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Biderivations of Simple Modular Lie Algebras of Cartan-Type

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Abstract

Assume that L is a simple Lie algebra of Cartan-type over an algebraically closed field with a characteristic $p > 3$. We demonstrate that all symmetric biderivations vanish by using weight space decompositions relative to a suitable torus and the standard $\mathbb{Z}$-grading structures of L. We then conclude that every biderivation of L is inner, based on a general result concerning skew-symmetric biderivations. As the direct applications, we determine the linear commuting maps and commutative post-Lie algebra structures on L completely.

Keywords: Lie algebras of cartan-type; biderivations; commuting maps; commutative post-Lie algebra structures

MSC: 17B50; 17B70; 17B05; 17B40

1. Introduction

Within the frameworks of ring and algebra theories, derivations and their various generalizations play an important role, because of their own significance as well as their wide applications in other related fields. The concept of a biderivation is applicable to any ring or algebra. In this general context, it was initially explored in [1]. In the context of the Lie algebra, biderivations made their first appearance in [2]. In recent years, there has been a growing interest in investigating biderivations of Lie algebras and related topics, including commuting maps and commutative post-Lie algebra structures (see [2–14]). Furthermore, an investigation has been undertaken on related issues concerning biderivations in pertinent algebra (see [15–18]). It is evident that biderivations have a number of applications in the field of physics. For example, they can be employed to elucidate the symmetry and inversion of forces between particles when delineating the interactions of elementary particles; in the quantum field theory, the examination of the behavior and laws of derivations and biderivations, as they evolve in conjunction with time and parameters, serves as a means to articulate the state and progression of the field. The concepts that are extremely closely related to the concept of biderivations are linear commuting maps and the commutative post-Lie algebra structures (see Section 5). The former was triggered by a famous theorem of Posner [19] and has been studied on various algebraic structures [1,3,20,21]. In [20], a variety of generalizations of the concept of a commuting map, along with applications of results concerning commuting maps to diverse areas, most notably Lie theory, are discussed. Post-Lie algebras, introduced by Valette, are associated with the homology of partition posets and the analysis of Koszul operads [22]. As noted by [23], post-Lie algebras serve as a natural generalization of pre-Lie algebras and LR-algebras within the geometric framework of nil-affine actions of Lie groups. Post-Lie

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Academic Editor: Ivan Kaygorodov

Received: 25 June 2025
Revised: 21 July 2025
Accepted: 30 July 2025
Published: 13 August 2025

Citation: Bai, W.; Liu, W. Biderivations of Simple Lie Algebras of Cartan-Type. *Mathematics* **2025**, *13*, 2596. https://doi.org/10.3390/math13162596

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algebras emerge in various fields, such as isospectral flows, Yang–Baxter equations, Lie–Butcher Series, and Moving Frames [24,25]. For recent developments in post-Lie algebras, interested readers are directed to the works of [23,26–28].

The current classification theory of Block–Wilson–Strade–Premet confirms that classical, Cartan-type, and Melikian algebras encompass all finite-dimensional simple Lie algebras with a characteristic $p > 3$. Indeed, additional examples of simple Lie algebras in characteristics 2 and 3 were identified as early as 1967. Y. Kotchetkov and D. Leites constructed simple Lie algebras over fields of characteristic 2 by leveraging superalgebra structures, suggesting a broader range of constructions could yield numerous examples in characteristic 2. The Lie algebras of Cartan-type are defined over all fields with characteristics $p \geqslant 3$. When $p = 3$, these algebras may exhibit some unexpected properties (see [29], Sections 4 and 7). Therefore, in this paper, we make the convention that L is a Cartan-type simple Lie algebra in characteristic $p > 3$. We will demonstrate that all the biderivations of L are inner. A standard fact is that every biderivation of a Lie algebra can be expressed as the sum of a symmetric biderivation and a skew-symmetric biderivation. In [3], skew-symmetric biderivations are uniformly characterized for a Lie algebra over fields of characteristic $p \neq 2$. It follows from a result of Corollary 2.4 in [3] that every skew-symmetric biderivation of L must be inner. We should point out that the above conclusion has been derived through a direct computation in [4,5] and with the additional condition that L is restricted. As a simple application of the conclusion on skew-symmetric biderivations, we find that all commuting maps on L are scalar multiples of the identity. The investigation of symmetric biderivations may be reduced to an analysis of their related linear maps (see Equation (2)), as both components of a biderivation are derivations (see Equation (1)) and L is centerless. According to the results on the derivations mentioned in [29], we can find that the symmetric biderivations of L are zero. We should mention that, in [23], using the Levi decompositions, it is proved that any commutative post-Lie algebra structure on a finite-dimensional, complex, perfect Lie algebra is trivial (i.e., it is zero). Notice that the aforementioned algebras are finite-dimensional perfect Lie algebras over an algebraically closed field of characteristic $p > 3$. We naturally want to know if these structures are also trivial. Using our results on the symmetric biderivations, we find that any commutative post-Lie algebra structure on L is trivial.

We conclude this introduction by outlining our conventions. Throughout, all vector spaces are defined over the underlying base field $\mathbb{F}$, an algebraically closed field with characteristic $p > 3$. Except for the standard notation $\mathbb{Z}$, we denote $\mathbb{N}^*$ and $\mathbb{N}$ as the sets of positive integers and nonnegative integers, respectively. For a proposition P, we define $\delta_P = 1$ if P holds and $\delta_P = 0$ otherwise.

2. Generalities and Main Results

2.1. Biderivations and Related Definitions

Assume that G is a Lie algebra and M is an G-module. Consider the following definitions relative to biderivations of G.

A linear map $\phi : G \longrightarrow M$ is termed a derivation from G to M if it satisfies

$$\phi([x, y]) = x \cdot \phi(y) - y \cdot \phi(x),\ x, y \in G.$$

A derivation from G to G is termed a derivation of G. Let Der$G$ denote the space consisting of all derivations of G.

A bilinear map $\vartheta : G \times G \longrightarrow M$ is termed a biderivation from G to M if it satisfies the following

$$\begin{aligned} x \cdot \vartheta(y,z) - y \cdot \vartheta(x,z) &= \vartheta([x,y],z), \\ y \cdot \vartheta(x,z) - z \cdot \vartheta(x,y) &= \vartheta(x,[y,z]),\ x,y \in G. \end{aligned} \tag{1}$$

A biderivation from $G \times G$ to G is termed a biderivation of G. Let BDer$G$ denote the space consisting of all biderivations of G.

A biderivation ϑ of G is termed weakly inner if there exists a Lie algebra $\widetilde{G}$ containing G and a unique linear map $\phi_\vartheta : G \longrightarrow \widetilde{G}$, referred to as related to ϑ, that satisfies

$$[\phi_\vartheta(x), y] = \vartheta(x,y),\ \ x,y \in G. \tag{2}$$

Moreover, ϑ is called inner if ϕ_ϑ is a scalar transformation.

A linear map $\varphi : G \longrightarrow G$ is termed commuting if $[x, \varphi(x)] = 0$ for all $x \in G$. A scalar transformation is commuting.

A bilinear multiplication $\cdot : G \times G \longrightarrow G$ is termed commutative post-Lie algebra structure on a Lie algebra G if it satisfies

$$\begin{aligned} x \cdot y &= y \cdot x; \\ x \cdot (y \cdot z) - y \cdot (x \cdot z) &= [x,y] \cdot z; \\ [x \cdot y, z] + [y, x \cdot z] &= x \cdot [y,z],\ x,y,z \in G. \end{aligned}$$

The commutative post-Lie algebra structure on G defined by $x \cdot y = 0$, for all $x, y \in G$, is said to be trivial.

2.2. Simple Lie Algebras of Cartan-Type

Review the notion of simple Lie algebras of Cartan-type [29,30]. Given $m \in \mathbb{N}^*$ and $\underline{t} = (t_1, t_2, \ldots, t_m) \in \mathbb{N}^{*m}$. Put $\pi = (p^{t_1} - 1, p^{t_2} - 1, \ldots, p^{t_m} - 1)$ and $\mathbf{I} = \{1, \ldots, m\}$. Put

$$\mathbf{A}(m;\underline{t}) = \{\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \mid 0 \leq \alpha_j \leq p^{t_j} - 1, j \in \mathbf{I}\}.$$

Denote $\mathcal{O}(m;\underline{t})$ as the divided power algebra with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbf{A}(m;\underline{t})\}$. We abbreviate $x^{(\varepsilon_i)}$ to x_i for $i \in \mathbf{I}$, where $\varepsilon_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$ and δ_{ij} is the Kronecker symbol. Denote $\partial_1, \ldots, \partial_m$ as the derivations of $\mathcal{O}(m;\underline{t})$ satisfying $\partial_i(x^{(\alpha)}) = x^{(\alpha-\varepsilon_i)}$.

Write

$$W(m;\underline{t}) = \mathrm{span}_{\mathbb{F}}\{f\partial_j \mid f \in \mathcal{O}(m;\underline{t}), j \in \mathbf{I}\},$$

termed the generalized Jacobson-Witt algebra.

Write

$$S(m;\underline{t}) = \mathrm{span}_{\mathbb{F}}\{D_{ij}(f) \mid f \in \mathcal{O}(m;\underline{t}), i,j \in \mathbf{I}\},$$

termed the special algebra, where

$$D_{ij}(f) = \partial_j(f)\partial_i - \partial_i(f)\partial_j.$$

For $i \in \{1, \ldots, 2\lfloor \frac{m}{2} \rfloor\}$, we put

$$\sigma(i) = \begin{cases} -1, & i \in \overline{\lfloor \frac{m}{2} \rfloor + 1, 2\lfloor \frac{m}{2} \rfloor}; \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad i' = \begin{cases} i + \lfloor \frac{m}{2} \rfloor, & i \in \overline{1, \lfloor \frac{m}{2} \rfloor}; \\ i - \lfloor \frac{m}{2} \rfloor, & i \in \overline{\lfloor \frac{m}{2} \rfloor + 1, 2\lfloor \frac{m}{2} \rfloor}; \\ i, & \text{otherwise.} \end{cases}$$

Assume that $m = 2r$ and provide a linear mapping

$$D_H : \mathcal{O}(m;\underline{t}) \longrightarrow W(m;\underline{t}), \quad D_H(f) = \sum_{j\in\mathbf{I}} \sigma(j)\partial_j(f)\partial_{j'},\ f \in \mathcal{O}(m;\underline{t}).$$

The image space of D_H is denoted as $\widetilde{H}(m;\underline{t})$. Put $\widetilde{\mathcal{O}}(m;\underline{t})$ as the quotient space $\mathcal{O}(m;\underline{t})/\mathbb{F}$, and define the bracket as follows:

$$[f,g] = D_H(f)(g) \ \text{ for } f,g \in \widetilde{\mathcal{O}}(m;\underline{t}).$$

Then, we obtain a Lie algebra isomorphism $\widetilde{\mathcal{O}}(m;\underline{t}) \cong \widetilde{H}(m;\underline{t})$.

Let $H(m;\underline{t}) := [\widetilde{H}(m;\underline{t}), \widetilde{H}m;\underline{t})]$, which is termed the Hamiltonian algebra.

Assume that $m = 2r+1$ and define a linear mapping

$$D_K : \mathcal{O}(m;\underline{t}) \longrightarrow W(m;\underline{t}), \quad D_K(f) = D_H(f) + \partial_m(f)\mathfrak{E} + \Theta(f)\partial_m,\ f \in \mathcal{O}(m;\underline{t}),$$

where $\mathfrak{E} = \sum_{j\in\mathbf{I}\setminus\{m\}} x_j\partial_j$ and $\Theta(g) = 2g - \mathfrak{E}(g)$. Let $\widetilde{K}(m;\underline{t})$ be the image space of D_K, which is a subalgebra of the Jacobson–Witt algebra. We have a Lie algebra isomorphism $\widetilde{K}(m;\underline{t}) \cong (\mathcal{O}(m;\underline{t}), [\ ,\ ])$, where the bracket is given by

$$[f,g] = D_H(f)g + \Theta(f)\partial_m(g) - \partial_m(f)\Theta(g) \ \text{ for } f,g \in \mathcal{O}(m;\underline{t}).$$

The derived subalgebra $K(m;\underline{t}) := [\widetilde{K}(m;\underline{t}), \widetilde{K}(m;\underline{t})]$ is called the contact algebra.

$W(m;\underline{t}), m \in \mathbb{N}^*$; $S(m;\underline{t}), m \geq 3$; $H(m;\underline{t}), K(m;\underline{t}), m \geq 2$ are simple. Usually we simply write X for $X(m;\underline{t})$, where $X = \mathcal{O}$, W, S, H or K. Notice that the Lie algebras W, S, H and K are finite-dimensional, which are said to be of Cartan-type.

2.3. Main Conclusions

The main conclusions of this paper are listed below.

Theorem 1. *Assume that L is a finite-dimensional simple Lie algebra of Cartan-type W, S, H or K. Then*

(1) Every biderivation of L is inner;
(2) A linear map on L is commuting if and only if it is a scalar transformation;
(3) Every commutative post-Lie algebra structure is trivial on L.

Proof. This follows directly from Propositions 1–4. □

3. Reductions and Technical Lemmas

3.1. Skew-Symmetric Biderivations

Let f be a bilinear map of a Lie algebra G. If it satisfies

$$f(x,y) = f(y,x) (\text{resp. } f(x,y) = -f(y,x)),\ x,y \in G,$$

then it is termed symmetric (resp. skew-symmetric). The following conclusion holds for any $\vartheta \in \mathrm{BDer}G$:

$$\vartheta = \rho + \tau,$$

where for $x, y \in G$,

$$\rho(x,y) = \frac{\vartheta(x,y) + \vartheta(y,x)}{2}, \tau(x,y) = \frac{\vartheta(x,y) - \vartheta(y,x)}{2}.$$

Note that $\rho \in \mathrm{BDer}G$ and is symmetric, $\tau \in \mathrm{BDer}G$ and is skew-symmetric.

Lemma 1 ([3], Corollary 2.4)**.** *Let G be a perfect and centerless Lie algebra over a field with characteristic $p \neq 2$. Assume that $\vartheta \in \mathrm{BDer}G$ and is skew-symmetric. Then it is of the form*

$$\vartheta(x,y) = \mu([x,y]),\ x,y \in G,$$

where μ is an G-module endomorphism on G.

Proposition 1. *Every skew-symmetric biderivation is inner on any of the Lie algebras W, S, H or K.*

Proof. By Lemma 1 and Schur's Lemma, the result is thus proven. □

3.2. Decompositions

Review the standard $\mathbb{Z}$-gradings of $\mathcal{O}$, W, S, H, K. Put the $\mathbb{Z}$-degrees of x_j and ∂_j to be $\mathrm{zd}(x_j) = -\mathrm{zd}(\partial_j) = 1 + \delta_{L=K}\delta_{jm}$, $j \in \mathbf{I}$. From now on, $\mathrm{zd}(x)$ means that x is $\mathbb{Z}$-homogeneous. Put $\eta = \sum_{j=1}^{m} p^{t_j} - m + \delta_{L=K}(p^{t_m} - 1)$. Then

$$\begin{aligned}
&\mathcal{O} = \oplus_{j=0}^{\eta}\mathcal{O}_j,\ \ \mathcal{O}_j = \mathrm{span}_{\mathbb{F}}\{f \in \mathcal{O} \mid \mathrm{zd}(f) = j\};\\
&W = \oplus_{j=-1}^{\eta-1}W_j,\ \ W_j = \mathrm{span}_{\mathbb{F}}\{f\partial_i \mid f \in \mathcal{O}_{j+1},\ i \in \mathbf{I}\};\\
&S = \oplus_{j=-1}^{\eta-2}S_j,\ \ S_j = \mathrm{span}_{\mathbb{F}}\{D_{ik}(f) \in W \mid f \in \mathcal{O}_{j+2},\ i,k \in \mathbf{I}\};\\
&H = \oplus_{j=-1}^{\eta-3}H_j,\ \ H_j = \mathrm{span}_{\mathbb{F}}\{f \mid f \in \widetilde{\mathcal{O}}_{j+2}\};\\
&K = \oplus_{j=-2}^{\eta-2-\delta_{m+3\equiv 0 \pmod p}}K_j,\ \ K_j = \mathrm{span}_{\mathbb{F}}\{f \mid f \in \mathcal{O}_{j+2}\}.
\end{aligned}$$

Put $h_j = x_j\partial_j$, $j \in \mathbf{I}$ and $T = \oplus_{j\in\mathbf{I}}\mathbb{F}h_j$.

Remark 1. *Every m-tuple $a = (a_1, \ldots, a_m) \in \mathbf{A}(m;\underline{t})$ is defined as a linear form on T by reduction module p as follows: $a(h_j) = a_j$.*

Put

$$\begin{aligned}
&T_L = \mathrm{Span}_{\mathbb{F}}\{h_i - h_j \mid i \neq j \in \mathbf{I}\},\ L = W, S;\\
&T_L = \mathrm{Span}_{\mathbb{F}}\{x_i x_{i'} \mid i \in \{1, \ldots, r\}\},\ L = H, K.
\end{aligned}$$

Note that T_L is a torus of L.

Remark 2. *Let $G = \oplus_{j=-r}^{s}G_j$ be an arbitrary $\mathbb{Z}$-graded Lie algebra and $T_G \subset G_0$ be a torus of G. Then there exist the weight space decompositions of G with respect to T_G:*

$$G = G^{\theta} \oplus \oplus_{\gamma\in\Omega_G}G^{\gamma},\ \ G_j = G_j^{\theta} \oplus \oplus_{\gamma\in\Omega_{G_j}}G_j^{\gamma},$$

where Ω_G (resp. Ω_{G_j}) is the weight set of G (resp. G_j) and θ is the zero weight. Clearly, $\Omega_{G_j} \subset \Omega_G \subset T_G^*$. The elements in G^{γ} are termed T_G^*-homogeneous with weight γ. Note that $\mathrm{BDer}G$ *carries over the T_G^*-grading and $\mathbb{Z}$-grading from G:*

$$\mathrm{BDer}G = \oplus_{a\in T_G^*}\mathrm{BDer}(G)^a, \quad \mathrm{BDer}G = \oplus_{j\in\mathbb{Z}}\mathrm{BDer}(G)_j,$$

where

$$\begin{aligned}
&\mathrm{BDer}(G)^a = \{\vartheta \in \mathrm{BDer}G \mid \vartheta(G^b, G^c) \subseteq G^{a+b+c},\ b,c \in T_G^*\},\\
&\mathrm{BDer}(G)_j = \{\vartheta \in \mathrm{BDer}G \mid \vartheta(G_i, G_k) \subseteq G_{i+j+k},\ i,k \in \mathbb{Z}\}.
\end{aligned}$$

A Lie algebra G is termed weakly complete if there exists a Lie algebra $\overline{G}$ such that G can be embedded as an ideal and satisfying

$$\mathrm{Der}G = \mathrm{ad}_G\overline{G};\ \ \mathrm{Ann}_{\overline{G}}G = \{x \in \overline{G} \mid [x, G] = 0\} = \{0\}.$$

Lemma 2. *Assume that G is a centerless Lie algebra. Then G is weakly complete and any biderivation of G is weakly inner. Furthermore, if $\vartheta \in \mathrm{BDer}(G)_i^a$, where $a \in T_G^*$, $i \in \mathbb{Z}$, then the linear map related to ϑ, ϕ_ϑ is T_G^*-homogeneous with weight a and $\mathrm{zd}(\phi_\vartheta) = i$.*

Proof. Obviously, $G \cong \mathrm{ad}(G) \rhd \mathrm{Der}G$ since G is centerless. Then choosing $\overline{G} = \mathrm{Der}G$ shows G is weakly complete. For $\vartheta \in \mathrm{BDer}(G)$, we know that ϑ is a derivation concerning both components. Since G is weakly complete, the results are thus proven. □

Put $\mathfrak{V} = \mathrm{Span}_{\mathbb{F}}\{\partial_j^{k_j} \mid j \in \mathbf{I}, 0 < k_j < t_j\}$ and $h = \sum_{j\in\mathbf{I}} h_j$.

Lemma 3 ([29])**.** *Assume that $L = W, S, H$ or K. Then*

$$\mathrm{Der}L = \mathrm{ad}_L(\mathfrak{V} \ltimes \widehat{L}),$$

where

$$\widehat{L} = \begin{cases} W, & L = W; \\ S + \sum_{j=1}^m \mathbb{F}x^{(\pi-(p^{t_j}-1)\varepsilon_j)}\partial_j + T, & L = S; \\ H + \mathbb{F}x^{(\pi)} + \sum_{j=1}^m \mathbb{F}x^{(p^{t_j}\varepsilon_j)} + \mathbb{F}h, & L = H; \\ K, & L = K,\ m+3 \not\equiv 0 \pmod p; \\ K + \mathbb{F}x^{(\pi)}, & L = K,\ m+3 \equiv 0 \pmod p. \end{cases}$$

Lemma 4. *Assume that $L = W, S, H$ or K.*

(1) *$\widehat{L}^\nu \neq 0$ if and only if $\nu \in \Omega_L \cup \theta$.*
(2) *$\widehat{L}$ is transitive.*

Proof. Through direct calculation, we get

$$\{x^{(\pi-(p^{t_j}-1)\varepsilon_j)}\partial_j \mid j \in \mathbf{I}\} \cup T \subset \widehat{S}^\theta;\ \ x^{(\pi)},\ x^{(p^{t_j}\varepsilon_j)},\ h \in \widehat{H}^\theta,\ j \in \mathbf{I};\ \ x^{(\pi)} \in \widehat{K}^\theta,$$

which shows (1) holds. Noting that $\widehat{L}_{-1} = L_{-1} \cong W_{-1}$, (2) holds. □

4. Symmetric Biderivations

Assume that $L = W, S, H$ or K. Lemma 2 shows that L is weakly complete; hence, all its biderivations are weakly inner. Note that if $\vartheta \in \mathrm{BDer}L$ is symmetric, then

$$[\phi_\vartheta(x), y] = [\phi_\vartheta(y), x] = \vartheta(x, y),\ x, y \in L \tag{3}$$

where ϕ_ϑ is the linear map related to ϑ.

Lemma 5. *Assume that $\vartheta \in \mathrm{BDer}L$ is symmetric. Then ϑ is zero if $\phi_\vartheta(L_{-1} \oplus V_L) = 0$, where*

$$V_L = \begin{cases} \mathrm{Span}_{\mathbb{F}}\{x^{(p^k\varepsilon_j)}\partial_i \mid i \neq j \in \mathbf{I},\ 0 < k < t_j\}, & L = W, S; \\ \mathrm{Span}_{\mathbb{F}}\{x^{(p^k\varepsilon_j)} \mid j \in \mathbf{I},\ 0 < k < t_j\}, & L = H, K. \end{cases}$$

Proof. By Remark 2, we can assume that $\vartheta \in \mathrm{BDer}(L)_r$, $r \in \mathbb{Z}$. When $\phi_\vartheta(L_{-1}) = 0$, we find that, for $E \in L$, $\phi_\vartheta(E) \in L_{-1} \oplus L_{-2} \oplus \mathfrak{V}$ by Equation (3) and the transitivity of $\widehat{L}$. And then, we get $\phi_\vartheta(E) = 0$ by $[\phi_\vartheta(E), V_L] = 0$, which implies $\vartheta = 0$. □

Lemma 6. *Let $\vartheta \in \mathrm{BDer}(L)^a$ be symmetric, where $a \in T_L^* \backslash (\Omega_L \cup \{\theta\})$. Then ϑ is zero.*

Proof. By Lemmas 2 and 4, we get $\phi_\vartheta(T_L) \subset \widehat{L}^a = \{0\}$. Then, for all $x \in T_L$, $b \in \Omega_L \cup \{\theta\}$, $y \in L^b$, we get

$$0 = [\phi_\vartheta(x), y] = -[x, \phi_\vartheta(y)],$$

which shows that $\phi_\vartheta(y) \in \widehat{L}^\theta \cap \widehat{L}^{a+b}$. Notice that

$$(L_{-1} \oplus V_L) \subset \oplus_{i\in\mathbf{I}} L^{\varepsilon_i} \oplus_{i\in\mathbf{I}} L^{-\varepsilon_i} \oplus L^\theta \text{ and } \pm\varepsilon_i \in \Omega_L,$$

which shows that $(L_{-1} \oplus V_L) \cap L^{-a} = \{0\}$. Thus we get $\phi_\vartheta(L_{-1} \oplus V_L) = 0$. Then the Lemma 5 confirms the conclusion. □

4.1. Special Algebras

Lemma 7. *Any symmetric biderivation ϑ of S is zero.*

Proof. According to Lemmas 5 and 6, it sufficed to consider $\vartheta \in \mathrm{BDer}(S)^a$, $a \in \Omega_S \cup \{\theta\}$. Notice that $\phi_\vartheta(S) \subset \widehat{S} \oplus \mathfrak{V} \subset W \oplus \mathfrak{V}$ and $W \oplus \mathfrak{V}$ is a module of S. Consider the weight space decompositions of W with respect to T_S. Put $\Lambda = \sum_{i\in\mathbf{I}} \varepsilon_i$. Then we can obtain that, for any $b \in \Omega_W \cup \{\theta\}$,

$$W^b = \mathrm{Span}_{\mathbb{F}}\{x^{(b+\lambda\Lambda+\varepsilon_k)}\partial_k \mid k \in \mathbf{I}, \lambda \in \mathbb{Z}\}.$$

When $i \neq j \in \mathbf{I}$, assume that

$$\begin{aligned}
\phi_\vartheta(x^{(l\varepsilon_i)}\partial_j) &= \sum_{k=1}^{m}\sum_{\lambda} a_{k,\lambda}^{l,i,j} x^{(a+\lambda\Lambda+\varepsilon_k+l\varepsilon_i-\varepsilon_j)}\partial_k + \delta_{a+l\varepsilon_i-\varepsilon_j=\theta}\mathfrak{a}^{l,i,j}, \ 0 \le l \le p^{t_i}-1,\\
\phi_\vartheta(\partial_i) &= \sum_{k=1}^{m}\sum_{\lambda} a_{k,\lambda}^{i} x^{(a+\lambda\Lambda+\varepsilon_k-\varepsilon_i)}\partial_k + \delta_{a-\varepsilon_i=\theta}\mathfrak{b}^{i},\\
\phi_\vartheta(h_i-h_j) &= \sum_{k=1}^{m}\sum_{\lambda} b_{k,\lambda}^{i,j} x^{(a+\lambda\Lambda+\varepsilon_k)}\partial_k + \delta_{a=\theta}\mathfrak{c}^{i,j},
\end{aligned}$$

where $\mathfrak{a}^{l,i,j}, \mathfrak{b}^i, \mathfrak{c}^{i,j} \in \mathfrak{V}$. For distinct $k, s \in \mathbf{I}$ and $0 \le r \le p^{t_k}-1$, the following equations hold

$$[\phi_\vartheta(x^{(r\varepsilon_k)}\partial_s), h_i - h_j] = [\phi_\vartheta(h_i-h_j), x^{(r\varepsilon_k)}\partial_s], \tag{4}$$

$$[\phi_\vartheta(\partial_t), x^{(r\varepsilon_k)}\partial_s] = [\phi_\vartheta(x^{(r\varepsilon_k)}\partial_s). \tag{5}$$

Notice that

$$[h_i - h_j, \phi_\vartheta(D^b)] = (a_i - a_j + b_i - b_j)\phi_\vartheta(D^b),\ D^b \in S^b. \tag{6}$$

The following cases are considered:

Case 1. $a_i - a_j \not\equiv -1, 0, 1 \pmod{p}$. For Equations (4) and (5), by choosing $r = 1$, we get

$$\begin{aligned}
\phi_\vartheta(\partial_i) =& a^i_{j,-a_j-1} x^{(a-(a_j+1)\Lambda+\varepsilon_j-\varepsilon_i)}\partial_j + \sum_{k\neq j} a^i_{k,-a_j} x^{(a-a_j\Lambda+\varepsilon_k-\varepsilon_i)}\partial_k,\\
\phi_\vartheta(\partial_j) =& a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i-\varepsilon_j)}\partial_i + \sum_{k\neq i} a^j_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k-\varepsilon_j)}\partial_k,\\
\phi_\vartheta(h_i - h_j) =& \sum_{k\neq j}(q-2) a^i_{k,-a_j} x^{(a-a_j\Lambda+\varepsilon_k)}\partial_k + (q-2) a^i_{j,-a_j-1} x^{(a-(a_j+1)\Lambda+\varepsilon_j)}\partial_j\\
&+ \sum_{k\neq i} q a^j_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k)}\partial_k + q a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i)}\partial_i,
\end{aligned}$$

where $q = a_i - a_j + 1$. Notice that $q, q-1, q-2 \neq 0$. From Equation (5), by choosing $r = 2$, we have

$$\begin{aligned}
&- q a^i_{i,-a_j} x^{(a-a_j\Lambda+\varepsilon_i)}\partial_j\\
=& a^{2,i,j}_{i,-a_i-2} x^{(a-(a_i+2)\Lambda+2\varepsilon_i-\varepsilon_j)}\partial_i + \sum_{k\neq i} a^{2,i,j}_{k,-a_i-3} x^{(a-(a_i+3)\Lambda+\varepsilon_i-\varepsilon_j+\varepsilon_k)}\partial_k\\
&+ \sum_{k=1}^{m} \sum_{\lambda\not\equiv -a_i-2,-a_i-3 \pmod{p}} a^{2,i,j}_{k,\lambda} x^{(a+\lambda\Lambda+\varepsilon_i-\varepsilon_j+\varepsilon_k)}\partial_k.
\end{aligned}$$

By considering the cases of $a_i - a_j \equiv -2, -3 \pmod{p}$ and $a_i - a_j \not\equiv -3, -2, -1, 0, 1 \pmod{p}$, respectively, we can obtain that

$$\phi_\vartheta(\partial_i) = \phi_\vartheta(\partial_j) = \phi_\vartheta(h_i - h_j) = 0.$$

Then, for distinct $k, l \in \mathbf{I}$, from Equation (6), we find that

$$\phi_\vartheta(x^{(p^{s_k}\varepsilon_k)}\partial_l) = 0, \ \ 0 < s_k < t_k.$$

Case 2. $a_i \equiv a_j \pmod{p}$. For Equations (4) and (5), by choosing $r = 1$, we get

$$\begin{aligned}
\phi_\vartheta(\partial_i) =& a^i_{i,-a_i} x^{(a-a_i\Lambda)}\partial_i + a^i_{j,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_j-\varepsilon_i)}\partial_j,\\
\phi_\vartheta(\partial_j) =& a^j_{j,-a_i} x^{(a-a_i\Lambda)}\partial_j + a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i-\varepsilon_j)}\partial_i,\\
\phi_\vartheta(h_i - h_j) =& - a^i_{i,-a_i} x^{(a-a_i\Lambda+\varepsilon_i)}\partial_i + a^j_{j,-a_i} x^{(a-a_i\Lambda+\varepsilon_j)}\partial_j + \sum_{k\neq i,j} b^{i,j}_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k)}\partial_k\\
&- a^i_{j,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_j)}\partial_j + a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i)}\partial_i + \delta_{a=\theta}\mathfrak{c}^{i,j}.
\end{aligned}$$

For Equations (4) and (5), by choosing $r = 2$, we get

$$\begin{aligned}
\phi_\vartheta(x^{(2\varepsilon_i)}\partial_j) =& \phi_\vartheta(x^{(2\varepsilon_j)}\partial_i) = \phi_\vartheta(\partial_i) = \phi_\vartheta(\partial_j) = 0,\\
\phi_\vartheta(h_i - h_j) =& \sum_{k\neq i,j} b^{i,j}_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k)}\partial_k + \delta_{a=\theta}\mathfrak{c}^{i,j}.
\end{aligned}$$

From Equation (4), for any $0 < s_i < t_i$, we find that there exists $e^{s_i} \in \mathbb{F}$ such that $\phi_\vartheta(x^{(p^{s_i}\varepsilon_i)}\partial_j) = e^{s_i}\partial_j$. Then $[\phi_\vartheta(x^{(p^{s_i}\varepsilon_i)}\partial_j), x^{(2\varepsilon_j)}\partial_i] = 0$ gives us $\phi_\vartheta(x^{(p^{s_i}\varepsilon_i)}\partial_j) = 0$. Similarly, we have $\phi_\vartheta(x^{(p^{s_j}\varepsilon_j)}\partial_i) = 0$, for any $0 < s_j < t_j$.

Case 3. $a_i - a_j \equiv 1 \pmod{p}$. For Equations (4) and (5), by choosing $r = 1$, we have

$$\begin{aligned}
\phi_\vartheta(\partial_i) =& a^i_{j,-a_i} x^{(a-a_i\Lambda+\varepsilon_j-\varepsilon_i)}\partial_j + \sum_{k\neq j} a^i_{k,-a_i+1} x^{(a-(a_i-1)\Lambda+\varepsilon_k-\varepsilon_i)}\partial_k + \delta_{a-\varepsilon_i=\theta}\mathfrak{b}^i,\\
\phi_\vartheta(\partial_j) =& \sum_{k\neq i,j} a^j_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k-\varepsilon_j)}\partial_k + a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i-\varepsilon_j)}\partial_i,\\
\phi_\vartheta(h_i - h_j) =& \sum_{k\neq i,j} 2a^j_{k,-a_i} x^{(a-a_i\Lambda+\varepsilon_k)}\partial_k + 2a^j_{i,-a_i-1} x^{(a-(a_i+1)\Lambda+\varepsilon_i)}\partial_i\\
&+ b^{i,j}_{j,-a_i} x^{(a-a_i\Lambda+\varepsilon_j)}\partial_j.
\end{aligned}$$

For Equations (4) and (5), by choosing $r = 2$, we have

$$\begin{aligned}
\phi_\vartheta(\partial_j) =& \phi_\vartheta(x^{(2\varepsilon_i)}\partial_j) = \phi_\vartheta(x^{(2\varepsilon_j)}\partial_i) = \phi_\vartheta(h_i - h_j) = 0,\\
\phi_\vartheta(\partial_i) =& \sum_{k\neq i,j} a^i_{k,-a_i+1} x^{(a-(a_i-1)\Lambda+\varepsilon_k-\varepsilon_i)}\partial_k + \delta_{a-\varepsilon_i=\theta}\mathfrak{b}^i.
\end{aligned}$$

It follows from Equation (6) that, for any $i, j \neq s \in \mathbf{I}$,

$$0 = [h_i - h_j, \phi_\vartheta(x_s\partial_j)] = (a_i - a_j + 1)\phi_\vartheta(x_s\partial_j) = 2\phi_\vartheta(x_s\partial_j),$$

which implies that $\phi_\vartheta(x_s\partial_j) = 0$. Then Equation (5) shows that $\phi_\vartheta(\partial_i) = \delta_{a-\varepsilon_i=\theta}\mathfrak{b}^i$. It follows from Equation (6) that, for $k \neq j \in \mathbf{I}$, $0 < s_k < t_k$,

$$\phi_\vartheta(x^{(p^{s_k}\varepsilon_k)}\partial_j) = \phi_\vartheta(x^{(p^{s_j}\varepsilon_j)}\partial_j - x^{((p^{s_j}-1)\varepsilon_j+\varepsilon_k))}\partial_k)) = 0. \tag{7}$$

Then we have $\phi_\vartheta(\partial_i) = 0$. Notice that $\phi_\vartheta(S_{-1}) = 0$. By the proof of Lemma 5, we can obtain that $\phi_\vartheta(x^{(p^{s_j}\varepsilon_j)}\partial_i) \in S_{-1} \oplus \mathfrak{V}$. Then we have $\phi_\vartheta(x^{(p^{s_j}\varepsilon_j)}\partial_i) = 0$ by Equation (7). From Lemma 5, the conclusion holds. □

4.2. Generalized Jacobson–Witt Algebra

Lemma 8. *Any symmetric biderivation of W is zero.*

Proof. For $\vartheta \in \mathrm{BDer}(W)$, we can infer from Lemma 7 and its proof that $\phi_\vartheta(S) = 0$. Note that

$$W_{-1} \oplus V_W = S_{-1} \oplus V_S \subset S.$$

From Lemma 5, the conclusion holds. □

4.3. Hamiltonian Algebras

Put

$$\begin{aligned}
\mathcal{A}(2r) &= \{\Gamma = (\gamma_i, \ldots, \gamma_{2r}) \in \mathbb{Z}^{2r} \mid \gamma_i \equiv \gamma_{i'} \pmod{p},\ \gamma_i \in \mathbb{Z},\ 1 \leq i \leq r\},\\
\Gamma^{i,k} &= \{\Gamma \in \mathcal{A}(2r) \mid \gamma_i = -a_i - k,\ i \in \mathbf{I},\ k \in \mathbb{Z}\}.
\end{aligned}$$

Lemma 9. *Any symmetric biderivation of H is zero.*

Proof. In consideration of the weight space decomposition of H with regard to T_H, for $b \in \Omega_H \cup \{\theta\}$, we get

$$\widehat{H}^b = \mathrm{Span}_{\mathbb{F}}\{x^{(b+\Gamma)} \mid \Gamma \in \mathcal{A}(2r)\} \oplus \delta_{b=\theta}\mathbb{F}h,$$

where $h = \sum_{i=1}^{m} x_i\partial_i$ and $[h, D_H(g)] = (\mathrm{zd}(g) - 2)D_H(g)$, for any $g \in \mathcal{O}$. According to Lemma 6, it suffices to consider $\vartheta \in \mathrm{BDer}(H)^a$, $a \in \Omega_H \cup \{\theta\}$. Assume that, for any $1 \le i \le 2r$,

$$\begin{aligned}
\phi_\vartheta(x^{(l\varepsilon_i)}) &= \sum_{\Gamma\in\mathcal{A}(2r)} a_\Gamma^{l,i} x^{(a+l\varepsilon_i+\Gamma)} + \delta_{a+l\varepsilon_i=\theta}(a^{l,i}h + \mathfrak{a}^{l,i}),\ 1 < l < p^{t_i},\\
\phi_\vartheta(x_i) &= \sum_{\Gamma\in\mathcal{A}(2r)} a_\Gamma^{i} x^{(a+\varepsilon_i+\Gamma)} + \delta_{a+\varepsilon_i=\theta}(a^{i}h + \mathfrak{b}^{i}),\\
\phi_\vartheta(x_i x_{i'}) &= \sum_{\Gamma\in\mathcal{A}(2r)} b_\Gamma^{i} x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)} + \delta_{a=\theta}(b^{i}h + \mathfrak{c}^{i}),
\end{aligned}$$

where $\mathfrak{a}^{l,i}, \mathfrak{b}^i, \mathfrak{c}^i \in \mathfrak{V}$.

For $1 \le i \le r$, $t \in \mathbf{I}$, the following equations hold:

$$[\phi_\vartheta(x_i), x_{i'}] = [\phi_\vartheta(x_{i'}), x_i], \tag{8}$$

$$[\phi_\vartheta(x^{(l\varepsilon_t)}), x_i x_{i'}] = [\phi_\vartheta(x_i x_{i'}), x^{(l\varepsilon_t)}],\ \ l = 1, 2, \tag{9}$$

$$[\phi_\vartheta(x_s), x^{(2\varepsilon_t)}] = [\phi_\vartheta(x^{(2\varepsilon_t)}), x_s],\ \ s = t, t'. \tag{10}$$

Note that

$$[x_i x_{i'}, \phi_\vartheta(g^b)] = (a_{i'} - a_i + b_{i'} - b_i)\phi_\vartheta(g^b),\ g^b \in H^b. \tag{11}$$

We consider the following cases.

Case 1. $a_{i'} \not\equiv a_i \pmod p$. If $\phi_\vartheta(x_i x_{i'}) = 0$, we can obtain that $\phi_\vartheta(V_H) = 0$ from Equation (11).

Subcase 1.1 $a_{i'} - a_i \equiv 1 \pmod p$. For Equations (8) and (9), we have

$$\begin{aligned}
\phi_\vartheta(x_{i'}) &= \sum_{\Gamma\in\Gamma^{i',1}} a_\Gamma^{i'} x^{(a+\varepsilon_{i'}+\Gamma)},\\
\phi_\vartheta(x_i x_{i'}) &= \sum_{\Gamma\in\Gamma^{i',1}} -2a_\Gamma^{i'} x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)},\\
\phi_\vartheta(x_i) &= \sum_{\Gamma\in\Gamma^{i,1}} a_\Gamma^{i} x^{(a+\varepsilon_i+\Gamma)} + \delta_{a+\varepsilon_i=\theta}(a_{\Gamma_0}^{i}(x_i x_{i'} + h) + \mathfrak{b}^{i}),
\end{aligned}$$

where Γ_0 satisfying $\gamma_i + a_i = 0, \gamma_{i'} = 1, \gamma_j = 0$, $j \in \mathbf{I}\backslash\{i, i'\}$. For Equations (9) and (10), we have

$$\begin{aligned}
\phi_\vartheta(x^{(2\varepsilon_{i'})}) =& \phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = 0,\\
\phi_\vartheta(x_i) &= \sum_{\Gamma\in\Gamma^{i,1}} a_\Gamma^{i} x^{(a+\varepsilon_i+\Gamma)}.
\end{aligned}$$

If $r = 1$, obviously, $\phi_\vartheta(x_i) = \sum_{\Gamma\in\Gamma^{i,1}} a_\Gamma^{i} x^{(a+\varepsilon_i+\Gamma)} \in \mathbb{F}$. Then $\phi_\vartheta(x_i) = 0$ in $\widehat{H}$. If $r > 1$, for any $s \in \mathbf{I}\backslash\{i, i'\}$, from Equation (9) we have $\phi_\vartheta(x_{s'}) = 0$ and then

$$\phi_\vartheta(x_i) = \sum_{\Gamma\in\Gamma^{i,1}\cap\bigcap\limits_{s\in\mathbf{I}\backslash\{i,i'\}}\Gamma^{s,0}} a_\Gamma^{i} x^{(a+\varepsilon_i+\Gamma)},$$

which is in $\mathbb{F}$. Then $\phi_\vartheta(x_i) = 0$ in $\widehat{H}$. Thus, we have

$$\phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = \phi_\vartheta(x_i) = \phi_\vartheta(V_H) = 0.$$

Subcase 1.2 $a_{i'} - a_i \equiv -1 \pmod{p}$. As in **Subcase 1.1**, we can prove that

$$\phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = \phi_\vartheta(x_i) = \phi_\vartheta(V_H) = 0.$$

Subcase 1.3 $a_{i'} - a_i \equiv 2 \pmod{p}$. For Equations (8) and (9), we have

$$\begin{aligned}
\phi_\vartheta(x_i) &= \sum_{\Gamma \in \Gamma^{i,1}} a_\Gamma^i x^{(a+\varepsilon_i+\Gamma)}, \\
\phi_\vartheta(x_{i'}) &= \sum_{\Gamma \in \Gamma^{i',1}} a_\Gamma^{i'} x^{(a+\varepsilon_{i'}+\Gamma)}, \\
\phi_\vartheta(x_i x_{i'}) &= \sum_{\Gamma \in \Gamma^{i,1}} a_\Gamma^i x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)} - \sum_{\Gamma \in \Gamma^{i',1}} 3a_\Gamma^{i'} x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)}.
\end{aligned}$$

For Equations (9) and (10), we have

$$\phi_\vartheta(x^{(2\varepsilon_{i'})}) = \phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = \phi_\vartheta(x_i) = 0,$$

and then

$$\phi_\vartheta(V_H) = 0.$$

Subcase 1.4 $a_{i'} - a_i \equiv -2 \pmod{p}$. As in **Subcase 1.3**, we can prove that

$$\phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = \phi_\vartheta(x_i) = \phi_\vartheta(V_H) = 0.$$

Subcase 1.5 $a_{i'} - a_i \not\equiv 0, \pm 1, \pm 2 \pmod{p}$. For Equations (8) and (9), we have

$$\begin{aligned}
\phi_\vartheta(x_i) &= \sum_{\Gamma \in \Gamma^{i,1}} a_\Gamma^i x^{(a+\varepsilon_i+\Gamma)}, \\
\phi_\vartheta(x_{i'}) &= \sum_{\Gamma \in \Gamma^{i',1}} a_\Gamma^{i'} x^{(a+\varepsilon_{i'}+\Gamma)}, \\
\phi_\vartheta(x_i x_{i'}) &= \sum_{\Gamma \in \Gamma^{i,1}} (a_{i'} - a_i - 1) a_\Gamma^i x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)} \\
&\quad - \sum_{\Gamma \in \Gamma^{i',1}} (a_{i'} - a_i + 1) a_\Gamma^{i'} x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)}.
\end{aligned}$$

For Equations (9) and (10), we have

$$\phi_\vartheta(x^{(2\varepsilon_i)}) = \phi_\vartheta(x^{(2\varepsilon_{i'})}) = \phi_\vartheta(x_i x_{i'}) = \phi_\vartheta(x_{i'}) = \phi_\vartheta(x_i) = 0,$$

and then

$$\phi_\vartheta(V_H) = 0.$$

Case 2. $a_{i'} \equiv a_i \pmod{p}$. For Equations (8) and (9), we have

$$\begin{aligned}
\phi_\vartheta(x_i) &= -\delta_{a=\theta} b^i x_i, \\
\phi_\vartheta(x_{i'}) &= \delta_{a=\theta} b^i x_{i'}, \\
\phi_\vartheta(x_i x_{i'}) &= \sum_{\Gamma \in \Gamma^{i,1}} b_\Gamma^i x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma)} + \delta_{a=\theta}(b^i h + \mathfrak{c}^i).
\end{aligned}$$

For Equations (9) and (10), we have

$$\phi_\vartheta(x^{(2\varepsilon_i)}) = \phi_\vartheta(x^{(2\varepsilon_{i'})}) = \phi_\vartheta(x_i) = \phi_\vartheta(x_{i'}) = 0.$$

As above, we find that $\phi_\vartheta(H_{-1}) = 0$. For all $D \in H$, it is evident, as demonstrated in the proof of Lemma 5, that $\phi_\vartheta(D) \in \mathfrak{V} \oplus \widehat{H}_{-1}$. Then $\phi_\vartheta(x_i x_{i'}) = 0$ from $[x_i x_{i'}, \phi_\vartheta(x^{(p^{s_k}\varepsilon_k)})] = 0$,

for any $k \in \mathbf{I}$, $0 < s_k < t_k$. Notice that, for $k = i, i'$, $\phi_\vartheta(x^{((p^{s_k}+1)\varepsilon_k)}) = 0$ by Equation (11). Thus, $\phi_\vartheta(D) = 0$. The result is thus proven. □

4.4. Contact Algebras

Lemma 10. *Any symmetric biderivation of K are zero.*

Proof. In consideration of the weight space decompositions of K with regard to T_K, for $b \in \Omega_K \cup \{\theta\}$, we get

$$\widehat{K}^b = \mathrm{Span}_{\mathbb{F}}\{x^{(b+\Gamma+\gamma_b\varepsilon_m)} \mid \Gamma \in \mathcal{A}(2r), \gamma_b \in \mathbb{Z}\}, \;\; x^{(a)} \in K^a.$$

According to Lemma 6, it suffices to consider $\vartheta \in \mathrm{BDer}(K)^a$, $a \in \Omega_K \cup \{\theta\}$. Assume that, for any $1 \le i \le 2r$,

$$\begin{aligned}
\phi_\vartheta(x^{(l\varepsilon_i)}) &= \sum_{\gamma_a}\sum_{\Gamma\in\mathcal{A}(2r)} a^{l,i}_{\gamma_a,\Gamma} x^{(a+l\varepsilon_i+\Gamma+\gamma_a\varepsilon_m)} + \delta_{a+l\varepsilon_i=\theta}\mathfrak{a}^{l,i}, \;\; 1 < l < p^{t_i},\\
\phi_\vartheta(x_i) &= \sum_{\gamma_a}\sum_{\Gamma\in\mathcal{A}(2r)} a^{i}_{\gamma_a,\Gamma} x^{(a+\varepsilon_i+\Gamma+\gamma_a\varepsilon_m)} + \delta_{a+\varepsilon_i=\theta}\mathfrak{b}^{i},\\
\phi_\vartheta(x_ix_{i'}) &= \sum_{\gamma_a}\sum_{\Gamma\in\mathcal{A}(2r)} b^{i}_{\gamma_a,\Gamma} x^{(a+\varepsilon_i+\varepsilon_{i'}+\Gamma+\gamma_a\varepsilon_m)} + \delta_{a=\theta}\mathfrak{c}^{i},\\
\phi_\vartheta(x_m) &= \sum_{\gamma_a}\sum_{\Gamma\in\mathcal{A}(2r)} a^{m}_{\gamma_a,\Gamma} x^{(a+\Gamma+\gamma_a\varepsilon_m)} + \delta_{a=\theta}\mathfrak{d},
\end{aligned}$$

where $\mathfrak{a}^{l,i}, \mathfrak{b}^i, \mathfrak{c}^i, \mathfrak{d} \in \mathfrak{V}$. For $i, j \in \mathbf{I}\backslash\{m\}$, $l = 1, 2$, the following equations hold

$$[\phi_\vartheta(x_m), x_ix_{i'}] = [\phi_\vartheta(x_ix_{i'}), x_m], \tag{12}$$
$$[\phi_\vartheta(x^{(l\varepsilon_j)}), x_ix_{i'}] = [\phi_\vartheta(x_ix_{i'}), x^{(l\varepsilon_j)}], \tag{13}$$
$$[\phi_\vartheta(x_m), x^{(l\varepsilon_j)}] = [\phi_\vartheta(x^{(l\varepsilon_j)}), x_m], \tag{14}$$
$$[\phi_\vartheta(x^{(l\varepsilon_j)}), x^{(2\varepsilon_i)}] = [\phi_\vartheta(x^{(2\varepsilon_i)}), x^{(l\varepsilon_j)}] \tag{15}$$
$$[x_ix_{i'}, \phi_\vartheta(g^b)] = (a_{i'} - a_i + b_{i'} - b_i)\phi_\vartheta(g^b), \; g^b \in K^b. \tag{16}$$

Notice that x_m is the $\mathbb{Z}$-degree derivation of K. Then, for any $\mathbb{Z}$-homogeneous element $D \in K$, we have

$$[x_m, \phi_\vartheta(D)] = (\mathrm{zd}(\vartheta) + \mathrm{zd}(D))\phi_\vartheta(D). \tag{17}$$

Assume that $\mathrm{zd}(\vartheta) = t$ and put $q = a_{i'} - a_i$. For Equations (12), (16) and (17), we have

$$t\phi_\vartheta(x_ix_{i'}) = q\phi_\vartheta(x_m). \tag{18}$$

Then we consider the following cases.
Case 1. $q \not\equiv 0 \pmod{p}$. From Equation (18), we have $\phi_\vartheta(x_m) = \frac{t}{q}\phi_\vartheta(x_ix_{i'})$.
Subcase 1.1 $t = 0$. Then we have $\phi_\vartheta(x_m) = 0$. For Equations (14) and (17), we have $\phi_\vartheta(K_{-1} \oplus V_K) = 0$.
Subcase 1.2 $t \neq 0$. From Equations (14), (16)–(18), we derive the following equations:

$$\phi_\vartheta(x^{(2\varepsilon_j)}) = 0, \; j = i, i'; \tag{19}$$
$$\phi_\vartheta(x_j) = 0, \; j \neq m, i, i'; \tag{20}$$
$$(q - t)\phi_\vartheta(x_i) = 0; \; (q + t)\phi_\vartheta(x_{i'}) = 0.$$

By Equations (13) and (19), we have $\phi_\vartheta(x_ix_{i'}) = 0$, and therefore $\phi_\vartheta(x_m) = 0$. We can

obtain that $(t-1)\phi_\vartheta(x_j)=0$, $j=i,i'$ by Equation (14). Then if $t\neq 1$ or $t=1, q\neq\pm 1$, we have $\phi_\vartheta(x_j)=0$, $j=i,i'$. If $t=1$, $q=1$, we have $\phi_\vartheta(x_{i'})=0$. From Equation (15), we have

$$\phi_\vartheta(x_i)=\sum_{\Gamma\in\Gamma^{i',0}} a^i_{-a_m,\Gamma}x^{(a+\varepsilon_i+\Gamma-a_m\varepsilon_m)}.$$

Notice that $[x_i,\phi_\vartheta(x_i)]=0$. From Equation (20) we have $[K_{-1},\phi_\vartheta(x_i)]=0$, which implies that $\phi_\vartheta(x_i)\in(K_{-1}\oplus K_{-2}\oplus\mathfrak{V})\cap K_0=0$. Thus, $\phi_\vartheta(x_i)=0$. If $t=1$, $q=-1$, we can also obtain that $\phi_\vartheta(x_i)=\phi_\vartheta(x_{i'})=0$. Then we have $\phi_\vartheta(K_{-1})=0$. By the proof of Lemma 5 and Equation (17), we can obtain that $\phi_\vartheta(V_K)=0$.

Case 2. $q\equiv 0 \pmod p$.

Subcase 2.1 $t\neq 0$. In this case, by Equation (18), we have $\phi_\vartheta(x_ix_{i'})=0$. By Equation (16), we have $\phi_\vartheta(x^{(l\varepsilon_k)})=0$, $0<l<p^{t_k}-1$, $k=i,i'$, which implies that $\phi_\vartheta(K_{-1}\oplus V_K)=0$.

Subcase 2.2 $t=0$. From Equations (13) and (15), we find that

$$\phi_\vartheta(x^{(2\varepsilon_i)})=\phi_\vartheta(x^{(2\varepsilon_{i'})})=\phi_\vartheta(x_i)=\phi_\vartheta(x_{i'})=0.$$

Then, we have $\phi_\vartheta(K_{-1})=0$. By the proof of Lemma 5, for any $D_l\in K_l$, we have $\phi_\vartheta(D_l)\subset(K_{-1}\oplus K_{-2}\oplus\mathfrak{V})\cap\widehat{K}_l$. Then we have $\phi_\vartheta(V_K)=0$. By Lemma 5, we get $\vartheta=0$. □

Proposition 2. *Assume that $L=W,S,H$ or K. Then any symmetric biderivation of L is zero.*

Proof. From Lemmas 7–10, the conclusion holds. □

5. Linear Commuting Maps, Commutative Post-Lie Algebra Structures

Let G be an arbitrary Lie algebra. Review that a linear map $\varphi: G\longrightarrow G$ is a commuting if $[\varphi(x),x]=0$ for all $x\in G$. This condition readily implies that $[\varphi(x),y]=[x,\varphi(y)]$ for all $x,y\in G$, which shows that $\vartheta(x,y)=[\varphi(x),y]$ is a skew-symmetric biderivation. Thus, skew-symmetric biderivations may be viewed as a generalization of commuting linear maps. Using Proposition 1, we obtain the following result.

Proposition 3. *A linear map on L is commuting if and only if it is a scalar transformation, where $L=W,S,H$ or K.*

Review the commutative post-Lie algebra structure " $\cdot$ " on the Lie algebra G. Note that

$$\psi: G\times G\longrightarrow G,\ \psi(x,y)=x\cdot y,\ x,y\in G$$

is a symmetric biderivation as stated in [8]. Utilising Proposition 2, the subsequent result is obtained.

Proposition 4. *Any commutative post-Lie algebra structures are trivial on any of the Lie algebras W, S, H or K.*

Author Contributions: Writing—original draft, W.B. and W.L. All authors have read and agreed to the published version of the manuscript.

Funding: Supported by NSF of China (12061029), the NSF of Heilongjiang Province of China (LH2022A019) and the Project of Higher Education Teaching Reform Research (Postgraduate) of Harbin Normal University (XJGYJSZ202406).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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