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A Half-Discrete Hardy–Mulholland-Type Inequality Involving One Multiple Upper Limit Function and One Partial Sum

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Abstract

In this paper, by using the techniques of real analysis, with the help of the Euler–Maclaurin summation formula, Abel's summation by parts formula, and the differentiation mid-value theorem, we establish a half-discrete Hardy–Mulholland-type inequality involving one multiple upper limit function and one partial sum. Based on the obtained inequality, we characterize the condition of the best possible constant factor related to several parameters. At the end of the paper, we illustrate that some new half-discrete Hardy–Mulholland-type inequalities can be deduced from the special values of the parameters. Our results enrich the current results in the study of half-discrete Hardy–Mulholland-type inequalities.

Keywords: half-discrete Hardy–Mulholland's inequality; multiple upper limit function; partial sum; best possible constant factor

MSC: 26D15; 26D10; 26A42

1. Introduction

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m, b_n \geq 0$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$. Then,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1) is known in the literature as the Hardy–Hilbert inequality ([1], Theorem 315). It is interesting that under a similar assumption condition to that above, one has the following Hardy–Mulholland's inequality ([1], Theorem 343):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is also the best possible.

In 1934, a half-discrete Hilbert-type inequality was presented ([1], Theorem 351), as follows:

$$\int_0^{\infty} x^{p-2}\left(\sum_{n=1}^{\infty} K(nx)a_n\right)^p \mathrm{d}x<\phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} a_n^p, \tag{3}$$

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where $K(t)$ $(t > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, the function $\phi(s)$ is defined as $0 < \phi(s) := \int_0^\infty K(t)t^{s-1}\mathrm{d}t < \infty$, $a_n \geq 0$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Since then, some half-discrete Hilbert-type inequalities have been proposed intermittently; we refer the interested reader to monograph [2] and the references cited therein.

In 2018, Batbold and Azar [3] gave a new form of half-discrete Hilbert inequality for three variables. In 2021, Huang and Yang [4] established a half-discrete Hardy–Mulholland-type inequality involving one multiple upper limit function, $F_m(x)$, as follows:

$$\begin{aligned}\int_0^\infty \sum_{n=2}^{\infty} \frac{a_n f(x)}{[x + \ln^\alpha (n-\xi)]^\lambda} dx < & \left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2) \\ \times \left[\int_0^\infty x^{p(1-m-\lambda_1)-1} F_m^p(x) dx\right]^{\frac{1}{p}} & \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\alpha\lambda_2)-1}(n-\xi)}{(n-\xi)^{1-q}} a_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{4}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = \lambda_1 + \lambda_2 > 0, \alpha \in (0,1], \lambda_1 \in (0,\lambda), \lambda_2 \in (0, \frac{1}{\alpha}] \cap (0,\lambda)$, and function $F_m(x)$ is defined by the form: $F_i(x) = \int_0^x \left(\int_0^{t_{i-1}} \cdots \int_0^{t_1} F_0(t_0) dt_0 \cdots dt_{i-2}\right) dt_{i-1}$ $(i = 1, 2, \cdots, m)$, $F_0(x) = f(x)$. The constant factor $\left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2)$ in (4) is the best possible. In 2023, Peng et al. [5] provided a reverse half-discrete Hardy–Mulholland-type inequality.

It is important to note that, in 2019, Adiyasuren et al. [6] presented a novel extension of the Hardy–Hilbert inequality by imbedding two partial sums, $A_m := \sum_{k=1}^m a_k$ and $B_n := \sum_{k=1}^n b_k$ $(m, n \in \{1, 2, \cdots\})$, in the right series of the following inequality, i.e.:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < \lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \left(\sum_{m=1}^{\infty} m^{-p\lambda_1 - 1} A_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\lambda_2 - 1} B_n^q\right)^{\frac{1}{q}} \tag{5}$$

where $\lambda_i \in (0,1] \cap (0,\lambda)$ $(\lambda \in (0,2]; i = 1,2), \lambda_1 + \lambda_2 = \lambda, a_m, b_n \geq 0$, $A_m = o(e^{tm})$, $B_n = o(e^{tn})$ $(t > 0; m, n \to \infty)$. The constant factor $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible.

To date, there have been lots of papers published on half-discrete Hardy–Hilbert-type and Mulholland-type inequalities (see [2,3,7–9]). However, the study of half-discrete Hardy–Mulholland-type inequalities involving partial sums has not been reported yet. This is the main motivation for us to carry out this work.

Hong et al. [10–15] gave equivalent parameter conditions for the construction of Hilbert-type integral inequalities. What has caught our attention is that Hong, Zhang, and Xiao [16] improve the condition for constructing a Hilbert-type integral inequality involving upper limit functions.

Inspired by the work of [6,16], in this paper, by replacing the term of series a_n with the partial sum A_n in the right-hand side of inequality (4), we establish a new half-discrete Hardy–Mulholland-type inequality. Our method is mainly based on the use of the Euler–Maclaurin summation formula, Abel's partial summation formula, and the differentiation mid-value theorem.

The rest of this paper is structured as follows. Firstly, we introduce some lemmas on the construction of weight function and related identities and inequalities. Secondly, we establish a half-discrete Hardy–Mulholland-type inequality involving one multiple upper limit function and one partial sum. For the obtained inequality, we deal with the equivalent conditions for the best possible constant factor associated with the parameters. Finally, we illustrate that the main results obtained can generate some new half-discrete Hardy–Mulholland-type inequalities.

2. Preliminaries and Lemmas

For convenience, we first specify the following assumption conditions (H1) because they will be used frequently in the subsequent analysis.

(H1). $p > 1$ $(q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta \in (0,1]$, $\eta \in [0, \frac{1}{4}]$, $\lambda > 0$, $\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \frac{8}{5\beta} - 1] \cap (0, \lambda)$, $\hat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$. *Let $f(x)$ be a nonnegative continuous function unless at finite points in $R_+ = (0, \infty)$, $m \in N_0 = \{0, 1, 2, \ldots\}$, $F_0(x) = f(x)$, inductively, for $m \in N$, we define the following multiple upper limit functions: $F_i(x) := \int_0^x F_{i-1}(t)dt$ $(x \geq 0)$, namely,*

$$F_i(x) = \int_0^x \left(\int_0^{t_{i-1}} \ldots \int_0^{t_1} F_0(t_0) dt_0 \ldots dt_{i-2} \right) dt_{i-1} (i = 1, 2, \ldots, m)$$

which satisfies $m \in N_0$

$$0 < c_{i-1} := \int_0^\infty t^{p(2-i-\hat{\lambda}_1)-1} F_{i-1}^p(t)dt < \infty \ (i = 1, 2, \ldots, m+1). \tag{6}$$

Let $b_n \geq 0$,$B_n := \sum_{k=2}^n b_k (n \in \mathrm{N}_2 = \{2, 3, \ldots\})$, which satisfies

$$B_n = o(e^{t \ln^\beta (n-\eta)}) \ (t > 0;\ n \to \infty) \text{ and } 0 < \sum_{n=2}^\infty \frac{\ln^{-q\beta\hat{\lambda}_2 - 1}(n-\eta)}{n-\eta} B_n^q < \infty. \tag{7}$$

Lemma 1. *(i) ([17], 2.2.13). If $k \in N_0$, $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [k, \infty)$ and $g^{(i)}(\infty) = 0$ $(i = 0, 1)$, $P_i(t), B_i (i \in \mathrm{N})$ are Bernoulli functions and Bernoulli numbers of i-order, then*

$$\int_m^\infty P_{2q-1}(t) g(t) dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) (0 < \varepsilon_q < 1; q = 1, 2, \ldots), \text{ and}$$

$$-\frac{1}{8} g(k) < \int_k^\infty P_1(t) g(t) dt < 0. \tag{8}$$

(ii) ([17], 2.3.2) If $k \in N_0$, $f(t) (> 0) \in C[k, \infty)$, $f^{(i)}(\infty) = 0$ $(i = 0, 1)$, then we have the following Euler–Maclaurin summation formula:

$$\sum_{i=k}^n f(i) = \int_k^n f(t)dt + \frac{1}{2}(f(k) + f(n)) + \int_k^n P_1(t) f'(t) dt \ (n \geq k). \tag{9}$$

Lemma 2. *Let $\eta \in [0, \frac{1}{4}]$, $s > 0$,$s_2 \in (0, \frac{8}{5\beta}] \cap (0, s)$,$k_s(s_2) := B(s_2, s - s_2)$. We can then define the following weight coefficient:*

$$\varpi_{(s,\eta)}(s_2, x) := x^{s-s_2} \sum_{n=2}^\infty \frac{\beta \ln^{\beta s_2 - 1}(n-\eta)}{[x + \ln^\beta (n-\eta)]^s (n-\eta)} (x > 0) \tag{10}$$

The following inequalities are valid:

$$0 < k_s(s_2)(1 - O_s(\frac{1}{x^{s_2}})) < \varpi_{(s,\eta)}(s_2, x) < k_s(s_2) (x > 0), \tag{11}$$

where $O_s(\frac{1}{x^{s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{\ln^\beta (2-\eta)}{x}} \frac{u^{s_2-1}}{(1+u)^s} du > 0.$

Proof. For fixed $x > 0$, we define a positive function $h(t)$ as follows:

$$h(t) := \frac{\ln^{\beta s_2 - 1}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s}(t-\eta)} (t > 1+\eta).$$

(i) For $s_2 \in (0, \frac{1}{\beta}] \cap (0, s), \eta \in [0, \frac{1}{4}], \beta \in (0, 1]$, it is easy to find

$$h'(t) = -\frac{(1-\beta s_2)\ln^{\beta s_2 - 2}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s}(t-\eta)^2} - \frac{\beta s \ln^{\beta(s_2+1)-2}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s+1}(t-\eta)^2} - \frac{\ln^{\beta s_2 - 1}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s}(t-\eta)^2} < 0.$$

Similarly, we can obtain

$$h''(t) = -[\frac{(1-\beta s_2)\ln^{\beta s_2 - 2}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s}(t-\eta)^2}]' - [\frac{\beta s \ln^{\beta(s_2+1)-2}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s+1}(t-\eta)^2}]' - [\frac{\ln^{\beta s_2 - 1}(t-\eta)}{[x + \ln^{\beta}(t-\eta)]^{s}(t-\eta)^2}]' > 0.$$

Now, using Hemite–Hadamard's inequality [18] and the decreasingness property of the series, we deduce that

$$\int_2^{\infty} h(t)dt < \sum_{n=2}^{\infty} h(n) < \int_{\frac{3}{2}}^{\infty} h(t)dt < \int_{1+\eta}^{\infty} h(t)dt. \tag{12}$$

(ii) For $s_2 \in (\frac{1}{\beta}, \frac{8}{5\beta}] \cap (0, s)$, in view of (8) (for $n \to \infty$), we have

$$\sum_{n=2}^{\infty} h(n) = \int_2^{\infty} h(t)dt + \frac{1}{2}h(2) + \int_2^{\infty} P_1(t)h'(t)dt$$

$$= \int_{1+\eta}^{\infty} h(t)dt - h_0(s_2, \eta),$$
$$h_0(s_2, \eta) := \int_{1+\eta}^{2} h(t)dt - \frac{1}{2}h(2) - \int_2^{\infty} P_1(t)h'(t)dt.$$

From the definition of $h(t)$, we derive $\frac{-1}{2}h(2) = \frac{-\ln^{\beta s_2 - 1}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)}$. Now, setting $u = \ln^{\beta}(t-\eta)$, integrating by parts gives

$$\int_{1+\eta}^{2} h(t)dt = \frac{1}{\beta}\int_0^{\ln^{\beta}(2-\eta)} \frac{n^{s_2-1}}{(x+u)^s}du = \frac{1}{\beta}\int_0^{\ln^{\beta}(2-\eta)} \frac{du^{s_2}}{s_2(x+u)^s}$$
$$= \frac{u^{s_2}}{\beta s_2(x+u)^s}\Big|_0^{\ln^{\beta}(2-\eta)} - \frac{1}{\beta}\int_0^{\ln^{\beta}(2-\eta)} \frac{u^{s_2}}{s_2} d\frac{1}{(x+u)^s} = \frac{\ln^{\beta s_2}(2-\eta)}{\beta s_2[x+\ln^{\beta}(2-\eta)]^s}$$
$$+ \frac{s}{\beta s_2(s_2+1)}\int_0^{\ln^{\beta}(2-\eta)} \frac{du^{s_2+1}}{(x+u)^{s+1}} > \frac{\ln^{\beta s_2}(2-\eta)}{\beta s_2[x+\ln^{\beta}(2-\eta)]^s} + \frac{s}{\beta s_2(s_2+1)} \frac{\ln^{\beta(s_2+1)}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}}.$$

If $(-1)^i f_j^{(i-1)}(t) > 0$ $(i, j = 1, 2)$, then we have the following inequality:

$$(-1)^i (f_1(t) f_2(t))^{(i-1)} > 0 \ (i = 1, 2).$$

Based on the above result, for $\eta \in [0, \frac{1}{4}]$, $\frac{\ln(t-\eta)}{(t-\eta)^2} > 0$, $(\frac{\ln(t-\eta)}{(t-\eta)^2})' = \frac{1-2\ln(t-\eta)}{(t-\eta)^3} < 0 (t > 2)$, with the aid of (8), it follows that

$$
\begin{aligned}
&h'(t) = \frac{(\beta s_2 - 1)\ln^{\beta s_2 - 2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}(t-\eta)^2} - \frac{\beta s \ln^{\beta s_2-(3-\beta)}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s+1}}\frac{\ln(t-\eta)}{(t-\eta)^2} - \frac{\ln^{\beta s_2-2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}}\frac{\ln(t-\eta)}{(t-\eta)^2},\\
&-\int_2^{\infty} P_1(t)\frac{(\beta s_2 - 1)\ln^{\beta s_2 - 2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}(t-\eta)^2}dt > 0 (1 < \beta s_2 \le \frac{8}{5} < 2),\\
&\int_2^{\infty} P_1(t)[\frac{\beta s \ln^{\beta s_2-(3-\beta)}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s+1}}\frac{\ln(t-\eta)}{(t-\eta)^2} + \frac{\ln^{\beta s_2-2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}}\frac{\ln(t-\eta)}{(t-\eta)^2}]dt\\
&> -\frac{1}{8}[\frac{\beta s \ln^{\beta s_2-(3-\beta)}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}}\frac{\ln(2-\eta)}{(2-\eta)^2} + \frac{\ln^{\beta s_2-2}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s}}\frac{\ln(2-\eta)}{(2-\eta)^2}](\beta s_2 - (3-\beta) \le \frac{8}{5} - 2 < 0),\\
&-\int_2^{\infty} P_1(t)h'(t)dt > -\frac{\beta s \ln^{\beta s_2-(2-\beta)}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s+1}(2-\eta)^2} - \frac{\ln^{\beta s_2-1}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)^2}.
\end{aligned}
$$

Hence, for $s_2 \in (\frac{1}{\beta}, \frac{8}{5\beta}) \cap (0, s)$, $\beta \in (0, 1]$, $\eta \in [0, \frac{1}{4}]$, we deduce that

$$
\begin{aligned}
h_0(s_2,\eta) &> \frac{\ln^{\beta s_2}(2-\eta)}{\beta s_2[x+\ln^{\beta}(2-\eta)]^{s}} + \frac{s}{\beta s_2(s_2+1)}\frac{\ln^{\beta s_2}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}}\\
&\quad - \frac{\ln^{\beta s_2-1}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)} - \frac{\beta s \ln^{\beta(s_2+1)-1}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s+1}(2-\eta)^2} - \frac{\beta \ln^{\beta s_2-1}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)^2}\\
&= \frac{\ln^{\beta s_2-1}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s}}[\frac{\ln(2-\eta)}{\beta s_2} - \frac{1}{2(2-\eta)} - \frac{1}{8(2-\eta)^2}]\\
&\quad + \frac{s\beta \ln^{\beta(s_2+1)-1}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}}[\frac{\ln(2-\eta)}{\beta s_2(\beta s_2+\beta)} - \frac{1}{8(2-\eta)^2}]\\
&\ge \frac{\ln^{\beta s_2-1}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s}}[\frac{5\ln(2-\frac{1}{4})}{8} - \frac{1}{2(2-\frac{1}{4})} - \frac{1}{8(2-\frac{1}{4})^2}]\\
&\quad + \frac{s\beta \ln^{\beta(s_2+1)-1}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}}[\frac{25\ln(2-\frac{1}{4})}{104} - \frac{1}{8(2-\frac{1}{4})^2}]\\
&> 0.023 \times \frac{\ln^{\beta s_2-1}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s}} + 0.093 \times \frac{s\beta \ln^{\beta(s_2+1)-2}(2-\eta)}{[x+\ln^{\beta}(2-\eta)]^{s+1}} > 0.
\end{aligned}
$$

We still have

$$
\begin{aligned}
\sum_{n=2}^{\infty} h(n) &= \int_2^{\infty} h(t)dt + \frac{1}{2}h(2) + \int_2^{\infty} P_1(t)h'(t)dt\\
&= \int_2^{\infty} h(t)dt + h_1(s_2,\eta),
\end{aligned}
$$

where $h_1(s_2,\eta) := \frac{1}{2}h(2) + \int_2^{\infty} P_1(t)h'(t)dt$.

For $s_2 \in (\frac{1}{\beta}, \frac{8}{5\beta}] \cap (0, s)$, by means of (8), we find

$$
\int_2^{\infty} P_1(t)\frac{(\beta s_2 - 1)\ln^{\beta s_2 - 2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}(t-\eta)^2}dt > \frac{-(\beta s_2 - 1)\ln^{\beta s_2 - 2}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)^2},
$$

$$
-\int_2^{\infty} P_1(t)\left\{\frac{\beta s \ln^{\beta s_2-(3-\beta)}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s+1}}\frac{\ln(t-\eta)}{(t-\eta)^2} + \frac{\ln^{\beta s_2-2}(t-\eta)}{[x+\ln^{\beta}(t-\eta)]^{s}}\frac{\ln(t-\eta)}{(t-\eta)^2}\right\}dt > 0,
$$

$$
\int_2^{\infty} P_1(t)h'(t)dt > -\frac{(\beta s_2 - 1)\ln^{\beta s_2 - 2}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)^2}.
$$

$$
\begin{aligned}
h_1(s_2,\eta) &> \frac{\ln^{\beta s_2-1}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)} - \frac{(\beta s_2 - 1)\ln^{\beta s_2 - 2}(2-\eta)}{8[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)^2}\\
&= \frac{\ln^{\beta s_2-2}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)}[\ln(2-\eta) - \frac{\beta s_2 - 1}{4(2-\eta)}]\\
&\ge \frac{\ln^{\beta s_2-2}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)}[\ln(2-\frac{1}{4}) - \frac{\frac{8}{5}-1}{4(2-\frac{1}{4})}]\\
&> 0.47 \times \frac{\ln^{\beta s_2-2}(2-\eta)}{2[x+\ln^{\beta}(2-\eta)]^{s}(2-\eta)} > 0.
\end{aligned}
$$

Hence, we derive inequality (12).

In conclusion, for $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s)$, by using (11) and setting $u = \frac{\ln^\beta(t-\eta)}{x}$, it follows that

$$\begin{aligned}\varpi_{(s,\eta)}(s_2, x) &= \beta x^{s-s_2} \sum_{n=2}^{\infty} h(n) < \beta x^{s-s_2} \int_{1+\eta}^{\infty} h(t)dt \\ &= \int_0^\infty \frac{u^{s_2-1}du}{(1+u)^s} = B(s_2, s-s_2) = k_s(s_2),\end{aligned}$$

$$\begin{aligned}\varpi_{(s,\eta)}(s_2, x) &> \beta x^{s-s_2} \int_2^\infty h(t)dt \\ &= \int_{\frac{\ln^\beta(2-\eta)}{x}}^\infty \frac{u^{s_2-1}}{(1+u)^s} du = k_s(s_2)(1 - O_s(\frac{1}{x^{s_2}})) > 0,\end{aligned}$$

where we indicate that $O_s(\frac{1}{x^{s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{\ln^\beta(2-\eta)}{x}} \frac{u^{s_2-1}}{(1+u)^s} du$ which satisfies

$$0 < \int_0^{\frac{\ln^\beta(2-\eta)}{x}} \frac{u^{s_2-1}}{(1+u)^s} du \le \int_0^{\frac{\ln^\beta(2-\eta)}{x}} u^{s_2-1} du = \frac{1}{s_2} [\frac{\ln^\beta(2-\eta)}{x}]^{s_2}.$$

Therefore, we obtain inequality (11). The proof of Lemma 1 is complete. □

Remark 1. *For $\eta = 0, 1 < \beta s_2 \le \frac{12}{5}$, we can still obtain*

$$h_0(s_2, 0) > 0.007 \times \frac{\ln^{\beta s_2 - 1} 2}{(x + \ln^\beta 2)^s} + 0.05 \times \frac{s\beta \ln^{\beta(s_2+1)-2} 2}{(x + \ln^\beta 2)^{s+1}} > 0,$$

$$h_1(s_2, 0) > 0.5 \times \frac{\ln^{\beta s_2 - 2} 2}{4(x + \ln^\beta 2)^s} > 0.$$

Thus, inequality (11) is still valid in the case that $\eta = 0, 0 < \beta s_2 \le \frac{12}{5}$.

Lemma 3. *Under assumption H1, we have the following half-discrete Hardy–Mulholland-type inequality:*

$$\begin{aligned}I_{\lambda+m+1} &:= \sum_{n=2}^{\infty} \int_0^\infty \frac{\ln^{\beta-1}(n-\eta)}{[x + \ln^\beta(n-\eta)]^{\lambda+m+1}\{n-\eta)} F_m(x) B_n dx \\ &< (\frac{1}{\beta} k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}} \\ &\times [\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} F_m^p(x)dx]^{\frac{1}{p}} [\sum_{n=2}^{\infty} \frac{\ln^{-q\beta\hat{\lambda}_2-1}(n-\eta)}{n-\eta} B_n^q]^{\frac{1}{q}}.\end{aligned} \tag{13}$$

Proof. Setting $y = \frac{x}{\ln^\beta(n-\eta)}$, for $s_1 \in (0, s)$, we can also obtain the following weight function:

$$\begin{aligned}\omega_{(s,\eta)}(s_1, n) &:= \ln^{\beta(s-s_1)}(n-\eta) \int_0^\infty \frac{x^{s_1-1}}{[x + \ln^\beta(n-\eta)]^s} dx = \int_0^\infty \frac{y^{s_1-1}}{(y+1)^s} dy \\ &= k_s(s_1) := B(s_1, s-s_1) \ (n \in N_2).\end{aligned} \tag{14}$$

By using Hölder's inequality [18], we have

$$\begin{aligned}
I_{\lambda+m+1} &= \sum_{n=2}^{\infty}\int_0^{\infty}\frac{1}{[x+\ln^{\beta}(n-\eta)]^{\lambda+m+1}}[\frac{\ln^{[\beta(\lambda_2+1)-1]/p}(n-\eta)}{(n-\eta)^{1/p}x^{(\lambda_1+m-1)/q}}F_m(x)]\\
&\quad\times[\frac{x^{(\lambda_1+m-1)/q}}{\ln^{\beta(\lambda_2+1)-1)/p}(n-\eta)}\frac{\ln^{\beta-1}(n-\eta)}{(n-\eta)^{1/q}}B_n]dx\\
&\le\left[\int_0^{\infty}\sum_{n=2}^{\infty}\frac{1}{[x+\ln^{\beta}(n-\eta)]^{\lambda+m+1}}\frac{\ln^{\beta(\lambda_2+1)-1}(n-\eta)}{(n-\eta)x^{(\lambda_1+m-1)(p-1)}}F_m^p(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=2}^{\infty}\int_0^{\infty}\frac{1}{[x+\ln^{\beta}(n-\eta)]^{\lambda+m+1}}\frac{x^{\lambda_1+m-1}}{\ln^{[\beta(\lambda_2+1)-1](q-1)}(n-\eta)}\frac{\ln^{q(\beta-1)}(n-\eta)}{(n-\eta)}B_n^q dx\right]^{\frac{1}{q}}\\
&=\left[\frac{1}{\beta}\int_0^{\infty}\varpi_{(\lambda+m+1,\eta)}(\lambda_2+1,x)x^{p(1-m-\hat{\lambda}_1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=2}^{\infty}\omega_{(\lambda+m+1,\eta)}(\lambda_1+m,n)\frac{\ln^{-q\beta\hat{\lambda}_2-1}(n-\eta)}{n-\eta}B_n^q\right].
\end{aligned}$$

By utilizing inequalities (11) and (14), based on the condition of parameters $s=\lambda+m+1>m+1, s_1=\lambda_1+m\in(m,\lambda+m)$, $s_2=\lambda_2+1\in(1,\frac{8}{5\beta}]\cap(1,\lambda+1)$, we have $\lambda>0,\lambda_1\in(0,\lambda)$, $\lambda_2\in(0,\frac{8}{5\beta}-1]\cap(0,\lambda)$, and then using (6) (for $i=m+1$) and (7), we derive inequality (13). This completes the proof of Lemma 3. □

Lemma 4. *Under assumption H1, for $t>0, m\in\mathrm{N}_0$, we have the following expression:*

$$\int_0^{\infty}e^{-tx}f(x)dx=t^m\int_0^{\infty}e^{-tx}F_m(x)dx. \tag{15}$$

Proof. For $m=0$ since $F_0(x)=f(x)$, equality (15) is naturally valid; for $m\in\mathrm{N}$, $\mathrm{i}\in\{1,2,\ldots,\mathrm{m}\}$, by applying Hölder's inequality and assumption H1, we have

$$\begin{aligned}
F_i(x)&=\int_0^x F_{i-1}(t)dt=\int_0^x(t^{\frac{1}{q}-i+1-\hat{\lambda}_1}F_{i-1}(t))(t^{\frac{-1}{q}+i-1+\hat{\lambda}_1})dt\\
&\le[\int_0^x t^{p(2-i-\hat{\lambda}_1)-1}F_{i-1}^p(t)dt]^{\frac{1}{p}}[\int_0^x t^{q(i-1+\hat{\lambda}_1)-1}dt]^{\frac{1}{q}}\\
&\le(c_{i-1})^{\frac{1}{p}}[\frac{1}{q(i-1+\hat{\lambda}_1)}]^{\frac{1}{q}}x^{i-1+\hat{\lambda}_1}(\hat{\lambda}_1>0).
\end{aligned}$$

It follows that

$$0\le\lim_{x\to\infty}e^{-tx}F_i(x)\le\lim_{x\to\infty}e^{-tx}(c_{i-1})^{\frac{1}{p}}[\frac{1}{q(i-1+\hat{\lambda}_1)}]^{\frac{1}{q}}x^{i-1+\hat{\lambda}_1}=0,$$

and then, from $F_i(0^+)=0$, we derive $e^{-tx}F_i(x)|_0^{\infty}=0$. Using integration by parts, we obtain

$$\begin{aligned}
&\int_0^{\infty}e^{-tx}F_{i-1}(x)dx=\int_0^{\infty}e^{-tx}dF_i(x)\\
&=e^{-tx}F_i(x)|_0^{\infty}-\int_0^{\infty}F_i(x)de^{-tx}=t\int_0^{\infty}e^{-tx}F_i(x)dx.
\end{aligned}$$

Substituting $i=1,2,\ldots,m(m\in\mathrm{N})$ in the above equality, by simplification, we get equality (15). Lemma 4 is proved. □

Remark 2. *For $i = 1, 2, \ldots, m (m \in \mathrm{N})$, by exchanging the assumption $0 < \int_0^\infty t^{p(2-i-\hat{\lambda}_1)-1} F_{i-1}^p(t)dt < \infty$ for the assumption $F_i(x) = o(e^{tx})(t > 0; x \to \infty)$, we still have equality (15).*

Lemma 5. *Under assumption H1, for $t > 0$, we have the following inequality:*

$$\sum_{n=2}^{\infty} e^{-t\ln^\beta(n-\eta)} b_n \le t\beta \sum_{n=2}^{\infty} e^{-t\ln^\beta(n-\eta)} \frac{\ln^{\beta-1}(n-\eta)}{n-\eta} B_n. \tag{16}$$

Proof. In view of $B_n e^{-t\ln^\beta(n-\eta)} = o(1)\ (n \to \infty)$, by using the Abel summation by parts formula, we deduce that

$$\begin{aligned}\sum_{n=2}^{\infty} e^{-t\ln^\beta(n-\eta)} b_n &= \lim_{n\to\infty} B_n e^{-t\ln^\beta(n-\eta)} + \sum_{n=2}^{\infty} B_n [e^{-t\ln^\beta(n-\eta)} - e^{-t\ln^\beta(n-\eta+1)}] \\ &= \sum_{n=2}^{\infty} B_n [e^{-t\ln^\beta(n-\eta)} - e^{-t\ln^\beta(n-\eta+1)}].\end{aligned}$$

We define a function $f(x) = e^{-t\ln^\beta(x-\eta)}, x \in (1+\eta, \infty)$. Then, we obtain

$$f'(x) = -t\beta \frac{\ln^{\beta-1}(x-\eta)}{x-\eta} e^{-t\ln^\beta(x-\eta)} = -t\beta h(x),$$

where $\beta \in (0,1], h(x) := \frac{\ln^{\beta-1}(x-\eta)}{x-\eta} e^{-t\ln^\beta(x-\eta)}$ is decreasing in $(1+\eta, \infty)$. By applying the differentiation mid-value theorem, we have the following equality

$$f(n+1) - f(n) = f'(n+\theta_n)(\theta_n \in (0,1), n \in N_2)$$

and then, we have

$$\begin{aligned}\sum_{n=2}^{\infty} e^{-t\ln^\beta(n-\eta)} b_n &= -\sum_{n=2}^{\infty} B_n (f(n+1) - f(n)) \\ &= -\sum_{n=2}^{\infty} B_n f'(n+\theta_n) = t\beta \sum_{n=2}^{\infty} h(n+\theta_n) B_n \\ &\le t\beta \sum_{n=2}^{\infty} h(n) B_n = t\beta \sum_{n=2}^{\infty} e^{-t\ln^\beta(n-\eta)} \frac{\ln^{\beta-1}(n-\eta)}{n-\eta} B_n,\end{aligned}$$

which leads to the required inequality (16). This completes the proof of Lemma 5. □

Lemma 6. *Under assumption H1, the following inequality holds:*

$$I_0 := \int_0^\infty \sum_{n=2}^{\infty} \frac{b_n f(x)}{[x+\ln^\beta(n-\eta)]^\lambda} dx \le \beta \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^{\infty} \frac{\ln^{\beta-1}(n-\eta) B_n F_m(x)}{[x+\ln^\beta(n-\eta)]^{\lambda+m+1}(n-\eta)} dx. \tag{17}$$

Proof. With the aid of the expression of the Gamma function [19]

$$\frac{1}{[x+\ln^\beta(n-\eta)]^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-[x+\ln^\beta(n-\eta)]t} dt,$$

and using the Lebesgue term-by-term theorem [20], from (15) and (16), we deduce that

$$\begin{aligned}
I_0 &= \frac{1}{\Gamma(\lambda)}\int_0^\infty b_n f(x)\int_0^\infty t^{\lambda-1}e^{-[x+\ln^\beta(n-\eta)]t}dtdx \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}(\int_0^\infty e^{-xt}f(x)dx)[\sum_{n=2}^\infty e^{-t\ln^\beta(n-\eta)}b_n]dt \\
&\le \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}(t^m\int_0^\infty e^{-xt}F_m(x)dx)[t\beta\sum_{n=2}^\infty e^{-t\ln^\beta(n-\eta)}\frac{\ln^{\beta-1}(n-\eta)}{n-\eta}B_n]dt \\
&= \beta\frac{1}{\Gamma(\lambda)}\int_0^\infty \sum_{n=1}^\infty (n-\eta)^{\beta-1}B_nF_m(x)\int_0^\infty t^{(\lambda+m+1)-1}e^{-[x+(n-\eta)^\beta]t}dtdx \\
&= \beta\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\int_0^\infty \sum_{n=2}^\infty \frac{\ln^{\beta-1}(n-\eta)B_n}{[x+\ln^\beta(n-\eta)]^{\lambda+m+1}(n-\eta)}F_m(x)dx.
\end{aligned}$$

This proves inequality (17). The proof of Lemma 6 is complete. □

Lemma 7. *Let $b \in (-1,1), n \in N_2$, then, there exists a constant C such that*

$$\sum_{k=2}^n \frac{\ln^b k}{k} = \frac{\ln^{b+1} n}{b+1} + C + O(\frac{1}{n}\ln^b n)(n\to\infty). \tag{18}$$

Proof. Let $G(t) := \frac{1}{t}\ln^b t\ (t \ge 2)$. Then, one has

$$G'(t) = \frac{b}{t^2}\ln^{b-1} t - \frac{1}{t^2}\ln^b t = bg_1(t) - g_2(t),$$

where $g_1(t) := \frac{1}{t^2}\ln^{b-1} t, g_2(t) := \frac{1}{t^2}\ln^b t\ (t \ge 2)$.

It is easy to see that $(-1)^i g_1^{(i)}(t) > 0 (t \ge 2; i = 0,1)$. Since

$$b < 1 < 2\ln 2 \le 2\ln t (t \ge 2), g_2'(t) = \frac{b - 2\ln t}{t^3}\ln^{b-1} t < 0 (t \ge 2),$$

we deduce that $(-1)^i g_2^{(i)}(t) > 0\ (i = 0,1)$.

In view of (8), we obtain

$$\int_2^n P_1(t)g_j(t)dt = \frac{\varepsilon_j}{8}g_j(t)|_2^n (0 < \varepsilon_j < 1; j = 0,1).$$

By utilizing (9), it follows that

$$\begin{aligned}
\sum_{k=2}^n G(k) &= \int_2^n G(t)dt + \frac{1}{2}(G(2)+G(n)) + \int_2^n P_1(t)G'(t)dt \\
&= \int_2^n G(t)dt + \frac{1}{2}(G(2)+G(n)) \\
&\quad + b\int_2^n P_1(t)g_1(t)dt - \int_2^n P_1(t)g_2(t)dt.
\end{aligned}$$

Substituting the above results, by simplification, we obtain (18), where

$$C := -\frac{1}{b+1}\ln^{b+1} 2 + (\frac{1}{4} + \frac{\varepsilon_2}{32})\ln^b 2 - \frac{\varepsilon_1 b}{32}\ln^{b-1} 2, \text{ and}$$

$$O(\frac{1}{n}\ln^b n) := \frac{\ln^b n}{2n} + \frac{\varepsilon_1 b}{8n^2}\ln^{b-1} n - \frac{\varepsilon_2}{8n^2}\ln^b n\ (n\to\infty).$$

This completes the proof of Lemma 7. □

3. Main Results

Theorem 1. *Under assumption H1, we have the following inequality:*

$$I_0 = \int_0^\infty \sum_{n=2}^{\infty} \frac{b_n f(x)}{[x+\ln^\beta(n-\eta)]^\lambda} dx < \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}} \times \Big[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} F_m^p(x)dx\Big]^{\frac{1}{p}} \Big[\sum_{n=2}^{\infty} \frac{\ln^{-q\beta\hat{\lambda}_2-1}(n-\eta)}{n-\eta} B_n^q\Big]^{\frac{1}{q}}. \quad (19)$$

In particular, if $\lambda_1+\lambda_2=\lambda$, this implies that

$$0 < \int_0^\infty x^{p(1-m-\lambda_1)-1} F_m^p(x)dx < \infty, 0 < \sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2-1}(n-\eta)}{n-\eta} B_n^q < \infty,$$

and we have the following inequality:

$$\int_0^\infty \sum_{n=2}^{\infty} \frac{b_n f(x)}{[x+\ln^\beta(n-\eta)]^\lambda} dx < \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1+m,\lambda_2+1) \times \Big[\int_0^\infty x^{p(1-m-\lambda_1)-1} F_m^p(x)dx\Big]^{\frac{1}{p}} \Big[\sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2-1}(n-\eta)}{n-\eta} B_n^q\Big]^{\frac{1}{q}}, \quad (20)$$

where the constant factor $\beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1+m,\lambda_2+1)$ is the best possible.

Proof. By applying inequalities (17) and (13), we acquire inequality (19).

For any $0<\varepsilon<\min\{p\lambda_1, q\lambda_2\}$, we set

$$\widetilde{F}_0(x) = \widetilde{f}(x) := \begin{cases} 0, & 0<x<1, \\ x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x\geq 1 \end{cases},$$
$$\widetilde{b}_n := \tfrac{1}{n} \ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} n, n \in \mathrm{N}_2,$$

and for $m \in \mathrm{N}$, we set

$$\widetilde{F}_m(x) := \int_0^x \Big(\int_0^{t_{m-1}} \dots \int_0^{t_1} \widetilde{f}(t_0)dt_0 \dots dt_{m-2}\Big)dt_{m-1}$$
$$= \begin{cases} 0, & 0<x<1 \\ \dfrac{1}{\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})} (x^{\lambda_1+m-1-\frac{\varepsilon}{p}} - p_{m-1}(x)), & x\geq 1 \end{cases}$$
$$\leq \begin{cases} 0, & 0<x<1 \\ \dfrac{1}{\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})} x^{\lambda_1+m-1-\frac{\varepsilon}{p}}, & x\geq 1 \end{cases},$$

where $p_{m-1}(x)$ is a positive polynomial of degree $m-1$ with $p_{m-1}(1)=1$

We denote $p_{-1}(x):=0, \prod_{i=0}^{1}(a+i):=1$ $(a>0)$. Then, the above expression satisfies $m \in \mathrm{N}_0$, with the following assumption:

$$0 < \int_0^\infty t^{p(2-i-\hat{\lambda}_1)-1} \widetilde{F}_{i-1}^p(t)dt < \infty (i=1,2,\dots,m+1)$$

For $-1 < b = \beta(\lambda_2 - \frac{\varepsilon}{q}) - 1 < 1$, with the aid of (18), we obtain

$$\begin{aligned} 0 < \widetilde{B}_n &:= \sum_{k=2}^{n} \widetilde{b}_k = \sum_{k=2}^{n} \frac{1}{k} \ln^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1} k \\ &= \frac{1}{\beta^{(\lambda_2 - \frac{\varepsilon}{q})}} \ln^{\beta(\lambda_2 - \frac{\varepsilon}{q})} n + C + o_1\left(\frac{1}{n} \ln^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1} n\right) \\ &\le \frac{\ln^{\beta(\lambda_2 - \frac{\varepsilon}{q})} n}{\beta^{(\lambda_2 - \frac{\varepsilon}{q})}} [1 + |C_1| \ln^{\beta(\lambda_2 - \frac{\varepsilon}{q})} n + |o_1(\frac{1}{n \ln n})|], \end{aligned}$$

which satisfies $\widetilde{B}_n = o(e^{t \ln^{\beta} n})$ $(t > 0; n \to \infty)$.

Note that for derivable function $a(x) = o(1)$ $(x \to \infty)$, we have

$$\lim_{x \to \infty} \frac{(1 + a(x))^q - 1}{a(x)} = \lim_{x \to \infty} \frac{q(1 + a(x))^{q-1} a'(x)}{a'(x)} = \lim_{x \to \infty} q(1 + a(x))^{q-1} = q,$$

and then we deduce $(1 + a(x))^q = 1 + o(a(x))$ $(x \to \infty)$ Hence, we obtain

$$\begin{aligned} &(1 + |C_1| \ln^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} n + |o_1(\tfrac{1}{n \ln n})|)^q \\ &= 1 + o(|C_1| \ln^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} n + |o_1(\tfrac{1}{n \ln n})|) \ (n \in N_2; n \to \infty). \end{aligned}$$

Further, it follows that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2 - 1} n}{n} \widetilde{B}_n^q &\le \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q \sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\varepsilon - 1} n (1 + |C_1| \ln^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} n + |o_1(\tfrac{1}{n \ln n})|)^q \\ &= \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q \sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\varepsilon - 1} n [1 + o(|C_1| \ln^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} n + |o_1(\tfrac{1}{n \ln n})|)] \\ &= \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q \left[\sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\varepsilon - 1} n + \sum_{n=2}^{\infty} o(|C_1| \frac{1}{n} \ln^{-\beta(\lambda_2 + \frac{\varepsilon}{p}) - 1} n + |o_1(\frac{1}{n} \ln^{-\beta\varepsilon - 2} n)|)\right] \\ &= \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q \left(\sum_{n=3}^{\infty} \frac{1}{n} \ln^{-\beta\varepsilon - 1} n + O_1(1)\right) \\ &< \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q \left(\int_2^{\infty} \ln^{-\beta\varepsilon - 1} x d \ln x + O_1(1)\right) = \left[\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}\right]^q (\frac{1}{\beta\varepsilon} \ln^{-\beta\varepsilon} 2 + O_1(1)). \end{aligned}$$

If there exists a positive constant $M \le \beta^{\frac{1}{q}} \frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1)$ such that inequality (20) is valid when we replace $\beta^{\frac{1}{q}} \frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1)$ by M, then, in particular, for $\eta = 0$ we still have

$$\widetilde{I} := \int_0^{\infty} \sum_{n=2}^{\infty} \frac{\widetilde{b}_n \widetilde{f}(x)}{(x + \ln^{\beta} n)^{\lambda}} dx < M [\int_0^{\infty} x^{p(1 - m - \lambda_1) - 1} \widetilde{F}_m^p(x) dx]^{\frac{1}{p}} (\sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2 - 1} n}{n} \widetilde{B}_n^q)^{\frac{1}{q}}. \tag{21}$$

In the following, we prove that $\beta^{\frac{1}{q}} \frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1) \le M$, which means that

$$M = \beta^{\frac{1}{q}} \frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1)$$

is the least value that makes (20) valid. In view of (21) and the above results, we acquire

$$\begin{aligned}\widetilde{I} &< \frac{M}{\beta(\lambda_2-\frac{\varepsilon}{q})\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}[\int_1^\infty x^{p(1-m-\lambda_1)-1}x^{p(\lambda_1+m-1)-\varepsilon}dx]^{\frac{1}{p}}(\frac{1}{\beta\varepsilon}\ln^{-\beta\varepsilon}2+O_1(1))^{\frac{1}{q}}\\ &= \frac{M}{\beta(\lambda_2-\frac{\varepsilon}{q})\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}(\int_1^\infty x^{-\varepsilon-1}dx)^{\frac{1}{p}}(\frac{1}{\beta\varepsilon}\ln^{-\beta\varepsilon}2+O_1(1))^{\frac{1}{q}}\\ &= \frac{M}{\varepsilon\beta(\lambda_2-\frac{\varepsilon}{q})\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}(\frac{1}{\beta}\ln^{-\beta\varepsilon}2+\varepsilon O_1(1))^{\frac{1}{q}}.\end{aligned}$$

By using (10) and (11), and setting $\eta = 0, s = \lambda \in (0,\infty)$, $s_2 = \widetilde{\lambda}_2 := \lambda_2 - \frac{\varepsilon}{q} \in (0, \frac{8}{5\beta}) \cap (0,\lambda)$, we get

$$\begin{aligned}\widetilde{I} &= \int_1^\infty [x^{\lambda_1+\frac{\varepsilon}{q}}\sum_{n=2}^{\infty}\frac{\ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}n}{(x+\ln^\beta n)^\lambda n}]x^{-\varepsilon-1}dx\\ &= \frac{1}{\beta}\int_1^\infty \varpi_{(\lambda,0)}(\widetilde{\lambda}_2,x)x^{-\varepsilon-1}dx > \frac{1}{\beta}k_\lambda(\widetilde{\lambda}_2)\int_1^\infty(1-O_\lambda(\frac{1}{x^{\widetilde{\lambda}_2}}))x^{-\varepsilon-1}dx\\ &= \frac{1}{\beta}B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})(\frac{1}{\varepsilon}-O(1)).\end{aligned}$$

Based on the above results, we obtain

$$\begin{aligned}&\frac{1}{\beta}B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})-\varepsilon O(1)\\ &< \varepsilon\widetilde{I} < \frac{M}{\beta(\lambda_2-\frac{\varepsilon}{q})\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}(\frac{1}{\beta}\ln^{-\beta\varepsilon}2+\varepsilon O_1(1))^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon \to 0^+$, with the help of the continuous property of the Beta function, we have

$$\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m,\lambda_2+1) = \beta^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i) \le M.$$

Consequently, $M = \beta^{\frac{1}{q}}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m,\lambda_2+1)$ is the best possible constant factor in inequality (20). The proof of Theorem 1 is complete. □

Theorem 2. *Under assumption H1, if the constant factor*

$$\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

in (19) is the best possible, then for $\lambda-\lambda_1-\lambda_2 \le q(\frac{8}{5\beta}-1-\lambda_2)$, *we have* $\lambda_1+\lambda_2=\lambda$.

Proof. Since $\hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, $0 < \lambda_1,\lambda_2 < \lambda$, it follows that $\hat{\lambda}_1+\hat{\lambda}_2=\lambda, 0<\hat{\lambda}_1,\hat{\lambda}_2<\lambda$, and $B(\hat{\lambda}_1+m,\hat{\lambda}_2+1) \in \mathrm{R}_+$.

For $\lambda-\lambda_1-\lambda_2 \le q(\frac{8}{5\beta}-1-\lambda_2)$, one has $\hat{\lambda}_2 \le \frac{8}{5\beta}-1$. Since $\hat{\lambda}_2$ satisfies the condition of λ_2 in (20), then, by virtue of (20), for $\lambda_i = \hat{\lambda}_i (i = 1, 2)$, we derive the following inequality:

$$\begin{aligned} \int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{[x+\ln^\beta(n-\eta)]^\lambda} dx &< \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) \\ &\times \Big[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} F_m^p(x) dx\Big]^{\frac{1}{p}} \Big[\sum_{n=2}^\infty \frac{\ln^{-q\beta\hat{\lambda}_2-1}(n-\eta)}{n-\eta} B_n^q\Big]^{\frac{1}{q}} \end{aligned} \tag{22}$$

By applying Hölder's inequality, we have

$$\begin{aligned} &B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) = k_{\lambda+m+1}\Big(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}+1\Big) \\ &= \int_0^\infty \frac{1}{(1+u)^{\lambda+m+1}} u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}} du = \int_0^\infty \frac{1}{(1+u)^{\lambda+m+1}} (u^{\frac{\lambda_2}{p}})(u^{\frac{\lambda-\lambda_1}{q}}) du \\ &\le \Big[\int_0^\infty \frac{1}{(1+u)^{\lambda+m+1}} u^{\lambda_2} du\Big]^{\frac{1}{p}} \Big[\int_0^\infty \frac{1}{(1+u)^{\lambda+m+1}} u^{\lambda-\lambda_1} du\Big]^{\frac{1}{q}} \\ &= \Big[\int_0^\infty \frac{1}{(1+u)^{\lambda+m+1}} u^{\lambda_2} du\Big]^{\frac{1}{p}} \Big[\int_0^\infty \frac{1}{(1+v)^{\lambda+m+1}} v^{(\lambda_1+m)-1} dv\Big]^{\frac{1}{q}} \\ &= (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}. \end{aligned} \tag{23}$$

According to the hypothesis that the constant factor

$$\beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

in (19) is the best possible, we observe that it is the least value that makes (19) valid. By comparing the constant factors in (19) and (22), we deduce that

$$\begin{aligned} &\beta^{\frac{1}{q}} \tfrac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}} \\ &\le \beta^{\frac{1}{q}} \tfrac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) (\in \mathrm{R}_+), \end{aligned}$$

which is

$$B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) \ge (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}.$$

Therefore, we conclude that inequality (23) keeps the form of an equality. Furthermore, we observe that (23) keeps the form of an equality if, and only if, constants A and B exist such that they are not both zero and $Au^{\lambda_2} = Bu^{\lambda-\lambda_1}$ *a.e.* in R_+. Assuming that $A \neq 0$, it follows that $u^{\lambda_2+\lambda_1-\lambda} = \frac{B}{A}$ *a.e.* in R_+, which implies that $\lambda_2+\lambda_1-\lambda = 0$. Hence, we acquire $\lambda_1+\lambda_2 = \lambda$. This completes the proof of Theorem 2. □

4. Application to Generating New Half-Discrete Hardy–Mulholland-Type Inequalities

As a direct application of the main result, we give here several new half-discrete Hardy–Mulholland-type inequalities by taking the special values of the parameters.

Remark 3. *(i) Taking $\eta = 0$ in (20), in view of Remark 1, we obtain*

$$\begin{aligned}\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\left(x+\ln^\beta n\right)^\lambda}dx <& \beta^{\frac{1}{q}}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m,\lambda_2+1)\\ &\times\left[\int_0^\infty x^{p(1-m-\lambda_1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1}n}{n}B_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{24}$$

where $\lambda_1 \in (0,\lambda), \lambda_2 \in (0,\frac{12}{5\beta}-1]\cap(0,\lambda)$, and the constant factor $\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m, \lambda_2+1)$ is the best possible.

In particular, for $\beta=\lambda=1, \lambda_1=\frac{1}{p}, \lambda_2=\frac{1}{q}(\le\frac{7}{5})$, we have

$$\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\ln e^x n}dx < \frac{\pi}{q\sin(\pi/p)}\prod_{i=0}^{m-1}(\frac{1}{p}+i)\left[\int_0^\infty x^{p(1-m)-2}F_m^p(x)dx\right]^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{1}{n\ln^2 n}B_n^q\right)^{\frac{1}{q}}. \tag{25}$$

For $\beta=\lambda=1, \lambda_1=\frac{1}{q}, \lambda_2=\frac{1}{p}(\le\frac{7}{5})$, we have

$$\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\ln e^x n}dx < \frac{\pi}{p\sin(\pi/p)}\prod_{i=0}^{m-1}(\frac{1}{q}+i)\left[\int_0^\infty x^{-pm}F_m^p(x)dx\right]^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{1}{n\ln^q n}B_n^q\right)^{\frac{1}{q}}. \tag{26}$$

For $p=q=2$, both inequalities (25) and (26) reduce to the following inequality:

$$\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\ln e^x n}dx < \frac{\pi(2m-1)!!}{2^{m+1}}\left(\int_0^\infty x^{-2m}F_m^2(x)dx\sum_{n=2}^\infty \frac{1}{n\ln^2 n}B_n^2\right)^{\frac{1}{2}}, \tag{27}$$

where the factorial $(2m-1)!!$ is defined as $(2m-1)!!=1$ for $m=0$; and $(2m-1)!!=1\cdot 3\cdot\ldots(2m-1)$ for $m\in\mathrm{N}$.

(ii) Choosing $\eta=\frac{1}{4}, \beta=1$ in (20), we acquire the following inequality:

$$\begin{aligned}\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\ln^\lambda e^x(n-\frac{1}{4})}dx <& \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m,\lambda_2+1)\\ &\times\left[\int_0^\infty x^{p(1-m-\lambda_1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{-q\lambda_2-1}(n-\frac{1}{4})}{n-\frac{1}{4}}B_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{28}$$

where $\lambda_1\in(0,\lambda), \lambda_2\in(0,\frac{3}{5}]\cap(0,\lambda)$, and the constant factor $\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}B(\lambda_1+m,\lambda_2+1)$ is the best possible.

(iii) Taking $m=0$ in (20), we have the following inequality involving one partial sum with the best possible constant factor $\beta^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)$:

$$\begin{aligned}\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{\left[x+\ln^\beta(n-\eta)\right]^\lambda}dx <& \beta^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)\\ &\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1}(n-\eta)}{n-\eta}B_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{29}$$

Remark 4. *For $u(x) > 0, u'(x) > 0$ $(x \in (b, \infty))$ with $u(b^+) = 0, u(\infty) = \infty$, by replacing x with t in (20) and setting $t = u(x)$, $t = u^{-1}(x)$, respectively, we obtain the following inequalities with the best possible constant factor $\beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1)$:*

$$\begin{aligned} &\int_b^\infty \sum_{n=2}^\infty \frac{b_n f(u(x))u'(x)}{[u(x) + \ln^\beta(n-\eta)]^\lambda} dx < \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1) \\ &\times \left[\int_b^\infty (u(x))^{p(1-m-\lambda_1)-1} u'(x) F_m^p(u(x)) dx\right]^{\frac{1}{p}} \left[\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1}(n-\eta)}{n-\eta} B_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{30}$$

$$\begin{aligned} &\int_b^\infty \sum_{n=2}^\infty \frac{b_n f(u^{-1}(x))u^{\lambda-2}(x)u'(x)}{[1 + u(x)\ln^\beta(n-\eta)]^\lambda} dx < \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2 + 1) \\ &\times \left[\int_b^\infty (u(x))^{p(m-1+\lambda_1)-1} u'(x) F_m^p(u^{-1}(x)) dx\right]^{\frac{1}{p}} \left[\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1}(n-\eta)}{n-\eta} B_n^q\right]^{\frac{1}{q}} \end{aligned} \tag{31}$$

5. Conclusions

In this work, by embedding multiple upper limit functions and partial sums, we have provided a new approach to producing an extension of half-discrete Hardy–Mulholland-type inequalities. By the clever use of the Euler–Maclaurin summation formula, Abel's summation by parts formula, and the differentiation mid-value theorem, we establish a new half-discrete Hardy–Mulholland-type inequality involving one multiple upper limit function and one partial sum in Theorem 1. Then, in Theorem 2, we characterize the condition for the best possible constant factor related to several parameters. Finally, in Remarks 3 and 4, we illustrate that some new half-discrete Hardy–Mulholland-type inequalities can be derived from the special values of the parameters. The main highlight of this work is that we introduce both multiple upper limit functions and partial sums in a half-discrete Hardy-Mulholland-type inequality; such a result has not been reported in previous studies. From the perspective of future research, we shall establish more half-discrete Hardy–Mulholland-type inequalities by constructing new kernel functions and improving the conditions related to the multiple upper limit function and partial sum. This will be interesting and challenging work.

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