

Article

Existence and Uniqueness of Solutions for Cohen–Grossberg BAM Neural Networks with Time-Varying Leakage, Neutral, Distributed, and Transmission Delays

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Abstract

This paper establishes a rigorous theoretical framework for analyzing the existence and uniqueness of solutions to Cohen–Grossberg bidirectional associative memory neural networks (CGBAMNNs) incorporating four distinct types of time-varying delays: leakage, neutral, distributed, and transmission delays. This study makes three key contributions to the field: First, it overcomes the fundamental challenge posed by the system’s inherent inability to be expressed in vector–matrix form, which previously limited the application of standard analytical techniques. Second, the work develops a novel and generalizable methodology that not only proves sufficient conditions for solution existence and uniqueness but also, for the first time in the literature, provides an explicit representation of the unique solution. Third, the proposed framework demonstrates remarkable extensibility, requiring only minor modifications to be applicable to a wide range of delayed system models. Theoretical findings are conclusively validated through numerical simulations, confirming both the robustness of the proposed approach and its practical relevance for complex neural network analysis.

Keywords: Cohen–Grossberg BAM neural networks; existence and uniqueness of solution; time-varying leakage; distributed delays

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1. Introduction

Neural networks (NNs) have garnered significant attention due to their broad applications in engineering and technical fields, such as signal processing, automatic control, image restoration, associative memory, and optimization [1–6]. As traditional NNs became increasingly prevalent, diverse types emerged, including Hopfield NNs, memristor-based NNs, cellular NNs, Cohen–Grossberg NNs (CGNNs), bidirectional associative memory NNs (BAMNNs), and fractional-order NNs. Among these, CGNNs were introduced in 1983 as a single-layer autoassociative Hebbian correlator [7]. Building on this, BAMNNs were proposed by Kosko in 1987 [8,9], extending CGNNs to a two-layer heteroassociative structure capable of retrieving complete stored patterns from incomplete or noisy inputs. In

practical applications, recurrent neural networks (RNNs) are used to process a group of input neurons and produce a related but distinct group of outputs. Meanwhile, the BAMNN architecture consists of two layers: the X-layer and the Y-layer. Neurons in one layer are fully connected to all neurons in the opposite layer, but there are no connections between neurons within the same layer.

In hardware implementation of NNs, time delay is an unavoidable factor in the characteristics of signal transmission between neurons. Time delay may lead to some complex dynamical behaviors of the whole network, for example, instability, chaos, periodic oscillations, and poor performance. Therefore, it is of prime importance to consider the delay effect on the dynamical behavior of NNs. In general, time delays in NNs can be classified as transmission, leakage, distributed, and proportional delays [10–13].

A considerable number of results have been reported regarding the combination of CGNNs and BAMNNs (i.e., CGBAMNNs) which have been applied to many areas, such as associative memory, signal processing, combinatorial optimization and parallel computing. Accordingly, many researchers have paid more attention to CGBAMNNs. The existence and uniqueness of solutions to recurrent neural networks under impulsive perturbations were investigated in [14,15]. In [14], Li et al. utilized the contraction mapping theorem to establish sufficient conditions for the global existence and uniqueness of solutions for neural networks with leakage delays. Building upon [14], the authors of [15] further incorporated distributed delays into the system and derived analogous sufficient conditions for the global existence and uniqueness of solutions by using the same methodology. In [16], the theoretical framework for reaction–diffusion stochastic Hopfield neural networks with S-type distributed delays was developed, where the existence and uniqueness of wild solutions was rigorously established using the homotopy invariance theorem combined with topological degree theory. In [17], fundamental global existence and uniqueness theorems were established through compatibility analysis of initial values, thereby laying theoretical foundations for applications of differential–algebraic NNs. The methodological spectrum expanded in [18] through discrete nabla operators of variable order, where Banach’s fixed-point technique was successfully applied to derive existence–uniqueness criteria for discrete-time fractional-order neural networks. In [19], sufficient conditions were established for the existence, uniqueness, and global Mittag-Leffler stability of solutions in fractional-difference BAMNN models. Ref. [20] proposed a novel five-neuron fractional delayed BAMNN, deriving solution existence and uniqueness through Lipschitz conditions. Concurrently, ref. [21] established criteria for solution boundedness, existence, and uniqueness in fractional delayed BAMNNs, while also proposing a delay-independent bifurcation criterion and stability conditions for the modeled systems. The study in [22] conducted qualitative bifurcation analysis of an incommensurate fractional-order five-neuron BAMNN incorporating four delays, including establishing the existence and uniqueness of solutions. Furthermore, ref. [23] employed equivalent integral equation formulation to systematically analyze BAM cellular NNs, achieving comprehensive existence–uniqueness proofs of solutions through functional analytic methods.

Based on the preceding analysis, this paper investigates the existence and uniqueness of solutions to CGBAMNNs with four types of time-varying delays: leakage, neutral, distributed, and transmission delays. Since the system model cannot be expressed in vector–matrix form, conventional analytical techniques for such dynamical system constructs are inapplicable. To overcome this hurdle, this study proposes a novel and more general methodology. Specifically, by defining a new norm, we establish a sufficient condition to guarantee a direct analysis approach proposing the uniqueness of solutions to CGBAMNNs. Furthermore, an explicit expression of the unique solutions is derived and presented.

Notation. The sets of real numbers and positive integer sets are denoted by $\mathbb{R}$ and $\mathbb{N}^+$, respectively. The symbol $C([- \sigma, 0], \mathbb{R})$ denotes the set that consists of all functions $\varphi : [- \sigma, 0] \rightarrow \mathbb{R}$. $\mathbb{Z}_k = \{1, 2, \dots, k\}, k \in \mathbb{N}^+$.

2. Problem Description

Consider a class of multiple-delay CGBAMNNs, which can be described as

$$\begin{aligned} \dot{\zeta}_i(t) = & \varrho_i(\zeta_i(t)) \left[-\rho_i(\zeta_i(t - p_i(t))) + \sum_{j \in \mathbb{Z}_m} [\mathfrak{b}_{ij} h_j(\vartheta_j(t)) + \mathfrak{c}_{ij} l_j(\vartheta_j(t - p_{ij}(t)))] \right. \\ & \left. + \sum_{j \in \mathbb{Z}_m} \mathfrak{d}_{ij} \int_{t-\delta_{ij}(t)}^t \mathfrak{h}_j(\alpha_j \vartheta_j(s)) ds \right] + \sum_{l \in \mathbb{Z}_n} \mathfrak{a}_{il} \omega_l(t) \\ & + \sum_{j \in \mathbb{Z}_m} \mathfrak{e}_{ij} \dot{\zeta}_j(t - \tau_{ij}(t)), i \in \mathbb{Z}_n, t \in [0, +\infty), \end{aligned} \quad (1a)$$

$$\begin{aligned} \dot{\vartheta}_j(t) = & \hat{\varrho}_j(\vartheta_j(t)) \left[-\hat{\rho}_j(\vartheta_j(t - q_j(t))) + \sum_{i \in \mathbb{Z}_n} [\hat{\mathfrak{b}}_{ji} \hat{h}_i(\zeta_i(t)) + \hat{\mathfrak{c}}_{ji} \hat{l}_i(\zeta_i(t - q_{ji}(t)))] \right. \\ & \left. + \sum_{i \in \mathbb{Z}_n} \hat{\mathfrak{d}}_{ji} \int_{t-\gamma_{ji}(t)}^t \hat{\mathfrak{h}}_i(\beta_i \zeta_i(s)) ds \right] + \sum_{k \in \mathbb{Z}_m} \hat{\mathfrak{a}}_{jk} \hat{\omega}_k(t) \\ & + \sum_{i \in \mathbb{Z}_n} \hat{\mathfrak{e}}_{ji} \dot{\vartheta}_i(t - \theta_{ji}(t)), j \in \mathbb{Z}_m, t \in [0, +\infty), \end{aligned} \quad (1b)$$

$$\zeta_i(s) = \varphi_i(s), \vartheta_j(s) = \psi_j(s), -\sigma \leq s \leq 0, i \in \mathbb{Z}_n, j \in \mathbb{Z}_m, \quad (1c)$$

$$\sigma = \max_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_m} \{p_i, q_j, p_{ij}, q_{ji}, \delta_{ij}, \gamma_{ji}, \tau_{ij}, \theta_{ji}\}, \quad (1d)$$

where $\zeta_i(t)$ and $\vartheta_j(t)$ denote the i th and j th neuronal state of layer- X and layer- Y , respectively; $\omega_l \in C([0, +\infty), \mathbb{R})$ and $\hat{\omega}_k \in C([0, +\infty), \mathbb{R})$ are the input of the neurons, respectively; $h_j(\cdot), \hat{h}_i(\cdot), l_j(\cdot), \hat{l}_i(\cdot), \mathfrak{h}_j(\cdot)$, and $\hat{\mathfrak{h}}_i(\cdot)$ are continuous activation functions; $\varrho_i(\cdot)$ and $\hat{\varrho}_j(\cdot)$ represent the amplification functions; $\rho_i(\cdot)$ and $\hat{\rho}_j(\cdot)$ denote appropriately behaved functions; the constants $\mathfrak{a}_i, \hat{\mathfrak{a}}_j, \mathfrak{b}_{ij}, \hat{\mathfrak{b}}_{ji}, \mathfrak{c}_{ij}, \hat{\mathfrak{c}}_{ji}, \mathfrak{d}_{ij}, \hat{\mathfrak{d}}_{ji}, \mathfrak{e}_{ij}$, and $\hat{\mathfrak{e}}_{ji}$ represent the connection weights; $\varphi_i(s), \psi_j(s) \in C([- \sigma, 0], \mathbb{R})$ stand for the initial functions; $p_i(\cdot)$ and $q_j(\cdot)$ are the time-varying leakage, $\check{p}_i \leq p_i(\cdot) \leq \hat{p}_i, \check{q}_j \leq q_j(\cdot) \leq \hat{q}_j$; $p_{ij}(\cdot)$ and $q_{ji}(\cdot)$ are the transmission delays, $\check{p}_{ij} \leq p_{ij}(\cdot) \leq \hat{p}_{ij}, \check{q}_{ji} \leq q_{ji}(\cdot) \leq \hat{q}_{ji}$; $\delta_{ij}(\cdot)$ and $\gamma_{ji}(\cdot)$ are the distributed delays, $0 \leq \delta_{ij}(\cdot) \leq \hat{\delta}_{ij}, 0 \leq \gamma_{ji}(\cdot) \leq \hat{\gamma}_{ji}$; and $\tau_{ij}(\cdot)$ and $\theta_{ji}(\cdot)$ are the neutral delays, $\check{\tau}_{ij} \leq \tau_{ij}(\cdot) \leq \hat{\tau}_{ij}, \check{\theta}_{ji} \leq \theta_{ji}(\cdot) \leq \hat{\theta}_{ji}$.

This paper will deal with the existence and uniqueness issues of solutions to CGBAMNNs (1) that involve time-varying leakage, neutral, distributed, and transmission delays. By proposing a direct analysis approach, we shall derive delay-dependent criteria to guarantee the existence and uniqueness of solutions to CGBAMNNs (1).

To realize the purpose, we require the assumptions below:

Assumption 1. There exist scalars $\mathfrak{A}_{1i}, \mathfrak{A}_{2i} > 0$ such that $|\varrho_i(a_1)\rho_i(a_2) - \varrho_i(b_1)\rho_i(b_2)| \leq \mathfrak{A}_{1i}|a_1 - b_1| + \mathfrak{A}_{2i}|a_2 - b_2|$ holds for all $i \in \mathbb{Z}_n$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Assumption 2. There exist scalars $\hat{\mathfrak{A}}_{1j}, \hat{\mathfrak{A}}_{2j} > 0$ such that $|\hat{\varrho}_j(a_1)\hat{\rho}_j(a_2) - \hat{\varrho}_j(b_1)\hat{\rho}_j(b_2)| \leq \hat{\mathfrak{A}}_{1j}|a_1 - b_1| + \hat{\mathfrak{A}}_{2j}|a_2 - b_2|$ holds for all $j \in \mathbb{Z}_m$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Assumption 3. There exist scalars $\tilde{q}_i, \tilde{h}_i, \tilde{l}_i, \tilde{h}_i, \tilde{q}_i, \tilde{h}_i, \tilde{l}_i, \tilde{h}_i > 0$ such that

$$\begin{aligned} |\varrho_i(a) - \varrho_i(b)| &\leq \tilde{q}_i |a - b|, |h_i(a) - h_i(b)| \leq \tilde{h}_i |a - b|, \\ |l_i(a) - l_i(b)| &\leq \tilde{l}_i |a - b|, |\tilde{h}_i(a) - \tilde{h}_i(b)| \leq \tilde{h}_i |a - b|, \\ |\varrho_i(\cdot)| &\leq \tilde{q}_i, |h_i(\cdot)| \leq \tilde{h}_i, |l_i(\cdot)| \leq \tilde{l}_i, |\tilde{h}_i(\cdot)| \leq \tilde{h}_i \end{aligned}$$

hold for all $i \in \mathbb{Z}_n$ and $a, b \in \mathbb{R}$.

Assumption 4. There exist scalars $\tilde{q}_j, \tilde{h}_j, \tilde{l}_j, \tilde{h}_j, \tilde{q}_j, \tilde{h}_j, \tilde{l}_j, \tilde{h}_j > 0$ such that

$$\begin{aligned} |\hat{q}_j(a) - \hat{q}_j(b)| &\leq \tilde{q}_j |a - b|, |\hat{h}_j(a) - \hat{h}_j(b)| \leq \tilde{h}_j |a - b|, \\ |\hat{l}_j(a) - \hat{l}_j(b)| &\leq \tilde{l}_j |a - b|, |\hat{\tilde{h}}_j(a) - \hat{\tilde{h}}_j(b)| \leq \tilde{h}_j |a - b|, \\ |\hat{q}_j(\cdot)| &\leq \tilde{q}_j, |\hat{h}_j(\cdot)| \leq \tilde{h}_j, |\hat{l}_j(\cdot)| \leq \tilde{l}_j, |\hat{\tilde{h}}_j(\cdot)| \leq \tilde{h}_j \end{aligned}$$

hold for all $j \in \mathbb{Z}_m$ and $a, b \in \mathbb{R}$.

Assumption 5. There exist scalars $\check{\tau}_{d1}, \check{\tau}_{d2}, \hat{\tau}_{d1}$, and $\hat{\tau}_{d2}$ such that $\check{\tau}_{d1} \leq \check{\tau}_{ij}(t) \leq \hat{\tau}_{d1} < 1$ and $\check{\tau}_{d2} \leq \check{\tau}_{ji}(t) \leq \hat{\tau}_{d2}$ for any $t \geq 0, i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$.

Assumption 6. There exist scalars $\check{\theta}_{d1}, \check{\theta}_{d2}, \hat{\theta}_{d1}$, and $\hat{\theta}_{d2}$ such that $\check{\theta}_{d1} \leq \check{\theta}_{ji}(t) \leq \hat{\theta}_{d1} < 1$ and $\check{\theta}_{d2} \leq \check{\theta}_{ij}(t) \leq \hat{\theta}_{d2}$ for any $t \geq 0, i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$.

3. Existence and Uniqueness of Solutions

In this section, we will prove a sufficient condition for the existence and uniqueness of the solutions to CGBAMNNs (1) by defining a new norm:

$$\begin{aligned} \|\varsigma(t)\|_* &= \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} (|\varsigma_i(t)| e^{-t}), \\ \|v(t)\|_* &= \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}). \end{aligned}$$

Theorem 1. Under Assumptions 1–6, if there exist $K_i > 0$ and $\hat{K}_j > 0$ ($i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$) that are to be solved, such that

$$\mathfrak{B}_{0i} + \frac{K_i + \mathfrak{B}_{1i} + \mathfrak{B}_{2i} + \mathfrak{B}_{3i}}{K_i + 1} + \frac{\mathfrak{C}_{4j} + \mathfrak{C}_{5j}}{\hat{K}_j + 1} \leq 1, \quad (2a)$$

$$\mathfrak{C}_{0j} + \frac{\hat{K}_j + \mathfrak{C}_{1j} + \mathfrak{C}_{2j} + \mathfrak{C}_{3j}}{\hat{K}_j + 1} + \frac{\mathfrak{B}_{4i} + \mathfrak{B}_{5i}}{K_i + 1} \leq 1, \quad (2b)$$

where

$$\begin{aligned} \mathfrak{B}_{0i} &= \sum_{j \in \mathbb{Z}_m} \frac{|\mathfrak{c}_{ij}|}{1 - \check{\tau}_{d1}} e^{-\check{\tau}_{ij}}, \\ \mathfrak{B}_{1i} &= \sum_{j \in \mathbb{Z}_m} \frac{|\mathfrak{c}_{ij}| \max\{|K_i(1 - \hat{\tau}_{d1}) + \check{\tau}_{d2}|, |K_i(1 - \check{\tau}_{d1}) + \hat{\tau}_{d2}|\}}{(1 - \hat{\tau}_{d1})^2} e^{-\check{\tau}_{ij}}, \\ \mathfrak{B}_{2i} &= \mathfrak{A}_{1i} + \mathfrak{A}_{2i} e^{-\check{\rho}_i}, \\ \mathfrak{B}_{3i} &= \sum_{j \in \mathbb{Z}_m} \tilde{q}_i (|\mathfrak{b}_{ij}| \tilde{h}_j + |\mathfrak{c}_{ij}| \tilde{l}_j + |\mathfrak{d}_{ij}| \tilde{h}_j \hat{\delta}_{ij}), \\ \mathfrak{B}_{4i} &= \sum_{j \in \mathbb{Z}_m} \tilde{q}_i (|\mathfrak{b}_{ij}| \tilde{h}_j + |\mathfrak{c}_{ij}| \tilde{l}_j e^{-\check{\rho}_{ij}}), \\ \mathfrak{B}_{5i} &= \sum_{j \in \mathbb{Z}_m} \tilde{q}_i |\mathfrak{d}_{ij}| \tilde{h}_j \alpha_j (1 - e^{-\hat{\delta}_{ij}}), \\ \mathfrak{C}_{0j} &= \sum_{i \in \mathbb{Z}_n} \frac{|\hat{\mathfrak{c}}_{ji}|}{1 - \check{\theta}_{d1}} e^{-\check{\theta}_{ji}}, \end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{1j} &= \sum_{i \in \mathbb{Z}_n} \frac{|\hat{\mathfrak{c}}_{ji}| \max\{|\hat{K}_j(1-\hat{\theta}_{d1}) + \check{\theta}_{d2}|, |\hat{K}_j(1-\check{\theta}_{d1}) + \hat{\theta}_{d2}|\}}{(1-\hat{\theta}_{d1})^2} e^{-\check{\theta}_{ji}}, \\
\mathfrak{C}_{2j} &= \sum_{i \in \mathbb{Z}_n} (\hat{\mathfrak{A}}_{1j} + \hat{\mathfrak{A}}_{2j} e^{-\check{q}_j}), \\
\mathfrak{C}_{3j} &= \sum_{i \in \mathbb{Z}_n} \check{\mathfrak{Q}}_j (|\hat{\mathfrak{b}}_{ji}| \tilde{h}_i + |\hat{\mathfrak{c}}_{ji}| \tilde{l}_i + |\hat{\mathfrak{d}}_{ji}| \tilde{h}_i \hat{\gamma}_{ji}), \\
\mathfrak{C}_{4j} &= \sum_{i \in \mathbb{Z}_n} \check{\mathfrak{Q}}_j (|\hat{\mathfrak{b}}_{ji}| \tilde{h}_i + |\hat{\mathfrak{c}}_{ji}| \tilde{l}_i e^{-\check{q}_{ji}}), \\
\mathfrak{C}_{5j} &= \sum_{i \in \mathbb{Z}_n} \check{\mathfrak{Q}}_j |\hat{\mathfrak{d}}_{ji}| \tilde{h}_i \beta_i (1 - e^{-\hat{\gamma}_{ji}}),
\end{aligned}$$

then for any given initial conditions $\varphi_i(t)$ and $\psi_j(t)$, the CGBAMNN (1) has a unique solution:

$$\zeta_i(t) = \begin{cases} \sum_{\kappa=0}^5 \mathfrak{R}_{i\kappa}(t), & t \in [0, +\infty), \\ \varphi_i(t), & t \in [-\sigma, 0), \end{cases} \quad i \in \mathbb{Z}_n, \quad (3)$$

$$\vartheta_j(t) = \begin{cases} \sum_{\hat{\kappa}=0}^5 \hat{\mathfrak{R}}_{j\hat{\kappa}}(t), & t \in [0, +\infty), \\ \psi_j(t), & t \in [-\sigma, 0), \end{cases} \quad j \in \mathbb{Z}_m, \quad (4)$$

Here,

$$\begin{aligned}
\mathfrak{R}_{i0}(t) &= \pi_{1i}(t) + e^{-K_i t} (\varphi_i(0) - \pi_{1i}(0)), \\
\mathfrak{R}_{i1}(t) &= \int_0^t e^{K_i(s-t)} K_i \zeta_i(s) ds, \\
\mathfrak{R}_{i2}(t) &= - \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \zeta_j(s - \tau_{ij}(s)) \pi_{2ij}(s) ds, \\
\mathfrak{R}_{i3}(t) &= - \int_0^t e^{K_i(s-t)} \mathfrak{Q}_i(\zeta_i(s)) \rho_i(\zeta_i(s - p_i(s))) ds, \\
\mathfrak{R}_{i4}(t) &= \int_0^t e^{K_i(s-t)} \mathfrak{Q}_i(\zeta_i(s)) \Pi_i(\vartheta, s) ds, \\
\mathfrak{R}_{i5}(t) &= \sum_{l \in \mathbb{Z}_n} \int_0^t e^{K_i(s-t)} \mathfrak{a}_{il} \omega_l(s) ds, \\
\pi_{1i}(s) &= \sum_{j \in \mathbb{Z}_m} \frac{\mathfrak{c}_{ij} \zeta_j(s - \tau_{ij}(s))}{1 - \tau_{ij}(s)}, \\
\pi_{2ij}(s) &= \frac{K_i \mathfrak{c}_{ij} (1 - \tau_{ij}(s)) + \mathfrak{c}_{ij} \tau_{ij}(s)}{(1 - \tau_{ij}(s))^2}, \\
\Pi_i(\vartheta, s) &= \sum_{j \in \mathbb{Z}_m} \left[\mathfrak{b}_{ij} h_j(\vartheta_j(s)) + \mathfrak{c}_{ij} l_j(\vartheta_j(s - p_{ij}(s))) + \mathfrak{d}_{ij} \int_{s - \delta_{ij}(s)}^s \hat{h}_j(\alpha_j \vartheta_j(u)) du \right], \\
\hat{\mathfrak{R}}_{j0}(t) &= \hat{\pi}_{1j}(t) + e^{-\hat{K}_j t} (\psi_j(0) - \hat{\pi}_{1j}(0)), \\
\hat{\mathfrak{R}}_{j1}(t) &= \int_0^t e^{\hat{K}_j(s-t)} \hat{K}_j \vartheta_j(s) ds, \\
\hat{\mathfrak{R}}_{j2}(t) &= - \sum_{i \in \mathbb{Z}_n} \int_0^t e^{\hat{K}_j(s-t)} \vartheta_i(s - \theta_{ji}(s)) \hat{\pi}_{2ji}(s) ds, \\
\hat{\mathfrak{R}}_{j3}(t) &= - \int_0^t e^{\hat{K}_j(s-t)} \hat{\mathfrak{Q}}_j(\vartheta_j(s)) \hat{\rho}_j(\vartheta_j(s - q_j(s))) ds, \\
\hat{\mathfrak{R}}_{j4}(t) &= \int_0^t e^{\hat{K}_j(s-t)} \hat{\mathfrak{Q}}_j(\vartheta_j(s)) \hat{\Pi}_j(\zeta, s) ds, \\
\hat{\mathfrak{R}}_{j5}(t) &= \sum_{k \in \mathbb{Z}_m} \int_0^t e^{\hat{K}_j(s-t)} \hat{\mathfrak{a}}_{jk} \hat{\omega}_k(s) ds, \\
\hat{\pi}_{1j}(s) &= \sum_{i \in \mathbb{Z}_n} \frac{\hat{\mathfrak{c}}_{ji} \vartheta_i(s - \theta_{ji}(s))}{1 - \theta_{ji}(s)}, \\
\hat{\pi}_{2ji}(s) &= \frac{\hat{K}_j \hat{\mathfrak{c}}_{ji} (1 - \theta_{ji}(s)) + \hat{\mathfrak{c}}_{ji} \theta_{ji}(s)}{(1 - \theta_{ji}(s))^2}, \\
\hat{\Pi}_j(\zeta, s) &= \sum_{i \in \mathbb{Z}_n} \left[\hat{\mathfrak{b}}_{ji} \hat{h}_i(\zeta_i(s)) + \hat{\mathfrak{c}}_{ji} \hat{l}_i(\zeta_i(s - q_{ji}(s))) + \hat{\mathfrak{d}}_{ji} \int_{s - \gamma_{ji}(s)}^s \hat{h}_i(\beta_i \zeta_i(u)) du \right].
\end{aligned}$$

Proof. (i) Existence. Next, we will first show that the set of functions defined in (3) and (4) are the solution to (1). According to (3), we can obtain

$$\begin{aligned}
 \dot{\zeta}_i(t) &= \sum_{j \in \mathbb{Z}_m} \mathbf{e}_{ij} \dot{\zeta}_j(t - \tau_{ij}(t)) + \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \zeta_j(t - \tau_{ij}(t)) \ddot{\tau}_{ij}(t)}{(1 - \dot{\tau}_{ij}(t))^2} \\
 &\quad - K_i e^{-K_i t} (\varphi_i(0) - \pi_{1i}(0)) \\
 &\quad - K_i \mathfrak{R}_{i1}(t) + K_i \zeta_i(t) \\
 &\quad - K_i \mathfrak{R}_{i2}(t) - \sum_{j \in \mathbb{Z}_m} \zeta_j(t - \tau_{ij}(t)) \frac{\mathbf{e}_{ij} (1 - \dot{\tau}_{ij}(t)) + \mathbf{e}_{ij} \ddot{\tau}_{ij}(t)}{(1 - \dot{\tau}_{ij}(t))^2} \\
 &\quad - K_i \mathfrak{R}_{i3}(t) - \varrho_i(\zeta_i(t)) \rho_i(\zeta_i(t - p_i(t))) \\
 &\quad - K_i \mathfrak{R}_{i4}(t) + \sum_{j \in \mathbb{Z}_m} \varrho_i(\zeta_i(t)) [\mathbf{b}_{ij} h_j(\vartheta_j(t)) + \mathbf{c}_{ij} l_j(\vartheta_j(t - p_{ij}(t)))] \\
 &\quad + \mathfrak{d}_{ij} \int_{t-\delta_{ij}(t)}^t h_j(\alpha_j \vartheta_j(u)) du - K_i \mathfrak{R}_{i5}(t) + \sum_{l \in \mathbb{Z}_n} \mathbf{a}_{il} \omega_l(t) \\
 &= \varrho_i(\zeta_i(t)) \left[-\rho_i(\zeta_i(t - p_i(t))) + \sum_{j \in \mathbb{Z}_m} [\mathbf{b}_{ij} h_j(\vartheta_j(t)) + \mathbf{c}_{ij} l_j(\vartheta_j(t - p_{ij}(t)))] \right. \\
 &\quad \left. + \sum_{j \in \mathbb{Z}_m} \mathfrak{d}_{ij} \int_{t-\delta_{ij}(t)}^t h_j(\alpha_j \vartheta_j(s)) ds \right] + \sum_{l \in \mathbb{Z}_n} \mathbf{a}_{il} \omega_l(t) \\
 &\quad + \sum_{j \in \mathbb{Z}_m} \mathbf{e}_{ij} \dot{\zeta}_j(t - \tau_{ij}(t)), i \in \mathbb{Z}_n, t \in [0, +\infty),
 \end{aligned}$$

That is, (1a) holds. Similarly, from (4), it can be obtained that (1b) holds. Therefore, the functions given in (3) and (4) are a solution to the CGBAMNN (1) corresponding to the initial function (1c). This shows the existence of the solution.

(ii) Uniqueness. Now we will prove the uniqueness of solutions of the CGBAMNN (1) corresponding to the initial functions $\varphi_i(t)$ and $\psi_j(t)$ in (1c). Assume that $\tilde{\zeta}_i(t)$ and $\tilde{\vartheta}_j(t)$ ($i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$) are also a solution of the CGBAMNN (1) corresponding to the initial functions $\varphi_i(t)$ and $\psi_j(t)$ in (1c), which differ from the ones given in (3) and (4).

According to (1a), we can obtain

$$\begin{aligned}
 &\frac{d}{dt} \left(\tilde{\zeta}_i(t) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t))}{1 - \dot{\tau}_{ij}(t)} \right) \\
 &= \dot{\tilde{\zeta}}_i(t) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \dot{\tilde{\zeta}}_j(t - \tau_{ij}(t)) (1 - \dot{\tau}_{ij}(t))^2 - \mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t)) \cdot (-\ddot{\tau}_{ij}(t))}{(1 - \dot{\tau}_{ij}(t))^2} \\
 &= \dot{\tilde{\zeta}}_i(t) - \sum_{j \in \mathbb{Z}_m} \mathbf{e}_{ij} \dot{\tilde{\zeta}}_j(t - \tau_{ij}(t)) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t)) \ddot{\tau}_{ij}(t)}{(1 - \dot{\tau}_{ij}(t))^2} \\
 &= \varrho_i(\tilde{\zeta}_i(t)) \left[-\rho_i(\tilde{\zeta}_i(t - p_i(t))) + \sum_{j \in \mathbb{Z}_m} [\mathbf{b}_{ij} h_j(\tilde{\vartheta}_j(t)) + \mathbf{c}_{ij} l_j(\tilde{\vartheta}_j(t - p_{ij}(t)))] \right. \\
 &\quad \left. + \sum_{j \in \mathbb{Z}_m} \mathfrak{d}_{ij} \int_{t-\delta_{ij}(t)}^t h_j(\alpha_j \tilde{\vartheta}_j(s)) ds \right] + \sum_{l \in \mathbb{Z}_n} \mathbf{a}_{il} \omega_l(t) \\
 &\quad - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t)) \ddot{\tau}_{ij}(t)}{(1 - \dot{\tau}_{ij}(t))^2}, i \in \mathbb{Z}_n, t \in [0, +\infty).
 \end{aligned}$$

Accordingly, we have

$$\begin{aligned}
 & \frac{d}{dt} \left[e^{K_i t} \left(\tilde{\zeta}_i(t) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t))}{1 - \dot{\tau}_{ij}(t)} \right) \right] \\
 = & K_i e^{K_i t} \left(\tilde{\zeta}_i(t) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t))}{1 - \dot{\tau}_{ij}(t)} \right) \\
 & + e^{K_i t} \mathbf{q}_i(\tilde{\zeta}_i(t)) \left[-\rho_i(\tilde{\zeta}_i(t - p_i(t))) + \sum_{j \in \mathbb{Z}_m} [\mathbf{b}_{ij} h_j(\tilde{\vartheta}_j(t)) + \mathbf{c}_{ij} l_j(\tilde{\vartheta}_j(t - p_{ij}(t)))] \right. \\
 & \left. + \sum_{j \in \mathbb{Z}_m} \mathbf{d}_{ij} \int_{t - \delta_{ij}(t)}^t h_j(\alpha_j \tilde{\vartheta}_j(s)) ds \right] + e^{K_i t} \sum_{l \in \mathbb{Z}_n} \mathbf{a}_{il} \omega_l(t) \\
 & - e^{K_i t} \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t)) \ddot{\tau}_{ij}(t)}{(1 - \dot{\tau}_{ij}(t))^2}, i \in \mathbb{Z}_n, t \in [0, +\infty).
 \end{aligned}$$

Taking the integrals from 0 to t on both sides and applying (1c), we obtain

$$\begin{aligned}
 \tilde{\zeta}_i(t) = & \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(t - \tau_{ij}(t))}{1 - \dot{\tau}_{ij}(t)} + e^{-K_i t} \left(\varphi_i(0) - \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(-\tau_{ij}(0))}{1 - \dot{\tau}_{ij}(0)} \right) \\
 & + \int_0^t e^{K_i(s-t)} K_i \tilde{\zeta}_i(s) ds \\
 & - \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \tilde{\zeta}_j(s - \tau_{ij}(s)) \frac{K_i \mathbf{e}_{ij} (1 - \dot{\tau}_{ij}(s)) + \mathbf{e}_{ij} \ddot{\tau}_{ij}(s)}{(1 - \dot{\tau}_{ij}(s))^2} ds \\
 & - \int_0^t e^{K_i(s-t)} \mathbf{q}_i(\tilde{\zeta}_i(s)) \rho_i(\tilde{\zeta}_i(s - p_i(s))) ds \\
 & + \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \mathbf{q}_i(\tilde{\zeta}_i(s)) [\mathbf{b}_{ij} h_j(\tilde{\vartheta}_j(s)) + \mathbf{c}_{ij} l_j(\tilde{\vartheta}_j(s - p_{ij}(s)))] \\
 & + \mathbf{d}_{ij} \int_{s - \delta_{ij}(s)}^s h_j(\alpha_j \tilde{\vartheta}_j(u)) du] ds + \sum_{l \in \mathbb{Z}_n} \int_0^t e^{K_i(s-t)} \mathbf{a}_{il} \omega_l(s) ds.
 \end{aligned}$$

Let

$$\begin{aligned}
 \tilde{\mathfrak{R}}_{i0}(t) &= \tilde{\pi}_{1i}(t) + e^{-K_i t} (\varphi_i(0) - \tilde{\pi}_{1i}(0)), \\
 \tilde{\mathfrak{R}}_{i1}(t) &= \int_0^t e^{K_i(s-t)} K_i \tilde{\zeta}_i(s) ds, \\
 \tilde{\mathfrak{R}}_{i2}(t) &= - \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \tilde{\zeta}_j(s - \tau_{ij}(s)) \pi_{2ij}(s) ds, \\
 \tilde{\mathfrak{R}}_{i3}(t) &= - \int_0^t e^{K_i(s-t)} \mathbf{q}_i(\tilde{\zeta}_i(s)) \rho_i(\tilde{\zeta}_i(s - p_i(s))) ds, \\
 \tilde{\mathfrak{R}}_{i4}(t) &= \int_0^t e^{K_i(s-t)} \mathbf{q}_i(\tilde{\zeta}_i(s)) \Pi_i(\tilde{\vartheta}, s) ds,
 \end{aligned}$$

where

$$\tilde{\pi}_{1i}(s) = \sum_{j \in \mathbb{Z}_m} \frac{\mathbf{e}_{ij} \tilde{\zeta}_j(s - \tau_{ij}(s))}{1 - \dot{\tau}_{ij}(s)}.$$

As a result, we have

$$\tilde{\zeta}_i(t) = \sum_{\kappa=0}^4 \tilde{\mathfrak{R}}_{i\kappa}(t) + \mathfrak{R}_{i5}(t). \quad (5)$$

Similarly, we can derive from (1b) and (1c) that

$$\tilde{\vartheta}_j(t) = \sum_{\hat{\kappa}=0}^4 \tilde{\mathfrak{R}}_{j\hat{\kappa}}(t) + \hat{\mathfrak{R}}_{j5}(t), \quad (6)$$

where

$$\begin{aligned}
\tilde{\mathfrak{H}}_{j0}(t) &= \tilde{\mathfrak{H}}_{1j}(t) + e^{-\hat{K}_j t} (\psi_j(0) - \tilde{\mathfrak{H}}_{1j}(0)), \\
\tilde{\mathfrak{H}}_{j1}(t) &= \int_0^t e^{\hat{K}_j(s-t)} \hat{K}_j \tilde{\vartheta}_j(s) ds, \\
\tilde{\mathfrak{H}}_{j2}(t) &= -\sum_{i \in \mathbb{Z}_m} \int_0^t e^{\hat{K}_j(s-t)} \tilde{\vartheta}_j(s - \theta_{ji}(s)) \hat{\pi}_{2ji}(s) ds, \\
\tilde{\mathfrak{H}}_{j3}(t) &= -\int_0^t e^{\hat{K}_j(s-t)} \hat{Q}_j(\tilde{\vartheta}_j(s)) \hat{\rho}_j(\tilde{\vartheta}_j(s - q_j(s))) ds, \\
\tilde{\mathfrak{H}}_{j4}(t) &= \int_0^t e^{\hat{K}_j(s-t)} \hat{Q}_j(\tilde{\vartheta}_j(s)) \hat{\Pi}_j(\tilde{\zeta}, s) ds, \\
\tilde{\mathfrak{H}}_{1j}(s) &= \sum_{i \in \mathbb{Z}_m} \frac{\hat{\mathfrak{e}}_{ji} \tilde{\vartheta}_i(s - \theta_{ji}(s))}{1 - \hat{\theta}_{ji}(s)}.
\end{aligned}$$

Let $\varsigma_i(t) = \tilde{\zeta}_i(t) - \zeta_i(t)$ and $v_j(t) = \tilde{\vartheta}_j(t) - \vartheta_j(t)$. In light of (3) and (5), for any $t \in [0, +\infty)$, $i \in \mathbb{Z}_m$, we have that

$$\begin{aligned}
& e^{-t} |\tilde{\zeta}_i(t) - \zeta_i(t)| \\
&= e^{-t} \left| \sum_{j \in \mathbb{Z}_m} \frac{\mathfrak{e}_{ij} \varsigma_j(t - \tau_{ij}(t))}{1 - \hat{\tau}_{ij}(t)} + \int_0^t e^{K_i(s-t)} K_i \varsigma_i(s) ds \right. \\
&\quad - \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \varsigma_j(s - \tau_{ij}(s)) \frac{K_i \mathfrak{e}_{ij} (1 - \hat{\tau}_{ij}(s)) + \mathfrak{e}_{ij} \tilde{\tau}_{ij}(s)}{(1 - \hat{\tau}_{ij}(s))^2} ds \\
&\quad - \int_0^t e^{K_i(s-t)} [q_i(\tilde{\zeta}_i(s)) \rho_i(\tilde{\zeta}_i(s - p_i(s))) - q_i(\zeta_i(s)) \rho_i(\zeta_i(s - p_i(s)))] ds \\
&\quad \left. + \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} [q_i(\tilde{\zeta}_i(s)) \Pi_i(\tilde{\vartheta}, s) - q_i(\zeta_i(s)) \Pi_i(\vartheta, s)] ds \right| \\
&\leq e^{-t} \left(\sum_{j \in \mathbb{Z}_m} \left| \frac{\mathfrak{e}_{ij} \varsigma_j(t - \tau_{ij}(t))}{1 - \hat{\tau}_{ij}(t)} \right| + \left| \int_0^t e^{K_i(s-t)} K_i \varsigma_i(s) ds \right| \right. \\
&\quad + K_i \sum_{j \in \mathbb{Z}_m} \left| \int_0^t e^{K_i(s-t)} \varsigma_j(s - \tau_{ij}(s)) \pi_{2ij}(s) ds \right| \\
&\quad + \left| \int_0^t e^{K_i(s-t)} [q_i(\tilde{\zeta}_i(s)) \rho_i(\tilde{\zeta}_i(s - p_i(s))) - q_i(\zeta_i(s)) \rho_i(\zeta_i(s - p_i(s)))] ds \right| \\
&\quad \left. + \sum_{j \in \mathbb{Z}_m} \left| \int_0^t e^{K_i(s-t)} [q_i(\tilde{\zeta}_i(s)) \Pi_i(\tilde{\vartheta}, s) - q_i(\zeta_i(s)) \Pi_i(\vartheta, s)] ds \right| \right).
\end{aligned}$$

Based on Assumptions 1–6, we have

$$\begin{aligned}
e^{-t} \left| \sum_{j \in \mathbb{Z}_m} \frac{\mathfrak{e}_{ij} \varsigma_j(t - \tau_{ij}(t))}{1 - \hat{\tau}_{ij}(t)} \right| &\leq e^{-t} \sum_{j \in \mathbb{Z}_m} \left| \frac{\mathfrak{e}_{ij}}{1 - \hat{\tau}_{d1}} \right| |\varsigma_j(t - \tau_{ij}(t))| \\
&= \sum_{j \in \mathbb{Z}_m} \frac{|\mathfrak{e}_{ij}|}{1 - \hat{\tau}_{d1}} e^{-\tau_{ij}(t)} |\varsigma_j(t - \tau_{ij}(t))| e^{-(t - \tau_{ij}(t))} \\
&\leq \sum_{j \in \mathbb{Z}_m} \frac{|\mathfrak{e}_{ij}|}{1 - \hat{\tau}_{d1}} e^{-\hat{\tau}_{ij}} \sup_{t \in [-\sigma, +\infty)} (|\varsigma_j(t)| e^{-t}) \\
&= \mathfrak{B}_{0i} \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\varsigma_j(t)| e^{-t}), \\
e^{-t} \left| \int_0^t e^{K_i(s-t)} K_i \varsigma_i(s) ds \right| &\leq \int_0^t e^{(K_i+1)(s-t)} ds K_i \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\varsigma_j(t)| e^{-t}),
\end{aligned}$$

$$\begin{aligned}
& e^{-t} \left| \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)} \zeta_j(s - \tau_{ij}(s)) \pi_{2ij}(s) ds \right| \\
& \leq \left| \sum_{j \in \mathbb{Z}_m} \int_0^t e^{K_i(s-t)-t} \zeta_j(s - \tau_{ij}(s)) \frac{K_i \mathfrak{e}_{ij}(1 - \check{\tau}_{ij}(s)) + \mathfrak{e}_{ij} \check{\tau}_{ij}(s)}{(1 - \check{\tau}_{ij}(s))^2} ds \right| \\
& \leq \sum_{j \in \mathbb{Z}_m} \int_0^t e^{(K_i+1)(s-t)} \left| \frac{K_i \mathfrak{e}_{ij}(1 - \check{\tau}_{ij}(s)) + \mathfrak{e}_{ij} \check{\tau}_{ij}(s)}{(1 - \check{\tau}_{ij}(s))^2} \right| ds e^{-\check{\tau}_{ij}} \sup_{t \in [-\sigma, +\infty)} (|\zeta_j(t)| e^{-t}) \\
& \leq \sum_{j \in \mathbb{Z}_m} \int_0^t e^{(K_i+1)(s-t)} ds \frac{|\mathfrak{e}_{ij}| \max\{|K_i(1 - \hat{\tau}_{d1}) + \check{\tau}_{d2}|, |K_i(1 - \check{\tau}_{d1}) + \hat{\tau}_{d2}|\}}{(1 - \hat{\tau}_{d1})^2} e^{-\check{\tau}_{ij}} \\
& \quad \times \sup_{t \in [-\sigma, +\infty)} (|\zeta_j(t)| e^{-t}) \\
& = \mathfrak{B}_{1i} \sum_{j \in \mathbb{Z}_m} \int_0^t e^{(K_i+1)(s-t)} ds \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\zeta_j(t)| e^{-t}), \\
& e^{-t} \left| \int_0^t e^{K_i(s-t)} [q_i(\check{\zeta}_i(s)) \rho_i(\check{\zeta}_i(s - p_i(s))) - q_i(\zeta_i(s)) \rho_i(\zeta_i(s - p_i(s)))] ds \right| \\
& \leq \int_0^t e^{(K_i+1)(s-t)} e^{-s} \left| q_i(\check{\zeta}_i(s)) \rho_i(\check{\zeta}_i(s - p_i(s))) - q_i(\zeta_i(s)) \rho_i(\zeta_i(s - p_i(s))) \right| ds \\
& \leq \int_0^t e^{(K_i+1)(s-t)} \mathfrak{A}_{1i} e^{-s} |\check{\zeta}_i(s) - \zeta_i(s)| + \mathfrak{A}_{2i} e^{-s} |\check{\zeta}_i(s - p_i(s)) - \zeta_i(s - p_i(s))| ds \\
& = \int_0^t e^{(K_i+1)(s-t)} e^{-s} [\mathfrak{A}_{1i} |\zeta_i(s)| + \mathfrak{A}_{2i} |\zeta_i(s - p_i(s))|] ds \\
& \leq \int_0^t e^{(K_i+1)(s-t)} ds [\mathfrak{A}_{1i} + \mathfrak{A}_{2i} e^{-\check{p}_i}] \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}) \\
& = \mathfrak{B}_{2i} \sum_{j \in \mathbb{Z}_m} \int_0^t e^{(K_i+1)(s-t)} ds \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}), \\
& e^{-t} \left| \int_0^t e^{K_i(s-t)} [q_i(\check{\zeta}_i(s)) \Pi_i(\check{\vartheta}, s) - q_i(\zeta_i(s)) \Pi_i(\vartheta, s)] ds \right| \\
& = e^{-t} \left| \int_0^t e^{K_i(s-t)} [q_i(\check{\zeta}_i(s)) \Pi_i(\check{\vartheta}, s) - q_i(\zeta_i(s)) \Pi_i(\check{\vartheta}, s) \right. \\
& \quad \left. + q_i(\zeta_i(s)) \Pi_i(\check{\vartheta}, s) - q_i(\zeta_i(s)) \Pi_i(\vartheta, s)] ds \right| \\
& = e^{-t} \left| \int_0^t e^{K_i(s-t)} \{ |q_i(\check{\zeta}_i(s)) - q_i(\zeta_i(s))| \Pi_i(\check{\vartheta}, s) + q_i(\zeta_i(s)) |\Pi_i(\check{\vartheta}, s) - \Pi_i(\vartheta, s)| \} ds \right| \\
& \leq \int_0^t e^{(K_i+1)(s-t)} ds \bar{q}_i (|\mathfrak{b}_{ij}| \bar{h}_j + |\mathfrak{c}_{ij}| \bar{l}_j + |\mathfrak{d}_{ij}| \bar{h}_j \hat{\delta}_{ij}) \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}) \\
& \quad + \int_0^t e^{(K_i+1)(s-t)} ds \bar{q}_i (|\mathfrak{b}_{ij}| \tilde{h}_j + |\mathfrak{c}_{ij}| \tilde{l}_j e^{-\check{p}_{ij}}) \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}) \\
& \quad + \int_0^t e^{(K_i+1)(s-t)} ds \bar{q}_i |\mathfrak{d}_{ij}| \tilde{h}_j \alpha_j (1 - e^{-\hat{\delta}_{ij}}) \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} |v_j(t)| e^{-t} \\
& = \mathfrak{B}_{3i} \int_0^t e^{(K_i+1)(s-t)} ds \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}) \\
& \quad + \mathfrak{B}_{4i} \int_0^t e^{(K_i+1)(s-t)} ds \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}) \\
& \quad + \mathfrak{B}_{5i} \int_0^t e^{(K_i+1)(s-t)} ds \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}).
\end{aligned}$$

So,

$$\begin{aligned} & \sup_{t \in [-\sigma, +\infty)} e^{-t} |\zeta_i(t) - \zeta_i(t)| \\ & \leq \left(\mathfrak{B}_{0i} + \frac{K_i + \mathfrak{B}_{1i} + \mathfrak{B}_{2i} + \mathfrak{B}_{3i}}{K_i + 1} \right) \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\zeta_j(t)| e^{-t}) \\ & \quad + \frac{\mathfrak{B}_{4i} + \mathfrak{B}_{5i}}{K_i + 1} \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}). \end{aligned} \quad (7)$$

Similarly, in light of (4) and (6), for any $t \in [0, +\infty)$, $j \in \mathbb{Z}_m$, we have that

$$\begin{aligned} & e^{-t} |\tilde{\vartheta}_j(t) - \vartheta_j(t)| \\ & = e^{-t} \left| \sum_{i \in \mathbb{Z}_n} \frac{\tilde{\mathfrak{e}}_{ji} v_i(t - \theta_{ji}(t))}{1 - \hat{\theta}_{ji}(t)} + \int_0^t e^{K_j(s-t)} \hat{K}_j v_j(s) ds \right. \\ & \quad - \sum_{i \in \mathbb{Z}_n} \int_0^t e^{\hat{K}_j(s-t)} v_i(s - \theta_{ji}(s)) \frac{\hat{K}_j \hat{\mathfrak{e}}_{ji} (1 - \hat{\theta}_{ji}(s)) + \hat{\mathfrak{e}}_{ji} \hat{\theta}_{ji}(s)}{(1 - \hat{\theta}_{ji}(s))^2} ds \\ & \quad - \int_0^t e^{\hat{K}_j(s-t)} [\hat{\varrho}_j(\tilde{\vartheta}_j(s)) \hat{\rho}_j(\tilde{\vartheta}_j(s - q_j(s))) - \hat{\varrho}_j(\vartheta_j(s)) \hat{\rho}_j(\vartheta_j(s - q_j(s)))] ds \\ & \quad \left. + \sum_{i \in \mathbb{Z}_n} \int_0^t e^{\hat{K}_j(s-t)} [\hat{\varrho}_j(\tilde{\vartheta}_j(s)) \hat{\Pi}_j(\tilde{\zeta}, s) - \hat{\varrho}_j(\vartheta_j(s)) \hat{\Pi}_j(\zeta, s)] ds \right| \\ & \leq e^{-t} \left(\sum_{i \in \mathbb{Z}_n} \left| \frac{\tilde{\mathfrak{e}}_{ji} v_i(t - \theta_{ji}(t))}{1 - \hat{\theta}_{ji}(t)} \right| + \left| \int_0^t e^{K_j(s-t)} \hat{K}_j v_j(s) ds \right| \right. \\ & \quad + \hat{K}_j \sum_{i \in \mathbb{Z}_n} \int_0^t \left| e^{\hat{K}_j(s-t)} v_i(s - \theta_{ji}(s)) \hat{\pi}_{2ji}(s) ds \right| \\ & \quad + \left| \int_0^t e^{\hat{K}_j(s-t)} [\hat{\varrho}_j(\tilde{\vartheta}_j(s)) \hat{\rho}_j(\tilde{\vartheta}_j(s - q_j(s))) - \hat{\varrho}_j(\vartheta_j(s)) \hat{\rho}_j(\vartheta_j(s - q_j(s)))] ds \right| \\ & \quad \left. + \sum_{i \in \mathbb{Z}_n} \left| \int_0^t e^{\hat{K}_j(s-t)} [\hat{\varrho}_j(\tilde{\vartheta}_j(s)) \hat{\Pi}_j(\tilde{\zeta}, s) - \hat{\varrho}_j(\vartheta_j(s)) \hat{\Pi}_j(\zeta, s)] ds \right| \right) \end{aligned}$$

Based on Assumptions 1–6, we obtain

$$\begin{aligned} & \sup_{t \in [-\sigma, +\infty)} e^{-t} |\tilde{\vartheta}_j(t) - \vartheta_j(t)| \\ & \leq \left(\mathfrak{C}_{0j} + \frac{\hat{K}_j + \mathfrak{C}_{1j} + \mathfrak{C}_{2j} + \mathfrak{C}_{3j}}{\hat{K}_j + 1} \right) \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} (|v_i(t)| e^{-t}) \\ & \quad + \frac{\mathfrak{C}_{4j} + \mathfrak{C}_{5j}}{\hat{K}_j + 1} \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}). \end{aligned} \quad (8)$$

By (7) and (8), we obtain

$$\begin{aligned}
 & \|\tilde{\zeta}(t) - \zeta(t)\|_* + \|\tilde{\vartheta}(t) - \vartheta(t)\|_* \\
 &= \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} |\tilde{\zeta}_i(t) - \zeta_i(t)| + \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} |\tilde{\vartheta}_j(t) - \vartheta_j(t)| \\
 &\leq \left(\mathfrak{B}_{0i} + \frac{K_i + \mathfrak{B}_{1i} + \mathfrak{B}_{2i} + \mathfrak{B}_{3i}}{K_i + 1} \right) \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|\zeta_j(t)| e^{-t}) \\
 &\quad + \frac{\mathfrak{B}_{4i} + \mathfrak{B}_{5i}}{K_i + 1} \max_{j \in \mathbb{Z}_m} \sup_{t \in [-\sigma, +\infty)} (|v_j(t)| e^{-t}) \\
 &\quad + \left(\mathfrak{C}_{0j} + \frac{\hat{K}_j + \mathfrak{C}_{1j} + \mathfrak{C}_{2j} + \mathfrak{C}_{3j}}{\hat{K}_j + 1} \right) \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} (|v_i(t)| e^{-t}) \\
 &\quad + \frac{\mathfrak{C}_{4j} + \mathfrak{C}_{5j}}{\hat{K}_j + 1} \max_{i \in \mathbb{Z}_n} \sup_{t \in [-\sigma, +\infty)} (|\zeta_i(t)| e^{-t}) \\
 &\leq \left(\mathfrak{B}_{0i} + \frac{K_i + \mathfrak{B}_{1i} + \mathfrak{B}_{2i} + \mathfrak{B}_{3i}}{K_i + 1} + \frac{\mathfrak{C}_{4j} + \mathfrak{C}_{5j}}{\hat{K}_j + 1} \right) \|\tilde{\zeta}(t) - \zeta(t)\|_* \\
 &\quad + \left(\mathfrak{C}_{0j} + \frac{\hat{K}_j + \mathfrak{C}_{1j} + \mathfrak{C}_{2j} + \mathfrak{C}_{3j}}{\hat{K}_j + 1} + \frac{\mathfrak{B}_{4i} + \mathfrak{B}_{5i}}{K_i + 1} \right) \|\tilde{\vartheta}_j(t) - \vartheta_j(t)\|_*.
 \end{aligned}$$

Through (2), we obtain a contradiction. Then we can conclude that for any given initial conditions, the CGBAMNN (1) has a unique solution. $\square$

4. Illustrative Examples

In this section we will present the effectiveness of the proposed method by numerical examples.

Example 1. Consider the CGBAMNN (1), where $n = 2, m = 2$, $[b_{ij}]_{2 \times 2} = \begin{bmatrix} -0.6 & 1 \\ 2 & 0.6 \end{bmatrix}$, $[\hat{b}_{ji}]_{2 \times 2} = \begin{bmatrix} 0.8 & 0.4 \\ 2 & 0.8 \end{bmatrix}$, $[c_{ij}]_{2 \times 2} = \begin{bmatrix} 0.4 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$, $[\hat{c}_{ji}]_{2 \times 2} = \begin{bmatrix} 2 & 1 \\ -0.3 & 0.2 \end{bmatrix}$, $[d_{ij}]_{2 \times 2} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$, $[\hat{d}_{ji}]_{2 \times 2} = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$, $[a_{ij}]_{2 \times 2} = \begin{bmatrix} 2.4 & 1.4 \\ 2.4 & 1.4 \end{bmatrix}$, $[\hat{a}_{ji}]_{2 \times 2} = \begin{bmatrix} 1.7 & 2.1 \\ 1.7 & 2.1 \end{bmatrix}$, $[e_{ij}]_{2 \times 2} = \begin{bmatrix} -0.3 & -0.2 \\ 0.2 & 0.2 \end{bmatrix}$, $[\hat{e}_{ji}]_{2 \times 2} = \begin{bmatrix} 0.2 & -0.2 \\ 0.2 & 0.2 \end{bmatrix}$, time-varying leakage $p_i(t) = q_i(t) = \frac{0.2}{1+t^2}$, transmission delays $p_{ij}(t) = q_{ji}(t) = 0.25 + 0.25 \cos(t)$, distributed delays $\delta_{i1}(t) = \delta_{i2}(t) = 0.2 + \frac{0.1}{1+t^2}$, $\gamma_{j1}(t) = \gamma_{j2}(t) = 0.1 + \frac{0.1}{1+t^2}$, neutral delays $\tau_{i1}(s) = \theta_{i1}(s) = 0.2 + 0.1 \sin(s)$, $\tau_{i2}(s) = \theta_{i2}(s) = 0.2 - 0.1 \cos(s)$,

$$q_i(s) = \hat{q}_i(s) = 0.2 + 0.1 \sin(s), \rho_i(s) = \hat{\rho}_i(s) = 0.2 - 0.1 \cos(s),$$

$$h_i(s) = \tilde{h}_i(s) = l_i(s) = \tilde{l}_i(s) = \hbar_i(s) = \tilde{\hbar}_i(s) = 0.3 \tanh(s),$$

$$s \in \mathbb{R}, t \geq 0, i = 1, 2, j = 1, 2.$$

Assumptions 1–6 are satisfied by taking $\mathfrak{A}_{1i} = \mathfrak{A}_{2i} = \hat{\mathfrak{A}}_{1j} = \hat{\mathfrak{A}}_{2j} = 0.03$, $\bar{h}_i = \bar{l}_i = \bar{\hbar}_i = \bar{\tilde{h}}_i = \bar{\tilde{l}}_i = \bar{\tilde{\hbar}}_i = 0.3$, $\tilde{h}_i = \tilde{l}_i = \tilde{\hbar}_i = \tilde{\tilde{h}}_i = \tilde{\tilde{l}}_i = \tilde{\tilde{\hbar}}_i = 0.3$, $\bar{q}_i = \bar{q}_i = 0.3$, $\bar{q}_j = \bar{q}_j = 0.3$, $\check{\tau}_{d1} = \hat{\tau}_{d1} = 0.1$, $\check{\tau}_{d2} = \hat{\tau}_{d2} = -0.1$, $\check{\theta}_{d1} = \hat{\theta}_{d1} = 0.1$, $\check{\theta}_{d2} = \hat{\theta}_{d2} = -0.1$, $\check{\tau}_{ij} = 0.1$, $\hat{\tau}_{ij} = 0.3$, $\check{\theta}_{ji} = 0.1$, $\hat{\theta}_{ji} = 0.3$, $\delta_{ij} = 0.3$, $\hat{\gamma}_{ji} = 0.2$, $\alpha_1 = 0.4$, $\alpha_2 = 0.03$, $\beta_1 = 0.4$, $\beta_2 = 0.03$.

According to Theorem 1, there exist $K_1 = 0.7694$, $K_2 = 0.5469$, $\hat{K}_1 = 0.8991$, and $\hat{K}_2 = 0.4314$ which satisfy (2a) and (2b), and we have $\mathfrak{B}_{01} = 0.5027$, $\mathfrak{B}_{02} = 0.4021$,

$\mathfrak{B}_{11} = 0.5286, \mathfrak{B}_{12} = 0.3135, \mathfrak{B}_{21} = 0.0546, \mathfrak{B}_{22} = 0.0546, \mathfrak{B}_{31} = 0.2223, \mathfrak{B}_{32} = 0.2952, \mathfrak{B}_{41} = 0.1768, \mathfrak{B}_{42} = 0.2613, \mathfrak{B}_{51} = 0.0049, \mathfrak{B}_{12} = 0.0039, \mathfrak{C}_{01} = 0.4021, \mathfrak{C}_{02} = 0.4021, \mathfrak{C}_{11} = 0.4866, \mathfrak{C}_{12} = 0.2567, \mathfrak{C}_{21} = 0.0546, \mathfrak{C}_{22} = 0.0546, \mathfrak{C}_{31} = 0.3906, \mathfrak{C}_{32} = 0.3042, \mathfrak{C}_{41} = 0.2718, \mathfrak{C}_{42} = 0.2793, \mathfrak{C}_{51} = 0.0028, \mathfrak{C}_{12} = 0.0020$. Then for any given initial conditions $\varphi_i(t)$ and $\psi_j(t)$, the CGBAMNN (1) has a unique solution. For different initial functions, Figures 1 and 2 show the state trajectories of the CGBAMNN (1). The figures clearly demonstrate that the system exhibits a unique state trajectory under different initial functions, providing visual confirmation that under the conditions of Theorem 1, the CGBAMNN (1) admits a unique solution.

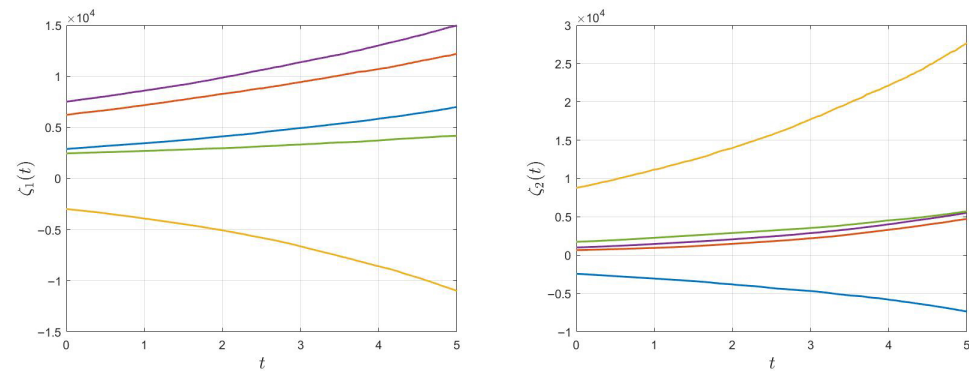


Figure 1. State trajectories of $\zeta_1(t)$ and $\zeta_2(t)$.

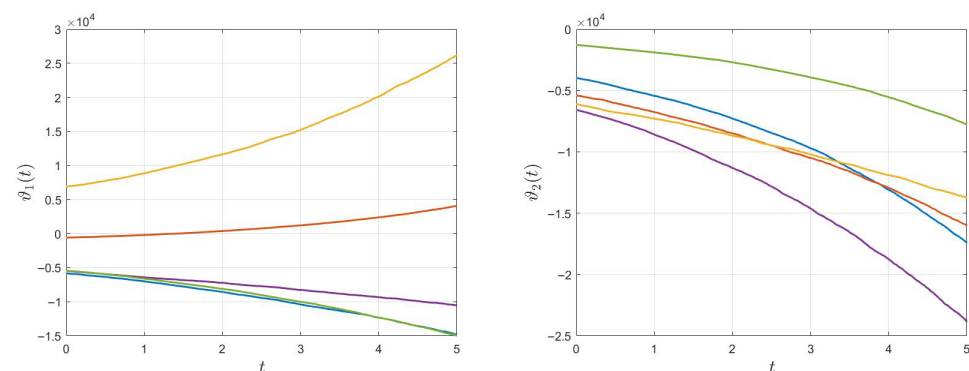


Figure 2. State trajectories of $\theta_1(t)$ and $\theta_2(t)$.

5. Conclusions

This paper delves into the issue of existence and uniqueness of solutions to CGBAMNNs featuring four distinct kinds of time-varying delays. These time-varying delays encompass leakage, neutral, distributed, and transmission delays. The involved system model cannot be formulated in vector–matrix form, so certain techniques for analyzing the performance of such models are not directly applicable. To overcome this hurdle, a novel and general approach is put forward. To establish a new norm, a sufficient condition guarantees the existence and uniqueness of solutions, and a representation of a unique solution is presented for the first time. Significantly, with slight adjustments, this method can be extended to a wide range of delayed system models. Finally, a numerical example is provided to verify the validity of the theoretical findings.

Based on our research foundation, the following issues can be further explored: relaxing the Lipschitz assumptions to handle more general activation functions, investigating stability and synchronization properties, developing efficient numerical algorithms based on our solution representation, and extending the approach to stochastic or impulsive delayed systems, which would significantly broaden the practical applications of these findings in real-world neural network implementations.

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