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Impact of Ambient Conformal Vector Fields on Yamabe Solitons on Riemannian Hypersurfaces

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Abstract

We investigate Yamabe solitons on Riemannian hypersurfaces induced by conformal vector fields in Riemannian and Lorentzian manifolds, with an emphasis on the tangential component. We show that these hypersurfaces are totally umbilical, and when the ambient manifold is Einstein, a rigidity condition emerges connecting the mean and scalar curvatures. Using this, we classify compact Yamabe solitons: each hypersurface is either totally geodesic or an extrinsic sphere. Additionally, we prove the non-existence of trivial Yamabe solitons on oriented hypersurfaces of higher dimension in Einstein manifolds. These results highlight the classification of compact hypersurfaces and rigidity phenomena in the ambient spaces, providing a clear understanding of the geometric structures associated with Yamabe solitons.

Keywords: Yamabe solitons; conformal vector fields; scalar curvature

MSC: 53A10; 53C40; 53C42; 53C50; 53C65

1. Introduction

Ricci solitons and Yamabe solitons arise as self-similar solutions to two fundamental geometric flows: the Ricci flow and the Yamabe flow. A Ricci soliton satisfies

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X h = \mu h, \tag{1}$$

where Ric is the Ricci tensor, and $\mathcal{L}_X h$ denotes the Lie derivative of the metric along a vector field X. Ricci solitons are central to the analysis of the Ricci flow, particularly in the study of singularities and the proof of the Poincaré conjecture.

In parallel, the Yamabe problem, a cornerstone in differential geometry, seeks to determine a metric conformal to a given Riemannian metric such that the scalar curvature becomes constant. This problem was fully resolved by R. Schoen, marking a major milestone in the field [1]. Building on this foundation, R. Hamilton introduced the Yamabe flow, a geometric evolution equation that deforms metrics over time toward constant scalar curvature [2,3]. A particularly important class of solutions to this flow are the so-called Yamabe solitons, which reflect self-similar behavior under the flow and are governed by the equation

$$\frac{1}{2}L_X h = (\rho - \mu)h, \tag{2}$$

where ρ is the scalar curvature, and μ is a real constant.

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Although Ricci and Yamabe solitons are governed by different curvature quantities—the full Ricci tensor versus the scalar curvature—both reflect self-similar evolution under their respective flows. In this work, we focus on Yamabe solitons on Riemannian and spacelike hypersurfaces embedded in Riemannian or Lorentzian manifolds, with particular emphasis on the influence of conformal vector fields in the ambient manifolds.

Over the past two decades, substantial effort has been devoted to understanding Yamabe solitons under various geometric conditions. For compact gradient Yamabe solitons, the scalar curvature is known to be constant, leading to trivial potential functions [4,5].

Moreover, specific classes of solutions have been studied in locally conformally flat settings, and in cases with positive sectional curvature or symmetry constraints [4]. Under some conditions, Yamabe solitons have been studied (cf. [6–9]).

In contrast to Ricci solitons, which coincide with Yamabe solitons in two dimensions, the higher-dimensional setting reveals structural differences that warrant independent investigation (see [10–13]).

The present work extends the study of Yamabe solitons to Riemannian and spacelike hypersurfaces embedded in Riemannian or Lorentzian manifolds, with particular emphasis on the influence of conformal vector fields in the ambient geometry. We derive geometric conditions under which such hypersurfaces admit a Yamabe soliton structure, focusing especially on the role of the tangential component of the conformal field. Our analysis also covers the case where the ambient manifold is Einstein, leading to a full characterization of totally umbilical hypersurfaces and relations between intrinsic and extrinsic curvature. For compact hypersurfaces, we further show that Yamabe solitons must be either totally geodesic or extrinsic spheres under suitable curvature constraints.

Section 2 provides key concepts and fundamental formulas related to Yamabe solitons and Riemannian (or spacelike) hypersurfaces within these manifolds.

Section 4 decisively establishes results regarding Yamabe solitons on Riemannian manifolds. We show that there are no steady Yamabe solitons on compact Riemannian manifolds with dimension $n \geq 3$ and nonzero scalar curvature. Furthermore, we provide a succinct proof of Theorem 1 in [5]. For $n \geq 3$, it is clear that there are no nontrivial Yamabe solitons on compact n-dimensional Riemannian manifolds. Additionally, trivial Yamabe solitons $(M, h, \mathcal{X}, \mu)$ cannot occur on an n-dimensional Riemannian manifold (M, h) with $n \geq 3$ where $Ric(\mathcal{X}, \mathcal{X}) \geq 0$.

Section 4, we examine Yamabe solitons on Riemannian hypersurfaces induced by a conformal vector field in either Riemannian or Lorentzian manifolds, with particular attention to the tangential component. Subsequently, we show that the hypersurface becomes totally umbilical. The analysis extends to the case where the ambient manifold is an Einstein. In such instances, we get a relation between the scalar curvature and the mean curvature of these hypersurfaces. In the case of a compact Yamabe soliton, the hypersurface is a totally geodesic or an extrinsic sphere. Also, we prove there is no trivial Yamabe soliton on an oriented Riemannian hypersurface of dimension $n \geq 3$ induced by a conformal vector field in the Einstein Riemannian or Lorentzian manifold with $\widetilde{Ric} = \widetilde{c}\gamma$, and $\widetilde{c} + \epsilon(n-1)\mathbb{H}^2 < 0$, where $\mathbb{H}$ is the mean curvature.

2. Preliminaries

In this section, we recall the necessary geometric background and fix the notation used throughout the paper, following the frameworks of [14,15].

Let (M, h) be an n-dimensional Riemannian manifold, and let ∇ be the Levi-Civita connection associated with h. The Riemann curvature tensor is defined for any $X, Y, Z \in \mathfrak{X}(M)$ by

$$\mathcal{R}(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z. \tag{3}$$

A Yamabe soliton is a quadruple $(M, h, \mathcal{X}, \mu)$, where $\mathcal{X}$ is a smooth vector field (called the soliton potential vector field) and $\mu \in \mathbb{R}$, satisfying (2), this means that $\mathcal{X}$ is a conformal vector field with conformal factor $\rho - \mu$. Depending on the sign of μ, the soliton is referred to as shrinking $(\mu > 0)$, steady $(\mu = 0)$, or expanding $(\mu < 0)$; see [2,16].

When the soliton potential vector field $\mathcal{X}$ is a Killing vector field—indicating that $\rho = \mu$—we classify the structure $(M, h, \mathcal{X}, \mu)$ as trivial. Furthermore, we consider $(M, h, \mathcal{X}, \mu)$ to be parallel if its potential vector field $\mathcal{X}$ is parallel.

If the vector field $\mathcal{X}$ arises from the gradient of a smooth function f, then the soliton is of gradient type and satisfies

$$\mathrm{Hess}_f = (\rho - \mu)h, \tag{4}$$

where Hess_f is the Hessian of f. Taking the trace of the soliton equation yields the relation

$$div(\mathcal{X}) = n(\rho - \mu), \tag{5}$$

and in the gradient case this is as follows:

$$\Delta f = n(\rho - \mu), \tag{6}$$

where Δ is the Laplace operator. These properties are discussed in [4].

To study Yamabe solitons on hypersurfaces, we recall some fundamental tools from submanifold theory. Let (M, h) be an oriented Riemannian or spacelike hypersurface in an ambient manifold $(\widetilde{M}, \gamma)$, which may be either Riemannian or Lorentzian. Let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections on M and $\widetilde{M}$, respectively, and let $\mathbb{U}$ be a unit normal vector field (chosen timelike if γ is Lorentzian).

The Gauss and Weingarten formulas are given by [14]:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \epsilon h(\mathcal{S}(X), Y)\mathbb{U}, \tag{7}$$

$$\mathcal{S}(X) = -\widetilde{\nabla}_X \mathbb{U}, \tag{8}$$

where $\mathcal{S}$ is the shape operator and $\varepsilon = \gamma(\mathbb{U}, \mathbb{U}) = \pm 1$. The mean curvature $\mathbb{H}$ of M is defined as

$$\mathbb{H} = \frac{\epsilon}{n} trace(\mathcal{S}). \tag{9}$$

The Gauss equation expresses the curvature tensor of M in terms of the ambient curvature and the shape operator:

$$(\mathcal{R}(X,Y)Z = \left(\widetilde{\mathcal{R}}(X,Y)Z\right)^{\top} + \epsilon(h(\mathcal{S}(Y), Z)\mathcal{S}(X) - h(\mathcal{S}(X), Z)\mathcal{S}(Y)), \tag{10}$$

and

$$Ric(X,Y) = \widetilde{Ric}(X,Y) - \epsilon\gamma(\widetilde{\mathcal{R}}(\mathbb{U}, X)Y, \mathbb{U}) + h(\mathcal{S}(X), n\mathbb{H}Y - \epsilon\mathcal{S}(Y)), \tag{11}$$

for all $X, Y \in \mathfrak{X}(M)$. So, the scalar curvature ρ of M satisfies the following:

$$\rho = \widetilde{\rho} - 2\epsilon\widetilde{Ric}(\mathbb{U}, \mathbb{U}) + \epsilon(n^2\mathbb{H}^2 - |\mathcal{S}|^2), \tag{12}$$

where $\|\mathcal{S}\|^2 = \mathrm{trace}(\mathcal{S}^2)$, and $\widetilde{\rho}$, $\widetilde{\mathrm{Ric}}$ are the scalar and Ricci curvatures of $(\widetilde{M}, \gamma)$, respectively.

3. Yamabe Solitons on Riemannian Manifolds

This section presents several findings regarding Yamabe solitons on Riemannian manifolds, along with various helpful formulas that facilitate deriving key results.

As stated in [17], formula 1.13 (see also [18]), for any conformal vector field $\mathcal{X}$ on an n-dimensional manifold (M, h) with a conformal function ψ, we have

$$\mathcal{X}(\rho) = 2(n-1)\Delta\psi - 2\psi\rho. \tag{13}$$

where $\Delta\psi$ is the Laplacian of ψ and ρ is the scalar curvature of M.

Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton, where (M, h) is an n-dimensional Riemannian manifold (M, h) and where $n \geq 3$. We know that the vector field $\mathcal{X}$ is conformal with the conformal function $\rho - \mu$. So, (13) yields

$$g(\nabla\rho, \mathcal{X}) = -2(n-1)\Delta\rho - 2(\rho-\mu)\rho. \tag{14}$$

On the other hand, by (5), we have

$$div(\rho\mathcal{X}) = n\rho(\rho-\mu) + h(\nabla\rho, \mathcal{X}). \tag{15}$$

From (14) and (15), it follows that

$$2(n-1)\Delta\rho + \frac{2}{n}div(\rho X) + \frac{n-2}{n}h(\nabla\rho, X) = 0. \tag{16}$$

Now, if M is compact, then by integration (16), we obtain

$$\int_M h(\nabla\rho, \mathcal{X})dv = 0. \tag{17}$$

By integrating (14) and taking into account (17), we get

$$\int_M (\rho-\mu)\rho dv = 0. \tag{18}$$

We can deduce an important fact from (18).

Theorem 1. *There is no steady Yamabe soliton in a compact Riemannian manifold with dimension $n \geq 3$ and nonzero scalar curvature.*

Remark 1. *Consider now the quadruple $(M, h, \mathcal{X}, \mu)$ representing a shrinking or expanding Yamabe soliton on a compact Riemannian manifold (M, h) with condition $0 < \rho < \mu$ for the shrinking case and $\mu < \rho < 0$ for the expanding case. According to Equation (14), ρ equals μ, and in this case, $\mathcal{X}$ is the Killing vector field.*

A brief proof of the following result was given in [5]. We will now offer an even shorter proof.

Theorem 2. *Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton on a compact n-dimensional Riemannian manifold (M, h), with $n \geq 3$. Then, $(M, h, \mathcal{X}, \mu)$ is trivial with $\rho = \mu$.*

Proof. Since $(\rho-\mu)^2 = \rho(\rho-\mu) - \mu(\rho-\mu)$, then using (5) and (18), we deduce that

$$\int_M (\rho-\mu)^2 dv = 0,$$

it follows that $\rho = \mu$, which implies that $\mathcal{X}$ is a Killing vector field. □

To illustrate the sharpness of the derived conditions, one can readily construct counterexamples. The most crucial assumption is the compactness of the hypersurface.

One can find examples of nontrivial Yamabe solitons on noncompact manifolds with $\rho \neq \mu$. A simple example is provided by Euclidean space $\mathbb{R}^n$ equipped with the position vector field X and $\mu = -1$. In this case, the scalar curvature is $\rho = 0$, so clearly this gives an example of a Yamabe soliton that is gradient and nontrivial.

Let $(\mathbb{R}^n, g_0)$ be Euclidean space with

$$g_0 = \sum_{i=1}^{n} dx_i^2,$$

then $\rho \equiv 0$. Define the position vector field

$$X = \sum_{i=1}^{n} x_i \, \partial x_i.$$

It is well-known that $\nabla_Y X = Y$ for any vector Y, hence

$$L_X g_0 = 2\, g_0 \quad \Longrightarrow \quad \tfrac{1}{2} L_X g_0 = g_0.$$

Substituting into (2) with $\rho = 0$ yields

$$g_0 = (0 - \mu)\, g_0 \quad \Longrightarrow \quad \mu = -1.$$

Therefore $(\mathbb{R}^n, g_0, X, \mu = -1)$ is a nontrivial Yamabe soliton. In the gradient formulation, set

$$f(x) = \tfrac{1}{2} |x|^2.$$

Then $\nabla f = X$ and $\mathrm{Hess}_f = g_0$. Since $\rho = 0$ and $\mu = -1$, (4) becomes

$$\mathrm{Hess}_f = (0 - (-1))\, g_0 = g_0,$$

which holds. Thus, the soliton is gradient and nontrivial (since $X \not\equiv 0$).

This example clearly shows that the compactness assumption in the compact case theorem cannot be dropped: in the noncompact setting (like $\mathbb{R}^n$), one can construct a nontrivial Yamabe soliton with $\rho = 0 \neq \mu = -1$, demonstrating the necessity of compactness for the rigidity result.

Corollary 1. *For $n \geq 3$, there is no nontrivial Yamabe soliton on a compact n-dimensional Riemannian manifold.*

Returning to Formula (2) and according to [19], the decomposition of $\nabla_X \mathcal{X}$ and both its symmetric component and antisymmetric component ϕ are presented as

$$L_{\mathcal{X}} h(X, Y) + d\theta_{\mathcal{X}}(X, Y) = 2h(\nabla_X \mathcal{X}, Y), \tag{19}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\theta_{\mathcal{X}}$ is the one-form dual to $\mathcal{X}$; that is, $\theta_{\mathcal{X}}(X) = h(X, \mathcal{X})$, $X \in \mathfrak{X}(M)$. Define a skew-symmetric tensor field ϕ of type $(1,1)$ on M by

$$d\theta_{\mathcal{X}}(X, Y) = 2h(\phi(X), Y), \tag{20}$$

for all $X, Y \in \mathfrak{X}(M)$. From (2) and (20), we have

$$\nabla_X \mathcal{X} = (\rho - \mu)X + \phi(X), \tag{21}$$

for all $X \in \mathfrak{X}(M)$. We see that $\phi = 0$ if and only if $\mathcal{X}$ is a closed conformal vector field. In this case, we will say that $\mathcal{X}$ is closed.

From (3), we obtain

$$\mathcal{R}(X,Y)\mathcal{X} = Y(\rho)X - X(\rho)Y + \nabla_Y(\phi(X)) - \phi(\nabla_Y X) - \nabla_X(\phi(Y)) + \phi(\nabla_X Y), \quad (22)$$

for all $X, Y \in \mathfrak{X}(M)$.

Now, for a Yamabe soliton on a Riemannian manifold, we provide a very useful formula.

Lemma 1. *Let (M, h, X, μ) be a Yamabe soliton on an n-dimensional Riemannian manifold (M, h). Then,*

$$Ric(\mathcal{X}, \mathcal{X}) = (1-n)h(\nabla\rho, \mathcal{X}) + |\phi|^2, \quad (23)$$

where ρ is the scalar curvature and ϕ is a skew-symmetric tensor field on M.

Proof. Let $\{E_1, \ldots, E_n\}$ be a parallel local orthonormal frame on M. Considering Equation (22), we obtain the following.

$$\begin{aligned}
Ric(\mathcal{X}, \mathcal{X}) &= \sum_{i=1}^{n} h(R(\mathcal{X}, E_i)\mathcal{X}, E_i) \\
&= \sum_{i=1}^{n} (h(E_i(\rho)\mathcal{X}, E_i) - h(\mathcal{X}(\rho)E_i, E_i) \\
&+ h(\nabla_{Ei}(\phi(\mathcal{X})), E_i) - h(\phi(\nabla_{Ei}\mathcal{X}), E_i)) \\
&= \sum_{i=1}^{n} (h(\nabla\rho, E_i)h(\mathcal{X}, E_i) - h(\nabla\rho, \mathcal{X})h(E_i, E_i) + h(\nabla_{Ei}\mathcal{X}, \phi(E_i))) \\
&= h(\nabla\rho, \sum_{i=1}^{n} h(\mathcal{X}, E_i)E_i) - h(\nabla\rho, \mathcal{X}) \sum_{i=1}^{n} h(E_i, E_i) \\
&- (\rho - \mu) \sum_{i=1}^{n} h(E_i, \phi(E_i)) + \sum_{i=1}^{n} h(\phi(E_i), \phi(E_i)) \\
&= h(\nabla\rho, \mathcal{X}) - nh(\nabla\rho, \mathcal{X}) + ||\phi||^2 \\
&= (1-n)h(\nabla\rho, \mathcal{X}) + ||\phi||^2,
\end{aligned}$$

where $||\phi||^2 = \sum_{i=1}^{n} h(\phi(E_i), \phi(E_i))$. □

Theorem 3. *Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton on an n-dimensional Riemannian manifold (M, h) with $n \geq 3$. If the scalar curvature ρ of (M, h) is constant along the integral curves of $\mathcal{X}$, then $Ric(\mathcal{X}, \mathcal{X}) \geq 0$, with equality holding if and only if $\mathcal{X}$ is closed.*

Proof. Using Lemma 3, we see that if ρ is constant, then $Ric(\mathcal{X}, \mathcal{X}) = ||\phi||^2$, which means that $Ric(\mathcal{X}, \mathcal{X}) \geq 0$, with equality holding if and only if $\phi = 0$, that is, $\mathcal{X}$ is closed. □

Corollary 2. *There is no trivial Yamabe soliton $(M, h, \mathcal{X}, \mu)$ on an n-dimensional Riemannian manifold (M, h) with $n \geq 3$ and $Ric(\mathcal{X}, \mathcal{X}) < 0$.*

If we define a function f on M by $f = \frac{1}{2}h(\mathcal{X}, \mathcal{X})$, we have

$$h(\nabla f, Y) = \frac{1}{2}Y \cdot h(\mathcal{X}, \mathcal{X}) = h(\nabla_Y \mathcal{X}, \mathcal{X}), \quad (24)$$

for all $Y \in \mathfrak{X}(M)$. It follows that

$$X \cdot h(\nabla f, Y) = X \cdot h(\nabla_Y \mathcal{X}, \mathcal{X}),$$

that is

$$h(\nabla_X \nabla f, Y) + h(\nabla f, \nabla_X Y) = h(\nabla_X \nabla_Y \mathcal{X}, \mathcal{X}) + h(\nabla_Y \mathcal{X}, \nabla_X \mathcal{X}).$$

Now, using (24), we obtain the following:

$$\mathrm{Hess}_f(X, Y) = (\rho - \mu)^2 h(X, Y) + X(\rho) h(Y, \mathcal{X}) + h(\phi(X), \phi(Y)), \tag{25}$$

Theorem 4. *Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton on an n-dimensional Riemannian manifold (M, h). If M is compact, then $\mathrm{Hess}_f(\mathcal{X}, \mathcal{X}) \geq 0$, with equality holding if and only if $\mathcal{X}$ is closed.*

Corollary 3. *There is no trivial Yamabe soliton $(M, h, \mathcal{X}, \mu)$ on a Riemannian manifold of dimension (M, h) with $n \geq 3$ with $\mathrm{Hess}_f(\mathcal{X}, \mathcal{X}) < 0$.*

Lemma 2. *Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton on an n-dimensional Riemannian manifold (M, h). Then,*

$$\Delta f = n(\rho - \mu)^2 + h(\nabla \rho, \mathcal{X}) + ||\phi||^2. \tag{26}$$

where ρ is the scalar curvature and ϕ is a skew-symmetric tensor field on M.

Proof. By taking the trace of (25) with respect to an orthonormal frame $\{E_1, \ldots, E_n\}$ on M, using the skew-symmetry of ϕ, and since $\Delta f = trace(\mathrm{Hess}_f)$, we deduce that

$$\begin{aligned} \Delta f &= (\rho - \mu)^2 \sum_{i=1}^{n} h(E_i, E_i) + \sum_{i=1}^{n} E_i(\rho) h(E_i, \mathcal{X}) + \sum_{i=1}^{n} h(\phi(E_i), \phi(E_i)) \\ &= n(\rho - \mu)^2 + h(\nabla \rho, \mathcal{X}) + ||\phi||^2. \end{aligned}$$

□

Theorem 5. *Let $(M, h, \mathcal{X}, \mu)$ be a Yamabe soliton on an n-dimensional Riemannian manifold (M, h), with $n \geq 3$. If M is compact, then $\mathcal{X}$ is of constant length.*

Proof. By Theorem 2 and (26), we have $\Delta f = ||\phi||^2$. By integrating this equation on M. We deduce that $\phi = 0$, meaning that $\mathcal{X}$ is closed at a constant length. □

4. Yamabe Solitons on Riemannian Hypersurfaces

In this section, we focus on the n-dimensional oriented Riemannian (or spacelike) hypersurface (M, h), which is embedded in a Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$ for $n \geq 3$.

We propose that $(\widetilde{M}, \gamma)$ has a nonzero conformal vector field $\widetilde{\mathcal{X}}$. This implies that the Lie derivative of the metric γ with respect to $\widetilde{\mathcal{X}}$ is given by the following:

$$L_{\widetilde{\mathcal{X}}} \gamma = 2\psi\gamma, \tag{27}$$

where ψ is a smooth function on $\widetilde{M}$. When $\left(\widetilde{M}, \gamma\right)$ is Lorentzian, we consider $\widetilde{\mathcal{X}}$ as timelike.

Select a unit normal vector field $\mathbb{U}$ on M. For Lorentzian $\left(\widetilde{M}, \gamma\right)$, choose $\mathbb{U}$ as a timelike vector field where $\theta = \gamma(\mathcal{X}, \mathbb{U}) < 0$. The restriction $\mathcal{X}$ of $\widetilde{\mathcal{X}}$ to M is expressed as follows:

$$\mathcal{X} = \mathcal{X}_M + \epsilon\theta\mathbb{U}, \tag{28}$$

where $\mathcal{X}_M$ the tangential component of $\mathcal{X}$ to M, and $\epsilon = \gamma(\mathbb{U}, \mathbb{U}) = \pm 1$.

The form $\theta_{\widetilde{\mathcal{X}}}$ is defined as the dual of $\widetilde{\mathcal{X}}$, meaning $\theta_{\widetilde{\mathcal{X}}}(X) = \gamma(X, \widetilde{\mathcal{X}})$, where X belongs to $\mathfrak{X}(\widetilde{M})$. So, according to [19] and (27), the decomposition of $\widetilde{\nabla}\widetilde{\mathcal{X}}$ into its symmetric component and antisymmetric component $\widetilde{\phi}$ is presented as

$$L_{\widetilde{\mathcal{X}}}\gamma(X, Y) = 2\gamma(\psi(X), Y), \tag{29}$$

and

$$d\theta_{\widetilde{\mathcal{X}}}(X, Y) = 2\gamma((\widetilde{\phi}(X), Y), \tag{30}$$

for all $X, Y \in \mathfrak{X}(\widetilde{M})$. From (27) and (30), we have

$$\widetilde{\nabla}_X\widetilde{\mathcal{X}} = \psi X + \widetilde{\phi}(X), \tag{31}$$

for all $X \in \mathfrak{X}(\widetilde{M})$.

Because $\mathcal{X} \in \mathfrak{X}(M)$, it follows that $\widetilde{\nabla}\mathcal{X}$ is expressible as

$$\widetilde{\nabla}_X\mathcal{X} = (\widetilde{\nabla}_X\mathcal{X})_M + \epsilon\alpha(X)\mathbb{U}, \tag{32}$$

where $(\widetilde{\nabla}_X\mathcal{X})_M$ is the tangential component of $\widetilde{\nabla}_X\mathcal{X}$ on M, and α is a one-form on M, specifically, $\alpha(X) = h(\eta, X)$ for some $\eta \in \mathfrak{X}(M)$. Consequently, we obtain

$$\begin{aligned}\alpha(X) &= \gamma(\widetilde{\nabla}_X\mathcal{X}, \mathbb{U}) \\ &= \gamma(\psi X + \widetilde{\phi}(X), \mathbb{U}) \\ &= \psi\gamma(X, \mathbb{U}) + \gamma(\widetilde{\phi}(X), \mathbb{U}) \\ &= -\gamma(X, \widetilde{\phi}(\mathbb{U})),\end{aligned}$$

for all $X \in \mathfrak{X}(M)$. Because $\widetilde{\phi}(\mathbb{U}) \in \mathfrak{X}(M)$, we conclude that $\eta = -\widetilde{\phi}(\mathbb{U})$.

Using Equation (28) for all $X \in \mathfrak{X}(M)$, we derive the following:

$$\widetilde{\nabla}_X\mathcal{X} = \widetilde{\nabla}_X(\mathcal{X}_M + \epsilon\theta\mathbb{U}).$$

By applying (7) and (8), and then comparing the results with Equation (32), we can obtain the following through direct calculations, achieved by equating the tangent and normal components on both sides of the equation:

$$\nabla_X\mathcal{X}_M = (\widetilde{\nabla}_X\mathcal{X})_M + \epsilon\theta\mathcal{S}(X), \tag{33}$$

and

$$\nabla\theta = \eta - \mathcal{S}(\mathcal{X}_M), \tag{34}$$

where $\mathcal{S}$ is the shape operator of M as a hypersurface on $\widetilde{M}$.

Also, since $\mathcal{X}$ is a conformal vector field, it is straightforward to derive

$$\mathrm{div}(\mathcal{X}_M) = n(\psi + \theta\mathbb{H}), \tag{35}$$

where $\mathbb{H}$ is the mean curvature of M.

A Yamabe soliton $(M, h, \mathcal{X}_M, \mu)$ on the hypersurface (M, h) is referred to as a Yamabe soliton on a Riemannian (or spacelike) hypersurface (M, h) induced by a conformal vector field $\widetilde{\mathcal{X}}$ within the Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$.

We shall now examine Yamabe solitons on Riemannian (or spacelike) hypersurfaces induced by a conformal vector field in either Riemannian (or Lorentzian) manifolds, with

particular attention to the tangential component. Subsequently, we will characterize these Yamabe solitons.

Theorem 6. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$, induced by a conformal vector field $\widetilde{\mathcal{X}}$ in either a Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$ ($\widetilde{\mathcal{X}}$ is timelike if $\widetilde{M}$ is Lorentzian). Then (M, h) is totally umbilical in $(\widetilde{M}, \gamma)$.*

Proof. Since for all $X \in \mathfrak{X}(M)$, we can write $\mathcal{X}$ as

$$(\widetilde{\nabla}_X \mathcal{X})_M = \psi X + \widetilde{\phi}(X)_M. \tag{36}$$

Also, by the property of the skew-symmetric of $(\widetilde{\nabla}\mathcal{X})_M$, we get

$$h(\widetilde{\phi}(X)_M, Y) + h(X, \widetilde{\phi}(Y)_M) = 0, \tag{37}$$

for any vector field $X, Y \in \mathfrak{X}(M)$. By using (33), (36) and (37), we obtain

$$\begin{aligned}(L_{\mathcal{X}_M})h(X,Y) &= h(\nabla_X \mathcal{X}_M, Y) + h(X, \nabla_Y \mathcal{X}_M)\\ &= h((\widetilde{\nabla}_Y \mathcal{X})_M, Y) + h(X, (\widetilde{\nabla}_Y \mathcal{X})_M) + 2\epsilon\theta h(\mathcal{S}(X), Y)\\ &= 2\psi h(X,Y) + 2\epsilon\theta h(\mathcal{S}(X), Y),\end{aligned}$$

for any pair of vector fields $X, Y \in \mathfrak{X}(M)$. It means that

$$\frac{1}{2}(L_{\mathcal{X}_M})h(X,Y) = \psi h(X,Y) + \epsilon\theta h(\mathcal{S}(X), Y). \tag{38}$$

Assume that $(M, h, \mathcal{X}_M, \mu)$ is a Yamabe soliton, satisfying the equation:

$$\frac{1}{2}L_{\mathcal{X}_M}h = (\rho - \mu)h, \tag{39}$$

where ρ is the scalar curvature on (M, h). Since

$$trace(\frac{1}{2}L_{\mathcal{X}_M}h) = div(\mathcal{X}_M),$$

then by taking the trace of (39) and then substituting into (35), it follows that

$$\rho - \mu = \psi + \theta\mathbb{H}. \tag{40}$$

Incorporating (38) and (40) into (39) yields

$$\mathcal{S} = \epsilon\mathbb{H}I, \tag{41}$$

where I represents the identity operator. Consequently, (M, h) is shown to be totally umbilical in $(\widetilde{M}, \gamma)$. □

Theorem 7. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an n-dimensional oriented minimal (or maximal) compact Riemannian hypersurface (M, h) induced by a conformal vector field $\widetilde{\mathcal{X}}$ in either a Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$ ($\widetilde{\mathcal{X}}$ is timelike if $\widetilde{M}$ is Lorentzian), with $n \geq 3$. Then $\mathcal{X}$ is a parallel vector field.*

Proof. Since (M, h) is compact, by Theorem 2, $\rho = \mu$, so $\psi = 0$. □

Corollary 4. *Let $\mathcal{X}$ not be parallel. Then, there exist no nontrivial compact Yamabe solitons on a minimal (or maximal) hypersurface on the Riemannian (or Lorentzian) $(\widetilde{M}, \gamma)$.*

Next, we will study Yamabe solitons $(M, h, \mathcal{X}_M, \mu)$ on the Riemannian (or spacelike) hypersurface (M, h) induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$.

Theorem 8. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. The soliton satisfies the following equation:*

$$\rho = (n-1)(\widetilde{c} + \epsilon n \mathbb{H}^2), \tag{42}$$

where ρ denotes the scalar curvature and $\mathbb{H}$ represents the mean curvature of (M, h).

Proof. Given that $(\widetilde{M}, \gamma)$ is an Einstein manifold with $\widetilde{Ric} = \widetilde{c}\gamma$, it follows that $\widetilde{\rho} = (n+1)\widetilde{c}$ and $\widetilde{Ric}(N, N) = \epsilon\widetilde{c}$. From Theorem 6, the shape operator $\mathcal{S} = \epsilon\mathbb{H}I$ implies $|\mathcal{S}|^2 = n\mathbb{H}^2$. Substituting this into (14), we obtain the following:

$$\rho = (n-1)(\widetilde{c} + \epsilon n \mathbb{H}^2). \tag{43}$$

□

Corollary 5. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. If $(M, h, \mathcal{X}_M, \mu)$ is a compact Yamabe soliton, it is a totally geodesic or an extrinsic sphere.*

Proof. From Theorem 1, we derive $\rho = \mu$, indicating that ρ is constant. Consequently, $\mathbb{H}$ remains constant by (42). Should $\mathbb{H} = 0$, Theorem 6 implies $\mathcal{S} = 0$, meaning (M, h) is totally geodesic. Conversely, for $\mathbb{H} \neq 0$, (M, h) is an umbilical with a nonzero mean curvature, hence an extrinsic sphere. □

Lemma 3. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. Then,*

$$Ric(\mathcal{X}_M, \mathcal{X}_M) = \left(\widetilde{c} + \epsilon(n-1)\mathbb{H}^2\right) h(\mathcal{X}_M, \mathcal{X}_M), \tag{44}$$

where $\mathbb{H}$ is the mean curvature of M.

Proof. Since $\gamma\left(\widetilde{R}(N, \mathcal{X}_M)\mathcal{X}_M, N\right) = 0$, then by (11), we will calculate

$$\begin{aligned} Ric(\mathcal{X}_M, \mathcal{X}_M) &= \widetilde{Ric}(\mathcal{X}_M, \mathcal{X}_M) - \epsilon\gamma(\widetilde{R}(N, \mathcal{X}_M)\mathcal{X}_M, N) + h(\mathcal{S}(\mathcal{X}_M), n\mathbb{H}\mathcal{X}_M) \\ &\quad - h(\mathcal{S}(\mathcal{X}_M), \epsilon\mathcal{S}(\mathcal{X}_M)) \\ &= \widetilde{c}h(\mathcal{X}_M, \mathcal{X}_M) + h(\epsilon\mathbb{H}\mathcal{X}_M, n\mathbb{H}\mathcal{X}_M - \mathbb{H}\mathcal{X}_M) \\ &= \widetilde{c}h(\mathcal{X}_M, \mathcal{X}_M) + \epsilon n\mathbb{H}^2 h(\mathcal{X}_M, \mathcal{X}_M) - \epsilon\mathbb{H}^2 h(\mathcal{X}_M, \mathcal{X}_M) \\ &= \left(\widetilde{c} + \epsilon(n-1)\mathbb{H}^2\right) h(\mathcal{X}_M, \mathcal{X}_M). \end{aligned}$$

□

Theorem 9. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. If $\mathcal{X}_M$ is a nonzero vector field and the scalar curvature ρ is constant, then $\widetilde{c} + \epsilon(n-1)\mathbb{H}^2 \geq 0$.*

Proof. Theorem 3 states that a constant ρ ensures $Ric(\mathcal{X}_M, \mathcal{X}_M) \geq 0$. Consequently, (44) yields $\widetilde{c} + \epsilon(n-1)\mathbb{H}^2 \geq 0$. □

Theorem 10. *There is no trivial Yamabe soliton $(M, h, \mathcal{X}_M, \mu)$ on an oriented Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$, and $\widetilde{c} + \epsilon(n-1)\mathbb{H}^2 < 0$.*

Proof. According to Corollary 2, a trivial Yamabe soliton $(M, h, \mathcal{X}_M, \mu)$ cannot exist if $Ric(\mathcal{X}_M, \mathcal{X}_M) < 0$. Thus, from Equation (44), we find $(\widetilde{c} + \epsilon(n-1)\mathbb{H}^2 < 0$. □

Corollary 6. *Let $(M, h, \mathcal{X}_M, \mu)$ be a Yamabe soliton on an oriented minimal (or maximal) Riemannian (or spacelike) hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. Then, $\widetilde{c} \geq 0$ and (M, h) are totally geodesic with non-negative constant sectional curvature. In particular, the Yamabe soliton is steady or shrinking.*

Proof. By Theorem 10, we have $\widetilde{c} \geq 0$. Also, by Theorem 8, $\rho = (n-1)\widetilde{c} \geq 0$ is a constant. □

Corollary 7. *Let (M, h) be an oriented minimal (or maximal) compact Riemannian (or spacelike) hypersurface of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Riemannian (or Lorentzian) manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$. Then, if $\widetilde{c} < 0$ or (M, h) has negative scalar curvature, then there is no trivial Yamabe soliton $(M, h, \mathcal{X}_M, \mu)$.*

Corollary 8. *There is no trivial Yamabe soliton $(M, h, \mathcal{X}_M, \mu)$ on an oriented spacelike hypersurface (M, h) of dimension $n \geq 3$ induced by a conformal vector field $\widetilde{\mathcal{X}}$ in the Einstein Lorentzian manifold $(\widetilde{M}, \gamma)$, with $\widetilde{Ric} = \widetilde{c}\gamma$ and $\widetilde{c} < 0$*

Remark 2. *We extend the findings to null hypersurfaces in Lorentzian manifolds. Consider the Minkowski space $(\mathbb{R}_1^{n+1}, \gamma)$ with the Lorentzian metric $\gamma = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$. Define $M = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}_1^{n+1} : x_0 = x_1\}$. M is a null hypersurface (i.e., h is the induced metric γ to M). Infact, $h = \sum_{i=2}^{n} dx_i^2$.*

On M, we use the coordinates $(t, x_2, \ldots, x_n)$, and since h does not depend on t, we get $L_X h = 0$. The scalar curvature ρ of M is zero since the metric is flat in the $x_2, \ldots, x_n$ directions. Then, the Yamabe soliton equation becomes

$$0 = L_X h = -\mu h, \tag{45}$$

which implies that $\mu = 0$. Hence, $(M, h, \partial t, 0)$ is a steady Yamabe soliton on the null hypersurface (M, h).

Remark 3. *If we consider $\widetilde{\mathcal{X}}$ as a closed conformal vector field rather than a general one, it asserts that in a Lorentzian manifold, the vector field $\widetilde{\mathcal{X}}$ is timelike. By restricting $\widetilde{\mathcal{X}}$ to M, denoted as $\mathcal{X}$, it can be expressed by the equation*

$$\mathcal{X} = \mathcal{X}_M + \epsilon\theta\mathbb{U}. \tag{46}$$

The tangential component $\mathcal{X}_M$, and $\theta = \gamma(\mathbb{U}, \mathcal{X}) < 0$. Since $\mathcal{X}$ is closed, Equations (27) and (46) imply certain relationships, leading directly to the equation

$$div(\mathcal{X}_M) = n(\psi + \theta\mathbb{H}). \tag{47}$$

Ultimately, the findings from Section 4 remain applicable for a closed conformal vector field.

5. Conclusions

In this study, we investigated the structure of Yamabe solitons on Riemannian (or spacelike) hypersurfaces (M, h) that are embedded in Riemannian or Lorentzian manifolds, with a particular focus on the role of ambient conformal vector fields $\mathcal{X}$. By analyzing the tangential component $\mathcal{X}_M$ of these fields on the hypersurface, we derived the necessary and sufficient conditions for the induced geometry to admit a Yamabe soliton structure.

Our results demonstrate that under specific curvature assumptions regarding the ambient manifold—especially when it is an Einstein manifold—the hypersurface exhibits strong rigidity properties, often becoming totally umbilical. In the case of compact manifolds, the hypersurface is further constrained, frequently reducing to either a totally geodesic surface or an extrinsic sphere. Additionally, we eliminated the possibility of nontrivial Yamabe solitons existing in certain geometric contexts, particularly when the interaction between scalar curvature and the curvature of the ambient manifold occurs in specific ways.

These findings highlight the geometric influence of ambient conformal symmetry on the soliton structure of submanifolds and open new avenues for exploring Yamabe solitons in broader geometric flows and ambient settings.

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