

Modular H -Irregularity Strength of Graphs

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Abstract

Two new graph characteristics, the modular edge H -irregularity strength and the modular vertex H -irregularity strength, are introduced. Lower bounds on these graph characteristics are estimated, and their exact values are determined for certain families of graphs. This demonstrates the sharpness of the presented lower bounds.

Keywords: (modular) vertex H -irregularity strength; (modular) edge H -irregularity strength; path; fan; friendship graph; ladder

MSC: 05C78

1. Introduction

Consider a simple graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. A labeling of a graph is a mapping that assigns numbers (usually positive integers) to its elements. The most common domains are the vertex set (yielding vertex labelings) and the edge set (yielding edge labelings).

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i , $i = 1, 2, \dots, t$. In this case, we say that G admits an $(H_1, H_2, \dots, H_t)$ -(edge) covering. If each subgraph H_i is isomorphic to a given graph H , then the graph G is said to admit an H -covering.

Let G be a graph admitting an H -covering. For a subgraph $H \subseteq G$, under an edge k -labeling $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$, where k is a positive integer, the associated H -weight is defined by

$$wt_{\varphi}(H) = \sum_{e \in E(H)} \varphi(e).$$

An edge k -labeling φ is called an *H -irregular edge k -labeling* of the graph G if for every pair of distinct subgraphs H_1, H_2 isomorphic to H , it holds that $wt_{\varphi}(H_1) \neq wt_{\varphi}(H_2)$. The smallest integer k for which such a labeling exists is called the *edge H -irregularity strength* of G and is denoted by $ehs(G, H)$.

Similarly, under a vertex k -labeling $\psi : V(G) \rightarrow \{1, 2, \dots, k\}$, the associated H -weight is defined by

$$wt_{\psi}(H) = \sum_{v \in V(H)} \psi(v).$$

A vertex k -labeling ψ is called an *H -irregular vertex k -labeling* of G if for every pair of distinct subgraphs H_1, H_2 isomorphic to H , there is $wt_{\psi}(H_1) \neq wt_{\psi}(H_2)$. The *vertex H -irregularity strength* of G , denoted by $vhs(G, H)$, is the minimum of the maximum label k over all such labelings. If no such labeling exists, we write $vhs(G, H) = \infty$.



Academic Editor: Darren Narayan

Received: 14 July 2025

Revised: 5 August 2025

Accepted: 12 August 2025

Published: 14 August 2025

Citation: Bača, M.; Lascsáková, M.; Semaničová-Feňovčíková, A. Modular H -Irregularity Strength of Graphs. *Mathematics* **2025**, *13*, 2599. <https://doi.org/10.3390/math13162599>

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The concepts of the edge H -irregularity strength and the vertex H -irregularity strength were introduced by Ashraf et al. in [1], who provided lower bounds for the parameter $\text{ehs}(G, H)$ and $\text{vhs}(G, H)$ as follows:

Theorem 1 ([1]). Let G be a graph admitting an H -covering given by t subgraphs isomorphic to H . Then,

$$\text{ehs}(G, H) \geq \left\lceil 1 + \frac{t-1}{|E(H)|} \right\rceil, \quad (1)$$

$$\text{vhs}(G, H) \geq \left\lceil 1 + \frac{t-1}{|V(H)|} \right\rceil. \quad (2)$$

The exact values of $\text{ehs}(G, H)$ and $\text{vhs}(G, H)$ for paths, fans and ladders were determined in [1,2], establishing the sharpness of bounds (1) and (2).

Theorem 2 ([1]). Let n and m be positive integers, $2 \leq m \leq n$. Then,

$$\text{ehs}(P_n, P_m) = \left\lceil \frac{n-1}{m-1} \right\rceil \quad \text{and} \quad \text{vhs}(P_n, P_m) = \left\lceil \frac{n}{m} \right\rceil.$$

Theorem 3 ([2]). Let F_n , $n \geq 2$, be a fan graph on $n+1$ vertices admitting F_m -covering, where m is a positive integer, $2 \leq m \leq n$. Then,

$$\text{ehs}(F_n, F_m) = \left\lceil \frac{n+m-1}{2m-1} \right\rceil \quad \text{and} \quad \text{vhs}(F_n, F_m) = \left\lceil \frac{n+1}{m+1} \right\rceil.$$

Theorem 4 ([2]). Let $L_n \cong P_n \square P_2$, $n \geq 2$, be a ladder admitting L_m -covering, where m is a positive integer, $2 \leq m \leq n$. Then,

$$\text{ehs}(L_n, L_m) = \left\lceil \frac{2m+n-2}{3m-2} \right\rceil \quad \text{and} \quad \text{vhs}(L_n, L_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Several papers have been published in this area over the past four years. For example, the exact values of the edge H -irregularity strength of hexagonal and octagonal grid graphs were computed in [3], and in [4], the vertex (edge) H -irregular labelings of graphs G are investigated, where both G and H are either a comb product or an edge comb product of graphs.

If H is isomorphic to K_2 then a K_2 -irregular vertex k -labeling is equivalent to an edge irregular k -labeling. Hence, the vertex K_2 -irregularity strength $\text{vhs}(G, K_2)$ is the edge irregularity strength $\text{es}(G)$. This graph characteristic was introduced by Ahmad et al. [5], who provided lower bounds and exact values for paths, stars and the Cartesian product of two paths. Additional results on the edge irregularity strength can be found in [6–8]. The edge irregularity strength is an edge modification of the concept of the irregularity strength $s(G)$, originally introduced by Chartrand et al. in [9] and further studied in [10–14].

A natural variation of the H -irregular edge (vertex) k -labeling is a modular H -irregular edge (vertex) k -labeling. Suppose that G admits an $(H_1, H_2, \dots, H_t)$ -covering, where each subgraph H_i is isomorphic to a given graph $H \subseteq G$. An edge labeling $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$ is called a *modular H -irregular edge k -labeling* if the weight function $\lambda : \{H_1, H_2, \dots, H_t\} \rightarrow \mathbb{Z}_t$ given by $\lambda(H_i) = wt_\varphi(H_i) = \sum_{e \in E(H_i)} \varphi(e)$ is bijective, where $\mathbb{Z}_t$ is the group of integers modulo t . The value $wt_\varphi(H_i)$ is called the *modular weight* of the subgraph H_i . The *modular edge H -irregularity strength*, $\text{mehs}(G, H)$, is defined as the smallest integer k for which such a labeling exists.

Analogously, a vertex labeling $\psi : V(G) \rightarrow \{1, 2, \dots, k\}$ is called a *modular H -irregular vertex k -labeling* if the weight function $\lambda : \{H_1, H_2, \dots, H_t\} \rightarrow \mathbb{Z}_t$ defined by $\lambda(H_i) =$

$wt_\psi(H_i) = \sum_{v \in V(H_i)} \psi(v)$ is bijective. Again, $wt_\psi(H_i)$ is called the *modular weight* of the subgraph H_i . The *modular vertex H -irregularity strength*, $mvhs(G, H)$, is defined as the minimum of the maximum vertex label k over all such labelings.

Note that the modular K_2 -irregular vertex k -labeling of a graph G is equivalent to a modular edge irregular k -labeling of G . Therefore, the modular vertex K_2 -irregularity strength $mvhs(G, K_2)$ is the modular edge irregularity strength $mes(G)$ introduced in [15], which also provides exact values for caterpillars, cycles, and friendship graphs. The modular irregularity strength $ms(G)$ as an edge modification of the modular edge irregularity strength $mes(G)$ was introduced in [16], where precise values of the modular irregularity strength for paths, cycles and stars are determined. The exact values of the modular irregularity strength for fan graphs and wheels are addressed in [17] and [18], respectively.

The main objective of the present paper is to estimate lower bounds on the modular edge H -irregularity strength and the modular vertex H -irregularity strength, to investigate the existence of corresponding labelings for various families of graphs, and to determine exact values of these parameters, thereby demonstrating the sharpness of the proposed bounds.

2. Lower Bounds for New Graph Characteristics

Evidently, every modular H -irregular edge k -labeling of a graph G is also an H -irregular edge k -labeling of G , and similarly, every modular H -irregular vertex k -labeling of G is also an H -irregular vertex k -labeling of G . Accordingly, for any simple graph G , the following inequalities hold:

$$ehs(G, H) \leq mehs(G, H), \quad (3)$$

$$vhs(G, H) \leq mvhs(G, H). \quad (4)$$

Of course, in general, the converses of (3) and (4) do not hold. Nevertheless, the validity of the following claims is immediate.

Theorem 5. *Let G be a simple graph admitting an H -covering with $ehs(G, H) = k$. If H -weights under the corresponding H -irregular edge k -labeling form a sequence of consecutive integers, then*

$$ehs(G, H) = mehs(G, H) = k. \quad (5)$$

Theorem 6. *Let G be a simple graph admitting an H -covering with $vhs(G, H) = k$. If H -weights under the corresponding H -irregular vertex k -labeling form a sequence of consecutive integers, then*

$$vhs(G, H) = mvhs(G, H) = k. \quad (6)$$

The existence of a P_m -irregular edge $\left\lceil \frac{n-1}{m-1} \right\rceil$ -labeling and a P_m -irregular vertex $\left\lceil \frac{n}{m} \right\rceil$ -labeling of the path on n vertices is established in the proof of Theorem 2, where the corresponding P_m -weights form a sequence of consecutive integers in both cases. In light of Theorems 5 and 6, we obtain the following corollary.

Corollary 1. *Let P_n , $n \geq 2$, be a path on n vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,*

$$mehs(P_n, P_m) = \left\lceil \frac{n-1}{m-1} \right\rceil \quad \text{and} \quad mvhs(P_n, P_m) = \left\lceil \frac{n}{m} \right\rceil.$$

The *fan graph* F_n , where $n \geq 2$, is the graph obtained by joining each vertex of a path on n vertices to an additional vertex. In the proof of Theorem 3, the existence of an F_m -irregular

edge $\left\lceil \frac{n+m-1}{2m-1} \right\rceil$ -labeling of F_n is established, where the corresponding F_m -weights form a sequence of consecutive integers. By Theorem 5, we obtain the following corollary:

Corollary 2. Let F_n , $n \geq 2$, be a fan graph on $n + 1$ vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mehs}(F_n, F_m) = \left\lceil \frac{n+m-1}{2m-1} \right\rceil.$$

To obtain the modular vertex F_m -irregularity strength of a fan graph, we first establish the following general theorem concerning the join of two graphs. The join $G + F$ of two disjoint graphs G and F is defined as the union of G and F , together with all edges connecting each vertex of G to each vertex of F .

Theorem 7. Let G and F be disjoint graphs such that G admits an H -covering consisting of exactly t subgraphs isomorphic to H . If the graph $G + F$ contains exactly t subgraphs isomorphic to $H + F$, then

$$\text{vhs}(G, H) = \text{vhs}(G + F, H + F).$$

Moreover, if there exists an $(H + F)$ -irregular vertex $\text{vhs}(G, H)$ -labeling of $G + F$, where the corresponding $(H + F)$ -weights form a sequence of consecutive integers, then

$$\text{mvhs}(G, H) = \text{mvhs}(G + F, H + F).$$

Proof. Let G admit an H -covering consisting of exactly t subgraphs isomorphic to H and let $\psi : V(G) \rightarrow \{1, 2, \dots, k\}$ be an H -irregular vertex k -labeling of G , where $k = \text{vhs}(G, H)$. Extend this labeling to the labeling of $G + F$ as follows.

Since $G + F$ contains exactly t subgraphs isomorphic to $H + F$, and each such subgraph must include all vertices of F , it follows that the contribution of labels from vertices in $V(F)$ is identical across all $(H + F)$ -weights. Therefore, without loss of generality, we may assign the label 1 to each vertex in $V(F)$. In this way, the distinguishing power of the labeling relies solely on the labels of vertices in G , implying that $\text{vhs}(G + F, H + F) \leq k$.

Assume, for the sake of contradiction, that $\text{vhs}(G + F, H + F) = k_0 < k$. Then, by restricting the corresponding $(H + F)$ -irregular vertex k_0 -labeling of $G + F$ to the subgraph G , we obtain an H -irregular vertex k_0 -labeling of G , contradicting the minimality of $k = \text{vhs}(G, H)$. Hence, $\text{vhs}(G + F, H + F) = k$.

If, in addition, there exists an $(H + F)$ -irregular vertex $\text{vhs}(G, H)$ -labeling of $G + F$, where the $(H + F)$ -weights form a sequence of consecutive integers, then Theorem 6 implies $\text{mvhs}(G, H) = \text{mvhs}(G + F, H + F)$, as claimed. $\square$

Since the fan graph F_n is isomorphic to the join $P_n + K_1$, Theorem 7 combined with Corollary 1 immediately yields the following result.

Corollary 3. Let F_n , $n \geq 2$, be a fan graph on $n + 1$ vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mvhs}(F_n, F_m) = \left\lceil \frac{n}{m} \right\rceil.$$

The ladder L_n , where $n \geq 2$, is defined as the Cartesian product of paths P_n and P_2 . In the proof of Theorem 4, the existence of an L_m -irregular edge $\left\lceil \frac{2m+n-2}{3m-2} \right\rceil$ -labeling and an L_m -irregular vertex $\left\lceil \frac{m+n}{2m} \right\rceil$ -labeling of the ladder L_n is established. In both cases, the corresponding L_m -weights form a sequence of consecutive integers. With respect to Theorems 5 and 6, we have the following result:

Corollary 4. Let $L_n \cong P_n \square P_2$, $n \geq 2$, be a ladder and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mehs}(L_n, L_m) = \left\lceil \frac{2m+n-2}{3m-2} \right\rceil \quad \text{and} \quad \text{mvhs}(L_n, L_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

The previous corollaries confirm that the lower bounds of the modular edge H -irregularity strength and the modular vertex H -irregularity strength are tight.

3. Friendship Graphs and Ladder-Wing Graphs

A friendship graph f_n , where $n \geq 1$, consists of n triangles sharing a common central vertex and is otherwise independent. Let w denote the central vertex and for the i th triangle, $1 \leq i \leq n$, let u_i and v_i denote the other two vertices. The friendship graph f_n contains $2n + 1$ vertices and $3n$ edges $u_i v_i$, $u_i w$ and $v_i w$ for $1 \leq i \leq n$.

The following theorem provides the exact values of the modular edge H -irregularity strength and the modular vertex H -irregularity strength of the friendship graph f_n , where the subgraph H is isomorphic to f_m , $1 \leq m \leq n$.

Theorem 8. Let f_n , $n \geq 1$, be a friendship graph on $2n + 1$ vertices and let m be a positive integer such that $1 \leq m \leq n$. Then,

$$\text{mehs}(f_n, f_m) = \left\lceil \frac{2m+n}{3m} \right\rceil \quad \text{and} \quad \text{mvhs}(f_n, f_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Proof. Clearly, for every positive integer m such that $1 \leq m \leq n$, the friendship graph f_n on $2n + 1$ vertices admits an f_m -covering consisting of exactly $t = n - m + 1$ subgraphs isomorphic to f_m .

We first consider the case of edge labeling. With respect to (1), we have that $\text{ehs}(f_n, f_m) \geq \left\lceil \frac{2m+n}{3m} \right\rceil$. To show that $\left\lceil \frac{2m+n}{3m} \right\rceil$ is also an upper bound of the edge f_m -irregularity strength, let us define an edge labeling φ as follows:

$$\begin{aligned} \varphi(u_i v_i) &= \left\lceil \frac{2m+i}{3m} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(u_i w) &= \left\lceil \frac{i}{3m} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(v_i w) &= \left\lceil \frac{m+i}{3m} \right\rceil, & \text{for } i = 1, 2, \dots, n. \end{aligned}$$

Evidently, φ is a $\left\lceil \frac{2m+n}{3m} \right\rceil$ -labeling. Let us now compute the f_m -weight of the subgraph f_m^j , $j = 1, 2, \dots, n - m + 1$, under the labeling φ :

$$\begin{aligned} \text{wt}_\varphi(f_m^j) &= \sum_{e \in E(f_m^j)} \varphi(e) = \sum_{i=j}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \varphi(u_j v_j) + \varphi(u_j w) + \varphi(v_j w) \\ &\quad + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \left\lceil \frac{2m+j}{3m} \right\rceil + \left\lceil \frac{j}{3m} \right\rceil + \left\lceil \frac{m+j}{3m} \right\rceil \\ &\quad + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)]. \end{aligned}$$

Similarly, for the weight of the next subgraph f_m^{j+1} , $j = 1, 2, \dots, n - m$, we have

$$\text{wt}_\varphi(f_m^{j+1}) = \sum_{e \in E(f_m^{j+1})} \varphi(e) = \sum_{i=j+1}^{m+j} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \varphi(u_{m+j} v_{m+j}) + \varphi(u_{m+j} w)$$

$$\begin{aligned}
& + \varphi(v_{m+j}w) + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \left\lceil \frac{2m+m+j}{3m} \right\rceil + \left\lceil \frac{m+j}{3m} \right\rceil \\
& + \left\lceil \frac{m+m+j}{3m} \right\rceil + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = 1 + wt_{\varphi}(f_m^j).
\end{aligned}$$

Thus, the f_m -weights under the labeling φ form the set of consecutive integers $\{3m, 3m+1, \dots, n+2m\}$. It follows that φ is an f_m -irregular edge $\left\lceil \frac{2m+n}{3m} \right\rceil$ -labeling. Combining this with Theorem 5, we conclude that $\text{ehs}(f_n, f_m) = \text{mehs}(f_n, f_m) = \left\lceil \frac{2m+n}{3m} \right\rceil$.

Now, we turn our attention to vertex labeling. From Inequality (2), we obtain the lower bound $\text{vhs}(f_n, f_m) \geq \left\lceil \frac{m+n+1}{2m+1} \right\rceil$. On the other hand, under any vertex labeling ψ of f_n , the smallest f_m -weight is at least $2m + \psi(w)$, and therefore, the largest f_m -weight is at least $2m + \psi(w) + t - 1 = n + m + \psi(w)$. Assuming that $\text{vhs}(f_n, f_m) = k$, it follows that the largest f_m -weight is at most $2mk + \psi(w)$. Combining these inequalities yields $n + m + \psi(w) \leq 2mk + \psi(w)$, which simplifies to $k \geq \left\lceil \frac{n+m}{2m} \right\rceil$. To show that $\left\lceil \frac{n+m}{2m} \right\rceil$ is also an upper bound for the vertex f_m -irregularity strength of f_n , we define a vertex labeling ψ as follows:

$$\begin{aligned}
\psi(u_i) &= \left\lceil \frac{m+i}{2m} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\
\psi(v_i) &= \left\lceil \frac{i}{2m} \right\rceil, & \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

Without loss of generality, we may assign the central vertex w the value $\psi(w) = 1$. Clearly, $\psi(v_i) \leq \psi(u_i) \leq \left\lceil \frac{n+m}{2m} \right\rceil$, and hence, ψ is a vertex $\left\lceil \frac{n+m}{2m} \right\rceil$ -labeling. Next, we evaluate the corresponding f_m -weights. For each $j = 1, 2, \dots, n-m+1$, we have

$$\begin{aligned}
wt_{\psi}(f_m^j) &= \sum_{v \in V(f_m^j)} \psi(v) = \sum_{i=j}^{m+j-1} [\psi(u_i) + \psi(v_i)] + \psi(w) = \psi(u_j) + \psi(v_j) + \psi(w) \\
&+ \sum_{i=j+1}^{m+j-1} [\psi(u_i) + \psi(v_i)] = \left\lceil \frac{m+j}{2m} \right\rceil + \left\lceil \frac{j}{2m} \right\rceil + 1 + \sum_{i=j+1}^{m+j-1} [\psi(u_i) + \psi(v_i)].
\end{aligned}$$

Similarly, for $j = 1, 2, \dots, n-m$, we obtain

$$\begin{aligned}
wt_{\psi}(f_m^{j+1}) &= \sum_{v \in V(f_m^{j+1})} \psi(v) = \sum_{i=j+1}^{m+j-1} [\psi(u_i) + \psi(v_i)] + \psi(w) + \psi(u_{m+j}) + \psi(v_{m+j}) \\
&= \sum_{i=j+1}^{m+j-1} [\psi(u_i) + \psi(v_i)] + 1 + \left\lceil \frac{2m+j}{2m} \right\rceil + \left\lceil \frac{m+j}{2m} \right\rceil = 1 + wt_{\psi}(f_m^j).
\end{aligned}$$

Thus, ψ is an f_m -irregular vertex $\left\lceil \frac{m+n}{2m} \right\rceil$ -labeling. Moreover, since the weights of subgraphs f_m^j for $j = 1, 2, \dots, n-m+1$ are not only distinct but form a sequence of consecutive integers $\{2m+1, 2m+2, \dots, n+m+1\}$, it follows from Theorem 6 that $\text{vhs}(f_n, f_m) = \text{mvhs}(f_n, f_m) = \left\lceil \frac{m+n}{2m} \right\rceil$. This completes the proof. $\square$

The ladder-wing graph LW_n , for $n \geq 2$, is obtained by joining all vertices of a ladder L_n to an additional vertex; let us denote it as w . In other words, $LW_n = L_n + K_1$. The following theorem determines the exact values of the modular edge H -irregularity strength and the modular vertex H -irregularity strength for the ladder-wing graph LW_n , where the subgraph H is isomorphic to LW_m , $2 \leq m \leq n$.

Theorem 9. Let LW_n , $n \geq 2$, be a ladder-wing graph on $2n + 1$ vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mehs}(LW_n, LW_m) = \left\lceil \frac{4m+n-2}{5m-2} \right\rceil \quad \text{and} \quad \text{mvhs}(LW_n, LW_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Proof. Let LW_n be a ladder-wing graph on $2n + 1$ vertices with vertex set $V(LW_n) = \{u_i, v_i : i = 1, 2, \dots, n\} \cup \{w\}$ and edge set $E(LW_n) = \{u_i u_{i+1}, v_i v_{i+1} : i = 1, 2, \dots, n-1\} \cup \{u_i v_i, u_i w, v_i w : i = 1, 2, \dots, n\}$. Clearly, for every m , $2 \leq m \leq n$, the ladder-wing graph LW_n admits a LW_m -covering consisting of exactly $t = n - m + 1$ subgraphs isomorphic to LW_m .

We first deal with the edge labeling. According to (1), we have $\text{ehs}(LW_n, LW_m) \geq \left\lceil \frac{4m+n-2}{5m-2} \right\rceil$. To prove the equality, it suffices to construct an edge $\left\lceil \frac{4m+n-2}{5m-2} \right\rceil$ -labeling of LW_n that is LW_m -irregular.

For $n \geq 2$, define an edge labeling φ on LW_n as follows:

$$\begin{aligned} \varphi(u_i u_{i+1}) &= \left\lceil \frac{3m+i}{5m-2} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(v_i v_{i+1}) &= \left\lceil \frac{4m-1+i}{5m-2} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(u_i v_i) &= \left\lceil \frac{2m+i}{5m-2} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(u_i w) &= \left\lceil \frac{i}{5m-2} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(v_i w) &= \left\lceil \frac{m+i}{5m-2} \right\rceil, & \text{for } i = 1, 2, \dots, n. \end{aligned}$$

All values are clearly bounded above by $\left\lceil \frac{4m+n-2}{5m-2} \right\rceil$; so, φ is an edge $\left\lceil \frac{4m+n-2}{5m-2} \right\rceil$ -labeling.

Now, consider the LW_m -subgraphs LW_m^j for $j = 1, 2, \dots, n - m + 1$. The weight of each such subgraph is

$$\begin{aligned} \text{wt}_\varphi(LW_m^j) &= \sum_{e \in E(LW_m^j)} \varphi(e) = \sum_{i=j}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\ &\quad + \sum_{i=j}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] = \varphi(u_j v_j) + \varphi(u_j w) + \varphi(v_j w) + \varphi(u_j u_{j+1}) \\ &\quad + \varphi(v_j v_{j+1}) + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\ &\quad + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] \\ &= \left\lceil \frac{2m+j}{5m-2} \right\rceil + \left\lceil \frac{j}{5m-2} \right\rceil + \left\lceil \frac{m+j}{5m-2} \right\rceil + \left\lceil \frac{3m+j}{5m-2} \right\rceil + \left\lceil \frac{4m-1+j}{5m-2} \right\rceil \\ &\quad + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})]. \end{aligned}$$

For the next subgraph LW_m^{j+1} , $j = 1, 2, \dots, n - m$, the analogous weight is

$$\begin{aligned} \text{wt}_\varphi(LW_m^{j+1}) &= \sum_{e \in E(LW_m^{j+1})} \varphi(e) = \sum_{i=j+1}^{m+j} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\ &\quad + \sum_{i=j+1}^{m+j-1} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] = \varphi(u_{m+j} v_{m+j}) + \varphi(u_{m+j} w) + \varphi(v_{m+j} w) \\ &\quad + \sum_{i=j+1}^{m+j-1} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] \end{aligned}$$

$$\begin{aligned}
& + \varphi(u_{m+j-1}u_{m+j}) + \varphi(v_{m+j-1}v_{m+j}) + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\
& + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] = \left\lceil \frac{3m+j}{5m-2} \right\rceil + \left\lceil \frac{m+j}{5m-2} \right\rceil + \left\lceil \frac{2m+j}{5m-2} \right\rceil \\
& + \left\lceil \frac{4m+j-1}{5m-2} \right\rceil + \left\lceil \frac{5m-2+j}{5m-2} \right\rceil + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\
& + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1})] = 1 + wt_{\varphi}(LW_m^j).
\end{aligned}$$

Thus, the LW_m -weights form a sequence of consecutive integers starting at $5m - 2$ and ending at $4m + n - 2$, i.e., $\{5m - 2, 5m - 1, \dots, 4m + n - 2\}$. By Theorem 5, it follows that $\text{ehs}(LW_n, LW_m) = \text{mehs}(LW_n, LW_m) = \left\lceil \frac{4m+n-2}{5m-2} \right\rceil$.

Now, consider the vertex case. By definition, the ladder-wing graph is obtained as the join of a ladder and an isolated vertex, that is, $LW_n = L_n + K_1$. Therefore, the desired result $\text{vhs}(LW_n, LW_m) = \text{mvhs}(LW_n, LW_m) = \left\lceil \frac{m+n}{2m} \right\rceil$ follows immediately by combining Theorem 7 and Corollary 4. $\square$

In the next result, we consider a graph obtained from the ladder graph by inserting both diagonals into every four-sided face of the ladder. We denote this graph by R_n . Alternatively, R_n can be defined as the strong product of the path on n vertices and the path on two vertices. Let the vertex set and edge set of R_n be given by $V(R_n) = \{u_i, v_i : i = 1, 2, \dots, n\}$ and $E(R_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1}, u_{i+1} v_i : i = 1, 2, \dots, n-1\} \cup \{u_i v_i : i = 1, 2, \dots, n\}$. The following theorem gives the exact values of the modular edge H -irregularity strength and the modular vertex H -irregularity strength for the graph R_n , where the subgraph H is isomorphic to R_m , with $2 \leq m \leq n$.

Theorem 10. Let R_n , $n \geq 2$, be the strong product of the path on n vertices and the path on two vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mehs}(R_n, R_m) = \left\lceil \frac{4m+n-4}{5m-4} \right\rceil \quad \text{and} \quad \text{mvhs}(R_n, R_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Proof. One can observe that the ladder $R_n \cong P_n \boxtimes P_2$, $n \geq 2$, admits an R_m -covering consisting of exactly $t = n - m + 1$ subgraphs isomorphic to R_m , where m is a positive integer satisfying $2 \leq m \leq n$. From Inequality (1), it follows that $\text{ehs}(R_n, R_m) \geq \left\lceil \frac{4m+n-4}{5m-4} \right\rceil$. Set $k = \left\lceil \frac{4m+n-4}{5m-4} \right\rceil$. To show that k is also an upper bound for the edge R_m -irregularity strength of R_n , we define an edge labeling φ as follows:

$$\begin{aligned}
\varphi(u_i u_{i+1}) &= \left\lceil \frac{4m-3+i}{5m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\
\varphi(v_i v_{i+1}) &= \left\lceil \frac{3m-2+i}{5m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\
\varphi(u_i v_i) &= \left\lceil \frac{2m-2+i}{5m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\
\varphi(u_i v_{i+1}) &= \left\lceil \frac{m-1+i}{5m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\
\varphi(u_{i+1} v_i) &= \left\lceil \frac{i}{5m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1.
\end{aligned}$$

It is straightforward to verify that all edge labels used by φ are at most k . For the R_m -weight of the subgraph R_m^j , $j = 1, 2, \dots, n - m + 1$, under the labeling φ , we obtain

$$\begin{aligned} wt_{\varphi}(R_m^j) &= \sum_{e \in E(R_m^j)} \varphi(e) = \sum_{i=j}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] \\ &\quad + \sum_{i=j}^{m+j-1} \varphi(u_i v_i) = \varphi(u_j u_{j+1}) + \varphi(v_j v_{j+1}) + \varphi(u_j v_{j+1}) + \varphi(u_{j+1} v_j) + \varphi(u_j v_j) \\ &\quad + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] + \sum_{i=j+1}^{m+j-1} \varphi(u_i v_i) \\ &= \left\lceil \frac{4m-3+j}{5m-4} \right\rceil + \left\lceil \frac{3m-2+j}{5m-4} \right\rceil + \left\lceil \frac{m-1+j}{5m-4} \right\rceil + \left\lceil \frac{j}{5m-4} \right\rceil + \left\lceil \frac{2m-2+j}{5m-4} \right\rceil \\ &\quad + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] + \sum_{i=j+1}^{m+j-1} \varphi(u_i v_i). \end{aligned}$$

For the R_m -weight of the subgraph R_m^{j+1} , $j = 1, 2, \dots, n - m$, we have

$$\begin{aligned} wt_{\varphi}(R_m^{j+1}) &= \sum_{e \in E(R_m^{j+1})} \varphi(e) = \sum_{i=j+1}^{m+j-1} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] \\ &\quad + \sum_{i=j+1}^{m+j} \varphi(u_i v_i) = \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] \\ &\quad + \varphi(u_{m+j-1} u_{m+j}) + \varphi(v_{m+j-1} v_{m+j}) + \varphi(u_{m+j-1} v_{m+j}) + \varphi(u_{m+j} v_{m+j-1}) \\ &\quad + \sum_{i=j+1}^{m+j-1} \varphi(u_i v_i) + \varphi(u_{m+j} v_{m+j}) = \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) \\ &\quad + \varphi(u_{i+1} v_i)] + \left\lceil \frac{5m-4+j}{5m-4} \right\rceil + \left\lceil \frac{4m-3+j}{5m-4} \right\rceil + \left\lceil \frac{2m-2+j}{5m-4} \right\rceil + \left\lceil \frac{m-1+j}{5m-4} \right\rceil \\ &\quad + \sum_{i=j+1}^{m+j-1} \varphi(u_i v_i) + \left\lceil \frac{3m-2+j}{5m-4} \right\rceil = 1 + wt_{\varphi}(R_m^j). \end{aligned}$$

This means that R_m -weights successively attain values $5m - 4, 5m - 3, \dots, 4m + n - 4$ and are distinct for all pairs of distinct subgraph R_m . The edge labeling φ thus provides an upper bound on the edge R_m -irregularity strength of the ladder R_n . Consequently, we obtain $\text{ehs}(R_n, R_m) = \left\lceil \frac{4m+n-4}{5m-4} \right\rceil$. According to Theorem 5, we obtain that $\text{ehs}(R_n, R_m) = \text{mehs}(R_n, R_m)$.

Since the graphs $P_n \boxtimes P_2$ and $P_n \square P_2$ have the same vertex set, and all corresponding subgraphs $P_m \boxtimes P_2$ and $P_m \square P_2$ likewise have identical vertex sets, it follows from Corollary 4 that $\text{mvhs}(R_n, R_m) = \text{mvhs}(L_n, L_m) = \left\lceil \frac{m+n}{2m} \right\rceil$. This concludes the proof. $\square$

Consider a new graph, called the *ladder-wing graph* and denoted by RW_n , $n \geq 2$, which is obtained by joining all vertices of the ladder $R_n \cong P_n \boxtimes P_2$ to an additional vertex w . This graph has $2n + 1$ vertices, with vertex set $V(RW_n) = \{u_i, v_i : i = 1, 2, \dots, n\} \cup \{w\}$ and edge set $E(RW_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1}, u_{i+1} v_i : i = 1, 2, \dots, n - 1\} \cup \{u_i v_i, u_i w, v_i w : i = 1, 2, \dots, n\}$.

The following theorem provides the exact values of the modular edge H -irregularity strength and the modular vertex H -irregularity strength for ladder-wing graph RW_n , where the subgraph H is isomorphic to RW_m with $2 \leq m \leq n$.

Theorem 11. Let RW_n , $n \geq 2$, be a ladder-wing graph on $2n + 1$ vertices and let m be a positive integer such that $2 \leq m \leq n$. Then,

$$\text{mehs}(RW_n, RW_m) = \left\lceil \frac{6m+n-4}{7m-4} \right\rceil \quad \text{and} \quad \text{mvhs}(RW_n, RW_m) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Proof. Clearly, the ladder-wing graph RW_n , $n \geq 2$, admits an RW_m -covering consisting of exactly $t = n - m + 1$ subgraphs for every m such that $2 \leq m \leq n$. According to (1), it follows that

$$\text{ehs}(RW_n, RW_m) \geq \left\lceil \frac{6m+n-4}{7m-4} \right\rceil. \quad (7)$$

To prove the converse inequality, we construct an edge labeling φ such that

$$\begin{aligned} \varphi(u_i u_{i+1}) &= \left\lceil \frac{6m-3+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(v_i v_{i+1}) &= \left\lceil \frac{5m-2+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(u_i v_i) &= \left\lceil \frac{4m-2+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(u_i v_{i+1}) &= \left\lceil \frac{3m-1+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(u_{i+1} v_i) &= \left\lceil \frac{2m+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n-1, \\ \varphi(u_i w) &= \left\lceil \frac{m+i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n, \\ \varphi(v_i w) &= \left\lceil \frac{i}{7m-4} \right\rceil, & \text{for } i = 1, 2, \dots, n. \end{aligned}$$

It is straightforward to verify that under the previously defined labeling, all edge labels are at most $\left\lceil \frac{6m+n-4}{7m-4} \right\rceil$. For the RW_m -weight of the subgraph RW_m^j , $j = 1, 2, \dots, n - m + 1$, we obtain

$$\begin{aligned} wt_{\varphi}(RW_m^j) &= \sum_{e \in E(RW_m^j)} \varphi(e) = \sum_{i=j}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] \\ &\quad + \sum_{i=j}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \varphi(u_j u_{j+1}) + \varphi(v_j v_{j+1}) + \varphi(u_j v_{j+1}) \\ &\quad + \varphi(u_{j+1} v_j) + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] \\ &\quad + \varphi(u_j v_j) + \varphi(u_j w) + \varphi(v_j w) + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\ &= \left\lceil \frac{6m-3+j}{7m-4} \right\rceil + \left\lceil \frac{5m-2+j}{7m-4} \right\rceil + \left\lceil \frac{3m-1+j}{7m-4} \right\rceil + \left\lceil \frac{2m+j}{7m-4} \right\rceil \\ &\quad + \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] + \left\lceil \frac{4m-2+j}{7m-4} \right\rceil \\ &\quad + \left\lceil \frac{m+j}{7m-4} \right\rceil + \left\lceil \frac{j}{7m-4} \right\rceil + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)]. \end{aligned}$$

For the RW_m -weight of the subgraph RW_m^{j+1} , $j = 1, 2, \dots, n - m$, we obtain

$$wt_{\varphi}(RW_m^{j+1}) = \sum_{e \in E(RW_m^{j+1})} \varphi(e) = \sum_{i=j+1}^{m+j-1} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)]$$

$$\begin{aligned}
& + \sum_{i=j+1}^{m+j} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] = \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) \\
& + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] + \varphi(u_{m+j-1} u_{m+j}) + \varphi(v_{m+j-1} v_{m+j}) \\
& + \varphi(u_{m+j-1} v_{m+j}) + \varphi(u_{m+j} v_{m+j-1}) + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] \\
& + \varphi(u_{m+j} v_{m+j}) + \varphi(u_{m+j} w) + \varphi(v_{m+j} w) = \sum_{i=j+1}^{m+j-2} [\varphi(u_i u_{i+1}) + \varphi(v_i v_{i+1}) \\
& + \varphi(u_i v_{i+1}) + \varphi(u_{i+1} v_i)] + \left\lceil \frac{7m-4+j}{7m-4} \right\rceil + \left\lceil \frac{6m-3+j}{7m-4} \right\rceil + \left\lceil \frac{4m-2+j}{7m-4} \right\rceil \\
& + \left\lceil \frac{3m-1+j}{7m-4} \right\rceil + \sum_{i=j+1}^{m+j-1} [\varphi(u_i v_i) + \varphi(u_i w) + \varphi(v_i w)] + \left\lceil \frac{5m-2+j}{7m-4} \right\rceil \\
& + \left\lceil \frac{2m+j}{7m-4} \right\rceil + \left\lceil \frac{m+j}{7m-4} \right\rceil = 1 + wt_{\varphi}(RW_m^j).
\end{aligned}$$

Observe that RW_m -weights successively receive values $7m-4, 7m-3, \dots, 6m+n-4$. This implies that the weights are distinct for all pairs of distinct subgraphs isomorphic to R_m , and therefore, edge labeling φ is a desired RW_m -irregular edge $\left\lceil \frac{6m+n-4}{7m-4} \right\rceil$ -labeling. Hence, this labeling provides an upper bound on $\text{ehs}(RW_n, RW_m)$. Combined with the lower bounds established in (7), we conclude that $\text{ehs}(RW_n, RW_m) = \left\lceil \frac{6m+n-4}{7m-4} \right\rceil$. The desired result then follows directly from Theorem 5.

Since the graphs LW_n and RW_n have the same vertex set, and each corresponding pair of subgraphs LW_m and RW_m likewise have identical vertex sets, it follows from Theorem 9 that $\text{mvhs}(LW_n, LW_m) = \text{mvhs}(RW_n, RW_m) = \left\lceil \frac{m+n}{2m} \right\rceil$. This completes the proof. $\square$

4. Conclusions

In this paper, we introduce two novel graph invariants: the modular edge H -irregularity strength, denoted $\text{mehs}(G, H)$, and the modular vertex H -irregularity strength, denoted $\text{mvhs}(G, H)$. These are natural extensions of existing irregularity strength concepts. We establish general lower bounds and determine exact values for several graph families, including fan graphs, friendship graphs, ladders, and ladder-wing graphs. In each case, the subgraphs H are isomorphic to graphs of the same type but on a smaller number of vertices. These results confirm the sharpness of the proposed bounds and demonstrate the utility of modular labelings in distinguishing subgraph structures.

It may be of interest to determine the exact values of these invariants in cases where G and H are not of the same type. In the examples presented, the subgraphs H are, in some sense, linearly ordered. It would be challenging to explore cases where the H -covering of G has a more complex structure. A simple example is the case where a cycle C_n is covered by paths P_m for $2 \leq m \leq n$. Considering the two extremal cases, covering the cycle C_n by P_2 and P_n , we trivially obtain $\text{mehs}(C_n, P_2) = \text{mehs}(C_n, P_n) = n$. This follows from the fact that the P_2 -weights correspond to the label of a single edge, which must be unique since all weights must differ. For P_n , the weights equal the sum of all but one edge label on C_n . For $m = 3$, we have that $\text{mehs}(C_n, P_3)$ equals the modular vertex-irregularity strength $\text{ms}(C_n)$, as defined by Bača et al. in [16]. This is because the weight of each path P_3 in C_n essentially corresponds to the vertex weight of the vertex of degree 2 in the path. This observation can be trivially generalized to any r -regular graph G , where $\text{mehs}(G, K_{1,r}) = \text{ms}(G)$. Moreover, for regular graphs, the concept of $\text{vhs}(G, K_{1,r})$ is closely related to the inclusive distance vertex irregularity strength introduced by Bača et al. [19].

Future work may focus on determining $\text{mehs}(G, H)$ and $\text{mvhs}(G, H)$ for other graph families, as well as exploring algorithmic aspects and investigating connections with other graph operations or labeling parameters.

All of these concepts open new avenues for studying labeling parameters that induce not only distinct subgraph weights, but weights that are distinct modulo the number of subgraphs. This implies that subgraphs can be uniquely identified by their weights, which may serve as a foundation for potential applications—an exciting direction we hope will be explored in future research.

Author Contributions: Conceptualization, M.B., M.L. and A.S.-F.; methodology, M.B., M.L. and A.S.-F.; software, A.S.-F.; validation, M.B., M.L. and A.S.-F.; formal analysis, M.B., M.L. and A.S.-F.; investigation, M.B., M.L. and A.S.-F.; resources, M.B., M.L. and A.S.-F.; data curation, M.B., M.L. and A.S.-F.; writing—original draft preparation, M.B.; writing—review and editing, M.B., M.L. and A.S.-F.; visualization, M.B., M.L. and A.S.-F.; supervision, M.B. and A.S.-F.; project administration, M.B. and A.S.-F.; funding acquisition, M.B. and A.S.-F. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Slovak Research and Development Agency under the contract No. APVV-23-0191 and by VEGA 1/0243/23.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors on request.

Conflicts of Interest: The authors declare no conflicts of interest.

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