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On the Cauchy Problem for a Simplified Compressible Oldroyd–B Model Without Stress Diffusion

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Abstract

In this paper, we are concerned with the Cauchy problem of the compressible Oldroyd-B model without stress diffusion in $\mathbb{R}^n$ ($n = 2, 3$). The absence of stress diffusion introduces significant challenges in the analysis of this system. By employing tools from harmonic analysis, particularly the Littlewood–Paley decomposition theory, we establish the global well-posedness of solutions with initial data in L^p critical spaces, which accommodates the case of large, highly oscillating initial velocity. Furthermore, we derive the optimal time decay rates of the solutions by a suitable energy argument.

Keywords: compressible Oldroyd-B model; global well-posedness; decay rates

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1. Introduction and the Main Results

The Oldroyd-B model serves as a fundamental framework for characterizing viscoelastic fluid dynamics, mathematically representing the behavior of certain viscoelastic flows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = \operatorname{div}(\mathbb{T} - H(\eta) \operatorname{Id}), \\ \partial_t \eta + \operatorname{div}(\eta \mathbf{u}) = \varepsilon \Delta \eta, \\ \partial_t \mathbb{T} + \operatorname{div}(\mathbb{T} \mathbf{u}) - (\nabla \mathbf{u} \mathbb{T} + \mathbb{T} \nabla^T \mathbf{u}) = \varepsilon \Delta \mathbb{T} + \frac{k A_0}{2\nu} \eta \operatorname{Id} - \frac{A_0}{2\nu} \mathbb{T}, \\ (\rho, \mathbf{u}, \eta, \mathbb{T})|_{t=0} = (\rho_0(x), \mathbf{u}_0(x), \eta_0(x), \mathbb{T}_0(x)), \end{cases} \quad (1)$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ ($n = 2, 3$). Here the unknown functions $\rho, \mathbf{u}, \eta$, and $\mathbb{T}$ represent the fluid density, the fluid velocity, the polymer number density, and the extra stress tensor in the fluid, respectively. The pressure $P = A\rho^\gamma$ for some $A > 0$ and $\gamma \geq 1$. The extra stress tensor $\mathbb{T} = (\mathbb{T}_{ij})$ is a positive definite symmetric matrix, and the notation $\operatorname{div}(\mathbb{T} \mathbf{u})$ is understood as

$$(\operatorname{div}(\mathbb{T} \mathbf{u}))_{ij} = \operatorname{div}(\mathbf{u} \mathbb{T}_{ij}).$$

Especially, η is the integral of a probability density function ψ with respect to the conformation vector, which is a microscopic variable in the modeling of dilute polymer chains, that is

$$\eta(t, x) = \int_{\mathbb{R}^n} \psi(t, x, q) dq \geq 0,$$

where ψ is governed by the Fokker–Planck equation. The term $H(\eta) = kL\eta + \zeta\eta^2$ in the momentum equation can be seen as the polymer pressure. Here μ and λ are both unknown viscosity constants that satisfy $\mu > 0$ and $\lambda + \frac{2\mu}{n} > 0$. The constant parameters k, A_0, ν are all positive numbers, whereas $\zeta \geq 0$ and $L \geq 0$ with $\zeta + L \neq 0$. More precisely, in the modeling of (1), $L = 2$ is the number of beads in the polymer chain in the classical Kramers expression. In particular, the parameter ε is the centre-of-mass diffusion coefficient, and system (1) is known as the diffusive Oldroyd-B model when the diffusion coefficient $\varepsilon > 0$. Furthermore, the Oldroyd–Jeffreys-type model, a constitutive equation in rheology, is derived as an extension or modification of the Oldroyd-B model, which can be used for the flow of viscoelastic liquids, but its linear characteristics limit its application in high strain rate or complex dynamic scenarios (e.g., the predicted damping factor for pendant droplet oscillation is an order of magnitude larger than the experimental value, which may underestimate the viscosity reduction caused by elasticity). For more details, please refer to [1].

In the case where both the additional stress tensor $\mathbb{T}$ and polymer number density η approach zero, the system (1) simplifies to the well-known compressible Navier–Stokes equations that have been thoroughly investigated in [2–9].

Apart from the aforementioned references, some interesting works on the absence of the polymer number density η in (1) were performed. Early investigations focused on the incompressible limit problem of the compressible Oldroyd-B model in $\mathbb{T}^n$ ($n = 2, 3$) [10]. Subsequent research extended this analysis to bounded domains [11]. A significant advancement was achieved by Baranovskii [12], who successfully resolved the limit problem for $\Omega \subset \mathbb{R}^n$ (a bounded domain with a locally Lipschitz boundary). Then the global results in $\mathbb{R}^n$ ($n \geq 2$) to small initial data in the critical L^2 Besov space was recently proved by Zi [13]; see [14] for the stability in $\mathbb{T}^n$ ($n \geq 2$). Subsequently, the result without the damping mechanism was generalized by Zhai and Chen [15] in the critical L^p spaces. Furthermore, some investigations on the compressible Oldroyd-type model at the basis of the deformation tensor were also reported in [16–18]. Furthermore, more interesting works (e.g., [19–24]) were devoted to the incompressible Oldroyd-B model, where ρ is constant.

Considering its physical significance and mathematical applications, the Oldroyd-B model has been considered by many researchers. We review key results for the compressible Oldroyd-B model, originally derived via micro–macro analysis of the compressible Navier–Stokes–Fokker–Planck system in [25]. Furthermore, by taking the limit $\zeta \rightarrow 0$, the existence of solutions for $\zeta = 0$ was proved in Section 12 of [25]. For more explanation on the physical background of the Navier–Stokes–Fokker–Planck system, one may refer to [25–27] and the other relevant references. Barrett and Süli [25] established the existence of global-in-time finite energy weak solutions for large initial data in $\mathbb{R}^2$, though the question of uniqueness for these weak solutions remained unresolved. Later, Lu and Zhang [28] derived local well-posedness, weak–strong uniqueness, and an improved blow-up criterion in $\mathbb{R}^2$, under the condition that the fluid density is bounded above. Wang and Wen [29] further established the global well-posedness in $\mathbb{R}^3$ for initial data close to a non-zero equilibrium state. Subsequently, Zhai and Li [30] extended the results of [29], demonstrating global well-posedness and long-time asymptotic behavior for the compressible Oldroyd-B model in $\mathbb{R}^n$ ($n = 2, 3$), with their analysis accommodating scenarios where the polymer number density may vanish and the stress tensor is near a zero equilibrium. Most recently, Liu, Wang, and Wen [31] achieved global well-posedness and optimal decay rates for the highest-

order derivatives of solutions to an inviscid Oldroyd-B model in $\mathbb{R}^3$, under the assumption of sufficiently small initial perturbations. Subsequently, Zhai and Zhao [32] extended these results to the periodic domain $\mathbb{T}^n$ ($n = 2, 3$), proving the global well-posedness of this model with small initial data. In addition, they derived the exponential decay of the solutions.

Research on the compressible Oldroyd-B model without stress diffusion remains relatively scarce. In their seminal work, Lu and Pokorný [33] established the global weak solutions without any restriction on the size of the data in a bounded open domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$). Subsequently, Liu, Lu, and Wen [34] established the global well-posedness of strong solutions near an equilibrium state using the standard continuity method, and the authors provided optimal decay rates for the highest-order derivatives of the solutions in $\mathbb{R}^3$. The goal of this paper, on the one hand, is to prove the global well-posedness of the Oldroyd-B model without stress diffusion for initial data in L^p critical Besov spaces on $\mathbb{R}^n$ ($n = 2, 3$) and, on the other hand, is to investigate the optimal time decay rates of the solutions using an energy framework, circumventing reliance on spectral analysis.

In this paper, we set $k = 1$, $L = 2$, and $\zeta = 0$, that is to say, $H(\eta) = 2\eta$. The system (1) without stress diffusion (i.e., $\varepsilon = 0$) can be rewritten as the following equation derived by introducing a transformation $\sigma_{i,j} = \mathbb{T}_{i,j} - \eta I_{i,j}$ in the Section 1.1 of Ref. [34]:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = \operatorname{div}(\sigma - \eta \operatorname{Id}), \\ \partial_t \eta + \operatorname{div}(\eta \mathbf{u}) = 0, \\ \partial_t \sigma + \operatorname{div}(\sigma \mathbf{u}) - (\nabla \mathbf{u} \sigma + \sigma \nabla^T \mathbf{u}) - \eta(\nabla \mathbf{u} + \nabla^T \mathbf{u}) = -\frac{A_0}{2\nu} \sigma. \end{cases}$$

We further make a serious simplifying assumption that the extra stress tensor σ is a scalar matrix:

$$\sigma = \tau \operatorname{Id} \quad \text{for some scalar function } \tau.$$

Letting $A_0 = 1$ and $\nu = \frac{1}{2}$ for brevity and neglecting the deformation tensor (i.e., $\frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} = 0$), the simplified compressible Oldroyd-B model without stress diffusion becomes

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = \nabla(\tau - \eta), \\ \partial_t \eta + \operatorname{div}(\eta \mathbf{u}) = 0, \\ \partial_t \tau + \operatorname{div}(\tau \mathbf{u}) + \tau = 0, \end{cases} \quad (2)$$

which is supplemented with far field behaviors

$$\rho \rightarrow \bar{\rho}, \quad \mathbf{u} \rightarrow \mathbf{0}, \quad \eta \rightarrow \bar{\eta}, \quad \tau \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where $\bar{\rho} > 0$, $\bar{\eta} > 0$.

Without loss of generality, we may assume $\bar{\rho} = \bar{\eta} = \mu = \gamma = 1$, $\lambda = 0$, since the other cases can be essentially reduced to this case, after omitting some high-order terms. Denoting $\rho = 1 + a$ and $\eta = 1 + b$, we can reformulate the system (2) into the following form:

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla a + a \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} - \nabla \operatorname{div} \mathbf{u} + \nabla a + \nabla b - \nabla \tau = \mathbf{F}(a, \mathbf{u}, b, \tau), \\ \partial_t b + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla b + b \operatorname{div} \mathbf{u} = 0, \\ \partial_t \tau + \tau + \operatorname{div}(\tau \mathbf{u}) = 0, \\ (a, \mathbf{u}, b, \tau)|_{t=0} = (a_0(x), \mathbf{u}_0(x), b_0(x), \tau_0(x)), \end{cases} \quad (3)$$

where

$$\mathbf{F}(a, \mathbf{u}, b, \tau) \stackrel{\text{def}}{=} -I(a)\nabla \tau + I(a)\nabla(a+b) - I(a)(\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u}) \quad \text{with} \quad I(a) = \frac{a}{1+a}.$$

Now, we state the first main theorem of the paper as follows.

Theorem 1 (Local well-posedness). *Let $n = 2, 3$ and $2 \leq p \leq 2n$. For any $(a_0, b_0, \tau_0) \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$ with $1 + a_0$ bounded away from zero, and $\mathbf{u}_0 \in \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$. Then there exists a positive time T such that the system (3) has a unique solution with*

$$\begin{aligned} (a, b) &\in C_b([0, T]; \dot{B}_{p,1}^{\frac{n}{p}}), \quad \mathbf{u} \in C_b([0, T]; \dot{B}_{p,1}^{\frac{n}{p}-1}) \cap L^1([0, T]; \dot{B}_{p,1}^{\frac{n}{p}+1}), \\ \tau &\in C_b([0, T]; \dot{B}_{p,1}^{\frac{n}{p}}) \cap L^1([0, T]; \dot{B}_{p,1}^{\frac{n}{p}}). \end{aligned}$$

Theorem 2 (Global well-posedness). *Let $n = 2, 3$ and $2 \leq p \leq 4$, and additionally, $p \neq 4$ if $n = 2$. For any $(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell) \in \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$, $\tau_0^\ell \in \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$, and $(a_0^h, b_0^h, \tau_0^h) \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$, $\mathbf{u}_0^h \in \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$, there exists a positive constant c_0 such that, if*

$$\|(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(a_0^h, b_0^h, \tau_0^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \leq c_0,$$

then the system (3) admits a unique solution $(a, \mathbf{u}, b, \tau)$ satisfying

$$\begin{aligned} (a^\ell, b^\ell) &\in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}-1}), & (a^h, b^h) &\in C_b(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}}), \\ \mathbf{u}^\ell &\in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}+1}), & \mathbf{u}^h &\in C_b(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}+1}), \\ \phi^\ell &\in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}+1}), & \phi^h &\in C_b(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}}) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}}), \\ \tau^\ell &\in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}}), & \tau^h &\in C_b(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}}) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}}), \end{aligned}$$

with $\phi \triangleq a + b$. Moreover, there exists some constant C such that

$$\begin{aligned} &\|(a, \mathbf{u}, b)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}^\ell + \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})}^\ell + \|\mathbf{u}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})}^h + \|(a, b, \tau)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}^h \\ &+ \|(\phi, \mathbf{u})\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}^\ell + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}^\ell + \|(\phi, \tau)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})}^h + \|\mathbf{u}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})}^h \leq Cc_0. \end{aligned} \quad (4)$$

Remark 1. For some basic knowledge on the Littlewood–Paley decomposition theory in the above theorem, see the beginning paragraph of Section 2.

Remark 2. Although there are no dissipations in the equations of a and b , by exploiting the structural characteristics of the system (3), we can also obtain the smoothing effect of $a + b$ in the low frequency and the damping effect of $a + b$ in the high frequency. Especially in the high frequency

part, we need two effective velocities, which are used to eliminate the coupling between the fluid density, the velocity, and the polymer number density.

Remark 3. Drawing upon the recent advancement presented in [35], we intend to investigate the global well-posedness of (3) with substantial initial data in our future research.

1.1. Scheme of the Proof of Theorem 2

To complete the proof of Theorem 2, we proceed in three key steps. Due to the absence of explicit dissipation or damping terms in the fluid density and polymer number density equations, it is essential to strategically leverage the system's hidden dissipative properties by exploiting the inherent structure of the system. A crucial aspect of our analysis involves decomposing the dynamics into low-frequency and high-frequency components, as the fluid density and polymer number density display markedly distinct dynamical responses across separated spectral bands.

Low frequency: To uncover the hidden dissipations in the fluid density and polymer number density, we primarily employ the cross-energy method. For this purpose, we first define the operator $\Lambda^s z \triangleq \mathcal{F}^{-1}(|\xi|^s \mathcal{F}z)$, $s \in \mathbb{R}$ and introduce the following key quantities:

$$\omega = \Lambda^{-1} \operatorname{curl} \mathbf{u}, \quad \mathbf{v} = \Lambda^{-1} \operatorname{div} \mathbf{u}.$$

For convenience, we denote

$$\phi \triangleq a + b, \quad \delta \triangleq \phi - 2a = b - a,$$

thus satisfying

$$\begin{cases} \partial_t \phi + 2\Lambda \mathbf{v} = f_1, \\ \partial_t \mathbf{v} - 2\Delta \mathbf{v} - \Lambda \phi = f_2, \end{cases}$$

with the nonlinear terms f_1 and f_2 defined in (8) and (9), respectively. Our main goal is to derive the dissipation estimates for a and b through careful analysis of the combined variables ϕ and δ . To achieve this, we construct the following “cross-energy” estimates:

$$\begin{aligned} & -\frac{d}{dt} \langle \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle - 2\|\Lambda \mathbf{v}_k^\ell\|_{L^2}^2 + \|\Lambda \phi_k^\ell\|_{L^2}^2 \\ & = -2\langle \Delta \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle - \langle (f_2)_k^\ell, \Lambda \phi_k^\ell \rangle - \langle \Lambda (f_1)_k^\ell, \mathbf{v}_k^\ell \rangle, \quad \text{see (12).} \end{aligned}$$

Combining with the estimates of τ , ω , $\mathbf{v}$, ϕ , and δ , we can infer the dissipation estimates for a , b , $\mathbf{u}$, τ , and ϕ in the low-frequency regime,

$$\begin{aligned} & \|(a^\ell, b^\ell, \mathbf{u}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \\ & \lesssim \|(a_0^\ell, b_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t (1 + \mathcal{E}_\infty(s)) \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \end{aligned}$$

For the definitions of $\mathcal{E}_\infty(s)$ and $\mathcal{E}_1(s)$, refer to Lemma 8 of Section 4 for more details.

High frequency: In the high-frequency regime, the primary objective arises from the absence of diffusion terms in the equations for a and b . The quasilinear terms in these equations lead to a loss of the derivatives. To capture the damping effect of $\phi = a + b$, we introduce the new auxiliary variable

$$\mathbf{G} \triangleq \mathbf{Q}\mathbf{u} - \frac{1}{2} \Delta^{-1} \nabla \phi.$$

Then we get the compressible part of Equation (3) as follows

$$\begin{cases} \partial_t \phi + \phi + \operatorname{div} \mathbf{G} = -\phi \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \phi, \\ \partial_t \mathbf{G} - 2\Delta \mathbf{G} = \mathbf{G} + \frac{1}{2} \Delta^{-1} \nabla \phi + \frac{1}{2} \mathbb{Q}(\phi \mathbf{u}) - \mathbb{Q}(\mathbf{u}, \nabla \mathbf{u}) + \nabla \tau + \mathbb{Q}\mathbf{F}(a, \mathbf{u}, b, \tau). \end{cases}$$

To establish the damping properties of ϕ and G , we construct the following estimates:

$$\begin{aligned} & \|\mathbf{G}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\phi^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|\phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ & \lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\phi_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t (\|\mathbf{G}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-2}}) ds + \int_0^t \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} ds \\ & \quad + \int_0^t (\|(\phi \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|(\mathbf{u} \cdot \nabla \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|(\phi \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) ds \\ & \quad + \int_0^t \left(\sum_{j \geq j_0} 2^{\frac{n}{p}j} \|\Delta^j \mathbf{u} \cdot \nabla \phi\|_{L^p} + \|\mathbf{F}(a, \mathbf{u}, b, \tau)^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \right) ds, \quad \text{see (41).} \end{aligned}$$

Based on the estimates for τ , the incompressible part $\mathbb{P}\mathbf{u}$, a , and b , the aforementioned estimate, and the nonlinear product laws in Besov spaces, we can now derive the dissipation estimates for $a, b, \mathbf{u}, \tau$, and ϕ in the high-frequency regime,

$$\begin{aligned} & \|(a^h, b^h, \tau^h)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|(\phi^h, \tau^h)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ & \lesssim \|(a_0^h, b_0^h, \tau_0^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \int_0^t \mathcal{E}_1(s) \mathcal{E}_\infty(s) ds. \end{aligned}$$

Through a continuity argument combined with the a priori estimates established above, we establish the global existence of solutions as presented in Theorem 2, with the detailed proof provided in Section 4.3.

Having established the existence of global solutions, we now turn to investigate their long-time asymptotic behavior. The third main result of this work characterizes these asymptotic properties as follows.

Theorem 3 (Optimal decay). *Let $(a, \mathbf{u}, b, \tau)$ be the global small solutions addressed by Theorem 2 with $p = 2$. Assume that the initial data $(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell) \in \dot{B}_{2,\infty}^\sigma(\mathbb{R}^n)$, $\tau_0^\ell \in \dot{B}_{2,\infty}^{\sigma+1}(\mathbb{R}^n)$ with $-\frac{n}{2} \leq \sigma < \frac{n}{2} - 1$. Then the following decay properties hold:*

$$\begin{aligned} \|\Lambda^{\gamma_1}(\phi, \mathbf{u})\|_{L^2} & \leq C(1+t)^{-\frac{\gamma_1-\sigma}{2}}, \quad \text{for any } \gamma_1 \in \left(\sigma, \frac{n}{2} - 1\right], \\ \|\Lambda^{\gamma_2} \tau\|_{L^2} & \leq C(1+t)^{-\frac{\gamma_2-\sigma-1}{2}}, \quad \text{for any } \gamma_2 \in \left(\sigma + 1, \frac{n}{2}\right]. \end{aligned}$$

1.2. Scheme of the Proof of Theorem 3

To complete the proof of Theorem 3, we consider the following asymptotic analysis features:

Regularity Propagation: Proposition 1 maintains initial data regularity with negative indices.

Lyapunov Dynamics: Constructing a Lyapunov-type inequality for $(\phi, \mathbf{u}, \tau)$ in time via pure energy methods, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \\ & \quad + c_1 (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}})^{1+\frac{4}{n-2\sigma-2}} \leq 0. \end{aligned}$$

Decay Estimates: The time-decay rates can be rigorously derived through interpolation techniques applied to the Lyapunov functional.

For a deeper understanding of the work, we provide a summary of the paper below. Inspired by Lu and Pokorný [33], we study the global well-posedness of the compressible Oldroyd-B model without stress diffusion in $\mathbb{R}^n$ ($n = 2, 3$). A key challenge arises from the lack of explicit dissipation or damping terms in the fluid density and polymer number density equations. The key issue lies in skillfully performing high–low frequency decomposition and establishing new effective fluxes to eliminate the coupling problems caused by ρ, u, η , thereby achieving global well-posedness. The Lyapunov functional constructed in this project is one of the key scientific problems for solving the global solution with large initial data and optimal decay estimates of the compressible Oldroyd-B system.

The remainder of this paper is organized as follows. After presenting the necessary preliminaries in Section 2, we provide a brief proof of Theorem 1, establishing the local well-posedness of solutions in Section 3. Section 4 develops the global well-posedness through the proof of Theorem 2. Finally, in Section 5, by investigating temporal decay characteristics through three analytical stages, we establish the optimal time decay rates, thereby completing the proof of Theorem 3.

Notations. For two operators A and B , we denote $[A, B] = AB - BA$ as the commutator between A and B . For two positive numbers, by $a \lesssim b$ we mean that there is a generic constant C which may be different on different lines, such that $a \leq Cb$. We shall denote $\langle f, g \rangle$ as the $L^2(\mathbb{R}^n)$ inner product of $f, g \in L^2(\mathbb{R}^n)$. For a Banach space X and an interval of $\mathbb{R}$ I , $C(I; X)$ is the set of continuous functions on I with values in X . For $q \in [1, +\infty]$, $L^q(I; X)$ is the space of measurable functions on I with values in X , such that $t \rightarrow \|f(t)\|_X$ belongs to $L^q(I)$. We always let $(d_j)_{j \in \mathbb{Z}}$ be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j \in \mathbb{Z}} d_j = 1$.

2. Preliminaries

In this section, we list some basic knowledge on the *Littlewood–Paley decomposition* theory, which plays a key role in our analysis. We set $\Delta_j \stackrel{\text{def}}{=} \varphi(2^{-j}D)$, $\dot{S}_j \stackrel{\text{def}}{=} \chi(2^{-j}D)$ with $\varphi(\xi) \stackrel{\text{def}}{=} \chi(\frac{\xi}{2}) - \chi(\frac{\xi}{4})$, and χ is a non-increasing nonnegative smooth function supported in $B(0, \frac{4}{3})$ with a value of 1 on $B(0, \frac{3}{4})$ (see [36] for more details).

Definition 1. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the homogeneous Besov space $\dot{B}_{p,1}^s(\mathbb{R}^n)$ is the set of tempered distributions f satisfying

$$\lim_{j \rightarrow -\infty} \|\dot{S}_j f\|_{L^\infty} = 0, \quad \text{and} \quad \|f\|_{\dot{B}_{p,1}^s} \triangleq \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{L^p} < \infty.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the lower and higher oscillation parts can be expressed as

$$f^\ell \stackrel{\text{def}}{=} \sum_{j \leq N_0} \Delta_j f \quad \text{and} \quad f^h \stackrel{\text{def}}{=} \sum_{j > N_0} \Delta_j f$$

for a large integer $N_0 \geq 0$. The corresponding truncated semi-norms are defined as follows:

$$\|f\|_{\dot{B}_{p,1}^s}^\ell \stackrel{\text{def}}{=} \|f^\ell\|_{\dot{B}_{p,1}^s} \quad \text{and} \quad \|f\|_{\dot{B}_{p,1}^s}^h \stackrel{\text{def}}{=} \|f^h\|_{\dot{B}_{p,1}^s}.$$

As we shall work with time-dependent functions valued in Besov spaces, we introduce the norms:

$$\|f\|_{L_T^q(\dot{B}_{p,1}^s)} \stackrel{\text{def}}{=} \| \|f(t, \cdot)\|_{\dot{B}_{p,1}^s} \|_{L^q(0,T)}.$$

Moreover, in this paper, we frequently use the so-called “time–space” Besov spaces or Chemin–Lerner space to solve parabolic estimates (see [36] for more details). Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define

$$\|f\|_{\tilde{L}_T^q(\dot{B}_{p,1}^s)} \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \left(\int_0^T \|\dot{\Delta}_j f(t)\|_{L^p}^q dt \right)^{\frac{1}{q}}$$

for $p, q \in [1, \infty)$ and with the standard modification for $p, q = \infty$.

Remark 4. For any $1 \leq q \leq \infty$, from the Minkowski’s inequality, one can deduce that

$$\|f\|_{L_T^q(\dot{B}_{p,1}^s)} \leq \|f\|_{\tilde{L}_T^q(\dot{B}_{p,1}^s)}.$$

The following lemma describes the way derivatives act on spectrally localized functions.

Lemma 1. Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^n$. A constant C exists such that for any positive real number λ , any nonnegative integer k , any smooth homogeneous function σ of degree m , and any couple of real numbers (p, q) with $1 \leq p \leq q$, the following inequalities hold:

$$\begin{aligned} \text{Supp } \hat{u} \subset \lambda \mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}, \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m} \lambda^{m+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}. \end{aligned}$$

Now we shall recall a few nonlinear estimates in Besov spaces that may be derived by means of paradifferential calculus. Here, we first recall the decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where $\dot{T}_u v \triangleq \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\dot{R}(u, v) \triangleq \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v$, and $\tilde{\Delta}_j v \triangleq \sum_{|j-j'| \leq 1} \dot{\Delta}_{j'} v$.

The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.

Lemma 2 ([35,36]). For all $s \in \mathbb{R}$, $\sigma \geq 0$, and $1 \leq p \leq p_1, p_2 \leq \infty$, the paraproduct $\dot{T}$ is a bilinear, continuous operator from $\dot{B}_{p_1,1}^{-\sigma} \times \dot{B}_{p_2,1}^s$ to $\dot{B}_{p,1}^{s-\sigma}$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The remainder $\dot{R}$ is bilinear continuous from $\dot{B}_{p_1,1}^{s_1} \times \dot{B}_{p_2,1}^{s_2}$ to $\dot{B}_{p,1}^{s_1+s_2}$ with $s_1 + s_2 > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Next, we give the important products that act on homogeneous Besov spaces as well as composition estimates, which will be also often used implicitly throughout the paper.

Lemma 3 ([20]). Let $s_1 \leq \frac{n}{q}$, $s_2 \leq \min\{\frac{n}{p}, \frac{n}{q}\}$, and $s_1 + s_2 > \max\{0, \frac{n}{p} + \frac{n}{q} - n\}$ for $1 \leq p, q \leq \infty$. For any $u \in \dot{B}_{q,1}^{s_1}(\mathbb{R}^n)$, $v \in \dot{B}_{p,1}^{s_2}(\mathbb{R}^n)$, we have

$$\|uv\|_{\dot{B}_{p,1}^{s_1+s_2-\frac{n}{q}}} \lesssim \|u\|_{\dot{B}_{q,1}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}}.$$

Lemma 4 ([30]). Let $n \geq 2$ and $2 \leq p \leq \min\{4, \frac{2n}{n-2}\}$, and additionally, $p \neq 4$ if $n = 2$. For any $u \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$ and $v^\ell \in \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$, $v^h \in \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$, the following holds:

$$\|(uv)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \left(\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|v^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \right) \|u\|_{\dot{B}_{p,1}^{\frac{n}{p}}}.$$

In order to overcome the nonlinear terms, we also need the classical commutator's estimate as follows.

Lemma 5 ([36]). Let $1 \leq p \leq \infty$, $\min\{-\frac{n}{p}, \frac{n}{p} - n\} < s \leq \frac{n}{p}$. For any $v \in \dot{B}_{p,1}^s(\mathbb{R}^n)$, $\nabla u \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$, we obtain

$$\|[\Delta_j, u \cdot \nabla]v\|_{L^p} \lesssim d_j 2^{-js} \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|v\|_{\dot{B}_{p,1}^s}.$$

Lemma 6 ([36]). Let $s > 0$, $1 \leq p, q \leq \infty$, and $a \in J$, where $J \subset I$ is a bounded interval. Let F with $F(0) = 0$ be a smooth function defined on an open interval I of $\mathbb{R}$ containing 0. Then we obtain the following estimates:

$$\|F(a)\|_{\dot{B}_{p,1}^s} \lesssim \|a\|_{\dot{B}_{p,1}^s}, \quad \|F(a)\|_{\tilde{L}_T^q(\dot{B}_{p,1}^s)} \lesssim \|a\|_{\tilde{L}_T^q(\dot{B}_{p,1}^s)}.$$

Lemma 7 ([36]). Let $\sigma \in \mathbb{R}$, $T > 0$, $1 \leq p, r \leq \infty$, and $1 \leq q_2 \leq q_1 \leq \infty$. Let u satisfy the heat equation

$$\partial_t u - \Delta u = f, \quad u|_{t=0} = u_0.$$

Then the following a priori estimate holds:

$$\|u\|_{\tilde{L}_T^{q_1}(\dot{B}_{p,r}^{\sigma+\frac{2}{q_1}})} \lesssim \|u_0\|_{\dot{B}_{p,r}^\sigma} + \|f\|_{\tilde{L}_T^{q_2}(\dot{B}_{p,r}^{\sigma-2+\frac{2}{q_2}})}.$$

3. Proof of Theorem 1

In this section, we shall give the brief proof for Theorem 1. Similar to the case of barotropic Navier–Stokes equations, we refer to [2,5,37] for more details. Define $\mathbf{u}_F = e^{t(\Delta + \nabla \operatorname{div})} \mathbf{u}_0$, and $\tau_F = e^{-t} \tau_0$. Then τ_F satisfies the linear system

$$\partial_t \tau_F + \tau_F = 0, \quad \tau_F(0) = \tau_0.$$

Let $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_F$, $\bar{\tau} = \tau - \tau_F$, then $(a, \bar{\mathbf{u}}, b, \bar{\tau})$ satisfies the following systems

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla a + a \operatorname{div} \mathbf{u} = 0, \\ \partial_t \bar{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \bar{\mathbf{u}} - \nabla \operatorname{div} \bar{\mathbf{u}} + \nabla a + \nabla b - \nabla \tau = \mathbf{F}(a, \mathbf{u}, b, \tau), \\ \partial_t b + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla b + b \operatorname{div} \mathbf{u} = 0, \\ \partial_t \bar{\tau} + \bar{\tau} = -\operatorname{div}(\tau \mathbf{u}). \end{cases} \quad (5)$$

Now we can prove the existence of the solutions by a standard scheme, which proceeds according to the following steps (to simplify the procedure, we omit the details of the process here; please refer to [2,5,37] for more details):

Step 1: Smoothing out the data and getting a sequence of smooth solutions $(a^n, \bar{\mathbf{u}}^n, b^n, \bar{\tau}^n)$ of an approximated system of (5), on a bounded interval $[0; T^n]$ which may depend on n .

Step 2: Exhibiting a positive bound T for T^n , and proving uniform estimates on $a^n, \bar{\mathbf{u}}^n, b^n, \bar{\tau}^n$.

Step 3: Applying compactness to show that the sequence $(a^n, \bar{\mathbf{u}}^n, b^n, \bar{\tau}^n)$ converges to the extracted solution of (5).

4. Proof of Theorem 2

In this section, we will prove Theorem 2 by using the bootstrap argument. First, we obtain a priori bounds by introducing a novel analytical framework. Meanwhile, we distinguish their different behaviors by separating the low frequency from the high frequency. The proof is organized into the following three subsections.

4.1. Low-Frequency Estimates

In this subsection, our goal is to obtain the low-frequency estimates of the solutions $(a, \mathbf{u}, b, \tau)$. Let $\Lambda^s z \triangleq \mathcal{F}^{-1}(|\xi|^s \mathcal{F}z)$, $s \in \mathbb{R}$. $\omega = \Lambda^{-1} \operatorname{curl} \mathbf{u}$ denotes the incompressible part of $\mathbf{u}$ and $\mathbf{v} = \Lambda^{-1} \operatorname{div} \mathbf{u}$ denotes the compressible part. Then, we see that ω satisfies the heat equation

$$\partial_t \omega - \Delta \omega = \Lambda^{-1} \operatorname{curl} g$$

with

$$g = -\mathbf{u} \cdot \nabla \mathbf{u} + \nabla \tau + \mathbf{F}(a, \mathbf{u}, b, \tau).$$

Applying Lemma 7 gives

$$\begin{aligned} \|\omega^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\omega^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} &\lesssim \|\omega_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} \\ &\quad + \|\mathbf{F}^\ell(a, \mathbf{u}, b, \tau)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}. \end{aligned} \quad (6)$$

Next, we consider the coupled system of $(a, \mathbf{v}, b)$. For convenience, we set

$$\phi \triangleq a + b$$

Then it follows from (3) that $(\phi, \mathbf{v})$ satisfies

$$\begin{cases} \partial_t \phi + 2\Lambda \mathbf{v} = f_1, \\ \partial_t \mathbf{v} - 2\Delta \mathbf{v} - \Lambda \phi = f_2, \end{cases} \quad (7)$$

with the nonlinear term f_1, f_2 defined by

$$f_1 \triangleq -\operatorname{div}(\phi \mathbf{u}), \quad (8)$$

$$f_2 \triangleq \Lambda^{-1} \operatorname{div}(-(\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \tau + \mathbf{F}(a, \mathbf{u}, b, \tau)). \quad (9)$$

Now we present the following lemma, which will be used in establishing the low-frequency estimates.

Lemma 8. Let k_0 be some integer, and $f^\ell \stackrel{\text{def}}{=} \dot{S}_{k_0} f$. For any $t \geq 0$, it holds that

$$\|(\phi^\ell, \mathbf{v}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(\phi^\ell, \mathbf{v}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} \lesssim \|(\phi_0^\ell, \mathbf{v}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|((f_1)^\ell, (f_2)^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}. \quad (10)$$

Proof. Let $f_k = \dot{\Delta}_k f$. Apply the operator $\dot{\Delta}_k S_{k_0}$ to the above equations and multiply (7)₁ by $\phi_k^\ell/2$ and (7)₂ by v_k^ℓ , respectively. Using the standard energy argument, then we deduce the following three equalities:

$$\frac{1}{2} \frac{d}{dt} (\|\phi_k^\ell\|_{L^2}^2/2 + \|\mathbf{v}_k^\ell\|_{L^2}^2) + 2\|\Lambda \mathbf{v}_k^\ell\|_{L^2}^2 = \langle (f_1)_k^\ell, \phi_k^\ell/2 \rangle + \langle (f_2)_k^\ell, \mathbf{v}_k^\ell \rangle, \quad (11)$$

$$- \frac{d}{dt} \langle \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle - 2\|\Lambda \mathbf{v}_k^\ell\|_{L^2}^2 + \|\Lambda \phi_k^\ell\|_{L^2}^2 = -2\langle \Delta \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle - \langle (f_2)_k^\ell, \Lambda \phi_k^\ell \rangle - \langle \Lambda (f_1)_k^\ell, \mathbf{v}_k^\ell \rangle, \quad (12)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\Lambda \phi_k^\ell\|_{L^2}^2 = \langle 2\Delta \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle + \langle (f_1)_k^\ell, \Lambda^2 \phi_k^\ell \rangle. \quad (13)$$

By multiplying both sides of (11) and (12) by $2, \frac{1}{2}$, respectively, then adding the result of multiplying both sides of (13) by $1/2$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \mathcal{L}_k^2 + 3\|\Lambda \mathbf{v}_k^\ell\|_{L^2}^2 + \frac{1}{2} \|\Lambda \phi_k^\ell\|_{L^2}^2 \\ &= \langle (f_1)_k^\ell, \phi_k^\ell \rangle + 2\langle (f_2)_k^\ell, \mathbf{v}_k^\ell \rangle - \frac{1}{2} \langle (f_2)_k^\ell, \Lambda \phi_k^\ell \rangle - \frac{1}{2} \langle \Lambda (f_1)_k^\ell, \mathbf{v}_k^\ell \rangle + \frac{1}{2} \langle (f_1)_k^\ell, \Lambda^2 \phi_k^\ell \rangle, \end{aligned}$$

where

$$\mathcal{L}_k^2 \triangleq \|\phi_k^\ell\|_{L^2}^2 + 2\|\mathbf{v}_k^\ell\|_{L^2}^2 - \langle \mathbf{v}_k^\ell, \Lambda \phi_k^\ell \rangle + \frac{1}{2} \|\Lambda \phi_k^\ell\|_{L^2}^2.$$

By Young's inequality we see that

$$\mathcal{L}_k^2 \approx \|(\phi_k^\ell, \Lambda \phi_k^\ell, \mathbf{v}_k^\ell)\|_{L^2}^2 \approx \|(\phi_k^\ell, \mathbf{v}_k^\ell)\|_{L^2}^2,$$

which leads to

$$\frac{d}{dt} \mathcal{L}_k + 2^{2k} \mathcal{L}_k \lesssim \|((f_1)_k^\ell, (f_2)_k^\ell)\|_{L^2}.$$

Multiplying the above inequality by $2^{(\frac{n}{2}-1)k}$ formally on both hand sides, and then integrating from 0 to t , we finally obtain the desired estimate (10) by summing for $k \leq k_0$. This completes the proof. $\square$

For simplicity, we define $\mathcal{E}_\infty(t), \mathcal{E}_1(t)$ as

$$\begin{aligned} \mathcal{E}_\infty(t) &\triangleq \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}, \\ \mathcal{E}_1(t) &\triangleq \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}. \end{aligned}$$

Combining (6) with (10) yields

$$\begin{aligned} & \|(\phi^\ell, \mathbf{u}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} \\ & \lesssim \|(\phi_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\operatorname{div}(\phi \mathbf{u}))^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} \\ & \quad + \|(\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\mathbf{F}^\ell(a, \mathbf{u}, b, \tau)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}. \end{aligned} \quad (14)$$

In the following, we estimate each term on the right-hand side of (14). By Lemma 4, we first estimate the term $\|(\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$ as follows

$$\begin{aligned} \|(\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} &\lesssim (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) \|\nabla \mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \\ &\lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t). \end{aligned} \quad (15)$$

Using Lemmas 4 and 7, the same strategy can be used for the term $\|(\operatorname{div}(\phi \mathbf{u}))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$, which is bounded by

$$\begin{aligned} \|(\operatorname{div}(\phi \mathbf{u}))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} &\lesssim \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) + \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \\ &\lesssim \|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 \\ &\lesssim \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 \\ &\lesssim (\mathcal{E}_1(t) + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t). \end{aligned} \quad (16)$$

We now begin to estimate terms in $\|\mathbf{F}^\ell(a, \mathbf{u}, b, \tau)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}$. It is easy to check that the first term $\|I(a) \nabla \tau\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell$ can be bounded by

$$\begin{aligned} \|I(a) \nabla \tau\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell &\lesssim \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|\nabla \tau\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell + \|\nabla \tau\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^h) \\ &\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}). \end{aligned} \quad (17)$$

Keeping in mind that

$$I(a) = a - aI(a),$$

we first use Lemmas 3 and 6 to get

$$\begin{aligned} \|(I(a))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} &\lesssim \|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(aI(a))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \\ &\lesssim \|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) \\ &\lesssim \|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \\ &\lesssim \|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})^2 \\ &\lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t). \end{aligned} \quad (18)$$

Similarly, we can infer from Lemmas 3 and 6 that

$$\begin{aligned} \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} &\lesssim \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|aI(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \\ &\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) + \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) + \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim (1 + \|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \\ &\lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t). \end{aligned} \quad (19)$$

Now, for the term $I(a)\nabla\phi$ in $\mathbf{F}(a, \mathbf{u}, b, \tau)$, in view of the fact that $\phi = \phi^\ell + \phi^h$, we can write

$$\|(I(a)\nabla\phi)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|(I(a)\nabla\phi^\ell)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(I(a)\nabla\phi^h)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}. \quad (20)$$

Thanks to Lemma 3 again, we have

$$\begin{aligned} \|(I(a)\nabla\phi^\ell)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} &\lesssim \|\nabla\phi^\ell\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|(I(a))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(I(a))^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) \\ &\lesssim \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} (\|(I(a))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(I(a))^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}). \end{aligned} \quad (21)$$

Combining (18), (19), and (21) leads to

$$\|(I(a)\nabla\phi^\ell)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t). \quad (22)$$

For the term $\|(I(a)\nabla\phi^h)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$ in (20), we use Bony's decomposition to write

$$\begin{aligned} \dot{S}_{j_0+1}(I(a)\nabla\phi^h) &= \dot{T}_{I(a)}\dot{S}_{j_0+1}\nabla\phi^h + [\dot{S}_{j_0+1}, \dot{T}_{I(a)}]\nabla\phi^h \\ &\quad + \dot{S}_{j_0+1}(\dot{T}_{\nabla\phi^h}I(a) + \dot{R}(I(a), \nabla\phi^h)). \end{aligned} \quad (23)$$

Applying Lemma 2, it holds that

$$\|\dot{T}_{I(a)}\dot{S}_{j_0+1}\nabla\phi^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|I(a)\|_{\dot{B}_{\infty,\infty}^{-1}} \|\dot{S}_{j_0+1}\nabla\phi^h\|_{\dot{B}_{2,1}^1} \lesssim \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\phi\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell,$$

from this result along with (19), we can further get

$$\|\dot{T}_{I(a)}\dot{S}_{j_0+1}\nabla\phi^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t) \|\phi\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell. \quad (24)$$

The last two terms in (23) can be estimated similarly so that

$$\begin{aligned} &\|\dot{S}_{j_0+1}(\dot{T}_{\nabla\phi^h}I(a) + \dot{R}(I(a), \nabla\phi^h))\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|[\dot{S}_{j_0+1}, \dot{T}_{I(a)}]\nabla\phi^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \\ &\lesssim \|\nabla\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim \mathcal{E}_\infty(t) \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}. \end{aligned} \quad (25)$$

From (24) and (25), we have

$$\|(I(a)\nabla\phi^h)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t) (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}).$$

which together with (20) and (22) give rise to

$$\|(I(a)\nabla\phi)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t) (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}). \quad (26)$$

Similarly,

$$\|I(a)(\Delta\mathbf{u} + \nabla\operatorname{div}\mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t). \quad (27)$$

Collecting (17), (26), and (27) together, we get

$$\|\mathbf{F}^\ell(a, \mathbf{u}, b, \tau)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (1 + \mathcal{E}_\infty(t)) \mathcal{E}_\infty(t) \mathcal{E}_1(t). \quad (28)$$

Thus plugging (15), (16), and (28) into (14) implies

$$\begin{aligned} & \|(\phi^\ell, \mathbf{u}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} \\ & \lesssim \|(\phi_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} + \int_0^t (1 + \mathcal{E}_\infty(s)) \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \end{aligned} \quad (29)$$

Now, applying $\dot{\Delta}_j$ to the fourth equation of (3) and taking the inner product L^2 with $\dot{\Delta}_j \tau$ gives

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j \tau\|_{L^2}^2 + c \|\dot{\Delta}_j \tau\|_{L^2}^2 \lesssim \|\dot{\Delta}_j \operatorname{div}(\tau \mathbf{u})\|_{L^2} \|\dot{\Delta}_j \tau\|_{L^2}. \quad (30)$$

Multiplying (30) by $2^{\frac{n}{2}j} / \|\dot{\Delta}_j \tau\|_{L^2}$ formally and integrating the resultant inequality from 0 to t , we can deduce by summing for $j \leq j_0$ that

$$\|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \lesssim \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t \|(\operatorname{div}(\tau \mathbf{u}))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} ds.$$

By Lemma 4, the last term on the right side of the above inequality can be controlled by

$$\begin{aligned} \|(\operatorname{div}(\tau \mathbf{u}))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} & \lesssim \|(\tau \operatorname{div} \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\mathbf{u} \cdot \nabla \tau)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \\ & \lesssim \|\tau\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) + \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \\ & \lesssim \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}). \end{aligned}$$

Thus

$$\begin{aligned} \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} & \lesssim \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2) ds \\ & \quad + \int_0^t (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) ds. \end{aligned} \quad (31)$$

Multiplying (31) by a suitable large constant, and then adding to (29), we finally get

$$\begin{aligned} & \|(\phi^\ell, \mathbf{u}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \\ & \lesssim \|(\phi_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t (1 + \mathcal{E}_\infty(s)) \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \end{aligned} \quad (32)$$

Subsequently, it is natural to derive the bound of $\|a\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}$. Due to the appearance of the unwanted term $\operatorname{div} \mathbf{u}$, the boundedness cannot be directly established through conventional methods. Thus, we introduce a novel truncation technique to circumvent this obstacle. Define

$$\delta \triangleq \phi - 2a = b - a,$$

which satisfies the following transport equation

$$\partial_t \delta + \mathbf{u} \cdot \nabla \delta + \delta \operatorname{div} \mathbf{u} = 0.$$

Now applying $\dot{\Delta}_j$ to the above equation and using the commutator argument yields

$$\partial_t \dot{\Delta}_j \delta + \mathbf{u} \cdot \nabla \dot{\Delta}_j \delta + [\dot{\Delta}_j, \mathbf{u} \cdot \nabla] \delta + \dot{\Delta}_j (\delta \operatorname{div} \mathbf{u}) = 0.$$

Taking the inner product L^2 of the resulting equation with $\dot{\Delta}_j \delta$, and applying the Hölder inequality and integrating the resultant inequality over $[0, t]$, we derive

$$\begin{aligned} \|\delta^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} &\lesssim \|\delta_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\delta \operatorname{div} \mathbf{u})^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\delta^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} \\ &\quad + \int_0^t \sum_{j \leq j_0} 2^{j(\frac{n}{2}-1)} \|[\dot{\Delta}_j, \mathbf{u} \cdot \nabla] \delta\|_{L^2} ds. \end{aligned} \quad (33)$$

Using Lemma 4, the second and third terms on the right-hand side of the inequality can be controlled as follows:

$$\begin{aligned} \|(\delta \operatorname{div} \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\delta^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} &\lesssim (\|\delta^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\delta^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) \|\operatorname{div} \mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \\ &\lesssim (\|(a^\ell, \phi^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a^h, \phi^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}). \end{aligned} \quad (34)$$

The last term in (33) can be bounded by a similar derivation of (4.9) in [38]:

$$\sum_{j \leq j_0} 2^{j(\frac{n}{2}-1)} \|[\dot{\Delta}_j, \mathbf{u} \cdot \nabla] \delta\|_{L^2} \lesssim (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) (\|(a^\ell, \phi^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a^h, \phi^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}). \quad (35)$$

Taking (34) and (35) into (33), we obtain

$$\|\delta^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|(a_0^\ell, \phi_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds.$$

which, combined with the definition $a = \frac{1}{2}(\phi - \delta)$, leads to

$$\begin{aligned} \|a^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} &\lesssim \|\delta^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\phi^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \\ &\lesssim \|(a_0^\ell, \phi_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \int_0^t \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \end{aligned} \quad (36)$$

In the same manner, we can infer from the forth equation of (3) that

$$\|b^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|(b_0^\ell, \phi_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \int_0^t \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \quad (37)$$

Consequently, combining (32), (36) and (37), we finally arrive at

$$\begin{aligned} \|(a^\ell, b^\ell, \mathbf{u}^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \\ \lesssim \|(a_0^\ell, b_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t (1 + \mathcal{E}_\infty(s)) \mathcal{E}_\infty(s) \mathcal{E}_1(s) ds. \end{aligned}$$

4.2. High-Frequency Estimates

In the high-frequency regime, the primary challenge arises from the absence of diffusion terms in the equations for a and b . The quasilinear terms in these equations lead to a loss of the derivatives. To overcome this difficulty, we introduce the new variable $\phi \triangleq a + b$ and apply the operators $\mathbb{Q}$ to the momentum equation, obtaining the compressible part of Equation (3) as follows

$$\begin{cases} \partial_t \phi + 2 \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla \phi + \phi \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbb{Q} \mathbf{u} - 2 \Delta \mathbb{Q} \mathbf{u} + \nabla \phi = -\mathbb{Q}(\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \tau + \mathbb{Q} \mathbf{F}(a, \mathbf{u}, b, \tau). \end{cases}$$

Similar to the approach used in [3,37], if we define an effective viscous flux

$$\mathbf{G} \triangleq \mathbb{Q} \mathbf{u} - \frac{1}{2} \Delta^{-1} \nabla \phi,$$

then $\mathbf{G}$ satisfies

$$\partial_t \mathbf{G} - 2 \Delta \mathbf{G} = \mathbf{G} + \frac{1}{2} \Delta^{-1} \nabla \phi + \frac{1}{2} \mathbb{Q}(\phi \mathbf{u}) - \mathbb{Q}(\mathbf{u}, \nabla \mathbf{u}) + \nabla \tau + \mathbb{Q} \mathbf{F}(a, \mathbf{u}, b, \tau).$$

Applying the heat estimate for the high frequencies of $\mathbf{G}$, we get

$$\begin{aligned} & \|\mathbf{G}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} \\ & \lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-2})} + \|\tau^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbb{Q}(\phi \mathbf{u})^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} \\ & \quad + \|\mathbb{Q}(\mathbf{u}, \nabla \mathbf{u})^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\mathbb{Q} \mathbf{F}(a, \mathbf{u}, b, \tau)^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})}. \end{aligned} \quad (38)$$

Furthermore, ϕ satisfies

$$\partial_t \phi + \phi + \operatorname{div} \mathbf{G} = -\phi \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \phi. \quad (39)$$

Applying Δ_j to (39) and taking the inner product L^p with $\Delta_j \phi$, then multiplying by $2^{\frac{n}{p}j} / \|\Delta_j \phi\|_{L^2}$ formally and integrating the resultant inequality from 0 to t , we deduce by summing for $j \geq j_0$ that

$$\begin{aligned} & \|\phi^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ & \lesssim \|\phi_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \int_0^t (\|(\phi \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) ds \\ & \quad + \int_0^t \sum_{j \geq j_0} 2^{\frac{n}{p}j} \|\Delta_j \mathbf{u} \cdot \nabla \phi\|_{L^p} ds, \end{aligned} \quad (40)$$

where we have use a commutator argument.

Then by a standard energy argument, we deduce from (38) and (40) that

$$\begin{aligned} & \|\mathbf{G}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\phi^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|\phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ & \lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\phi_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t (\|\mathbf{G}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-2}}) ds + \int_0^t \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} ds \\ & \quad + \int_0^t (\|(\phi \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|(\mathbf{u} \cdot \nabla \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|(\phi \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) ds \\ & \quad + \int_0^t (\sum_{j \geq j_0} 2^{\frac{n}{p}j} \|\Delta_j \mathbf{u} \cdot \nabla \phi\|_{L^p} + \|\mathbf{F}(a, \mathbf{u}, b, \tau)^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) ds. \end{aligned} \quad (41)$$

To control the right-hand side of (41), we have to estimate the nonlinear terms. Applying Lemma 3 and the interpolation inequality, we treat the nonlinear terms as follows

$$\begin{aligned} & \|(\phi \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} \lesssim (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}), \\ & \|(\mathbf{u} \cdot \nabla \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \lesssim \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} \lesssim (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}), \end{aligned} \quad (42)$$

and

$$\begin{aligned} \|(\phi \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} &\lesssim \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 \\ &\lesssim \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\phi^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}. \end{aligned}$$

By Lemma 5 we infer that

$$\sum_{j \geq j_0} 2^{\frac{n}{p}j} \|[\Delta, \mathbf{u} \cdot \nabla] \phi\|_{L^p} \lesssim \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} \lesssim (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}).$$

The term $I(a)(\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u})$ in $\mathbf{F}(a, \mathbf{u}, b, \tau)$ can be estimated in a similar way to (42). Here we just give some representative term $I(a)(\nabla \tau - \nabla \phi)$ as follows, and, with the aid of Lemma 6, we get

$$\begin{aligned} \|I(a)(\nabla \tau - \nabla \phi)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^h &\lesssim \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\nabla \tau - \nabla \phi\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^h \lesssim \|I(a)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\nabla \tau - \nabla \phi\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^2 \\ &\lesssim \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|(\nabla \tau - \nabla \phi)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^2 + \|(\nabla \tau - \nabla \phi)^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^2 \\ &\lesssim \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\phi^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^2 + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}^2. \end{aligned}$$

Next, we deal with the estimation on τ . From the fourth equation of (3) we get

$$\begin{aligned} &\|\tau^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\tau^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ &\lesssim \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t \|(\tau \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} ds + \int_0^t \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} ds + \int_0^t \sum_{j \geq j_0} 2^{\frac{n}{p}j} \|[\Delta, \mathbf{u} \cdot \nabla] \tau\|_{L^p} ds \\ &\lesssim \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) ds, \end{aligned} \quad (43)$$

where we used the fact

$$\|(\tau \operatorname{div} \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \sum_{j \geq j_0} 2^{\frac{n}{p}j} \|[\Delta, \mathbf{u} \cdot \nabla] \tau\|_{L^p} \lesssim (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) (\|\mathbf{u}^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}).$$

Applying the operator $\mathbb{P}$ to the momentum equation, we get the equation for the incompressible part of $\mathbf{u}$ as follows

$$\partial_t \mathbb{P} \mathbf{u} - \Delta \mathbb{P} \mathbf{u} + \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}) = \mathbb{P} \mathbf{F}(a, \mathbf{u}, b, \tau).$$

By using a standard energy estimation, we get

$$\begin{aligned} \|\mathbb{P} \mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\mathbb{P} \mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} &\lesssim \|\mathbb{P} \mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \int_0^t \|(\mathbf{u} \cdot \nabla \mathbf{u})^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} ds \\ &\quad + \int_0^t \|\mathbf{F}^h(a, \mathbf{u}, b, \tau)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} ds. \end{aligned} \quad (44)$$

Summing up (41), (43), and (44), and noticing that $\mathbb{Q} \mathbf{u} = \mathbf{G} + \frac{1}{2} \Delta^{-1} \nabla \phi$, we obtain

$$\begin{aligned} &\|\mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|(\phi^h, \tau^h)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|(\phi^h, \tau^h)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ &\lesssim \|\mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|(\phi_0^h, \tau_0^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t \mathcal{E}_1(s) \mathcal{E}_\infty(s) ds. \end{aligned} \quad (45)$$

Finally, we combine the estimates above to obtain the bound of $\|a^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}$. It directly follows from a similar derivation of (40) that

$$\begin{aligned}\|a^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} &\lesssim \|a_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \int_0^t \|\nabla \mathbf{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|a\|_{\dot{B}_{p,1}^{\frac{n}{p}}} ds \\ &\lesssim \|a_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \int_0^t \mathcal{E}_1(s) \mathcal{E}_\infty(s) ds.\end{aligned}\quad (46)$$

In the same manner, we can infer that

$$\|b^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} \lesssim \|b_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \int_0^t \mathcal{E}_1(s) \mathcal{E}_\infty(s) ds. \quad (47)$$

Multiplying by a suitable large constant on both sides of (45) and then adding (46) and (47), we finally achieve

$$\begin{aligned}\|(a^h, b^h, \tau^h)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|(\phi^h, \tau^h)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} \\ \lesssim \|(a_0^h, b_0^h, \tau_0^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \|\mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \int_0^t \mathcal{E}_1(s) \mathcal{E}_\infty(s) ds.\end{aligned}$$

4.3. Proof of Theorem 2

In this subsection, we shall prove Theorem 2 by applying the local existence result and the continuous arguments. Denote

$$\begin{aligned}\mathcal{X}(t) &\triangleq \|(a^\ell, \mathbf{u}^\ell, b^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(\phi^\ell, \mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|\tau^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \\ &\quad + \|\mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|(a^h, b^h, \tau^h)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\mathbf{u}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})} + \|(\phi^h, \tau^h)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})}, \\ \mathcal{X}_0 &\triangleq \|(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|(a_0^h, b_0^h, \tau_0^h)\|_{\dot{B}_{p,1}^{\frac{n}{p}}}.\end{aligned}$$

Subsequently, we deduce from (32) and (45) that

$$\mathcal{X}(t) \leq \mathcal{X}_0 + C(\mathcal{X}(t))^2(1 + C\mathcal{X}(t)).$$

Let $\mathcal{X}_0 \leq c_0$ as stated in Theorem 2. Using the local existence in Theorem 1, we deduce that there exists a positive time T such that

$$\mathcal{X}(t) \leq 2c_0, \quad \forall t \in [0, T]. \quad (48)$$

By using a standard continuation argument, we can show that, if c_0 is small enough, then $T = \sup\{T^* > 0 \mid \mathcal{X}(t) \leq 2c_0 \text{ holds}\} = \infty$. This completes the proof of Theorem 2.

5. Proof of Theorem 3

The goal of this section is to obtain the decay rate of the solutions by the method (which is different from the spectral analysis) used in [39,40]. This approach relies on constructing a Lyapunov function for $(\phi, \mathbf{u}, \tau)$, which is derived from the energy inequality of the solutions, along with the interpolation and embedding theorem. To achieve this, we first establish the propagation of the initial data's regularity in Besov space with low regularity, as shown in the following Proposition 1.

Proposition 1. Let $(a, \mathbf{u}, b, \tau)$ be the solution of (3) constructed in Theorem 2 with $p = 2$. For any $-\frac{n}{2} \leq \sigma < \frac{n}{2} - 1$ such that $(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell) \in \dot{B}_{2,\infty}^\sigma(\mathbb{R}^n)$, $\tau_0^\ell \in \dot{B}_{2,\infty}^{\sigma+1}(\mathbb{R}^n)$, then there exists a constant $C_0 > 0$ depending on the norm of the initial data such that for all $t \geq 0$,

$$\|(a, \mathbf{u}, b)(t, \cdot)\|_{\dot{B}_{2,\infty}^\sigma}^\ell + \|\tau(t, \cdot)\|_{\dot{B}_{2,\infty}^{\sigma+1}}^\ell \leq C_0. \quad (49)$$

Proof. The proof of this proposition can be obtained the same as Proposition 5.1 in [41]. We omit the details here. $\square$

In the following, our main goal is to establish a Lyapunov-type inequality. From the proof of Theorem 2, we can get the following inequality

$$\begin{aligned} & \frac{d}{dt} (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \\ & + \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \\ & \lesssim (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}) (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}) \\ & + (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(a^h, b^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}). \end{aligned} \quad (50)$$

Since the solution which we have proved in (4) satisfies

$$\|(a^\ell, \mathbf{u}^\ell, b^\ell)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\mathbf{u}^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(a^h, b^h, \tau^h)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} \leq Cc_0,$$

this, along with (50), allows us to obtain

$$\begin{aligned} & \frac{d}{dt} (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \\ & + \|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \leq 0. \end{aligned} \quad (51)$$

By applying the interpolation inequality, for any $-\frac{n}{2} \leq \sigma < \frac{n}{2} - 1$, it holds that

$$\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell \leq C(\|(\phi, \mathbf{u})\|_{\dot{B}_{2,\infty}^\sigma}^\ell)^{\theta_1} (\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}^\ell)^{1-\theta_1},$$

with $\theta_1 = \frac{4}{n-2\sigma+2} \in (0, 1)$. By using (49), we directly obtain that

$$\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}^\ell \geq c_0 (\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell)^{\frac{1}{1-\theta_1}}. \quad (52)$$

Subsequently, again by the interpolation inequality, we have

$$\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell \leq C(\|\tau\|_{\dot{B}_{2,\infty}^{\sigma+1}}^\ell)^{\theta_1} (\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}+2}}^\ell)^{1-\theta_1} \leq c_0 (\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}+2}}^\ell)^{1-\theta_1}.$$

Hence, we obtain

$$\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell \geq C\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}+2}}^\ell \geq (\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell)^{\frac{1}{1-\theta_1}}. \quad (53)$$

In addition, thanks to $\|(\phi, \mathbf{u}, \nabla \tau)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^h \ll 1$, it follows by taking $\alpha = 1 + \theta_1$ that

$$C(\|(\phi, \tau)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h + \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^h) \geq (\|(\phi, \tau)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h + \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^h)^{1+\alpha}. \quad (54)$$

Finally, substituting (52), (53), and (54) into (51), we infer that

$$\begin{aligned} & \frac{d}{dt} (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \\ & + c_1 (\|(\phi^\ell, \mathbf{u}^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\phi^h, \tau^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}})^{1+\frac{4}{n-2\sigma-2}} \leq 0. \end{aligned}$$

Thus, by solving this differential inequality we obtain

$$\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell + \|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell + \|(a, \tau)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h + \|(\mathbf{u}, b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^h \leq C(1+t)^{-\frac{n-2\sigma-2}{4}}.$$

Choosing any γ_1 satisfying $\sigma < \gamma_1 < \frac{n}{2} - 1$, then it follows from (49) and the interpolation inequality that

$$\begin{aligned} \|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1}}^\ell & \leq C \|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1+\frac{n}{2}-\frac{n}{2}}}^\ell \\ & \leq C (\|(\phi, \mathbf{u})\|_{\dot{B}_{2,\infty}^\ell})^{\theta_2} (\|a, \mathbf{u}, b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^\ell)^{1-\theta_2}, \\ & \leq C(1+t)^{-\frac{(\frac{n}{2}-\sigma-1)(1-\theta_2)}{2}} \\ & \leq C(1+t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})-\frac{\gamma_1-\sigma}{2}}, \quad \theta_2 = \frac{\frac{n}{2}-\gamma_1-1}{\frac{n}{2}-\sigma-1} \in (0,1). \end{aligned} \quad (55)$$

For the high frequency, we have

$$\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1}}^h \leq C (\|a\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h + \|(\mathbf{u}, b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}^h) \leq C(1+t)^{-\frac{n-2\sigma-2}{4}}. \quad (56)$$

Then combining (55) and (56) yields

$$\begin{aligned} \|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1}} & \leq C (\|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1}}^\ell + \|(\phi, \mathbf{u})\|_{\dot{B}_{2,1}^{\gamma_1}}^h) \\ & \leq C(1+t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})-\frac{\gamma_1-\sigma}{2}} + C(1+t)^{-\frac{n-2\sigma-2}{4}} \\ & \leq C(1+t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})-\frac{\gamma_1-\sigma}{2}}. \end{aligned}$$

Subsequently, using the embedding $\dot{B}_{2,1}^0 \hookrightarrow L^2$, it implies

$$\|\Lambda^{\gamma_1}(\phi, \mathbf{u})\|_{L^2} \leq C(1+t)^{-\frac{\gamma_1-\sigma}{2}}.$$

Choosing γ_2 such that $\sigma + 1 < \gamma_2 < \frac{n}{2}$, it follows from the interpolation inequality that

$$\begin{aligned} \|\tau\|_{\dot{B}_{2,1}^{\gamma_2}} & \leq C (\|\tau\|_{\dot{B}_{2,1}^{\gamma_2}}^\ell + \|\tau\|_{\dot{B}_{2,1}^{\gamma_2}}^h) \\ & \leq C \|\tau\|_{\dot{B}_{2,1}^{\gamma_2}}^\ell + C \|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h \\ & \leq C (\|\tau\|_{\dot{B}_{2,\infty}^\ell})^{\theta_2} (\|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^\ell)^{1-\theta_2} + C \|\tau\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^h \\ & \leq C(1+t)^{-\frac{\gamma_2-\sigma-1}{2}}. \end{aligned}$$

Thus, again by using the embedding $\dot{B}_{2,1}^0 \hookrightarrow L^2$, we finally get

$$\|\Lambda^{\gamma_2}\tau\|_{L^2} \leq C(1+t)^{-\frac{\gamma_2-\sigma-1}{2}}.$$

This completes the proof of Theorem 3.

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