theta}} \left\| |x|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta}}}\right)^{\delta p'+N+2\kappa},$$

provides the second inequality.

Finally, given $\theta = \frac{2\kappa+N}{p'}$, $\delta > 0$ and $r > 0$. Then from Theorem 16 and Lemma 3, we have

$$\left\|\chi_{B_r}\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} \leq \sqrt{\sigma_k}(C_3(\theta,b))^{p'} r^{\kappa+N/2} \|f\|_p^{\frac{p'}{2}} \left\| |x|^\theta f\right\|_p^{\frac{p'}{2}}. \tag{169}$$

Combining the relations (165), (167), and (169), we get

$$\begin{aligned}\left\|\mathfrak{D}_k^M(f)\right\|_{p'}^{p'} &\leq \sqrt{\sigma_k} C_3^{p'} r^{N/2+\kappa} \|f\|_p^{\frac{p'}{2}} \left\| |x|^\theta f\right\|_p^{\frac{p'}{2}} \\ &+ r^{-\delta p'} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}^{p'}.\end{aligned}$$

The last inequality follows by choosing

$$r = \left(\frac{2\delta N p'}{2+(2\kappa-1)N}\frac{1}{\sqrt{\sigma_k}C_3^{p'}}\right)^{\frac{2N}{2+(2\kappa-1+2\delta p')N}} \left(\frac{\left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_{p'}}{\|f\|_p^{\frac{1}{2}} \left\| |x|^\theta f\right\|_p^{\frac{1}{2}}}\right)^{\frac{2Np'}{(2\kappa-1+2\delta p')N+2}},$$

□

Corollary 14. *Let $\theta, \delta > 0$. Then for every $f \in L_k^2(\mathbb{R}^N)$,*

$$\|f\|_2 \leq \mathfrak{C}(\theta,\delta,k) \left\| |x|^\theta f\right\|_2^{\frac{\delta}{\theta+\delta}} \left\| |\xi|^\delta \mathfrak{D}_k^M(f)\right\|_2^{\frac{\theta}{\theta+\delta}},$$

where

$$\mathfrak{C}(\theta,\delta,k)=\begin{cases}\tilde{C}_1(\theta,\delta,b,2), & 0<\theta<\kappa+N/2,\\ (\tilde{C}_2(\theta,\delta,b,2))^{\frac{\theta(N+2\kappa+2\delta)}{(N+2\kappa+2\delta)(\theta+\delta)}}, & \theta>\kappa+N/2,\\ (\tilde{C}_3(\theta,\delta,b,2))^{\frac{\theta+2\delta}{\theta+\delta}}, & \theta=\kappa+N/2.\end{cases}$$

Building on the ideas of Faris [60] and Price [61,62] in the classical setting, we will provide another local-type uncertainty inequality for the LCDT.

Theorem 18. *Let $0<\eta<\kappa+N/2$ and $p\in[1,\infty)$. Then there is $C(p,k)$, such that for all $f\in L^1_k(\mathbb{R}^N)$ and every subset $S\subset\mathbb{R}^N$ of finite measure $\gamma_k(S)<\infty$,*

$$\begin{aligned}\left(\int_S|\mathfrak{D}^M_k(f)(y)|^p d\gamma_k(y)\right)^{\frac{1}{p}} &\leqslant C(p,k)(\gamma_k(S))^{\frac{1}{p(p+1)}}\big\||y|^{\eta}\mathfrak{D}^M_k(f)\big\|_2^{\frac{2(N+2\kappa)}{(N+2\kappa+2\eta)(p+1)}}\\ &\times \big\|f\big\|_1^{\frac{(N+2\kappa+2\eta)(p+1)-2(N+2\kappa)}{(N+2\kappa+2\eta)(p+1)}}.\end{aligned}\tag{170}$$

Proof. Suppose that $\big\|f\big\|_1=1$. We have for every $s>1$,

$$\big\|\mathfrak{D}^M_k(f)\big\|_{L^p_k(S)}\leqslant\big\|\mathfrak{D}^M_k(f)\chi_{B_s}\big\|_{L^p_k(S)}+\big\|\mathfrak{D}^M_k(f)\chi_{B_s^c}\big\|_{L^p_k(S)}.$$

From (23) we have

$$\begin{aligned}\big\|\mathfrak{D}^M_k(f)\chi_{B_s}\big\|_{L^p_k(S)} &= \left(\int_{\mathbb{R}^N}|\mathfrak{D}^M_k(f)(y)|^p\chi_{B_s}(y)\chi_S(y)d\gamma_k(y)\right)^{\frac{1}{p}}\\ &\leqslant \big\|\mathfrak{D}^M_k(f)\big\|_\infty^{\frac{p}{p+1}}\left(\int_{\mathbb{R}^N}|\mathfrak{D}^M_k(f)(y)|^{\frac{p}{p+1}}\chi_{B_s}(y)\chi_S(y)d\gamma_k(y)\right)^{\frac{1}{p}}\\ &\leqslant |b|^{-\frac{(2+N(2\kappa-1))p}{2N(p+1)}}(\gamma_k(S))^{\frac{1}{p(p+1)}}\big\|\mathfrak{D}^M_k(f)\chi_{B_s}\big\|_1^{\frac{1}{p+1}}\\ &\leqslant |b|^{-\frac{p(N+2\kappa)}{2(p+1)}}(\gamma_k(S))^{\frac{1}{p(p+1)}}\big\||y|^{\eta}\mathfrak{D}^M_k(f)\big\|_2^{\frac{1}{p+1}}\big\||y|^{-\eta}\chi_{B_s}\big\|_2^{\frac{1}{p+1}}.\end{aligned}$$

Moreover, by Lemma 3, we have

$$\big\||y|^{-\eta}\chi_{B_s}\big\|_2\leq\left(\frac{2M_k}{N+2\kappa-2\eta}\right)^{\frac{1}{2}}s^{\kappa+N/2-\eta}.$$

Thus, we get

$$\big\|\mathfrak{D}^M_k(f)\chi_{B_s}\big\|_{L^p_k(S)}\leqslant|b|^{-\frac{p(N+2\kappa)}{2(p+1)}}\left(\frac{2M_k}{N+2\kappa-2\eta}\right)^{\frac{1}{2(p+1)}}s^{\frac{N+2\kappa-2\eta}{2(p+1)}}\big\||y|^{\eta}\mathfrak{D}^M_k(f)\big\|_2^{\frac{1}{p+1}}.$$

On the other hand, by (23), we have

$$\begin{aligned}\big\|\mathfrak{D}^M_k(f)\chi_{B_s^c}\big\|_{L^p_k(S)} &\leqslant \big\|\mathfrak{D}^M_k(f)\big\|_\infty^{\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^N}|\mathfrak{D}^M_k(f)(y)|^{\frac{2p}{p+1}}\chi_{B_s^c}(y)\chi_S(y)d\gamma_k(y)\right)^{\frac{1}{p}}\\ &\leqslant |b|^{-\frac{(p-1)(N+2\kappa)}{2(p+1)}}(\gamma_k(S))^{\frac{1}{p(p+1)}}\left(\int_{\mathbb{R}^N}|\mathfrak{D}^M_k(f)(y)|^2\chi_{B_s^c}(y)d\gamma_k(y)\right)^{\frac{1}{p+1}}\\ &\leqslant |b|^{-\frac{(p-1)(N+2\kappa)}{2(p+1)}}(\gamma_k(S))^{\frac{1}{p(p+1)}}\big\||y|^{\eta}\mathfrak{D}^M_k(f)\big\|_2^{\frac{2}{p+1}}s^{-\frac{2\eta}{p+1}}.\end{aligned}$$

Hence, for every $\eta \in (0, \kappa + N/2)$,

$$\left(\int_S |\mathfrak{D}_k^M(f)(y)|^p d\gamma_k(y)\right)^{\frac{1}{p}} \leq |b|^{-\frac{(p-1)(N+2\kappa)}{2(p+1)}} \left(\gamma_k(S)\right)^{\frac{1}{p(p+1)}} \left\| |y|^\eta \mathfrak{D}_k^M(f)\right\|_2^{\frac{1}{p+1}} s^{-\frac{\eta}{p+1}}$$
$$\times \left(\left(\frac{2M_k}{N+2\kappa-2\eta}\right)^{\frac{1}{2(p+1)}} s^{\frac{N+2\kappa}{2(p+1)}} |b|^{-\frac{N+2\kappa}{2(p+1)}} + \left\| |y|^\eta \mathfrak{D}_k^M(f)\right\|_2^{\frac{1}{p+1}} s^{-\frac{\eta}{p+1}}\right).$$

Choosing

$$s_0 = \left(\frac{N+2\kappa-2\eta}{2M_k}\right)^{\frac{1}{N+2\kappa+2\eta}} \left(\frac{2\eta}{N+2\kappa}\right)^{\frac{2(p+1)}{N+2\kappa+2\eta}} |b|^{\frac{N+2\kappa}{N+2\kappa+2\eta}} \left\| |y|^\eta \mathfrak{D}_k^M(f)\right\|_2^{\frac{2}{N+2\kappa+2\eta}}$$

we derive the result. □

Corollary 15. *If* $0 < \gamma_k(S) < \infty$, *then for* $f \in L_k^p(\mathbb{R}^N)$, $p \in (1,2]$, *we have*

$$\left\|\mathfrak{D}_k^M(\chi_S \mathfrak{D}_k^M(f))\right\|_{p'} \leq \begin{cases} \frac{C_1}{|b|^{(\kappa+N/2)(1-2/p')}} \left(\gamma_k(S)\right)^{\frac{\theta}{N+2\kappa}+2/p-1} \left\| |y|^\theta f\right\|_{p'}, & 0 < \theta < \frac{N+2\kappa}{p'}, \\ \frac{C_2}{|b|^{(\kappa+N/2)(1-2/p')}} \left(\gamma_k(S)\right)^{\frac{1}{p}} \|f\|_p^{1-\frac{N+2\kappa}{\theta p'}} \left\| |y|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, & \theta > \frac{N+2\kappa}{p'}, \\ \frac{C_3}{|b|^{(\kappa+N/2)(1-2/p')}} \left(\gamma_k(S)\right)^{\frac{3}{2p}-\frac{1}{2}} \|f\|_p^{\frac{1}{2}} \left\| |y|^\theta f\right\|_p^{\frac{1}{2}}, & \theta = \frac{N+2\kappa}{p'}, \end{cases}$$

where C_j, $j = 1,2,3$ *are the constants given in Theorem 16.*

Proof. By (28) we have

$$\begin{aligned} \left\|\mathfrak{D}_k^M(\chi_S \mathfrak{D}_k^M(f))\right\|_{p'} &\leq |b|^{(2/p'-1)(\kappa+N/2)} \left\|\chi_S \mathfrak{D}_k^M(f)\right\|_p \\ &\leq |b|^{(2/p'-1)(\kappa+N/2)} \left(\gamma_k(S)\right)^{2/p-1} \left\|\chi_S \mathfrak{D}_k^M(f)\right\|_{p'}. \end{aligned}$$

Theorem 16 allows to conclude. □

Proposition 3. *If* $0 < \gamma_k(S), \gamma_k(\Sigma) < \infty$, *then for* $f \in B^p(S)$, $p \in (1,2]$, *we have*

$$\left\|\chi_\Sigma f\right\|_p \leq \begin{cases} \frac{C_1}{|b|^{\kappa+N/2}} \left(\gamma_k(\Sigma)\right)^{\frac{1}{p}} \left(\gamma_k(S)\right)^{\frac{\theta}{N+2\kappa}+\frac{1}{p}} \left\| |y|^\theta f\right\|_{p'}, & 0 < \theta < \frac{N+2\kappa}{p'}, \\ \frac{C_2}{|b|^{\kappa+N/2}} \left(\gamma_k(\Sigma)\right)^{\frac{1}{p}} \gamma_k(S) \|f\|_p^{1-\frac{N+2\kappa}{\theta p'}} \left\| |y|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, & \theta > \frac{N+2\kappa}{p'}, \\ \frac{C_3}{|b|^{\kappa+N/2}} \left(\gamma_k(\Sigma)\right)^{\frac{1}{p}} \left(\gamma_k(S)\right)^{\frac{1}{2p}+\frac{1}{2}} \left(\|f\|_p^{\frac{1}{2}} \left\| |y|^\theta f\right\|_p^{\frac{1}{2}}\right), & \theta = \frac{N+2\kappa}{p'}, \end{cases}$$

where C_j, $j = 1,2,3$ *are the constants given by Theorem 16.*

Proof. For $f \in B^p(S)$, $p \in (1,2]$ we have

$$f = \mathfrak{D}_k^{M^{-1}} \circ \mathfrak{D}_k^M(f).$$

By (23),

$$|f| \leq \frac{1}{|b|^{\kappa+N/2}} \left(\gamma_k(S)\right)^{\frac{1}{p}} \left\|\chi_S \mathfrak{D}_k^M(f)\right\|_{p'}.$$

Therefore,

$$\left\|\chi_\Sigma f\right\|_p \leq \frac{1}{|b|^{N/2+\kappa}}\left(\gamma_k(\Sigma)\right)^{\frac{1}{p}}\left(\gamma_k(S)\right)^{\frac{1}{p}}\left\|\chi_S \mathfrak{D}_k^M(f)\right\|_{p'},$$

which implies the desired inequality from Theorem 16. □

Proposition 4. *Let* $0 < \theta < \frac{N+2\kappa}{p'}$, $p \in (1,2]$. *If* $f \in B^p(S)$ *is* ε_Σ*-concentrated on* Σ *for the* $L_k^p(\mathbb{R}^N)$*-norm, then*

$$\left\|f\right\|_p \leq \begin{cases} \frac{C_1}{(1-\varepsilon_\Sigma)|b|^{\kappa+N/2}}\left(\gamma_k(\Sigma)\right)^{\frac{1}{p}}\left(\gamma_k(S)\right)^{\frac{\theta}{N+2\kappa}+\frac{1}{p}}\left\|\,|y|^\theta f\right\|_p, & 0<\theta<\frac{N+2\kappa}{p'}, \\ \frac{C_2}{(1-\varepsilon_\Sigma)|b|^{\kappa+N/2}}\left(\gamma_k(\Sigma)\right)^{\frac{1}{p}}\gamma_k(S)\left\|f\right\|_p^{1-\frac{N+2\kappa}{\theta p'}}\left\|\,|y|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, & \theta>\frac{N+2\kappa}{p'}, \\ \frac{C_3}{(1-\varepsilon_\Sigma)|b|^{\kappa+N/2}}\left(\gamma_k(\Sigma)\right)^{\frac{1}{p}}\left(\gamma_k(S)\right)^{\frac{1}{2p}+\frac{1}{2}}\left\|f\right\|_p^{\frac{1}{2}}\left\|\,|y|^\theta f\right\|_p^{\frac{1}{2}}, & \theta=\frac{N+2\kappa}{p'}. \end{cases}$$

Proof. Using the fact that f is ε_Σ-concentrated on Σ, and Theorem 3, we obtain the result. □

In a similar way we have the following results.

Proposition 5. *If* $0 < \gamma_k(\Sigma) < \infty$, *then for all* $f \in L_k^p(\mathbb{R}^N)$, $p \in (1,2]$, *such that* $\mathfrak{D}_k^M(f)$ *is* ε_Σ*-concentrated on* Σ *for the* $L_k^{p'}(\mathbb{R}^N)$, *we have*

$$\left\|\mathfrak{D}_k^M(f)\right\|_{p'} \leq \begin{cases} \frac{C_1}{1-\varepsilon_\Sigma}\left(\gamma_k(\Sigma)\right)^{\frac{\theta}{N+2\kappa}}\left\|\,|y|^\theta f\right\|_p, & 0<\theta<\frac{N+2\kappa}{p'}, \\ \frac{C_2}{1-\varepsilon_\Sigma}\left(\gamma_k(\Sigma)\right)^{\frac{1}{p'}}\left\|f\right\|_p^{1-\frac{N+2\kappa}{\theta p'}}\left\|\,|y|^\theta f\right\|_p^{\frac{N+2\kappa}{\theta p'}}, & \theta>\frac{N+2\kappa}{p'}, \\ \frac{C_3}{1-\varepsilon_\Sigma}\left(\gamma_k(\Sigma)\right)^{\frac{1}{2p'}}\left\|f\right\|_p^{\frac{1}{2}}\left\|\,|y|^\theta f\right\|_p^{\frac{1}{2}}, & \theta=\frac{N+2\kappa}{p'}. \end{cases}$$

3. Conclusions and Perspectives

In the present paper, we proved some quantitative uncertainty principles for functions in the L_k^p-spaces, comparing time or frequency dispersions, the supports or the essential supports of these functions. All these uncertainty inequalities show the impossibility of a function and its LCDT to be both well concentrated in the time-frequency plane.

In future work, we will study some qualitative uncertainty principles for the LCDT, including in particular Hardy and Beurling uncertainty principles. Moreover, we will extend some results proved in the recent paper [63] to arbitrary dimension.

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