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Improved Probability-Weighted Moments and Two-Stage Order Statistics Methods of Generalized Extreme Value Distribution

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Abstract

This study evaluates six parameter estimation methods for the generalized extreme value (GEV) distribution: maximum likelihood estimation (MLE), two probability-weighted moments (PWM-UE and PWM-PP), and three robust two-stage order statistics estimators (TSOS-ME, TSOS-LMS, and TSOS-LTS). Their performance was assessed using simulation experiments under varying tail behaviors, represented by three types of GEV distributions: Weibull (short-tailed), Gumbel (light-tailed), and Fréchet (heavy-tailed) distributions, based on the mean squared error (MSE) and mean absolute percentage error (MAPE). The results showed that TSOS-LTS consistently achieved the lowest MSE and MAPE, indicating high robustness and forecasting accuracy, particularly for short-tailed distributions. Notably, PWM-PP performed well for the light-tailed distribution, providing accurate and efficient estimates in this specific setting. For heavy-tailed distributions, TSOS-LTS exhibited superior estimation accuracy, while PWM-PP showed a better predictive performance in terms of MAPE. The methods were further applied to real-world monthly maximum PM_{2.5} data from three air quality stations in Bangkok. TSOS-LTS again demonstrated superior performance, especially at Thon Buri station. This research highlights the importance of tailoring estimation techniques to the distribution's tail behavior and supports the use of robust approaches for modeling environmental extremes.

Keywords: generalized extreme value; maximum likelihood estimation; probability-weighted moments; two-stage order statistics

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1. Introduction

Extreme value analysis (EVA) plays a crucial role in anticipating rare, high-impact events that traditional statistical methods often overlook, as they primarily model average behavior rather than tail extremes. Developed initially in hydrology and meteorology, EVA is now widely applied in fields such as structural engineering, risk management, and public health. By focusing on the distributional extremes, such as maximum or minimum EVA, it addresses key questions, including the probability of severe floods [1], record-high temperatures [2], structural wind resistance [3], and worst-case financial losses [4].

The generalized extreme value (GEV) distribution is a cornerstone of extreme value analysis, widely used for modeling block maxima due to its ability to encompass the Gumbel, Fréchet, and Weibull families of distributions. This flexibility allows it to represent a broad spectrum of tail behaviors. Accurate estimation of its parameters (location, scale, and shape) is essential for reliable inference, such as return level estimation and risk

assessment. Since these parameters govern the distribution's center, spread, and tail characteristics, the choice of estimation method critically affects the robustness and accuracy of the results.

Accurate parameter estimation is essential for applying the GEV distribution in climate-related analyses. Unlike traditional methods, GEV-based approaches more effectively capture extremes and tail risks [5], providing a reliable basis for estimating return levels—vital for infrastructure design and climate adaptation planning. The model's flexibility, governed by its shape parameter, allows it to accommodate diverse tail behaviors. Common estimation techniques include maximum likelihood estimation (MLE), L-moments, and Bayesian methods. Hossain et al. [6] compared these approaches for modeling extreme rainfall, while Rai et al. [7] integrated MLE with neural networks to enhance parameter estimation for modeling extreme events.

Alternative estimation methods, such as probability-weighted moments (PWMs) [8] and the two-stage order statistics estimator (TOSE) [9], have demonstrated strong potential in improving parameter accuracy and enhancing the reliability of extreme value forecasts. These techniques are particularly effective in modeling tail behavior, which is critical for risk assessment and return level estimation. PWMs, in particular, offer a robust performance under heavy-tailed distributions and small samples, reducing estimation bias and improving efficiency even in the presence of outliers [10]. Helu [11] further emphasized the value of moment-based approaches by proposing a modified PWM technique and comparing it favorably with classical moments and MLE for the Kumaraswamy distribution, illustrating their broader applicability in complex data structures.

Probability-weighted moments (PWMs) can be further refined through unbiased estimation techniques, thereby enhancing their theoretical accuracy and flexibility in empirical modeling. Caeiro and Mateus [12] proposed unbiased PWMs for estimating parameters of the Pareto Type I distribution, demonstrating a strong performance under heavy-tailed conditions and sample variability. These estimators have been effectively applied in modeling extremes related to rainfall, financial risk, and system reliability. Chen et al. [13] also utilized higher-order PWMs to estimate flood parameters accurately.

Although the GEV distribution has been widely used in modeling extreme events, several studies have highlighted the limitations of classical estimation approaches such as MLE and PWM. MLE, while asymptotically efficient, can be highly unstable for small sample sizes and in the presence of outliers, as shown by Roslan et al. [14]. PWM estimators, although more robust in some scenarios, tend to lose performance when the shape parameter exceeds 0.5, which is commonly observed in heavy-tailed distributions. To address this, Diebolt et al. [15] introduced an improved version of PWM—generalized PWM (GPWM)—to enhance estimation in such conditions. More recently, Caeiro et al. [16] further explored tail inference under PWM-based methods, emphasizing the need for more robust techniques in practice.

The two-stage order statistics estimator (TSOSE) offers notable advantages in estimating GEV parameters by emphasizing extreme order statistics and iteratively refining estimates to reduce the influence of outliers. This method enhances accuracy in small samples—a common challenge in extreme value analysis. Its second-stage iteration improves the precision of location, scale, and shape parameters by reducing bias. TSOSE can also be integrated with robust regression techniques such as the least median of squares (LMS) and least trimmed squares (LTS). LMS, which minimizes the median of squared residuals, is well-suited for heavy-tailed and skewed data. In contrast, LTS enhances robustness by reducing the sum of the smallest squared residuals, making it effective under moderate outlier contamination.

A core concept in extreme value theory is the return level, which represents the expected magnitude of an event that occurs once in a specified return period. Estimating return levels is essential for environmental risk assessment and infrastructure planning. However, accurate estimation becomes increasingly difficult for long return periods due to the limited availability of extreme-value data [17]. Tanprayoon et al. [18] applied the GEV distribution to estimate rainfall return levels in Lopburi Province, while Rydén [19] focused on estimating return levels associated with long-term environmental extremes. Such analyses are critical, as future increases in extreme weather events, such as heavy rainfall or drought, pose growing risks to infrastructure and public safety [20].

PM2.5 refers to particulate matter with aerodynamic diameters less than 2.5 μm . Due to their small size, these particles can penetrate the lungs and even enter the bloodstream, leading to cardiovascular and respiratory diseases. Air pollution, excellent particulate matter (PM2.5), poses serious environmental and health risks in urban areas. Klinjan et al. [21] highlighted the development and application of the Garima-generalized extreme value distribution in modeling the extreme values of PM2.5 and PM10 in Pathum Thani, Thailand.

In Bangkok, Thailand, PM2.5 concentrations often exceed safe thresholds, particularly during the dry seasons. This study applies the GEV distribution to model the extreme behavior of PM2.5 concentrations in Bangkok. The GEV distribution framework captures the stochastic nature of extreme pollution events, providing critical insights into their frequency and magnitude, and offering practical implications for risk management and environmental policy. Deng and Liu [22] classified cities across China into eight categories based on the generalized extreme value GEV distribution, using hourly station-level PM2.5 concentration data.

In the context of environmental applications, such as air pollution monitoring, Zhao et al. [23] proposed robust estimation strategies for GEV modeling of PM2.5 extremes in China, noting that classical estimators often fail to capture the data's skewness and extremal behavior. Building upon these works, our study introduces robust two-stage order statistics (TSOS) estimators designed to integrate order statistics with robust regression principles, such as the least median of squares (LMS) and the least trimmed squares (LTS) method. These estimators offer improved accuracy and stability across varying tail behaviors and are evaluated through extensive simulations and real-world applications of PM2.5 data from Bangkok.

The remainder of this paper is organized as follows: Section 2 introduces the GEV distribution and reviews the estimation methods under study; Section 3 describes the simulation design and performance metrics and presents the simulation results; Section 4 provides a real-data application using monthly PM2.5 data from Bangkok; finally, Section 5 concludes the findings and discusses future research directions.

2. Materials and Methods

The GEV distribution arises from the theoretical foundation of EVA, which focuses on the statistical behavior of extreme observations in a dataset. Unlike classical statistical models that emphasize central tendencies, EVA deals with the tail behavior of distributions, particularly the modeling of the maxima or minima of sequences of random variables.

The theoretical basis of the GEV distribution stems from the extreme value limit theorem, which is analogous to the central limit theorem but for extreme values. Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with a common cumulative distribution function. Define the block maximum as:

$$M_n = \max\{X_1, X_2, \dots, X_n\} \quad (1)$$

To avoid degeneracy, it is necessary to normalize the maxima using linear transformations. That is, if there exist sequences of constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that:

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow F(x), \quad n \rightarrow \infty, \quad (2)$$

then the limiting distribution F is a GEV distribution [24]. This result, first proven by Fisher and Tippett [25] and later formalized by Gnedenko [26], classifies all possible non-degenerate limiting distributions for the normalized maxima into three types, which the GEV distribution unifies. The cumulative distribution function defines the GEV distribution as:

$$F(x; \mu, \sigma, \xi) = \begin{cases} \exp\left\{-\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right\} & , \xi \neq 0 \\ \exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\} & , \xi = 0 \end{cases}. \quad (3)$$

The probability density function [27] is given by:

$$f(x; \mu, \sigma, \xi) = \begin{cases} \frac{1}{\sigma} \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{(-1/\xi)-1} \exp\left\{-\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right\} & , \xi \neq 0 \\ \frac{1}{\sigma} \exp\left(\frac{x-\mu}{\sigma}\right)^{-1} \exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\} & , \xi = 0 \end{cases}, \quad (4)$$

for $\left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\} > 0$, $-\infty < \mu$, $\xi < \infty$, $\sigma > 0$ [28], where μ (location parameter) defines the center of the distribution, σ (scale parameter) controls the dispersion, and ξ (shape parameter) determines the tail behavior. The GEV distribution can be expressed in three different forms, depending on the value of the shape parameter, where $\xi > 0$ represents the Fréchet type as heavy-tailed, $\xi = 0$ represents the Gumbel type as light-tailed, and $\xi < 0$ represents the Weibull type as a bounded tail.

Parameter estimation in the GEV distribution is critical in modeling extreme events. Several estimation methods are available, each with their strengths and limitations. The choice of method depends on the sample size, data characteristics, and analysis objectives. Maximum likelihood estimation (MLE) is the most commonly used estimation method due to its desirable statistical properties, such as consistency, asymptotic normality, and efficiency. Probability-weighted moments (PWMs) and the two-stage order statistics estimator (TSOE) have been developed to estimate the GEV distribution parameters.

2.1. Maximum Likelihood Estimation (MLE)

MLE is a widely adopted approach due to its asymptotic properties. However, the likelihood equations derived from the GEV distribution do not have closed-form solutions. Therefore, numerical methods such as the Newton–Raphson algorithm are employed to find the values of the parameters that maximize the log-likelihood function.

Let $X_1, X_2, \dots, X_n$ be a sample of observed block maxima assumed to follow a GEV distribution. The likelihood function $L(\mu, \sigma, \xi)$ is the product of the individual probability density function, given by:

$$L(\mu, \sigma, \xi) = \prod_{i=1}^n f(x_i; \mu, \sigma, \xi). \quad (5)$$

Then, the log-likelihood function, which is typically easier to maximize numerically from (4) and (5), is:

$$\log L(\mu, \sigma, \xi) = -n \log(\sigma) - \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^n \log\left[1 + \xi\left(\frac{x_i - \mu}{\sigma}\right)\right] - \sum_{i=1}^n \left(1 + \xi\left(\frac{x_i - \mu}{\sigma}\right)\right)^{-1/\xi}, \quad (6)$$

provided $1 + \zeta \frac{x_i - \mu}{\sigma} > 0$ for all i .

When $\zeta \neq 0$, the log-likelihood function involves the term $\left[1 + \zeta \frac{x_i - \mu}{\sigma}\right]^{-1/\zeta}$, and the corresponding score functions and Hessian require derivative expressions that account for this nonlinearity. When $\zeta \rightarrow 0$, this expression becomes undefined unless handled via the limit theory. In this case, the GEV model converges to the Gumbel distribution, and the log-likelihood simplifies accordingly to avoid singularities. To ensure numerical stability and unbiased inference near the boundary, we implemented a separate analytical treatment for $\zeta = 0$ using the Gumbel log-likelihood form and its associated gradients.

Since this function is often nonlinear and may not have a closed-form solution, iterative numerical optimization methods maximize the log-likelihood function. The Newton–Raphson method [29] is an iterative technique that is used to find the root of the first derivative. In the context of MLE, it seeks the parameter vector $\theta = (\mu, \sigma, \zeta)^T$ that maximizes the log-likelihood function $\ell(\theta) = \log L(\mu, \sigma, \zeta)$ by solving $\nabla \ell(\theta) = 0$. Using a second-order Taylor expansion, the Newton–Raphson update rule is:

$$\theta^{(k+1)} = \theta^{(k)} - [H(\theta^{(k)})]^{-1} \nabla \ell(\theta^{(k)}) \quad (7)$$

where $\nabla \ell(\theta^{(k)})$ is the gradient vector (first derivative) of the log-likelihood and $[H(\theta^{(k)})]^{-1}$ is the Hessian matrix (second derivatives). The components of the gradient vector are given by $\frac{\partial \ell}{\partial \mu}$, $\frac{\partial \ell}{\partial \sigma}$, and $\frac{\partial \ell}{\partial \zeta}$. The Hessian matrix [30] contains second-order partial derivatives as:

$$H(\theta) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \mu^2} & \frac{\partial^2 \ell}{\partial \mu \partial \sigma} & \frac{\partial^2 \ell}{\partial \mu \partial \zeta} \\ \frac{\partial^2 \ell}{\partial \sigma \partial \mu} & \frac{\partial^2 \ell}{\partial \sigma^2} & \frac{\partial^2 \ell}{\partial \sigma \partial \zeta} \\ \frac{\partial^2 \ell}{\partial \zeta \partial \mu} & \frac{\partial^2 \ell}{\partial \zeta \partial \sigma} & \frac{\partial^2 \ell}{\partial \zeta^2} \end{bmatrix}. \quad (8)$$

These expressions also do not simplify easily and are calculated numerically at each iteration of the Newton–Raphson process. The step-by-step procedure of the iterative algorithm can be shown to be:

- (1) Initialize parameters $\theta^{(0)} = (\mu^{(0)}, \sigma^{(0)}, \zeta^{(0)})$.
- (2) Evaluate the gradient vector $\nabla \ell(\theta^{(k)})$ and Hessian matrix $[H(\theta^{(k)})]$.
- (3) Update using $\theta^{(k+1)} = \theta^{(k)} - [H(\theta^{(k)})]^{-1} \nabla \ell(\theta^{(k)})$.
- (4) Check convergence by stopping if the parameter change is below a tolerance level or the gradient norm is near zero.
- (5) Repeat steps 2–4 until convergence.

The Newton–Raphson method offers rapid convergence and high accuracy in parameter estimation when the starting values are close to the true parameters.

2.2. Probability-Weighted Moments Estimation (PWME)

PWME [31] is a valuable method for estimating parameters in extreme value analysis, providing robust and efficient estimates in various statistical models. PWME can be categorized into two primary estimation approaches: unbiased estimators and plotting-position estimators.

The cumulative distribution function of the GEV distribution from (3) is defined as $F(x)$. The r -th order PWME of a random variable X is written as:

$$\beta_r = E[X \cdot F(X)^r], \quad r = 0, 1, 2. \quad (9)$$

These moments capture the values and ranks of the data and are used to infer distribution parameters.

2.2.1. Probability-Weighted Moments in Unbiased Estimators (PWM-UEs)

PWM-UEs offer accurate parameter estimation with minimal bias, making them particularly useful in statistical modeling. They are also particularly effective in extreme value analysis, improving the estimation of location, scale, and shape parameters.

Hosking [32] derived closed-form expressions for the first three PWMs of the GEV distribution:

$$\beta_r = \mu + \frac{\sigma}{\xi} \left(\Gamma(1 - \xi) - \sum_{k=1}^{r+1} \binom{r}{k-1} (-1)^{k+1} k^{-\xi} \right). \quad (10)$$

But for practical use, the specific cases from (10) can be considered as:

$$\beta_0 = \mu + \frac{\sigma}{\xi} (\Gamma(1 - \xi) - 1), \beta_1 = \mu + \frac{\sigma}{\xi} (\Gamma(1 - \xi) - 2^{-\xi}), \text{ and } \beta_2 = \mu + \frac{\sigma}{\xi} (\Gamma(1 - \xi) - 3^{-\xi}).$$

These equations are nonlinear in ξ , but they can be manipulated to extract parameter estimates.

Let $X_1, X_2, \dots, X_n$ be an i.i.d sample, and let $X_1 \leq X_2 \leq \dots \leq X_n$ denote the order statistics. The unbiased sample estimators of β_r are given by:

$$\hat{\beta}_r = \frac{1}{n} \sum_{i=1}^n b_{i:n}^{(r)} X_{(i)}, \quad (11)$$

where the weights $b_{i:n}^{(r)}$ are $b_{i:n}^{(r)} = \frac{(n-1)(n-2)\dots(n-r)}{(i-1)(i-2)\dots(i-r)}$, for $r = 0, 1, 2$.

Thus, it can be computed that

$$\hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n X_{(i)}, \hat{\beta}_1 = \frac{1}{n} \sum_{i=1}^n \frac{i-1}{n-1} X_{(i)}, \text{ and } \hat{\beta}_2 = \frac{1}{n} \sum_{i=1}^n \frac{(i-1)(i-2)}{(n-1)(n-2)} X_{(i)}.$$

These are unbiased estimators of the theoretical PWME.

From the specific cases and the weights, using the relationships between the PWME and the GEV distribution parameters, the shape, scale, and location of PWME-UEs are estimated as follows:

$$\text{Shape parameter } (\xi): \hat{\xi} = \frac{\hat{\beta}_0 - 2\hat{\beta}_1 + \hat{\beta}_2}{2\hat{\beta}_1 - \hat{\beta}_0} - 2,$$

$$\text{Scale parameter } (\sigma): \hat{\sigma} = \frac{\Gamma(1-\hat{\xi})(2^{\hat{\xi}}-1)}{(\hat{\beta}_0 - 2\hat{\beta}_1 + \hat{\beta}_2)(\hat{\xi}+1)(\hat{\xi}+2)}, \text{ and}$$

$$\text{Location parameter } (\mu): \hat{\mu} = \hat{\beta}_0 - \frac{\hat{\sigma}}{\hat{\xi}} (1 - \Gamma(1 - \hat{\xi})).$$

2.2.2. Probability-Weighted Moments in Plotting Position (PWM-PPs)

The plotting position-based PWME, also known as PWM-PP, provides a practical approach for the PWM-UE to estimate GEV distribution parameters. It uses ranked data and predefined empirical plotting positions to approximate the theoretical PWME. Define the plotting position p_i for each order statistic as:

$$p_i = \frac{n+1-2a}{i-a}, \quad \text{for } i = 1, 2, \dots, n, \quad (12)$$

where a is typically chosen as: $a = 0$ defines the Weibull plotting position, $a = 0.35$ defined Hazen's rule, and $a = 0.5$ defines Blom's rule. The PWME of order r is then estimated as:

$$\hat{\beta}_r = \frac{1}{n} \sum_{i=1}^n X_{(i)} p_i^r. \quad (13)$$

This is known as the PWME-PP to estimate $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$. The GEV parameters are calculated using the shape, scale, and location parameters via the PWME-UE formula.

2.3. Two-Stage Order Statistics Estimation (TSOSE)

TSOSE has been developed as a robust alternative for parameter estimation in the GEV distribution based on order statistics. It works in two steps: estimating using order statistics and refining using robust statistical techniques. The TSOSE, incorporating median, least median of squares (LMS), and least trimmed squares (LTS) estimators, offers a strong alternative to traditional parameter estimation methods.

2.3.1. Two-Stage Order Statistics in Median Estimation (TSOS-ME)

The theoretical foundation is derived from the properties of order statistics, while the computational steps involve an iterative procedure combining initial quantile-based estimation with nonlinear optimization. The inverse of the GEV distribution, also called the quantile function, is written as:

$$Q(p; \mu, \sigma, \xi) = \begin{cases} \mu + \frac{\sigma}{\xi} \left[(-\log p)^{-\xi} - 1 \right], & \text{if } \xi \neq 0 \\ \mu - \sigma \log(-\log p), & \text{if } \xi = 0 \end{cases}, \quad (14)$$

where the selected quartiles (p_1, p_2, p_3) are defined as: the first quartile is $Q(p_1) = x_{(q_1)}$, the median is $Q(p_2) = \hat{r}$, and the third quartile is $Q(p_3) = x_{(q_3)}$. Let the specific cases of the quantile function as $\beta_0 = Q(p_1) = x_{[0.25n]}$, $\beta_1 = Q(p_2) = \hat{r}$, and $\beta_2 = Q(p_3) = x_{[0.75n]}$.

Assume the GEVD quantile function holds for three quantiles:

$$\begin{aligned} \beta_0 &= \mu + \frac{\sigma}{\xi} \left[(-\log p_1)^{-\xi} - 1 \right], \\ \beta_1 &= \mu + \frac{\sigma}{\xi} \left[(-\log p_2)^{-\xi} - 1 \right], \\ \beta_2 &= \mu + \frac{\sigma}{\xi} \left[(-\log p_3)^{-\xi} - 1 \right]. \end{aligned}$$

Subtracting pairs of equations and taking the ratio, it can be obtained that:

$$R = \frac{\beta_2 - \beta_1}{\beta_1 - \beta_0} = \frac{(-\log p_3)^{-\xi} - (-\log p_2)^{-\xi}}{(-\log p_2)^{-\xi} - (-\log p_1)^{-\xi}}. \quad (15)$$

Now take logarithms of both sides and solve numerically for ξ . Alternatively, use a symmetric form to approximate the shape parameter as:

$$\hat{\xi} = \frac{\beta_0 - 2\beta_1 + \beta_2}{2\beta_1 - \beta_0 - \beta_2}.$$

This is derived from an approximate second-order central difference on the quantile scale, assuming symmetry around the median. Next, from the GEV distribution quantile function, subtracting pairs:

$$\beta_0 - 2\beta_1 + \beta_2 = \frac{\xi}{\sigma} \left[(-\log p_1)^{-\xi} - 2(-\log p_2)^{-\xi} + (-\log p_3)^{-\xi} \right]$$

Let $\Delta = (-\log p_1)^{-\xi} - 2(-\log p_2)^{-\xi} + (-\log p_3)^{-\xi}$, then the scale parameter is given by:

$$\hat{\sigma} = \frac{\xi(\beta_0 - 2\beta_1 + \beta_2)}{\Delta}.$$

From the quantile function at $p = 0.5$, it can be evaluated that:

$$\beta_1 = \mu + \frac{\xi}{\sigma} \left[(\log 2)^{-\xi} - 1 \right] \Rightarrow \mu = \beta_1 - \frac{\xi}{\sigma} \left[(\log 2)^{-\xi} - 1 \right].$$

So, the location parameter is written as:

$$\hat{\mu} = \hat{\tau} - \frac{\hat{\xi}}{\hat{\sigma}} \left[(\log 2)^{-\hat{\xi}} - 1 \right].$$

2.3.2. Two-Stage Order Statistics in Least Median of Squares (TSOS-LMS)

TSOS-LMS combines order statistics to focus on extreme value analysis, such as the GEV distribution, and the least median of squares (LMS) method, enhancing robustness by minimizing the median of squared residuals rather than the mean.

The computation is to select order statistics by choosing a small number k of upper order statistics to emphasize extremes as $X_{(1)}, X_{(2)}, \dots, X_{(n)}$. Next, the order statistics transform to a linear model by assuming a linear relationship based on the GEV distribution quantile function as:

$$X_{(i)} = \mu + \frac{\sigma}{\xi} \left[(-\log p_i)^{-\xi} - 1 \right], \quad i = 1, 2, \dots, k, \quad (16)$$

where $p_i = \frac{i-0.5}{n}$ are the plotting positions.

The least median of squares objective function is defined by setting up the least median of squares criterion as:

$$\min \text{Median} \left\{ \left(X_{(i)} - \hat{X}_{(i)}(\mu, \sigma, \xi) \right)^2 \right\},$$

where $\hat{X}_{(i)}(\mu, \sigma, \xi) = \mu + \frac{\xi}{\sigma} \left[(-\log p_i)^{-\xi} - 1 \right]$.

Since no closed-form solution exists, parameter estimation minimizes the objective function using numerical optimization methods as:

$$\text{Objective}(\mu, \sigma, \xi) = \text{Median} \left[\left(X_{(i)} - \hat{X}_{(i)} \right)^2 \right], \quad (17)$$

where the ensuring constraints are $\sigma > 0$, for all i and $1 + \xi \frac{X_{(i)} - \mu}{\sigma} > 0$.

The optimal values $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\xi}$ are minimizing the median of the squared residuals. These are the TSOS-LMS parameter estimates for the GEV distribution.

2.3.3. Two-Stage Order Statistics in Least Trimmed of Squares (TSOS-LTS)

The computational aspects of the shape, scale, and location parameters in the TSOS estimator of least trimmed squares, also known as TSOS-LTS, for the GEV distribution combine robust estimation and order statistics-based modeling, which is explicitly designed for extreme-value data.

The fundamental structure of the LMS and LTS methods is the same. Each begins with a two-stage approach: in the first stage, a subset of upper-order statistics is selected to emphasize the tail behavior of the distribution, which is central to the GEV distribution framework. In the second stage, these selected order statistics are used to estimate the location parameters by minimizing a measure of the residuals between the observed order statistics and the theoretical quantiles derived from the GEV distribution's quantile function.

In contrast, the LTS-based estimator minimizes the sum of the smallest h squared residuals, where $h < k$, such as $h = [\alpha \cdot k]$, $\alpha \in (0.5, 1)$. Let the residuals of the LTS criterion be defined as:

$$r_i(\mu, \sigma, \xi) = X_{(i)} - \left[\mu + \frac{\xi}{\sigma} \left((-\log p_i)^{-\xi} - 1 \right) \right] \quad (18)$$

The LTS objective function is written as:

$$L_{\text{LTS}}(\mu, \sigma, \xi) = \sum_{i=1}^h (r^2)_{(i)}, \quad (19)$$

where $(r^2)_{(1)}, (r^2)_{(2)}, \dots, (r^2)_{(h)}$ are the smallest h squared residuals sorted from the full set $\{r_1^2, \dots, r_k^2\}$. The goal is to find the parameters $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\xi}$ that minimize the trimmed sum of squares by solving numerical optimization techniques in this formula $(\hat{\mu}, \hat{\sigma}, \hat{\xi}) = \arg_{\mu, \sigma, \xi} \min L_{\text{LTS}}(\mu, \sigma, \xi)$.

3. Simulation Studies

To evaluate the performance of parameter estimation methods, synthetic data were generated from the GEV distribution with a location parameter set to two and a scale parameter set to one. Three scenarios were considered based on different shape parameter values: -0.8 corresponding to the Weibull distribution or short-tailed distribution, zero corresponding to the Gumbel distribution or light-tailed distribution, and 0.8 corresponding to the Fréchet distribution or heavy-tailed distribution. The estimation procedures employed these parameter configurations; the results are presented in Figure 1.

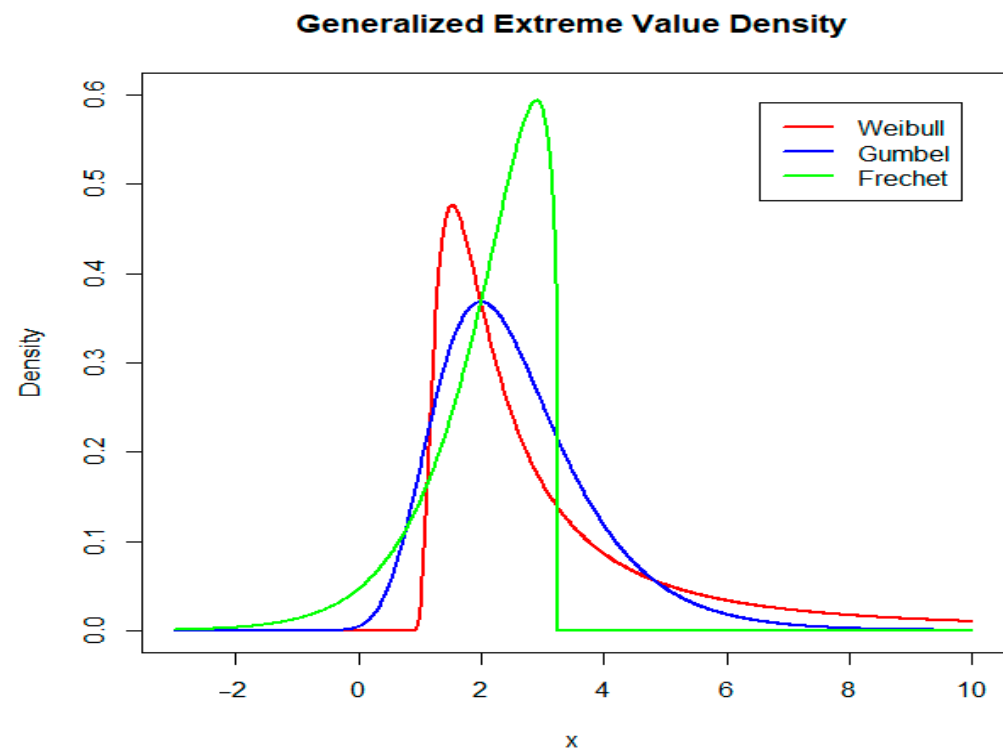


Figure 1. The GEV density for various shape parameters, including the Weibull, Gumbel, and Fréchet distributions.

Sample sizes of 20, 40, and 80 were considered in the study, with each scenario replicated 1000 times. These sample sizes and replication settings were used to estimate the GEV distribution parameters, such as the location parameter (μ), scale parameter (σ), and shape parameter (ξ). Parameter estimation was performed using the three primary methods: maximum likelihood estimation (MLE), probability-weighted moments estimation (PWME), and two-stage order statistics estimation (TSOSE). The PWME can also be considered in two additional approaches: probability-weighted moments in unbiased estimators (PWM-UEs) and probability-weighted moments in plotting position (PWM-PPs). Moreover, the TOSE can be further extended into three improved approaches: two-stage order statistics in

median estimation (TSOS-ME), two stage order statistics in least median of squares (TSOS-LMS), and two-stage order statistics in least trimmed squares (TSOS-LTS). A total of five methods were employed in this study. The dataset was partitioned into two subsets: 90% for training and 10% for testing. The training data were used to estimate the parameters and compute the corresponding mean and mean squared error (MSE). The testing data were then utilized to calculate the return levels, which were subsequently used to compute the mean absolute percentage error (MAPE).

The return level analysis examines historical data to assess the frequency of extreme events within a probabilistic framework, aiming to estimate the likelihood of such events recurring. Extreme value theory quantifies the magnitude of extreme occurrences expected to occur within a given return period. It represents the value that is expected to be exceeded, on average, once every T observations or time units, where T denotes the return period.

The return level x_T associated with a return period T is defined as the quantile satisfying $P(X > x_T) = \frac{1}{T}$ or equivalently $F(x_T) = \frac{1}{T}$. Solving the equation $F(x_T) = \frac{1}{T}$ leads to the following closed-form expression for the return level:

$$x_T = \mu + \frac{\sigma}{\xi} \left[\left(-\log \left(1 - \frac{1}{T} \right) \right)^{-\xi} - 1 \right].$$

The results, including the mean and standard deviation (SD) of the parameter estimates (Tables 1, 3 and 5), the mean squared error (MSE) for different shape parameters (Tables 2, 4 and 6), and the mean absolute percentage error (MAPE) for all scenarios (Table 7), are summarized in Tables 1–7.

Table 1. Comparison of parameter estimates, mean, and standard deviation (SD) of the parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = -0.8$.

Method	Location	n = 20 Scale	Shape	Location	n = 40 Scale	Shape	Location	n = 80 Scale	Shape
MLE	2.019 (0.306)	0.944 (0.335)	−0.857 (0.391)	2.001 (0.201)	0.966 (0.229)	−0.828 (0.224)	2.013 (0.169)	0.957 (0.198)	−0.709 (0.228)
PWM-UE	2.123 (0.350)	1.180 (0.576)	−0.533 (0.226)	2.077 (0.224)	1.124 (0.352)	−0.605 (0.178)	2.068 (0.160)	1.119 (0.255)	−0.647 (0.143)
PWM-PP	2.207 (0.604)	1.475 (0.812)	−0.491 (0.204)	2.119 (0.250)	1.270 (0.733)	−0.582 (0.170)	2.087 (0.161)	1.182 (0.271)	−0.635 (0.140)
TSOS-ME	2.080 (0.640)	0.967 (0.642)	−0.645 (0.362)	2.036 (0.224)	0.960 (0.238)	−0.696 (0.290)	2.037 (0.164)	0.972 (0.175)	−0.709 (0.235)
TSOS-LMS	2.035 (0.339)	0.935 (0.349)	−0.616 (0.357)	2.005 (0.236)	0.937 (0.254)	−0.697 (0.280)	2.013 (0.164)	0.958 (0.194)	−0.710 (0.228)
TSOS-LTS	2.035 (0.338)	0.933 (0.351)	−0.649 (0.353)	2.001 (0.230)	0.935 (0.258)	−0.696 (0.280)	2.002 (0.135)	0.991 (0.163)	−0.815 (0.143)

Table 1 presents the mean and standard deviation of the parameter estimates—location, scale, and shape for the GEV distribution with shape parameter $\xi = -0.8$ —across sample sizes $n = 20, 40$, and 80 . Across all sample sizes, the MLE, TSOS-LMS, and TSOS-LTS methods consistently yield estimates that are closer to the true parameter values with relatively lower standard deviations, indicating a robust performance under the short-tailed distribution scenario. In contrast, the PWM-PP method tends to produce higher bias and variability, particularly in estimating the scale parameter. As the sample size increases, the variability in estimates generally decreases across all methods, suggesting consistency and improved accuracy with larger sample sizes.

Table 2. Comparison of the mean squared error (MSE) of parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = -0.8$.

Method	Location	n = 20		Location	n = 40		Location	n = 80	
		Scale	Shape		Scale	Shape		Scale	Shape
MLE	0.094	0.115	0.156	0.040	0.053	0.050	0.028	0.041	0.060
PWM-UE	0.137	0.364	0.122	0.056	0.139	0.069	0.030	0.079	0.043
PWM-PP	0.408	0.127	0.136	0.076	0.610	0.076	0.033	0.106	0.046
TSOS-ME	0.122	0.118	0.154	0.051	0.058	0.095	0.028	0.031	0.063
TSOS-LMS	0.116	0.125	0.150	0.055	0.068	0.089	0.027	0.039	0.060
TSOS-LTS	0.115	0.128	0.147	0.047	0.071	0.089	0.018	0.002	0.020

Note: The underlined number indicates the minimum MSE.

From Table 2, when the sample size increases, the MSE values decrease across all methods, indicating improved estimation accuracy. The TSOS-LTS method consistently yields the lowest MSEs for all parameters at $n = 80$. In contrast, the MLE method exhibits the lowest MSEs at $n = 20$ and 40 , particularly in location and scale estimations, suggesting a highly reliable performance under short-tailed conditions. Overall, robust MLE and TSOS-LTS methods demonstrate superior efficiency and stability compared with traditional estimators.

Table 3 summarizes that all methods produce estimates that are close to the true values, with improved precision as the sample size increases. The MLE method performs well overall, especially in estimating large sample sizes. However, robust TSOS-based methods, particularly TSOS-LTS, provide comparable or better performance with slightly lower variability in some parameters. PWM-PP shows slightly higher bias in the location and shape estimates, but with acceptable variability. Overall, all methods are consistent under the light-tailed scenario, with TSOS variants showing stable and competitive results.

Table 3. Comparison of parameter estimates, mean, and standard deviation (SD) of the parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = 0$.

Method	n = 20			n = 40			n = 80		
	Location	Scale	Shape	Location	Scale	Shape	Location	Scale	Shape
MLE	2.031 (0.278)	0.945 (0.201)	0.018 (0.267)	2.011 (0.195)	0.966 (0.139)	0.010 (0.153)	2.009 (0.137)	0.982 (0.100)	0.011 (0.094)
PWM-UE	2.005 (0.263)	0.994 (0.202)	0.022 (0.195)	2.000 (0.193)	0.990 (0.143)	0.009 (0.129)	2.002 (0.137)	0.994 (0.106)	0.010 (0.089)
PWM-PP	1.988 (0.253)	0.982 (0.183)	−0.005 (0.161)	1.991 (0.190)	0.981 (0.137)	−0.004 (0.117)	1.997 (0.136)	0.989 (0.104)	0.002 (0.085)
TSOS-ME	2.014 (0.266)	0.945 (0.208)	0.066 (0.199)	2.010 (0.195)	0.962 (0.166)	0.048 (0.127)	2.017 (0.142)	0.970 (0.129)	0.046 (0.096)
TSOS-LMS	2.003 (0.289)	0.949 (0.217)	0.065 (0.202)	2.002 (0.213)	0.963 (0.165)	0.048 (0.127)	2.011 (0.150)	0.970 (0.129)	0.045 (0.094)
TSOS-LTS	2.000 (0.288)	0.949 (0.215)	0.064 (0.200)	2.004 (0.215)	0.963 (0.167)	0.048 (0.127)	2.010 (0.155)	0.969 (0.129)	0.045 (0.094)

Table 4. Comparison of the mean squared error (MSE) of parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = 0$.

Method	Location	n = 20			n = 40			n = 80	
		Scale	Shape	Location	Scale	Shape	Location	Scale	Shape
MLE	0.078	0.043	0.071	0.038	0.020	0.023	0.019	<u>0.010</u>	0.008
PWM-UE	0.069	0.041	0.038	0.037	0.020	0.016	<u>0.018</u>	0.011	0.008
PWM-PP	<u>0.064</u>	<u>0.034</u>	<u>0.026</u>	<u>0.036</u>	<u>0.019</u>	<u>0.013</u>	<u>0.018</u>	<u>0.010</u>	<u>0.007</u>
TSOS-ME	0.071	0.046	0.044	0.038	0.029	0.018	0.020	0.017	0.011
TSOS-LMS	0.084	0.049	0.045	0.045	0.028	0.018	0.022	0.017	0.010
TSOS-LTS	0.082	0.048	0.044	0.046	0.029	0.018	0.024	0.017	0.011

Note: The underlined number indicates the minimum MSE.

As expected, all methods show decreasing MSEs with increasing sample size, indicating improved estimation accuracy. The PWM-PP method yields the lowest MSEs across all parameters and sample sizes. While MLE and PWM-UE also perform well, especially for larger samples, the TSOS-based methods (TSOS-ME, TSOS-LMS, and TSOS-LTS) provide competitive accuracy with stable results, especially for the scale and shape parameters. These findings support the reliability of the PWM-PP method under light-tailed conditions.

Table 5 displays that the MLE method shows noticeably higher variability, particularly in the scale and shape estimates, especially for smaller sample sizes. In contrast, the TSOS-based methods (TSOS-ME, TSOS-LMS, and TSOS-LTS) produce more stable and accurate estimates, with lower standard deviations across parameters as the sample size increases. Notably, PWM-UE also performs well with low bias, while PWM-PP underestimates the shape parameter in small samples. Overall, the TSOS-LTS method demonstrates a balance of accuracy and robustness, especially under conditions of extreme tail behavior.

Table 5. Comparison of parameter estimates, mean, and standard deviation (SD) of the parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = 0.8$.

Method	n = 20			n = 40			n = 80		
	Location	Scale	Shape	Location	Scale	Shape	Location	Scale	Shape
MLE	1.725 (0.536)	1.199 (0.899)	0.968 (0.949)	1.977 (0.320)	1.121 (0.522)	0.972 (0.936)	1.977 (0.185)	1.189 (0.631)	0.940 (0.308)
PWM-UE	2.012 (0.279)	0.974 (0.227)	0.807 (0.278)	2.005 (0.191)	0.986 (0.159)	0.800 (0.181)	2.004 (0.134)	0.994 (0.116)	0.801 (0.127)
PWM-PP	1.924 (0.274)	0.981 (0.217)	0.610 (0.206)	1.959 (0.189)	0.989 (0.157)	0.695 (0.157)	1.980 (0.134)	0.996 (0.115)	0.747 (0.119)
TSOS-ME	1.982 (0.270)	0.982 (0.269)	0.832 (0.239)	1.995 (0.188)	0.985 (0.211)	0.815 (0.165)	1.990 (0.131)	0.998 (0.171)	0.807 (0.121)
TSOS-LMS	1.984 (0.292)	0.993 (0.275)	0.827 (0.243)	1.997 (0.205)	0.993 (0.218)	0.811 (0.161)	1.988 (0.149)	1.004 (0.177)	0.803 (0.115)
TSOS-LTS	1.984 (0.291)	0.994 (0.274)	0.827 (0.239)	1.998 (0.207)	0.994 (0.219)	0.810 (0.161)	1.989 (0.150)	1.005 (0.177)	0.804 (0.115)

Table 6. Comparison of the mean squared error (MSE) of parameter estimates across various sample sizes (n), under the specified parameter settings $\mu = 2, \sigma = 1$, and $\xi = 0.8$.

Method	Location	n = 20		Location	n = 40		Location	n = 80	
		Scale	Shape		Scale	Shape		Scale	Shape
MLE	0.095	0.082	0.089	0.102	0.287	0.906	0.034	0.434	0.114
PWM-UE	0.077	0.052	0.077	0.036	0.025	0.033	0.018	<u>0.013</u>	0.016
PWM-PP	0.081	<u>0.047</u>	0.078	0.037	<u>0.024</u>	0.035	0.018	<u>0.013</u>	0.016
TSOS-ME	<u>0.073</u>	0.072	0.059	<u>0.035</u>	0.045	0.027	<u>0.017</u>	0.029	0.014
TSOS-LMS	0.085	0.075	0.058	0.042	0.047	<u>0.026</u>	0.022	0.031	<u>0.013</u>
TSOS-LTS	0.085	0.075	<u>0.057</u>	0.042	0.048	<u>0.026</u>	0.022	0.031	<u>0.013</u>

Note: The underlined number indicates the minimum MSE.

The TSOS-LTS method consistently achieves the lowest MSEs for the shape parameter across all sample sizes, highlighting its robustness in modeling extreme tail behavior. In contrast, MLE shows the highest MSEs for the scale and shape parameters, particularly in larger sample sizes, indicating its reduced efficiency under heavy-tailed conditions. TSOS-ME and PWM-PP perform relatively well for the location parameter, but their shape estimates are less reliable. Overall, TSOS-based methods, especially TSOS-LTS, demonstrate a superior performance in terms of estimation accuracy and stability under extreme conditions.

Table 7. Comparison of the mean absolute percentage error (MAPE) of parameter estimates across various sample sizes (n).

Method	$\mu = 2, \sigma = 1$ and $\xi = -0.8$			$\mu = 2, \sigma = 1$ and $\xi = 0$			$\mu = 2, \sigma = 1$ and $\xi = 0.8$		
	n = 20	n = 40	n = 80	n = 20	n = 40	n = 80	n = 20	n = 40	n = 80
MLE	41.383	72.987	123.213	49.589	61.128	74.325	117.353	101.732	142.757
PWM-UE	<u>40.913</u>	70.615	118.959	49.540	61.983	75.339	116.881	100.002	138.111
PWM-PP	42.563	75.840	124.729	<u>49.517</u>	62.082	75.581	<u>114.021</u>	<u>98.554</u>	<u>135.343</u>
TSOS-ME	41.132	68.604	115.608	50.086	<u>59.380</u>	<u>71.987</u>	120.119	100.391	165.350
TSOS-LMS	41.426	<u>68.310</u>	113.943	50.386	59.740	72.204	121.370	100.734	164.593
TSOS-LTS	41.426	68.326	<u>113.713</u>	50.447	59.765	72.219	122.595	100.335	166.217

Note: The underlined number indicates the minimum MAPE.

Table 7 compares the mean absolute percentage error (MAPE) of parameter estimates across sample sizes and three tail scenarios: short-tailed ($\xi = -0.8$), light-tailed ($\xi = 0$), and heavy-tailed ($\xi = 0.8$). For all methods, the MAPE increases with larger sample sizes under short-tailed and heavy-tailed conditions, reflecting the greater difficulty in accurately capturing tail behavior. TOSE-ME yields the lowest MAPE under the Gumbel case ($\xi = 0$), while PWM-PP performs consistently well in heavy-tail scenarios. TSOS-based methods, particularly TSOS-LMS and TSOS-LTS, show a stable performance in light-tailed cases but exhibit higher MAPE under short-tailed conditions. These results indicate that while TSOS estimators are robust in parameter estimation, their predictive accuracy under extreme tails may be limited compared with PWM methods.

Under short-tailed conditions ($\xi = -0.8$), the TSOS-LTS estimator yields the lowest MSE and standard deviation across all parameters, along with relatively low MAPE values, indicating strong reliability in estimating rare minimum extremes. For light-tailed distributions ($\xi = 0$), most methods perform comparably; however, PWM-PP and TSOS-ME exhibit superior stability and accuracy, particularly in estimating the shape parameter. In the heavy-tailed case ($\xi = 0.8$), PWM-based methods, especially PWM-PP, achieve the lowest MAPE values; however, TSOS-LTS consistently maintains the highest estimation accuracy with the lowest MSE and standard deviation for the shape parameter.

These results support a context-specific ranking of the six estimation methods. The TSOS-LTS method consistently demonstrates the best overall estimation performance, exhibiting the lowest mean squared errors and standard deviations across large sample sizes and shape parameters. In terms of predictive accuracy, as measured by the mean absolute percentage error (MAPE), PWM-PP yields the most favorable results, particularly under heavy-tailed conditions. When considering both accuracy and robustness, TSOS-LTS and TSOS-LMS show the most consistent performance across varying tail behaviors and sample sizes. Meanwhile, the traditional MLE method serves as a valuable benchmark and performs reasonably well in short-tailed scenarios; however, it tends to be less reliable when estimating extreme values in light- and heavy-tailed distributions.

Compared with the traditional maximum likelihood estimation (MLE) method, the TSOS-LTS approach offers notable advantages. By incorporating order statistics with robust regression techniques, particularly the least trimmed squares (LTS) method, it demonstrates greater resilience to outliers and sample variability. While MLE performs adequately under ideal conditions, its sensitivity to deviations from distributional assumptions, especially in the presence of extreme or skewed observations, limits its robustness in practical applications.

4. Real-Life Example

This section outlines the research methodology used to estimate the parameters of the GEV distribution using five statistical approaches: maximum likelihood estimation (MLE), probability-weighted moments in unbiased estimators (PWM-UEs), probability-weighted Moments in plotting position (PWM-PPs), two-stage order statistics in median estimation (TSOS-ME), two-stage order statistics in least median of squares (TSOS-LMS), and two-stage order statistics in least trimmed of squares (TSOS-LTS). These methods are subsequently applied to the analysis of monthly maximum PM_{2.5} data from various districts in Bangkok.

The PM_{2.5} data used in this study are secondary data obtained from <http://air4thai.pcd.go.th/webV3/#/History> (accessed on 16 July 2025), covering the period from January 2017 to December 2024, a total of 96 months, which serve as the training data. The data were collected from Bang Na station, Phaya Thai station, and Thon Buri station. The data visualization for each station is presented in Figure 2.

All three stations show a clear seasonal pattern, with PM_{2.5} concentrations peaking at around the same time each year, likely during the dry season (December to March), when air pollution typically intensifies due to weather conditions and human activities.

The study separates the period from January 2017 to December 2023, a total of 84 months, as the training data. The testing data are designated from February to December 2024 to compute the return levels. Descriptive statistics of the train data, including the mean, standard deviation, skewness, kurtosis, minimum, and maximum values, and the Kolmogorov–Smirnov (KS) statistic are presented in Table 8.

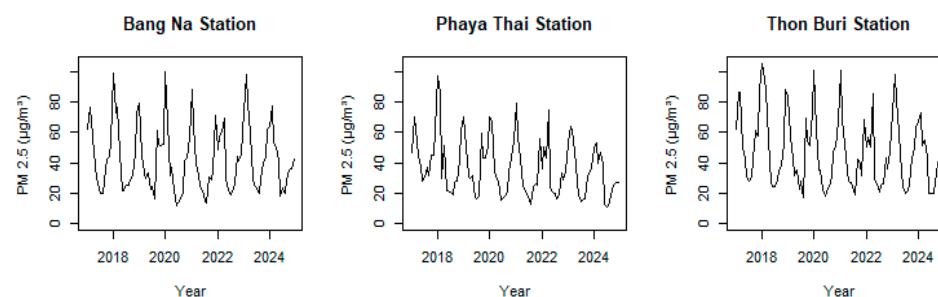
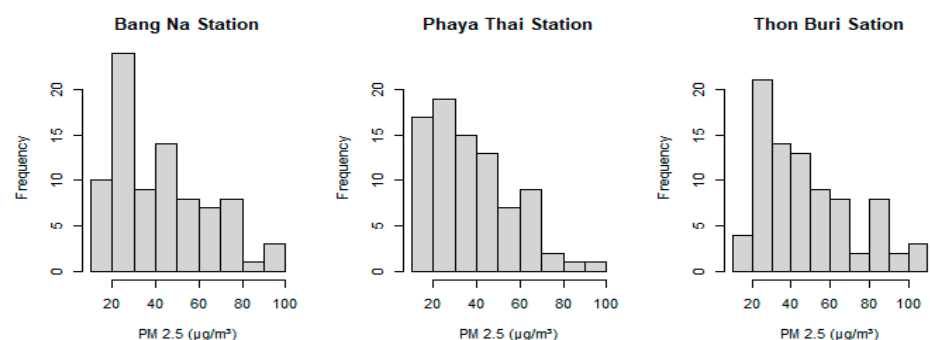


Figure 2. The time series plots of monthly maximum PM_{2.5} concentrations from 2017 to 2024.

Table 8. Descriptive statistics for each station's monthly maximum PM2.5 concentrations.

Descriptive Statistics	Bang Na Station	Phaya Thai Station	Thon Buri Station
Mean	42.640	37.736	47.980
Standard Deviation	21.910	18.894	23.760
Max	100	97	105
Min	12	13	17
Skewness	0.802	0.886	0.755
Kurtosis	−0.118	0.257	−0.443
KS Statistic (<i>p</i> -value)	0.064 (0.814)	0.081 (0.540)	0.088 (0.440)

The mean values indicate that Thon Buri station recorded the highest average PM2.5 concentration ($47.980 \mu\text{g}/\text{m}^3$), followed by Bang Na station ($42.640 \mu\text{g}/\text{m}^3$), and Phaya Thai station ($37.736 \mu\text{g}/\text{m}^3$). Regarding variability, Thon Buri station also exhibited the highest standard deviation ($23.760 \mu\text{g}/\text{m}^3$), indicating greater fluctuation in the monthly maximum values. The maximum recorded PM2.5 concentration was highest at Thon Buri station ($105 \mu\text{g}/\text{m}^3$) and lowest at Phaya Thai station ($97 \mu\text{g}/\text{m}^3$). Minimum values ranged from $12 \mu\text{g}/\text{m}^3$ at Bang Na to $17 \mu\text{g}/\text{m}^3$ at Thon Buri. Skewness values for all three stations were positive, indicating a right-skewed distribution of PM2.5 concentrations, with the highest skewness observed at Phaya Thai station (0.886). The kurtosis values suggest near-normal distributions, although slight deviations are noted: Bang Na and Thon Buri showed negative kurtosis (−0.118 and −0.443, respectively), while Phaya Thai exhibited a slightly leptokurtic distribution (0.257). In addition, Table 8 presents the results of the extreme value (EV) goodness-of-fit test. The obtained *p*-value exceeds 0.05, indicating that the null hypothesis cannot be rejected and confirming that the GEV model provides an adequate fit to the data. The histograms of monthly maximum PM2.5 concentrations at three monitoring stations exhibit right skewness, indicating that extremely high values occur less frequently but significantly affect the overall distribution, as displayed in Figure 3.

**Figure 3.** The histograms of monthly maximum PM2.5 concentrations at three monitoring stations.

The estimation methods included MLE, PWM-UE, PWM-PP, TSOS-ME, TSOS-LMS, and TSOS-LTS. The monthly maximum PM2.5 values of the train data were assessed using location, scale, and shape parameters based on the GEV distribution to estimate the parameters. The return levels are computed from the parameter estimation and compared with the test data and the predicted return levels to evaluate MAPE. The parameter estimates for each parameter, categorized by station, are presented in Table 9.

Table 9. The estimated parameters of GEVD and MAPE based on the return levels.

Method	Parameters	Bang Na Station	Phaya Thai Station	Thon Buri Station
MLE	$\hat{\mu}$	31.352	27.885	35.048
	$\hat{\sigma}$	15.266	12.850	15.750
	$\hat{\xi}$	−0.155	−0.180	−0.233
	MAPE	81.247	151.338	59.706
PWM-UE	$\hat{\mu}$	32.097	28.642	36.462
	$\hat{\sigma}$	16.880	14.394	18.235
	$\hat{\xi}$	−0.045	−0.052	−0.052
	MAPE	77.225	143.972	51.743
PWM-PP	$\hat{\mu}$	32.054	28.605	36.413
	$\hat{\sigma}$	16.807	14.342	18.170
	$\hat{\xi}$	−0.050	−0.056	−0.056
	MAPE	79.063	146.931	54.427
TSOS-ME	$\hat{\mu}$	33.410	28.805	37.715
	$\hat{\sigma}$	13.206	10.484	12.910
	$\hat{\xi}$	0.030	−0.060	0.018
	MAPE	52.426	110.428	25.996
TSOS-LMS	$\hat{\mu}$	32.472	26.259	33.315
	$\hat{\sigma}$	12.803	12.113	10.773
	$\hat{\xi}$	0.0139	0.009	−0.076
	MAPE	49.427	107.104	14.411
TSOS-LTS	$\hat{\mu}$	28.905	27.005	32.421
	$\hat{\sigma}$	11.421	11.197	10.417
	$\hat{\xi}$	−0.072	−0.030	−0.084
	MAPE	37.947	106.635	11.530

Table 9 presents the estimated parameters among the methods considered. The maximum likelihood estimation (MLE) produced moderate parameter estimates but yielded a relatively high MAPE, particularly at Phaya Thai (151.338). The PWM methods, including PWM-UE and PWM-PP, offered slightly better accuracy, with MAPE values generally lower than those from MLE, especially at Thon Buri. The two-stage order statistics estimator (TSOSE) showed notable improvements in forecasting performance. TSOS-ME and TSOS-LMS reduced MAPE considerably at all stations, particularly at Thon Buri, where TSOS-LMS achieved a MAPE of 14.411. The most accurate results were obtained using the TSOS-LTS method, which produced the lowest MAPE values across all stations: 37.947 at Bang Na, 106.635 at Phaya Thai, and only 11.530 at Thon Buri, indicating its robustness and superior forecasting performance. The consistent negative or near-zero shape parameter estimates across most methods suggest a short-tailed distribution for the extreme values of PM2.5. Overall, the results confirm that robust estimators such as TSOS-LTS outperform traditional methods in modeling and forecasting extreme PM2.5 levels in urban areas.

In the descriptive statistics for the monthly maximum PM2.5 data from three monitoring stations in Bangkok—Bang Na, Phaya Thai, and Thon Buri—the Thon Buri station exhibited the highest mean and standard deviation values, while Phaya Thai had the lowest mean and standard deviation values. The skewness values across all stations were positive, indicating right-skewed distributions. The kurtosis values ranged from platykurtic (Thon Buri) to slightly leptokurtic (Phaya Thai), supporting the assumption of non-normality and the appropriateness of using the GEV distribution for modeling the extreme value analy-

sis [33,34]. The extreme value analysis (EVA) provides a rigorous framework for modeling rare events through specialized probability distributions. It enables the reliable estimation of return levels, even within relatively short observational windows. Interestingly, two independent modeling approaches yielded consistent outcomes, suggesting convergence in their predictive performance and reinforcing the robustness of the underlying statistical methods [35].

Histograms of monthly maximum PM_{2.5} values further confirm the positively skewed nature of the data, with a higher frequency observed in lower concentration ranges and long right tails, especially evident at the Thon Buri and Bang Na stations. These distributional characteristics underscore the need for flexible and robust estimation methods, such as TSOS and PWM, in capturing the heavy-tailed nature of air pollution data [18]. The application of the TSOS-LTS estimation method consistently outperformed other methods, yielding the lowest MAPE values at all three stations. Notably, the lowest MAPE was observed at Thon Buri, consistent with its high variability and heavy tail, as shown in Table 4 and the histogram. While traditional and widely used due to its asymptotic properties, MLE produced the highest MAPE at Phaya Thai (151.338), reaffirming its sensitivity to data irregularities and extreme values. PWM-based methods demonstrated a moderate performance but were outperformed by TSOS-based approaches, particularly those incorporating robust techniques such as LMS and LTS, which are known to mitigate the influence of outliers and yield reliable estimates in the presence of heavy-tailed data.

In summary, simulation and real-world analyses consistently demonstrate that the TSOS-LTS estimator delivers the most reliable parameter estimates and forecasting performance for modeling extreme PM_{2.5} concentrations. Its ability to effectively capture tail behavior, while maintaining a low mean squared error (MSE) and mean absolute percentage error (MAPE), underscores its suitability for applications involving environmental data characterized by high variability. These results highlight the importance of employing robust estimation techniques within the GEV framework for air quality modeling and ecological risk assessment [36].

5. Conclusions

This study evaluated the performance of six parameter estimation methods—MLE, PWM-UE, PWM-PP, TSOS-ME, TSOS-LMS, and TSOS-LTS—for the GEV distribution using simulated and real PM_{2.5} data. The simulation results demonstrate that the choice of estimation method has a significant impact on both the estimation accuracy and predictive performance. Among the methods evaluated, TSOS-LTS consistently provides the most accurate and robust parameter estimates across all scenarios, with the lowest MSE values and stable standard deviations. PWM-PP excels in terms of predictive accuracy (MAPE), particularly in distributions with heavy tails. TSOS-ME also performs reliably, particularly in moderate tail conditions and small sample sizes. In contrast, the traditional MLE method is primarily suitable for light-tailed cases and larger samples, but its performance degrades in extreme conditions. When applied to actual monthly maximum PM_{2.5} concentrations from three air quality monitoring stations in Bangkok, the TSOS-LTS method outperformed other approaches by achieving the lowest MAPE values across all locations, with particularly outstanding accuracy at Thon Buri station. The findings further reveal that while traditional methods, such as maximum likelihood estimation (MLE), remain widely adopted, their effectiveness diminishes when applied to real-world datasets that exhibit skewness and heavy tails.

The contribution of this study is to advance parameter estimation in extreme value modeling by highlighting the strengths of robust alternatives, particularly for environmental applications such as PM_{2.5} analysis. The research offers practical insights into method selection based on tail characteristics through a systematic comparison of classical and robust techniques under varied distributional scenarios. In particular, the two-stage order statistics estimator with least trimmed squares (TSOS-LTS) demonstrates substantial improvement in the reliability of return level forecasts—an essential component in air pollution risk assessment. These results have a significant impact on urban air quality monitoring, evidence-based environmental policy formulation [37], and public health preparedness in regions that are increasingly affected by severe air pollution events [38].

As a future research direction, it is recommended to investigate the performance of the estimation methods under varying sample sizes and levels of data contamination, such as the presence of outliers or missing values. This would enable a more comprehensive evaluation of the robustness and adaptability of each method in real-world applications where data imperfections are prevalent.

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Abbreviations

The following abbreviations are used in this manuscript:

MLE	Maximum Likelihood Estimation
PWM	Probability-Weighted Moment
PWM-UE	Probability-Weighted Moment in Unbiased Estimators
PWM-PP	Probability-Weighted Moment in Plotting Position
TSOS	Two-Stage Order Statistics
TSOS-ME	Two-Stage Order Statistics in Median Estimation
TSOS-LMS	Two-Stage Order Statistics in Least Median of Squares
TSOS-LTS	Two-Stage Order Statistics in Least Trimmed of Squares

References

1. Tabari, H. Extreme value analysis dilemma for climate change impact assessment on global flood and extreme precipitation. *J. Hydrol.* **2021**, *593*, 125932. [CrossRef]
2. Chukwudum, Q.C.; Nadarajah, S. Bivariate extreme value analysis of rainfall and temperature in Nigeria. *Environ. Model. Assess.* **2021**, *27*, 343–362. [CrossRef]
3. Pryor, S.C.; Barthelmie, R.J. A global assessment of extreme wind speeds for wind energy applications. *Nat. Energy* **2021**, *6*, 268–276. [CrossRef]
4. Candis, C.; Herrera, R. An empirical review of dynamic extreme value models for forecasting value at risk, expected shortfall and expectile. *J. Empir. Finance* **2024**, *77*, 101488. [CrossRef]
5. Abdulali, B.A.A.; Bakar, M.A.B.; Ibrahim, K.; Ariff, N.M. Extreme Value Distributions: An Overview of Estimation and Simulation. *J. Probab. Stat.* **2022**, *2022*, 5449751. [CrossRef]
6. Hossain, I.; Imteaz, M.A.; Khastagir, A. Effects of estimation techniques on generalised extreme value distribution (GEVD) parameters and their spatio-temporal variations. *Stoch. Environ. Res. Risk Assess* **2021**, *35*, 2303–2312. [CrossRef]
7. Rai, S.; Hoffman, A.; Lahiri, S.; Nychka, D.W.; Sain, S.R.; Bandyopadhyay, S. Fast parameter estimation of generalized extreme value distribution using neural networks. *Environmetrics* **2024**, *35*, e2845. [CrossRef]

8. Greenwood, J.A.; Landwehr, J.M.; Matalas, N.C.; Wallis, J.R. Probability weighted moments: Definition and relation to parameters of several distributions expressible in inverse form. *Water Resour. Res.* **1979**, *15*, 1049–1054. [\[CrossRef\]](#)
9. Castillo, E.; Hadi, A. Parameter and quantile estimation for the generalized extreme-value distribution. *Environmetrics* **1994**, *5*, 417–432. [\[CrossRef\]](#)
10. Caeiro, F.; Gomes, M.I.; Vandewalle, B. Semi-parametric probability-weighted moments estimation revisited. *Methodol. Comput. Appl. Probab.* **2014**, *16*, 1–29. [\[CrossRef\]](#)
11. Helu, A. The principle of maximum entropy and the probability-weighted moments for estimating the parameters of the Kumaraswamy distribution. *PLoS ONE* **2022**, *17*, e0268602. [\[CrossRef\]](#)
12. Caeiro, F.; Mateus, A. A new class of generalized probability-weighted moment estimators for the Pareto distribution. *Mathematics* **2023**, *11*, 1076. [\[CrossRef\]](#)
13. Chen, F.; Sang, Y.-F.; Xie, P.; Wu, L.; Hua, J.; Singh, V.P. Coupling higher-order probability weighted moments with norming constants method for non-stationary annual maximum flood frequency analysis. *J. Hydrol.* **2024**, *641*, 131832. [\[CrossRef\]](#)
14. Roslan, R.A.; Su Na, C.; Gabda, D. Parameter Estimations of the Generalized Extreme Value Distributions for Small Sample Size. *Math. Stat.* **2020**, *8*, 47–51. [\[CrossRef\]](#)
15. Diebolt, J.; Guillou, A.; Naveau, P.; Ribereau, P. Improving Probability-Weighted Moment Methods for the Generalized Extreme Value Distribution. *Revstat Stat. J.* **2008**, *6*, 199–220.
16. Caeiro, F.; Mateus, A.; Gomes, D.P. Tail inference with probability weighted moments. *Math. Methods Appl. Sci.* **2025**, *10*, 10922. [\[CrossRef\]](#)
17. Rydén, J. Estimation of return levels with long return periods for extreme sea levels by the Average Conditional Exceedance rate method. *GeoHazards* **2024**, *5*, 166–175. [\[CrossRef\]](#)
18. Tanprayoon, E.; Tonggumnead, U.; Aryuyuen, S. A new extension of generalized extreme value distribution: Extreme value analysis and return level estimation of the rainfall data. *Trends Sci.* **2023**, *20*, 4034. [\[CrossRef\]](#)
19. Rydén, J. Estimation of return levels with long return periods for extreme sea levels in a time-varying framework. *Environ. Syst. Decis.* **2024**, *44*, 1019–1028. [\[CrossRef\]](#)
20. Milojevic, T.; Blanchet, J.; Lehning, M. Determining return levels of extreme daily precipitation, reservoir inflow, and dry spells. *Front. Water* **2023**, *5*, 1141786. [\[CrossRef\]](#)
21. Klinjan, K.; Sottiwat, T.; Aryuyuen, S. Extreme value analysis with new generalized extreme value distributions: A case study for risk analysis on PM2.5 and PM10 in Pathum Thani, Thailand. *Commun. Math. Biol. Neurosci.* **2024**, *2024*, 100. [\[CrossRef\]](#)
22. Deng, L.; Liu, X. Features of extreme PM 2.5 pollution and its influencing factors: Evidence from China. *Environ. Monit. Assess.* **2024**, *196*, 272371365. [\[CrossRef\]](#)
23. Zhao, X.; Wang, C.; Liu, Y.; Wei, C.; Song, J. A Robust Estimator for the Generalized Extreme Value Distribution with Application to PM2.5 Concentration in China. *Atmosphere* **2023**, *14*, 1197.
24. Bali, T.G. The generalized extreme value distribution. *Econ. Lett.* **2003**, *79*, 423–427. [\[CrossRef\]](#)
25. Fisher, R.A.; Tippett, L.H.C. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Math. Proc. Camb. Philos. Soc.* **1928**, *24*, 180–190. [\[CrossRef\]](#)
26. Gnedenko, B.V. Sur la distribution limite du terme maximum d’une série aléatoire. *Ann. Math.* **1943**, *44*, 423–453. [\[CrossRef\]](#)
27. Beirlant, J.; Goegebeur, Y.; Segers, J.; Teugels, J. *Statistics of Extremes: Theory and Applications*; Wiley: Chichester, UK, 2004; pp. 11–15.
28. Coles, S. *An Introduction to Statistical Modeling of Extreme Values*; Springer: London, UK, 2001; pp. 23–24.
29. Dennis, J.E., Jr.; Schnabel, R.B. *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*; SIAM: Philadelphia, PA, USA, 1996; pp. 17–19.
30. Nocedal, J.; Wright, S.J. *Numerical Optimization*, 2nd ed.; Springer: New York, NY, USA, 2006; pp. 6–10.
31. Hosking, J.R.M.; Wallis, J.R. Parameter and quantile estimation for the Generalized Extreme Value Distribution. *Technometrics* **1987**, *29*, 339–349. [\[CrossRef\]](#)
32. Hosking, J.R.M.; Wallis, J.R.; Wood, E.F. Estimation of the generalized extreme-value distribution by the method of probability-weighted moments. *Technometrics* **1985**, *27*, 251–261. [\[CrossRef\]](#)
33. Zhang, L.; Gong, P.; Zhao, X.; Liu, Y. Extreme value analysis of PM2.5 concentrations in China. *Atmos. Pollut. Res.* **2017**, *8*, 475–483.
34. Warsona, W.; Antonio, Y.; Yuwono, S.B.; Kurniasari, D.; Suroso, E.; Yushananta, P.; Usman, M.; Hadi, S. Modeling generalized statistical distributions of PM2.5 concentrations during the COVID-19 pandemic in Jakarta, Indonesia. *Decis. Sci. Lett.* **2021**, *10*, 393–400. [\[CrossRef\]](#)
35. Martins, L.D.; Wikuats, C.F.H.; Capucim, M.N.; Almeida, D.S.; Costa, S.C.; Albuquerque, T.; Carvalho, V.S.B.; Freitas, E.D.; Andrade, M.F.; Martins, J.A. Extreme value analysis of air pollution data and their comparison between two large urban regions of South America. *Weather Clim. Extrem.* **2017**, *18*, 44–54. [\[CrossRef\]](#)
36. Hyndman, R.J.; Koehler, A.B. Another look at measures of forecast accuracy. *Int. J. Forecast.* **2006**, *22*, 679–688. [\[CrossRef\]](#)

37. Aguirre-Salado, A.I.; Venancio-Guzmán, S.; Aguirre-Salado, C.A.; Santiago-Santos, A. A novel tree ensemble model to approximate the generalized extreme value distribution parameters of the PM2.5 maxima in the Mexico City metropolitan area. *Mathematics* **2022**, *10*, 2056. [[CrossRef](#)]
38. Wook, O.J.; Jin, L.T. Regional analysis of extreme values by particulate Matter (PM2.5) concentration in Seoul, Korea. *J. Korean Soc. Qual. Manag.* **2019**, *47*, 47–57.

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