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Quadratic BSDEs with Singular Generators and Unbounded Terminal Conditions: Theory and Applications

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Abstract

We investigate a class of quadratic backward stochastic differential equations (BSDEs) with generators that are singular in y. First, we establish the existence of solutions and a comparison theorem, thereby extending the existing results in the literature. Furthermore, we analyze the stability properties, derive the Feynman–Kac formula, and prove the uniqueness of viscosity solutions for the corresponding singular semi-linear partial differential equations (PDEs). Finally, we demonstrate applications in the context of robust control linked to stochastic differential utility and the certainty equivalent based on g-expectation. In these applications, the quadratic coefficients in the generators, respectively, quantify ambiguity aversion and absolute risk aversion.

Keywords: quadratic backward stochastic differential equation; singular generators; unbounded terminal conditions; viscosity solution; comparison theorem; stochastic differential utility

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1. Introduction

This paper focuses on a one-dimensional backward stochastic differential equation (BSDE):

$$Y_t = \xi + \int_t^T f(u, Y_u, Z_u)du - \int_t^T Z_u dW_u, \quad t \in [0, T], \tag{1}$$

where W denotes a standard d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, \mathbb{P})$. Here, $(\mathcal{F}_s)_{s\in[0,T]}$ is the $\mathbb{P}$-completion of the filtration generated by W. The terminal condition ξ is an $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable, and the generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$ is a progressively measurable process. Nonlinear BSDEs were first pioneered under the Lipschitz condition by [1]. BSDEs with generators exhibiting quadratic growth in the variable z have been extensively researched, with significant contributions from [2–6] and others since [7] first investigated the case of bounded terminal conditions. This paper characterizes a specific category of BSDEs with the form of generators given by $g(y)|z|^2$, which, to our knowledge, first appeared in [8]. The existence and uniqueness results for the case where g is globally integrable on $\mathbb{R}$ were demonstrated by [9], through the employment of Itô-Krylov's formula and a "domination method" (see Lemma 1) derived from the existence result for reflected BSDEs obtained in [10]. Another intriguing case is $f(y, z) = |z|^2/y$, meaning that the generator f is singular at $y = 0$. In this instance, ref. [11] assumed that the generator f satisfied

$$0 \leq f(s, \omega, y, z) \leq a_s + b_s y + \gamma_s z + \frac{\delta}{2y}|z|^2, (s, \omega, y, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d}, \quad (2)$$

for some processes a, b, γ and a constant δ. By utilizing the domination method, they established the existence of solutions in $\mathcal{S}^p \times \mathcal{L}^2$. Furthermore, they proved a uniqueness result for bounded solutions using techniques from convex duality. In this paper, we prove that solutions exist in $\mathcal{S}^p \times \mathcal{M}^q$, and we establish a comparison theorem for L^p solutions through θ-techniques (see [3]). Moreover, our approach allows for a straightforward extension of these results to more generalized BSDEs with generators satisfying

$$0 \leq f(s, \omega, y, z) \leq a_s + b_s\phi(y) + \gamma_s|z| + \frac{|z|^2}{2\psi(y)}, (s,\omega,y,z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d},$$

under appropriate assumptions. Recent advances in this field include works by [12,13]. Ref. [12] pioneered the study of geometric BSDEs and included an investigation into a class of BSDEs with generators that satisfy the following structural condition:

$$0 \leq f(s, \omega, y, z) \leq a_s y + b_s y|\ln(y)| + \lambda_s z + \frac{\delta}{y}|z|^2, (s,\omega,y,z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d},$$

where the processes a, b, and λ must be bounded for uniqueness and stability. In contrast, our methodology employs a series of auxiliary functions (see Remark 2) leading to a non-trivial generalization of the related results obtained in [12], as shown in Remark 3. Furthermore, ref. [13] investigated BSDEs and reflected BSDEs with generators exhibiting one-sided growth in y and general quadratic growth in z. However, the assumptions in [13] cannot be directly applied to the equations presented in this study, and we provide further conclusions about the integrability of the solutions. Additionally, we derive the stability property for L^p solutions and the Feynman–Kac formula within our framework, and we verify the uniqueness of viscosity solutions to the related singular semi-linear quadratic PDEs.

More importantly, this class of quadratic BSDEs with singularities and related partial differential equations (PDEs) has significant applications in fields such as physics, quantitative finance, and biology (see [11,14–16]). For instance, in finance the conditional generalized entropic risk measure (see [17]) can be expressed via Y, the value process of a solution to such a BSDE. Ref. [11] illustrated that Y can be ordinally equivalent to stochastic differential utility of the Epstein–Zin type with relative risk aversion represented by $0 \leq \delta \neq 1$ in (2). In Section 5.1, we demonstrate that Y can represent stochastic differential utility concerning robustness, while $1/\psi(y)$, the coefficient of $|z|^2/2$, quantifies the level of ambiguity aversion. In Example 2, we show that Y can depict the certainty equivalent of the terminal value based on g-expectation, while $1/\psi(y)$ represents the absolute risk aversion coefficient.

The structure of this paper is as follows. Sections 2 and 3 introduce the notations, present the existence results, and establish the comparison theorem. In Section 4, we prove a stability result for the L^p solutions, derive the Feynman–Kac formula within our theoretical framework, and verify the uniqueness of viscosity solutions to the associated PDE. Having established the theoretical foundations, Section 5 presents an application linking robust control to stochastic differential utility, slightly generalizing the results of [18,19], while including an example of the certainty equivalent based on g-expectation.

2. Notations and Existence Results

This section establishes the existence results for a class of quadratic BSDEs with singular generators. First, we introduce the notations used in this paper. We say a process or random variable satisfies some property if this property holds except on a $\mathbb{P}$-null subset of Ω, and, thus, we occasionally omit the notation $\mathbb{P}$-a. s.. We define $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$ as the conditional mathematical expectation with respect to $\mathcal{F}_t$. The set composed of stopping times τ satisfying $0 \leq \tau \leq T$ is represented by $\mathcal{T}_{0,T}$. If N is an adapted and càdlàg process, we define $N^* := \sup_{t\in[0,T]}|N_t|$. We recall that class (D) comprises progressively measurable processes X satisfying that $\{X_\tau \mid \tau \in \mathcal{T}_{0,T}\}$ is uniformly integrable. Occasionally, we denote the stochastic integral $\int_0^{\cdot} Z_s dW_s$ by $Z \circ W$ and denote by $\mathscr{E}(X)$ the stochastic exponential of a one-dimensional local martingale X. For any real number $p \geq 1$ and subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R}^n$, let us define the following spaces and notations.

$\mathcal{C}^p(U)$: the space of functions from U to $\mathbb{R}$ having a continuous p-th derivative.

$L^p_{loc}(U)$: the space of locally L^p integrable functions from U to $\mathbb{R}$.

$\mathcal{W}^2_{p,loc}(U)$: the Sobolev space of functions $h : U \mapsto \mathbb{R}$, such that h and its generalized derivatives h' and h'' belong to $L^p_{loc}(U)$.

$\mathcal{S} := \mathcal{S}(\mathbb{R})$: the space of adapted and continuous processes valued in $\mathbb{R}$.

$\mathbb{L}^p := \mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$: the space of random variables η fulfilling η takes values in $\mathbb{R}$ and $\mathbb{E}[|\eta|^p] < +\infty$, with η measurable with respect to $\mathcal{F}_T$.

$\mathcal{S}^\infty := \mathcal{S}^\infty(\mathbb{R})$: the space of bounded processes in $\mathcal{S}$.

$\mathcal{S}^p := \mathcal{S}^p(\mathbb{R})$: the space of all processes $Y \in \mathcal{S}$, such that $\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{0\leq t\leq T}|Y_t|^p]\right)^{\frac{1}{p}} < +\infty$.

$\mathcal{L}^2 := \mathcal{L}^2(\mathbb{R}^{1\times d})$: the space of $\mathbb{R}^{1\times d}$-valued processes N satisfying N is progressively measurable and $\mathbb{P}$-a. s., $\int_0^T |N_s|^2 ds < \infty$.

$\mathcal{M}^p := \mathcal{M}^p(\mathbb{R}^{1\times d})$: the space of progressively measurable processes G fulfilling that G is valued in $\mathbb{R}^{1\times d}$ and $\|G\|_{\mathcal{M}^p} := \left(\mathbb{E}\left[\left(\int_0^T |G_s|^2 ds\right)^{p/2}\right]\right)^{1/p} < +\infty$.

$\mathcal{K}$: the set of $\mathbb{R}$-valued processes K satisfying $K_0 = 0$, K is continuous, non-decreasing, progressively measurable, and $\mathbb{P}$-a. s., $K_T < +\infty$.

BMO: the space of all uniformly integrable martingales N satisfying

$$\|N\|_{BMO} := \sup_{\tau\in\mathcal{T}_{0,T}} \left\|\mathbb{E}\left[|N_T - N_\tau|\middle|\mathcal{F}_\tau\right]\right\|_\infty < \infty.$$

For any function $u : V \mapsto \mathbb{R}$, we define $D^{2,\pm}_V u$ as a mapping from V to the subsets of $\mathbb{R}^n \times S(n)$, where $S(n)$ is the set of real symmetric $n \times n$ matrices. For $\hat{x} \in V$, we define the second-order "superjet" (see [20]) of u at $\hat{x}$ as

$$D^{2,+}_V u(\hat{x}) := \{(p, K) \in \mathbb{R}^n \times S(n) \mid$$
$$u(x) \leq u(\hat{x}) + p^T(x - \hat{x}) + \frac{1}{2}(x - \hat{x})^T K(x - \hat{x}) + o(|x - \hat{x}|^2) \text{ as } V \ni x \to \hat{x}\},$$

the second-order "subjet" of u at $\hat{x}$ as

$$D^{2,-}_V u(\hat{x}) := \{(p, K) \in \mathbb{R}^n \times S(n) \mid$$
$$u(x) \geq u(\hat{x}) + p^T(x - \hat{x}) + \frac{1}{2}(x - \hat{x})^T K(x - \hat{x}) + o(|x - \hat{x}|^2) \text{ as } V \ni x \to \hat{x}\},$$

and their closures as

$$\bar{D}_V^{2,\pm}u(\hat{x}) := \Big\{(p,\, K) \in \mathbb{R}^n \times S(n) \mid \exists x_n \in V \text{ and } (p_n,\, K_n) \in D_V^{2,\pm}u(x_n),$$
$$\text{such that } (x_n,\, u(x_n),\, p_n,\, K_n) \to (\hat{x},\, u(\hat{x}),\, p,\, K) \text{ as } n \to \infty\}.$$

We sometimes describe the BSDE with generator f and terminal condition ξ as BSDE(ξ, f), rather than (1), for notational simplicity. A solution of BSDE(ξ, f) is defined as a pair of processes $(Y,\, Z) := \{(Y_t,\, Z_t)\}_{t\in[0,T]} \in \mathcal{S} \times \mathcal{L}^2$ fulfilling that $\mathbb{P}$-a. s., (1) holds and $\int_0^T |f(u,\, Y_u,\, Z_u)|du < \infty$. Furthermore, we designate $(Y,\, Z) := \{(Y_t,\, Z_t)\}_{t\in[0,T]}$ as an L^p solution, provided that $(Y,\, Z) \in \mathcal{S}^p \times \mathcal{M}^q$ for some $p > 1$, $q > 1$. Now, we establish the existence theorem for BSDE (1).

Theorem 1. *Assume BSDE (1) satisfies that $\xi > 0$, f is continuous in $(y,\, z)$ for each fixed $(s,\, \omega)$, and for any $(s,\, \omega,\, y,\, z) \in [0,\, T] \times \Omega \times (0,\, +\infty) \times \mathbb{R}^{1\times d}$,*

$$0 \le f(s,\, \omega,\, y,\, z) \le a_s + b_s y + \gamma_s |z| + \frac{\delta}{2y}|z|^2, \tag{3}$$

where $\delta \ge 0$ is a constant, and a, b, and γ are non-negative and progressively measurable processes. Then, BSDE (1) admits a solution $(Y,\, Z)$, such that $Y \in \mathcal{S}^{2p(1+\delta)}$ for some $p > 1$ if

$$\mathbb{E}\left[\exp\left\{\frac{p}{p-1}\int_0^T \gamma_s^2 ds + 2p(\delta+1)\int_0^T b_s ds\right\}\left(1 + \xi^{1+\delta} + \int_0^T a_s^{\delta+1} ds\right)^{2p}\right] < +\infty.$$

Moreover, $Z \in \mathcal{M}^{2p}$ if $0 < \delta \ne 1$.

To prove Theorem 1, we recall the domination method, which was first employed by [21] and further developed in [9,11,22].

Lemma 1 ([22], Lemma 3.1)**.** *We say that the BSDE(ξ, f) satisfies the domination conditions if there exist two BSDEs, (ξ^1, f^1) and (ξ^2, f^2), such that*

(1) $\xi^1 \le \xi \le \xi^2$;
(2) *BSDE(ξ^1, f^1) and BSDE(ξ^2, f^2) have solutions $(Y^1,\, Z^1)$ and $(Y^2,\, Z^2)$ in $\mathcal{S} \times \mathcal{L}^2$, respectively, such that $Y_s^1 \le Y_s^2$, $\forall s \in [0,\, T]$. For every $(s,\, \omega) \in [0,\, T] \times \Omega$, $y \in [Y_s^1(\omega),\, Y_s^2(\omega)]$, and $z \in \mathbb{R}^d$, the following inequalities hold:*

$$f^1(s,\, \omega,\, y,\, z) \le f(s,\, \omega,\, y,\, z) \le f^2(s,\, \omega,\, y,\, z),$$

and

$$|f(s,\, \omega,\, y,\, z)| \le \eta_s(\omega) + \frac{\zeta_s(\omega)}{2}|z|^2,$$

where η and ζ are two $\mathcal{F}_t$-adapted processes satisfying that ζ is continuous and $\forall\omega$, $\int_0^T |\eta_t(\omega)|dt < +\infty$.

Assume that the BSDE(ξ, f) satisfies the domination conditions and f is continuous in $(y,\, z)$ for each fixed $(s,\, \omega)$. Then, the BSDE(ξ, f) admits at least one solution $(Y,\, Z)$, such that

$$Y_s^1 \le Y_s \le Y_s^2,\ \forall s \in [0,\, T], \tag{4}$$

and a maximal and a minimal solution exist among all the solutions satisfying (4).

The proof of Theorem 1 relies on the following technical lemma:

Lemma 2. *Let $\zeta > 0$ be an $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable, and let γ be a non-negative and progressively measurable process. Then, the following BSDE,*

$$Y_t = \zeta + \int_t^T \gamma_s |Z_s| ds - \int_t^T Z_s dW_s, \tag{5}$$

admits a solution (Y, Z) satisfying for some $p > 1$,

$$\forall t \in [0, T], (\mathbb{E}[\zeta \mid \mathcal{F}_t])^p \leq Y_t^p \leq \mathbb{E}\left[\exp\left\{\frac{p}{2(p-1)} \int_t^T \gamma_s^2 ds\right\} \zeta^p \mid \mathcal{F}_t\right],$$

if

$$\mathbb{E}\left[\exp\left\{\frac{p}{2(p-1)} \int_0^T \gamma_s^2 ds\right\} \zeta^p\right] < +\infty.$$

Proof. Consider the following BSDEs:

$$\bar{Y}_t = \zeta^p + \int_t^T \left(\gamma_s |\bar{Z}_s| - \frac{p-1}{2p} \frac{|\bar{Z}_s|^2}{\bar{Y}_s}\right) ds - \int_t^T \bar{Z}_s dW_s \tag{6}$$

and

$$y_t = \zeta^p - \int_t^T \frac{p-1}{2p} \frac{|z_s|^2}{y_s} ds - \int_t^T z_s dW_s. \tag{7}$$

Since the BSDE (7) has a solution (y_t, z_t) denoted by $y_t = (\mathbb{E}[\zeta \mid \mathcal{F}_t])^p$, we infer from Young's inequality and Lemma 1 that BSDE (6) admits a solution $(\bar{Y}, \bar{Z})$ satisfying $\bar{Y}_t \geq (\mathbb{E}[\zeta \mid \mathcal{F}_t])^p$ if the BSDE

$$Y'_t = \zeta^p + \int_t^T \frac{p\gamma_s^2}{2(p-1)} Y'_s ds - \int_t^T Z'_s dW_s \tag{8}$$

admits a solution satisfying $Y'_t \geq y_t$. It is clear from the integrability assumption that (8) admits a solution (Y', Z') given by

$$Y'_t = \mathbb{E}\left[\exp\left\{\frac{p}{2(p-1)} \int_t^T \gamma_s^2 ds\right\} \zeta^p \mid \mathcal{F}_t\right] \geq y_t.$$

Therefore, BSDE (6) admits a solution $(\bar{Y}, \bar{Z})$ satisfying $\bar{Y}_t \geq (\mathbb{E}[\zeta \mid \mathcal{F}_t])^p$. Consequently, BSDE (5) admits a solution (Y, Z) given by

$$(Y'_t)^{\frac{1}{p}} \geq Y_t = (\bar{Y}_t)^{\frac{1}{p}} \geq \mathbb{E}[\zeta \mid \mathcal{F}_t],$$

which completes the proof. □

Proof of Theorem 1. Without loss of generality, we assume $\delta > 0$. Consider the BSDEs

$$Y_t^U = \xi + \int_t^T a_s + b_s Y_s^U + \gamma_s |Z_s^U| + \frac{\delta}{2Y_s^U} |Z_s^U|^2 ds - \int_t^T Z_s^U dW_s \tag{9}$$

and

$$Y_t^L = \xi - \int_t^T Z_s^L dW_s.$$

Since ξ is integrable, the second BSDE has a solution (Y^L, Z^L) denoted by $Y_t^L = \mathbb{E}[\xi \mid \mathcal{F}_t]$ according to Proposition 1.1 in [22]. Using Lemma 1 (with $Y_t^2 = Y_t^U$ and $Y_t^1 = Y_t^L$), we infer that BSDE (1) has a solution (Y, Z) satisfying $Y_t \geq \mathbb{E}[\xi \mid \mathcal{F}_t]$ if the BSDE (9) admits a solution (Y^U, Z^U) satisfying $Y_t^U \geq Y_t^L$.

Define an invertible function

$$u(y) := \frac{y^{1+\delta}}{1+\delta}, \quad y > 0.$$

Then, $u(\cdot)$, $u^{-1}(\cdot) \in \mathcal{C}^2((0, +\infty))$. By Itô's formula and Lemma 1, the BSDE (9) admits a solution (Y_t^U, Z_t^U) satisfying $Y_t^U \geq Y_t^L$ if and only if

$$Y_t^0 = u(\xi) + \int_t^T \left(a_s \left((\delta+1) Y_s^0 \right)^{\frac{\delta}{\delta+1}} + b_s(\delta+1) Y_s^0 + \gamma_s |Z_s^0| \right) ds - \int_t^T Z_s^0 dW_s \tag{10}$$

admits a solution (Y_t^0, Z_t^0), such that $Y_t^0 \geq u(Y_t^L)$. So far, the proof is similar to that in [11]. The results in [23] prove that Equation (10) has a solution (Y^0, Z^0), such that $Y_t^0 \geq u(Y_t^L)$, while we continue to use Lemma 1. From Young's inequality, Equation (10) has a solution (Y^0, Z^0), such that $Y_t^0 \geq u(Y_t^L)$ if

$$Y_t^1 = u(\xi) + \int_t^T \left(\frac{a_s^{\delta+1}}{\delta+1} + (\delta + b_s(\delta+1)) Y_s^1 + \gamma_s |Z_s^1| \right) ds - \int_t^T Z_s^1 dW_s \tag{11}$$

admits a solution (Y^1, Z^1), such that $Y_t^1 \geq u(Y_t^L)$. By using Itô's formula and Lemma 1 again, this holds if

$$Y_t^2 = e^{\delta T + (\delta+1)\int_0^T b_s ds} \left(u(\xi) + \int_0^T \frac{a_s^{\delta+1}}{\delta+1} ds \right) + \int_t^T \gamma_s |Z_s^2| ds - \int_t^T Z_s^2 dW_s \tag{12}$$

admits a solution (Y^2, Z^2), such that $Y_t^2 \geq e^{\delta t + (\delta+1)\int_0^t b_s ds} \left(u(\mathbb{E}[\xi \mid \mathcal{F}_t]) + \int_0^t \frac{a_s^{\delta+1}}{\delta+1} ds \right)$, which holds if (12) admits a solution (Y^2, Z^2), such that $Y_t^2 \geq \mathbb{E}\left[Y_T^2 \mid \mathcal{F}_t \right]$ according to Jensen's inequality.

From Lemma 2 and the integrability assumption, (12) admits a solution (Y^2, Z^2), such that

$$\mathbb{E}\left[Y_T^2 \mid \mathcal{F}_t \right] \leq Y_t^2 \leq \left(\mathbb{E}\left[\exp\left\{ \frac{p}{2(p-1)} \int_t^T \gamma_s^2 ds \right\} \left(Y_T^2 \right)^p \mid \mathcal{F}_t \right] \right)^{\frac{1}{p}}.$$

Going back to BSDE (1), we conclude that BSDE (1) has a solution (Y, Z), such that

$$\mathbb{E}[\xi \mid \mathcal{F}_t] \leq Y_t \leq u^{-1}\left(Y_t^2 \right).$$

By Doob's L^p inequality and Jensen's inequality, we have $Y \in \mathcal{S}^{2p(1+\delta)}$.

Now, we prove $Z \in \mathcal{M}^{2p}$. Assume that $\delta > 1$. We use C to denote a generic positive constant throughout the proof and subsequent sections, which may change from line to line. We define a function:

$$v(y) := \begin{cases} \frac{1}{\delta-1} \left(\frac{y^{1+\delta}-1}{\delta+1} - \frac{y^2-1}{2} \right), & y > 1, \\ \frac{y^2-1}{2} - (y-1), & 0 < y \leq 1. \end{cases}$$

It is clear that $v(\cdot) \in \mathcal{C}^2((0, +\infty))$. Applying Itô's formula to $v(Y_t)$ derives the following:

$$v(Y_t) = v(Y_0) + \int_0^t \left(v'(Y_r)(-f(r, Y_r, Z_r)) + \frac{1}{2} v''(Y_r) |Z_r|^2 \right) dr + \int_0^t v'(Y_r) Z_r dW_r.$$

By the Cauchy–Schwarz inequality and the BDG inequality, we have

$$\mathbb{E}\left[\left(\int_0^T |Z_u|^2 du\right)^p\right] \leq C\mathbb{E}\left[1+\sup_{0\leq s\leq T}(Y_s)^{2p(1+\delta)}+\left(\int_0^T (a_s+b_s)ds\right)^{2p}+\left(\int_0^T \gamma_t^2 dt\right)^{p(1+\delta)}\right].$$

According to the integrability assumption, $Z \in \mathcal{M}^{2p}$. The case of $0 < \delta < 1$ can be proved similarly by taking $v(y) = \frac{y^2}{2(1-\delta)}$ for $y > 0$. This completes the proof. □

Remark 1. *We give an a priori estimate for solutions to the BSDE (1) according to the proof of Theorem 1. Assume the BSDE (1) satisfies that $\xi > 0$, f is continuous in (y, z) for each fixed (s, ω), and for any $(s, \omega, y, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d}$,*

$$0 \leq f(s, \omega, y, z) \leq a_s + b_s y + \gamma_s |z| + \frac{\delta}{2y}|z|^2,$$

where $\delta \geq 0$ is a constant, and a, b, and γ are non-negative and progressively measurable processes. For notational simplicity, we let (for $p > 1$)

$$\bar{A}_t := \exp\left\{\frac{p}{2(p-1)}\int_0^t \gamma_s^2 ds + p\delta t + p(\delta+1)\int_0^t b_s ds\right\}.$$

If $(y, z) \in \mathcal{S} \times \mathcal{L}^2$ is a solution to BSDE (1), such that $y_t > 0$, $t \in [0, T]$ and

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \bar{A}_t\left(y_t^{1+\delta} + \int_0^t a_s^{\delta+1} ds\right)^p\right] < \infty,$$

then we have

$$\mathbb{E}_t[\xi] \leq y_t \leq \left[\left(\mathbb{E}_t\left[\frac{\bar{A}_T}{\bar{A}_t}\left(\xi^{1+\delta} + \int_0^T a_s^{\delta+1} ds\right)^p\right]\right)^{\frac{1}{p}} - \int_0^t a_s^{\delta+1} ds\right]^{\frac{1}{1+\delta}} := \bar{y}_t(\xi). \tag{13}$$

This result can be directly verified through Itô's formula.

Remark 2. *We extend Theorem 1 to a more generalized BSDE (1) with the generator satisfying that f is continuous in (y, z) for each fixed (s, ω), and*

$$0 \leq f(s, \omega, y, z) \leq a_s + b_s\phi(y) + \gamma_s|z| + \frac{|z|^2}{2\psi(y)},\ (s, \omega, y, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d}, \tag{14}$$

where a, b, and γ are non-negative and progressively measurable processes with $a > 0$ or $b > 0$; $\phi : [0, +\infty) \mapsto [0, +\infty)$ is convex and increasing and has a continuous first derivative on $(0, +\infty)$; $\phi(x) > 0$, $\forall x > 0$. $\psi : [0, +\infty) \mapsto [0, +\infty)$ satisfies $\psi(0) = 0$; ϕ/ψ is increasing, and $1/\psi(x)$ is continuous on $(0, +\infty)$. For example, one can take the following functions, ϕ and ψ:

- $\phi(y) = \psi(y) = \exp(y) - 1$, $y > 0$; $\phi(0) = \psi(0) = 0$.
- $\phi(y) = \psi(y) = y^2$, $y \geq 0$.
- $\phi(y) = y$, $\psi(y) = \sqrt{y}$, $y \geq 0$.

To present our results, we define auxiliary functions:

$$\hat{u}(y) := \int_1^y \exp\left(\int_1^x \frac{dr}{\psi(r)}\right) dx,\ y > 0.$$

$$\tilde{u}(y) := \int_1^y \exp\left(\int_1^x \frac{\mathbb{1}_{r>0}}{\psi(r)}\, dr\right) dx,\ y > 0.$$

$$\hat{H}(y) := \int_2^{2+y} \frac{dr}{\phi(\hat{u}^{-1}(r))\hat{u}'(\hat{u}^{-1}(r))},\ y > -2.$$

$$K(x) := \int_0^x \exp\left(-\int_1^z \frac{\mathbb{1}_{r>0}}{\psi(r)}\,dr\right) dz,\ x \geq 0,$$

$$\hat{v}(y) := \int_0^y K(x) \exp\left(\int_1^x \frac{\mathbb{1}_{r>0}}{\psi(r)}\,dr\right) dx,\ y > 0.$$

For $p > 1$, $0 \leq s \leq T$ and $y \in (0, +\infty)$, we define

$$\Phi(s, y; a., b., \gamma.) := \exp\left\{\frac{p}{2(p-1)}\int_0^s \gamma_r^2 ds\right\}\left(\hat{H}^{-1}\left(\hat{H}\left((\hat{u}(y))^+\right) + \int_0^s \left(\frac{a_r}{\phi(1)} + b_r\right)dr\right)\right)^p.$$

$$X_t := \hat{u}^{-1}\left(\left\{\mathbb{E}\left[\Phi(T, \xi; a., b., \gamma.) \mid \mathcal{F}_t\right]\right\}^{1/p}\right).$$

We conclude the following results:

- *If $\xi > 0$ and $\mathbb{E}\left[\Phi^k(T, \xi; a., b., \gamma.)\right] < +\infty$ for some $k > 1$ then BSDE (1) has a solution (Y, Z) satisfying $\hat{u}(Y) \in \mathcal{S}^{kp}$;*
- *If $\xi > 0$, $(\mathbb{1}_{x>0}/\psi(x))_{x\in\mathbb{R}}$ is locally integrable on $\mathbb{R}$, $\mathbb{E}\left[\left(\int_0^T \gamma_t^2 dt\right)^p\right] < +\infty$, $\mathbb{E}\left[\phi^p\left(\hat{X}^*\right)\right] < +\infty$, and $\mathbb{E}\left[\left(\hat{X}^*\hat{u}'(\hat{X}^*)\right)^{2p}\right] < +\infty$, then BSDE (1) has a solution (Y, Z) satisfying $\hat{u}(Y) \in \mathcal{S}^{2p}$ and $Z \in \mathcal{M}^p$.*

These results can be verified using the same method as in Theorem 1. It is worth noting that we have checked that $\hat{u}(\cdot) \in \mathcal{C}^1((0, +\infty)) \cap \mathcal{W}^2_{1,loc}((0, +\infty))$, which is critical for applying Itô–Krylov's formula. The functions $\hat{u}^{-1}(\cdot)$, $\tilde{u}(\cdot)$, $\tilde{u}^{-1}(\cdot)$, $\hat{H}(\cdot)$, $\hat{H}^{-1}(\cdot)$, and $\hat{v}(\cdot)$ also satisfy this property.

3. Comparison Theorem

To obtain the uniqueness result, we assume a convexity condition in y, z so as to exploit the θ-technique proposed by [3], which has proved to be convenient for dealing with quadratic generators.

Theorem 2. *Assume that BSDE(ξ^1, f^1) and BSDE(ξ^2, f^2) satisfy that $\mathbb{P}$-a. s., $0 < \xi^1 \leq \xi^2$, and $d\mathbb{P} \otimes du$-a.s., $f^1(u, Y_u^2, Z_u^2) \leq f^2(u, Y_u^2, Z_u^2)$. Moreover, assume that $f^1(t, \cdot, \cdot)$ is convex for any $t \in [0, T]$, and for any $(s, \omega, y, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d}$,*

$$0 \leq f^1(s, \omega, y, z) \leq a_s + b_s y + \gamma_s|z| + \frac{\delta}{2y}|z|^2,$$

where $\delta \geq 0$ is a constant, and a, b, and γ are non-negative and progressively measurable processes satisfying for some $p > 1$,

$$\mathbb{E}\left[\left(\int_0^T a_s^{2+\delta} ds\right)^{2p}\right] < +\infty, \text{ and } \mathbb{E}\left[\exp\left\{\frac{p}{p-1}\int_0^T \gamma_s^2 ds + 2p(2+\delta)\int_0^T b_t dt\right\}\right] < +\infty.$$

Let (Y^1, Z^1) and (Y^2, Z^2) be solutions to BSDE(ξ^1, f^1) and BSDE(ξ^2, f^2), respectively, such that both Y^1 and Y^2 belong to $\mathcal{S}^{2p(2+\delta)}$; then, $\mathbb{P}$-a. s., $\forall\, 0 \leq t \leq T$, $Y_t^1 \leq Y_t^2$.

Proof. For any given $\theta \in (0, 1)$, set

$$\Delta_\theta Y_t := \frac{Y_t^1 - \theta Y_t^2}{1-\theta},\ \Delta_\theta Z_t := \frac{Z_t^1 - \theta Z_t^2}{1-\theta} \text{ and } \Delta_\theta \xi := \frac{\xi^1 - \theta\xi^2}{1-\theta}.$$

Suppose $L^0_\cdot(\Delta_\theta Y)$ represents the local time of $\Delta_\theta Y$ at 0. We obtain by Tanaka's formula that

$$\begin{aligned}(\Delta_\theta Y_t)^+ =&(\Delta_\theta \xi)^+ + \int_t^T \mathbb{1}_{\Delta_\theta Y_u>0}\frac{1}{1-\theta}\Big[f^1\Big(u, Y^1_u, Z^1_u\Big) - \theta f^2\Big(u, Y^2_u, Z^2_u\Big)\Big]du\\ &- \int_t^T \mathbb{1}_{\Delta_\theta Y_u>0}\Delta_\theta Z_u dW_u - \frac{1}{2}\int_t^T dL^0_u(\Delta_\theta Y),\end{aligned}$$

where

$$\begin{aligned}&\mathbb{1}_{\Delta_\theta Y_u>0}\frac{1}{1-\theta}\Big[f^1\Big(u, Y^1_u, Z^1_u\Big) - \theta f^2\Big(u, Y^2_u, Z^2_u\Big)\Big]\\ =&\mathbb{1}_{\Delta_\theta Y_u>0}\frac{1}{1-\theta}\Big[f^1\Big(u, Y^1_u, Z^1_u\Big) - \theta f^1\Big(u, Y^2_u, Z^2_u\Big) + \theta f^1\Big(u, Y^2_u, Z^2_u\Big) - \theta f^2\Big(u, Y^2_u, Z^2_u\Big)\Big]\\ \le&\mathbb{1}_{\Delta_\theta Y_u>0}\Big[a_u + b_u\Delta_\theta Y_u + \gamma_u|\Delta_\theta Z_u| + \frac{\delta|\Delta_\theta Z_u|^2}{2\Delta_\theta Y_u}\Big].\end{aligned}$$

For ease of notation, let us denote

$$\begin{aligned}C_t &:= \exp\Big\{\frac{1}{2(p-1)}\int_0^t \gamma_s^2 ds + (2+\delta)\int_0^t b_s ds + (1+\delta)t\Big\},\\ P^0_t &:= C_t\Big(\frac{((\Delta_\theta Y_t)^+)^{2+\delta}}{2+\delta} + \int_0^t \frac{a_s^{2+\delta}}{2+\delta}ds\Big),\\ G_t &:= \Big(P^0_t\Big)^p, \text{ and } R_t := C_t p\Big(P^0_t\Big)^{p-1}\Big((\Delta_\theta Y_t)^+\Big)^{1+\delta}\mathbb{1}_{\Delta_\theta Y_t>0}\Delta_\theta Z_t.\end{aligned}$$

Due to $P^0_t \ge 0$ and $p > 1$, we cannot directly apply Itô's formula to G_t. Instead, we invoke Lemma 3.1 in [24], which provides an Itô's-type formula for $p \ge 1$, $y \to |y|^p$ based on the technique introduced in [25]. By applying Itô's formula to P^0_t and using Lemma 3.1 in [24], we deduce that

$$G_t \le G_T - \int_t^T R_s dW_s.$$

Let us define $\sigma^t_n := \inf\{s > t : \int_t^s |R_u|^2 du > n\} \wedge T$, $t \in [0, T]$, $n \in \mathbb{N}$. Therefore, $G_t \le \mathbb{E}\Big[G_{\sigma^t_n} \mid \mathcal{F}_t\Big]$. Noting that $(\Delta_\theta \xi)^+ \le \xi^1$, the dominated convergence theorem gives

$$\left(\frac{(Y^1_t - \theta Y^2_t)^+}{1-\theta}\right)^{p(2+\delta)} \le \mathbb{E}_t\Big[C_T^p\Big((\xi^1)^{2+\delta} + \int_0^T a_s^{2+\delta}ds\Big)^p\Big].$$

Moving $(1-\theta)$ to the right-hand side and sending $\theta \to 1$, we complete the proof. □

Corollary 1. *There is only one solution $(Y, Z) \in \mathcal{S}^{2p(2+\delta)} \times \mathcal{M}^{2p}$ for BSDE(ξ, f), if $\xi > 0$, $f(t, \cdot, \cdot)$ is convex for any $t \in [0, T]$, and for any $(s, \omega, y, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^{1\times d}$,*

$$0 \le f(s, \omega, y, z) \le a_s + b_s y + \gamma_s|z| + \frac{\delta}{2y}|z|^2,$$

where $0 \le \delta \ne 1$ is a constant, a, b, and γ are non-negative and progressively measurable processes, and for some $p > 1$,

$$\mathbb{E}\left[\exp\Big\{\frac{p}{p-1}\int_0^T \gamma_s^2 ds + 2p(2+\delta)\int_0^T b_t dt\Big\}\Big(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta}ds\Big)^{2p}\right] < +\infty.$$

More precisely,

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T} Y_t^{2p(2+\delta)} + \left(\int_0^T |Z_u|^2 du\right)^p\right]\\ \le& C\mathbb{E}\left[\exp\left\{\frac{p}{p-1}\int_0^T \gamma_s^2 ds + 2p(2+\delta)\int_0^T b_t dt\right\}\left(1+\xi^{2+\delta}+\int_0^T a_s^{2+\delta} ds\right)^{2p}\right],\end{aligned}$$

where C is a constant depending on T, p, and δ.

Remark 3. *For the more generalized BSDEs in Remark 2, a similar comparison theorem holds. In fact, let (Y^1, Z^1) and (Y^2, Z^2) be solutions to BSDE(ξ^1, f^1) and BSDE(ξ^2, f^2), respectively, fulfilling for a given $p > 1$,*

$$\forall \lambda > 0,\ \mathbb{E}\left[\sup_{0\le t\le T} \Phi(t, \lambda Y_t^i; a_\cdot, b_\cdot, \gamma_\cdot)\right] < +\infty,\ i = 1, 2.$$

Assume $d\mathbb{P}\otimes du$-a.s., $f^1(u, Y_u^2, Z_u^2) \le f^2(u, Y_u^2, Z_u^2)$, and $\mathbb{P}$-a. s., $0 < \xi^1 \le \xi^2$. If f^1 satisfies (14), $(\mathbb{1}_{x>0}/\psi(x))_{x\in\mathbb{R}}$ is locally integrable on $\mathbb{R}$ and f^1 is convex in (y, z) for all $t \in [0, T]$ then $\mathbb{P}$-a. s., $\forall\, 0 \le t \le T$, $Y_t^1 \le Y_t^2$.

Moreover, if BSDE(ξ, f) satisfies (14), $(\mathbb{1}_{x>0}/\psi(x))_{x\in\mathbb{R}}$ is locally integrable on $\mathbb{R}$, f is convex in (y, z) for all $t \in [0, T]$, for some $p > 1$,

$$\mathbb{E}\left[\sup_{0\le t\le T} \phi^{p/2}(X_t)\right] < +\infty,$$

and for any $\lambda \ge 1$,

$$\mathbb{E}\left[\sup_{0\le t\le T} \exp\left\{\frac{p}{2(p-1)}\int_0^t \gamma_s^2 ds\right\}\left(\hat{H}^{-1}\left(\hat{H}(\lambda X_t \hat{u}'(\lambda X_t)) + \int_0^t \left(\frac{a_s}{\phi(1)} + b_s\right) ds\right)\right)^p\right] < \infty,$$

then BSDE(ξ, f) has a unique solution (Y, Z) in $\mathcal{S}^p \times \mathcal{M}^{p/2}$.

Corollary 2 (BMO property)**.** *Assume that BSDE(ξ, f) satisfies that $\xi > 0$ is bounded, (3) holds with $0 \le \delta \ne 1$, $\int_0^T a_s^{2+\delta} ds$, $\int_0^T b_s ds$, and $\int_0^T \gamma_s^2 ds$ are bounded. If $f(t, \cdot, \cdot)$ is convex for any $t \in [0, T]$ then there is only one solution (Y, Z) for BSDE(ξ, f) fulfilling that Y is bounded and $Z \circ W \in BMO$.*

Proof. Corollary 1 provides the existence, uniqueness of solution (Y, Z), and presents the boundedness of Y_t. To prove that $Z \circ W \in BMO$, we apply Itô's formula to $v(Y_t)$, where $v(\cdot)$ is defined in the proof of Theorem 1. Taking the conditional expectation and employing a classical localization argument alongside the dominated convergence theorem yields the desired result. □

Remark 4. *It is clear that for the more generalized BSDEs in Remark 2, Corollary 2 still holds.*

4. Stability Result and Application to PDEs

4.1. Stability Result

The following result establishes a stability result for the solution to BSDE(ξ, f). To be more detailed, suppose that f is a generator satisfying (3) with coefficient processes $a_\cdot$, $b_\cdot$, and $\gamma_\cdot$, and $f(t, \cdot, \cdot)$ is convex for any $t \in [0, T]$. For each $n \ge 1$, f_n is a generator fulfilling (3) with coefficient processes $a_\cdot^n$, $b_\cdot^n$, and $\gamma_\cdot^n$, and $f_n(t, \cdot, \cdot)$ is convex for any

$t \in [0,\, T]$; $0 \leq \delta \neq 1$ is the constant in (3). For simplicity of representation, we denote for $p > 1$,

$$A_t := \exp\left\{ \frac{p}{p-1} \int_0^t \gamma_s^2 ds + p(2+\delta) \int_0^t b_s ds \right\},$$

and

$$A_t^n := \exp\left\{ \frac{p}{p-1} \int_0^t (\gamma_s^n)^2 ds + p(2+\delta) \int_0^t b_s^n ds \right\}.$$

Assume that ξ and $(\xi_n)_{n\geq 1}$ are strictly positive terminal conditions, such that

$$\mathbb{E}\left[A_T^4 \left(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta} ds \right)^{4p} \right] + \sup_{n\geq 1} \mathbb{E}\left[(A_T^n)^4 \left(1 + (\xi_n)^{2+\delta} + \int_0^T (a_s^n)^{2+\delta} ds \right)^{4p} \right] < +\infty. \quad (15)$$

Owing to Corollary 1, let $(Y,\, Z) \in \mathcal{S}^{4p(2+\delta)} \times \mathcal{M}^{4p}$ solve BSDE $(\xi,\, f)$, and let $(Y^n,\, Z^n) \in \mathcal{S}^{4p(2+\delta)} \times \mathcal{M}^{4p}$ solve BSDE $(\xi_n,\, f_n)$ for each $n \geq 1$.

Theorem 3. *If $\xi_n \to \xi$ in $L^{2p(\delta+2)}$ and $\int_0^T |f_n(r,\, Y_r,\, Z_r) - f(r,\, Y_r,\, Z_r)|\, dr \to 0$ in $L^{2p(\delta+2)}$ as $n \to \infty$ then Y^n converges to Y in $\mathcal{S}^{4p}$ and Z^n converges to Z in $\mathcal{M}^{2p}$.*

As a preliminary step, we establish the following lemma:

Lemma 3. *There exists a constant $C > 0$ independent of n, such that*

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T f(s,\, Y_s\, Z_s) + f_n(s,\, Y_s^n,\, Z_s^n)\, ds \right)^{2p} \right] \\
\leq & C\mathbb{E}\left[A_T^4 \left(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta} ds \right)^{4p} \right] + C \sup_{n\geq 1} \mathbb{E}\left[(A_T^n)^4 \left(1 + (\xi_n)^{2+\delta} + \int_0^T (a_s^n)^{2+\delta} ds \right)^{4p} \right].
\end{aligned}$$

Proof. Since f and f_n satisfy (3), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T f(s,\, Y_s\, Z_s) + f_n(s,\, Y_s^n,\, Z_s^n)\, ds \right)^{2p} \right] \\
\leq & C\mathbb{E}\left[\left(\int_0^T a_s + a_s^n + b_s Y_s + b_s^n Y_s^n + \gamma_s |Z_s| + \gamma_s^n |Z_s^n| + \frac{\delta |Z_s|^2}{2Y_s} + \frac{\delta |Z_s^n|^2}{2Y_s^n}\, ds \right)^{2p} \right] \\
\leq & C\left\{ \mathbb{E}\left[A_T + \left(\int_0^T a_s ds \right)^{2p} + \sup_{0\leq s\leq T} Y_s^{4p(2+\delta)} + \left(\int_0^T |Z_s|^2 ds \right)^{2p} + \left(\int_0^T \frac{\delta |Z_s|^2}{Y_s} ds \right)^{2p} \right] \right. \\
& \left. + \sup_{n\geq 1} \mathbb{E}\left[A_T^n + \left(\int_0^T a_s^n ds \right)^{2p} + \sup_{0\leq s\leq T} (Y_s^n)^{4p(2+\delta)} + \left(\int_0^T |Z_s^n|^2 ds \right)^{2p} + \left(\int_0^T \frac{\delta |Z_s^n|^2}{Y_s^n}\, ds \right)^{2p} \right] \right\}.
\end{aligned}$$

From Theorem 1, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds \right)^{2p} \right] \leq & C\mathbb{E}\left[A_T + \left(\int_0^T a_s ds \right)^{4p} + \sup_{0\leq s\leq T} Y_s^{4p(2+\delta)} \right] \\
\leq & C\mathbb{E}\left[A_T^4 \left(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta} ds \right)^{4p} \right],
\end{aligned}$$

and, similarly,

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^T |Z_s^n|^2 ds \right)^{2p} \right] \leq C \sup_{n\geq 1} \mathbb{E}\left[(A_T^n)^4 \left(1 + (\xi_n)^{2+\delta} + \int_0^T (a_s^n)^{2+\delta} ds \right)^{4p} \right].$$

To estimate $\mathbb{E}\left[\left(\int_0^T \frac{\delta|Z_s|^2}{Y_s}ds\right)^{2p}\right]$, we define for $\delta > 0$,

$$I(y) := \begin{cases} \frac{y^{\delta+1}}{\delta(\delta+1)} - \frac{y}{\delta} - \frac{\delta}{\delta+1}, & y \geq 1, \\ y\ln y - y, & 0 < y < 1. \end{cases}$$

Then, $I(\cdot) \in \mathcal{C}^2((0, +\infty))$. Applying Itô's formula to $I(Y_t)$ and noting the fact that $(\ln x)^2 \mathbb{1}_{0<x\leq 1} \leq \frac{1}{x}\mathbb{1}_{0<x\leq 1}$, we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \frac{|Z_s|^2}{Y_s}ds\right)^{2p}\right] \leq & C\mathbb{E}\left[A_T + \left(\int_0^T a_s ds\right)^{2p(2+\delta)} + \sup_{0\leq s\leq T}(Y_s)^{4p(2+\delta)} + \left(\int_0^T |Z_s|^2 ds\right)^{2p}\right] \\ \leq & C\mathbb{E}\left[A_T^4\left(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta}ds\right)^{4p}\right].\end{aligned}$$

Analogously,

$$\sup_{n\geq 1}\mathbb{E}\left[\left(\int_0^T \frac{|Z_s^n|^2}{Y_s^n}ds\right)^{2p}\right] \leq C\sup_{n\geq 1}\mathbb{E}\left[(A_T^n)^4\left(1 + (\xi_n)^{2+\delta} + \int_0^T (a_s^n)^{2+\delta}ds\right)^{4p}\right].$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T f(s, Y_s Z_s) + f_n(s, Y_s^n, Z_s^n)\, ds\right)^{2p}\right] \\ \leq & C\mathbb{E}\left[A_T^4\left(1 + \xi^{2+\delta} + \int_0^T a_s^{2+\delta}ds\right)^{4p}\right] + C\sup_{n\geq 1}\mathbb{E}\left[(A_T^n)^4\left(1 + (\xi_n)^{2+\delta} + \int_0^T (a_s^n)^{2+\delta}ds\right)^{4p}\right].\end{aligned}$$

We complete the proof. □

Proof of Theorem 3. First, we need to verify that $\left\{\sup_{0\leq t\leq T}|Y_t^n - Y_t|^{4p}\right\}_{n\geq 1}$ is uniformly integrable. According to Corollary 1,

$$\sup_{n\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^n - Y_t|^{4p(2+\delta)}\right] \leq C\sup_{n\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^n|^{4p(2+\delta)}\right] + C\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^{4p(2+\delta)}\right] < \infty;$$

thus, $\left\{\sup_{t\in[0,T]}|Y_t^n - Y_t|^{4p}\right\}_{n\geq 1}$ is uniformly integrable by de la Vallée Poussin's criterion.

Our task now is to prove that as $n \to \infty$, $\sup_{0\leq t\leq T}|Y_t^n - Y_t| \to 0$ in $\mathbb{P}$. For any given $0 \leq s \leq T$, $\theta \in (0, 1)$ and $n \geq 1$, we define the following notation:

$$\begin{aligned}&\Delta_\theta\xi := \frac{\xi - \theta\xi_n}{1-\theta}, \Delta_\theta Y_s := \frac{Y_s - \theta Y_s^n}{1-\theta}, \ \Delta_\theta Z_s := \frac{Z_s - \theta Z_s^n}{1-\theta}, \\ &\Delta f_s := f(s, Y_s, Z_s) - f_n(s, Y_s, Z_s), \\ &R_s := \frac{((\Delta_\theta Y_s)^+)^{2+\delta}}{2+\delta} + \int_0^s \frac{(a_r^n)^{2+\delta}}{2+\delta}dr, \\ &D_s^n := \exp\left\{\frac{p}{p-1}\int_0^s (\gamma_r^n)^2 dr + p(2+\delta)\int_0^s b_r^n dr + (1+\delta)sp\right\}.\end{aligned}$$

Applying the Tanaka–Meyer–Itô formula and Lemma 3.1 in [24], we obtain from assumptions on f_n and Young's inequality that

$$\begin{aligned} R_t^p D_t^n \leq & R_T^p D_T^n + \int_t^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \frac{|\Delta f_s|}{1-\theta} \\ & - \frac{p(p-1)}{4} D_s^n \mathbb{1}_{R_s>0} R_s^{p-2} \big((\Delta_\theta Y_s)^+\big)^{2+2\delta} |\Delta_\theta Z_s|^2 ds \\ & + \int_t^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \Delta_\theta Z_s dW_s, \ 0 \leq t \leq T. \end{aligned}$$

Let

$$N_t := \int_0^t p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \Delta_\theta Z_s dW_s.$$

Due to (15) and the integrability properties of Y, Z, and Z^n, we derive that N is a martingale. Thus,

$$\begin{aligned} & \frac{p(p-1)}{4} \mathbb{E}\left[\int_t^T D_s^n \mathbb{1}_{R_s>0} R_s^{p-2} \big((\Delta_\theta Y_s)^+\big)^{2+2\delta} |\Delta_\theta Z_s|^2 ds\right] \\ \leq & \mathbb{E}\left[R_T^p D_T^n\right] + \mathbb{E}\left[\int_t^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \frac{|\Delta f_s|}{1-\theta} ds\right], \end{aligned}$$

and it follows from the BDG inequality that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} R_t^p D_t^n\right] \leq \mathbb{E}\left[R_T^p D_T^n\right] + \mathbb{E}\left[\int_0^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \frac{|\Delta f_s|}{1-\theta} ds\right] + c(1)\mathbb{E}\left[\langle N\rangle_T^{1/2}\right]. \quad (16)$$

Then, Young's inequality yields that

$$\begin{aligned} & c(1)\mathbb{E}\left[\langle N\rangle_T^{1/2}\right] \\ \leq & \frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T} R_t^p D_t^n\right] + \frac{2pc^2(1)}{p-1}\left\{\mathbb{E}\left[R_T^p D_T^n\right] + \mathbb{E}\left[\int_0^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \frac{|\Delta f_s|}{1-\theta} ds\right]\right\}. \end{aligned} \quad (17)$$

By substituting (17) to (16) and denoting $C_1 := 2(1 + \frac{2pc^2(1)}{p-1})$, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} R_t^p D_t^n\right] \leq C_1 \mathbb{E}\left[R_T^p D_T^n + \int_0^T p D_s^n R_s^{p-1} \big((\Delta_\theta Y_s)^+\big)^{1+\delta} \frac{|\Delta f_s|}{1-\theta} ds\right]. \quad (18)$$

Analogously, employing Young's inequality to the last term in (18) yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T} R_t^p D_t^n\right] \leq C\mathbb{E}\left[R_T^p D_T^n + \left(\int_0^T (D_s^n)^{\frac{1}{p(2+\delta)}} \frac{|\Delta f_s|}{1-\theta} ds\right)^{p(2+\delta)}\right].$$

Since $\sup_{n\geq 1} \mathbb{E}\left[(D_T^n)^4 \left(1 + \int_0^T (a_s^n)^{2+\delta} ds\right)^{4p}\right] < +\infty$, we deduce from Jensen's inequality and Young's inequality that

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\leq t\leq T} (Y_t - \theta Y_t^n)^+\right] \\ \leq & (1-\theta) C \left\{1 + \mathbb{E}\left[\left(\frac{(\xi - \theta\xi_n)^+}{1-\theta}\right)^{2p(2+\delta)} + \left(\int_0^T \frac{|\Delta f_s|}{1-\theta} ds\right)^{2p(2+\delta)}\right]\right\}^{\frac{1}{p(2+\delta)}}. \end{aligned}$$

Interchanging Y^n and Y and similar deductions then leads to

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}(Y_t^n-\theta Y_t)^+\right]\\ \le&(1-\theta)C\left\{1+\mathbb{E}\left[\left(\frac{(\xi_n-\theta\xi)^+}{1-\theta}\right)^{2p(2+\delta)}+\left(\int_0^T\frac{|\Delta f_s|}{1-\theta}ds\right)^{2p(2+\delta)}\right]\right\}^{\frac{1}{p(2+\delta)}}.\end{aligned}$$

Finally, it can be easily checked that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|\right]\\ \le&(1-\theta)C\left\{1+\mathbb{E}\left[\left(\int_0^T\frac{|\Delta f_s|}{1-\theta}ds\right)^{2p(2+\delta)}+\left(\frac{|\xi_n-\xi|}{1-\theta}\right)^{2p(2+\delta)}+\xi^{2p(2+\delta)}+\xi_n^{2p(2+\delta)}\right]\right\}.\end{aligned}$$

Recall that $|\xi-\xi_n|$ and $\int_0^T|\Delta f_s|ds$ converge to 0 in $L^{2p(2+\delta)}$ space as $n\to\infty$; consequently,

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|\right]\le(1-\theta)C\left\{1+\mathbb{E}\left[\xi^{2p(2+\delta)}\right]\right\}.$$

By sending $\theta\to1$, we have $\sup_{0\le t\le T}|Y_t^n-Y_t|$ converging to 0 in probability. Thus, we arrive at the conclusion that $\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|^{4p}\right]=0$.

Now, we prove that $\lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^T|Z_t^n-Z_t|^2\,dt\right)^p\right]=0$. Applying Itô's formula to $|Y_t^n-Y_t|^2$, we have

$$\begin{aligned}\int_0^T|Z_t^n-Z_t|^2\,dt\le&(\xi_n-\xi)^2+2\sup_{0\le t\le T}|Y_t^n-Y_t|\int_0^T(f(s,Y_s\,Z_s)+f_n(s,Y_s^n,Z_s^n))\,ds\\ &-\int_0^T2(Y_s^n-Y_s)(Z_s^n-Z_s)dW_s.\end{aligned}$$

By Hölder's inequality and the BDG inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T|Z_t-Z_t^n|^2\,dt\right)^p\right]\\ \le&C\left\{\mathbb{E}\left[|\xi_n-\xi|^{2p}\right]+\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|^{2p}\right]\right.\\ &\left.+\left(\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|^{2p}\right]\right)^{\frac12}\left(\mathbb{E}\left[\left(\int_0^T(f(s,Y_s\,Z_s)+f_n(s,Y_s^n,Z_s^n))\,ds\right)^{2p}\right]\right)^{\frac12}\right\}.\end{aligned}$$

From Lemma 3 and (15), it follows that

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T|Z_t-Z_t^n|^2\,dt\right)^p\right]\\ \le&C\left\{\mathbb{E}\left[|\xi_n-\xi|^{2p}\right]+\left(\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|^{2p}\right]\right)^{\frac12}+\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|^{2p}\right]\right\}.\end{aligned}$$

Note that the generic constant C is independent of n. Consequently, we infer that Z^n converges to Z in $\mathcal{M}^{2p}$. Now, the proof is complete. □

4.2. Application to Singular Quadratic PDEs

In this subsection, we investigate singular quadratic PDEs associated with the BSDEs in our study. To be specific, we prove the related nonlinear Feynman–Kac formula, as well as the uniqueness of viscosity solutions. Consider the semi-linear PDE for $z \in \mathbb{R}^n$, $0 \le s \le T$,

$$-\mathscr{L}u(s, z) - \partial_s u(s, z) - f\Big(s, z, u(s, z), ((\partial_z u)^\top \sigma)(s, z)\Big) = 0, \quad u(T, \cdot) = \Psi(\cdot), \tag{19}$$

in which $\mathscr{L}$ refers to the infinitesimal generator of $X^{t', x'}$, the solution to the following SDE for each fixed $(t', x') \in [0, T] \times \mathbb{R}^n$,

$$X_t = x' + \int_{t'}^{t} B(r, X_r)dr + \int_{t'}^{t} \sigma(r, X_r)dW_r, \ t' \le t \le T, \quad X_t = x', \ 0 \le t \le t'. \tag{20}$$

In addition, let $(Y^{t', x'}, Z^{t', x'})$ solve the BSDE

$$Y_s = \Psi(X_T^{t', x'}) + \int_s^T f(r, X_r^{t', x'}, Y_r, Z_r)dr - \int_s^T Z_r dW_r, \ 0 \le s \le T. \tag{21}$$

Our first goal is to verify that (19) has a viscosity solution defined by

$$u(s, x) := Y_s^{s, x}, \ s \in 0, \le s \le T, \ x \in \mathbb{R}^n. \tag{22}$$

First, we retrospect the definition of viscosity solutions to (19).

Definition 1. *A function u that is continuous on $[0, T] \times \mathbb{R}^n$ is defined as a viscosity subsolution (resp. supersolution) to (19) if for each $(s, x) \in [0, T) \times \mathbb{R}^n$ and $\varpi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$ satisfying (s, x) is a local minimum (resp. maximum) of $\varpi - u$,*

$$-\partial_s \varpi(s, x) - \mathscr{L}\varpi(s, x) - f\Big(s, x, \varpi(s, x), ((\partial_x \varpi)^\top \sigma)(s, x)\Big) \le 0 \quad (\textit{ resp. } \ge 0)$$

as well as

$$\forall x \in \mathbb{R}^n, \ u(T, x) \le \Psi(x) \quad (\textit{ resp. } \ge).$$

Furthermore, u is termed as a viscosity solution if it is a viscosity supersolution as well as a viscosity subsolution.

Now, we give our assumptions.

(A.1) $B : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ as well as $\sigma : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times d}$ are continuous functions fulfilling for all $(s, z, z^1, z^2) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|B(s, 0)| + |\sigma(s, z)| \le J$$

and

$$|B(s, z^1) - B(s, z^2)| + |\sigma(s, z^1) - \sigma(s, z^2)| \le J|z^1 - z^2|$$

are met for some constant $J > 0$.

(A.2) $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \mapsto \mathbb{R}$ and $\Psi : \mathbb{R}^n \mapsto (0, +\infty)$ are continuous functions fulfilling for all $(s, x, y, z) \in [0, T] \times \mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^{1 \times d}$,

$$\begin{aligned} &(y, z) \mapsto f(s, y, z) \text{ is convex }, \\ &\Psi(x) \le k(1 + |x|^q), \\ \text{and} \quad &0 \le f(s, x, y, z) \le k(1 + |x|^q + y + |z|) + \frac{\delta|z|^2}{2y} \end{aligned}$$

are met for some constants $0 \leq \delta \neq 1$, $q \geq 1$ and $k \geq 0$.

Given (A.1) and constants $p \geq 1$, $t' \in [0, T]$ and $x' \in \mathbb{R}^n$, there exists a unique solution $X^{t',x'} \in \mathcal{S}^p(\mathbb{R}^n)$ to SDE (20). Consequently, we infer from (A.2) together with Corollary 1 that BSDE (21) admits only one solution, $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$, $\forall p \geq 1$.

Theorem 4. *Assuming that conditions (A.1) and (A.2) are met, the function u defined in (22) is a viscosity solution for (19) and fulfills the following inequality:*

$$0 < u(t', x') \leq C\big(1 + |x'|^q\big), \quad \forall t' \in [0, T], x' \in \mathbb{R}^n. \tag{23}$$

Proof. Let us first prove the continuity of the map $(t, x) \mapsto u(t, x)$. Consider a sequence $(t_n, x_n)_{n\geq 1}$ converging to (t, x) as $n \to +\infty$. We begin by observing that

$$\forall p > 0, \lim_{n\to\infty} \mathbb{E}\Big[|\Psi(X_T^{t_n,x_n}) - \Psi(X_T^{t,x})|^p\Big] = 0,$$

which follows directly from $\lim_{n\to\infty} \mathbb{E}\Big[|X_T^{t_n,x_n} - X_T^{t,x}|^p\Big] = 0$, $\forall p > 0$ and the Vitali convergence theorem. Now, we establish that for any $p > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T \big|f(s, X_s^{t_n,x_n}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\big| ds\right)^p\right] = 0. \tag{24}$$

Since f is continuous and $\mathbb{E}\left[\sup_{n\geq 1}\sup_{0\leq s\leq T} |X_s^{t_n,x_n}|^p\right] \leq Ce^{Cp|x|}$, $\forall p > 0$ (see [26]), the dominated convergence theorem gives that

$$\lim_{n\to\infty} \int_0^T \big|f(s, X_s^{t_n,x_n}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\big| ds = 0, \ \mathbb{P}\text{-a. s.}.$$

Note the fact that

$$\begin{aligned}
&\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^T \big|f(s, X_s^{t_n,x_n}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\big| ds\right)^p\right] \\
\leq & C\left\{1 + |x|^{q(p\vee 2)} + \sup_{n\geq 1}|x_n|^{q(p\vee 2)} + \mathbb{E}\left[\left(\int_0^T \frac{\delta|Z_s^{t,x}|^2}{Y_s^{t,x}} ds\right)^p\right] + \mathbb{E}\left[\sup_{0\leq s\leq T}\big(Y_s^{t,x}\big)^{p(2+\delta)}\right]\right\}.
\end{aligned}$$

From Lemma 3, we obtain that for any $p > 0$ and $\delta > 0$,

$$\mathbb{E}\left[\left(\int_0^T \frac{|Z_s^{t,x}|^2}{Y_s^{t,x}} ds\right)^p\right] \leq C\left\{1 + \mathbb{E}\left[\sup_{0\leq s\leq T}\big(Y_s^{t,x}\big)^{2p(2+\delta)}\right] + \mathbb{E}\left[\left(\int_0^T |Z_s^{t,x}|^2 ds\right)^p\right]\right\}.$$

Thus, (24) follows immediately from the Vitali convergence theorem. The integrable condition (15) still holds, and the continuity of u is derived from Theorem 3. The growth of u follows routinely from (A.2) and Corollary 1. Finally, we prove that u defined in (22) is a viscosity solution to PDE (19) via a classical approximation technique together with the stability results of viscosity solutions (see [26,27]). For $n \geq 1$, $(t', x') \in [0, T] \times \mathbb{R}^n$, $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d}$, we define

$$\sigma_n := \inf\Big\{s > 0 : |X_s^{t',x'}|^q > n\Big\} \wedge T, \text{ and}$$

$$f^n(t, x, y, z) := \mathbb{1}_{t\leq\sigma_n} \ \inf\Big\{f(t, x, p, q) \vee 0 + n|p - y| + n|q - z| : (p, q) \in \mathbb{Q} \times \mathbb{Q}^{1\times d}\Big\}.$$

By [28], each f^n is uniformly Lischitz continuous in (y, z), and $\{f^n\}_{n\geq 1}$ is increasing and converges uniformly on compact sets to $f \vee 0$. Moreover, for any $(t', x') \in [0, T] \times \mathbb{R}^n$ and $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d}$, (A.2) implies the following inequalities:

1. $|f^n\left(t, X_t^{t',x'}, y, z\right)| \leq k(2+n) + n(1+|y|+|z|)$;
2. $0 \leq f^n(t, x, y, z) \leq f(t, x, y, z) \vee 0$;
3. If $y > 0, 0 \leq f^n(t, x, y, z) \leq f(t, x, y, z) \leq k(1+|x|^q + y + |z|) + \frac{\delta|z|^2}{2y}$.

For any fixed $l > 1$, we set $\xi^l := (\frac{1}{l}) \vee \Psi(X_T^{t',x'})$, and we consider the BSDE,

$$y_t^{n,l} = \xi^l + \int_t^T f^n\left(s, X_s^{t',x'}, y_s^{n,l}, z_s^{n,l}\right) ds - \int_t^T z_s^{n,l}\, dW_s,\ t \in [0, T]. \tag{25}$$

By [25], (A.1), and (A.2), this BSDE admits a unique solution $(y^{n,l}, z^{n,l}) \in \mathcal{S}^p \times \mathcal{M}^p$, $\forall p \geq 1$. From Remark 1, we have that for any l, $n > 1$,

$$\mathbb{E}_t\left[\xi^l\right] \leq y_t^{n,l} \leq \bar{y}_t(\xi^l), \tag{26}$$

where $\bar{y}_t(\xi^l)$ is defined in (13).

Additionally, $\{y^{n,l}\}_{n\geq 1}$ is increasing by the classical comparison theorem in [29]. For any $l > 1$, we define $y_t^l := \sup_{n\geq 1} y_t^{n,l}$. Following [2], we will prove there exists an appropriate process z^l, such that (y^l, z^l) solves the BSDE

$$y_t^l = (\frac{1}{l}) \vee \Psi(X_T^{t',x'}) + \int_t^T f\left(s, X_s^{t',x'}, y_s^l, z_s^l\right) ds - \int_t^T z_s^l\, dW_s,\ t \in [0, T]. \tag{27}$$

Then, it follows from Theorems 2 and 3 that y_t^l decreasingly converges to the value process Y_t of the solution to BSDE (21). By (13) and (26), it can be checked that $\lim_{t\to T} y_t^l = \xi^l$. Thus, $\lim_{t\to T} y_t^l = y_T^l$. To apply the monotone stability theorem in [7], we define the stopping time for any $m > 1$,

$$\tau_m := \inf\left\{s > 0 : \bar{y}_s(\xi^l) + |X_s^{t',x'}|^q > m\right\} \wedge T,$$

and the truncated function $\rho(y) := \frac{1}{l}\mathbb{1}_{y<\frac{1}{l}} + y\mathbb{1}_{\frac{1}{l}\leq y\leq m} + m\mathbb{1}_{y>m}$, $y \in \mathbb{R}$. Then, $(y_t^{m,n,l}, z_t^{m,n,l})$ $:= (y_{t\wedge\tau_m}^{n,l}, \mathbb{1}_{t\leq\tau_m} z_t^{n,l})$ solves the BSDE,

$$y_t^{m,n,l} = y_{\tau_m}^{n,l} + \int_t^T \mathbb{1}_{s\leq\tau_m} f^n\left(s, X_s^{t',x'}, \rho(y_s^{m,n,l}), z_s^{m,n,l}\right) ds - \int_t^T z_s^{m,n,l}\, dW_s,\ t \in [0, T].$$

Since for any $(s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$,

$$\begin{aligned}
0 \leq & \mathbb{1}_{s\leq\tau_m} f^n\left(s, X_s^{t',x'}, \rho(y), z\right) \\
\leq & \mathbb{1}_{s\leq\tau_m} f\left(s, X_s^{t',x'}, \rho(y), z\right) \\
\leq & \mathbb{1}_{s\leq\tau_m} k(1+|X_s^{t',x'}|^q + \rho(y) + |z|) + \frac{\delta|z|^2}{2\rho(y)} \\
\leq & k(1+m+m+\frac{1}{2}) + \frac{k+l\delta}{2}|z|^2,
\end{aligned}$$

$\mathbb{1}_{s\leq\tau_m} f^n\left(s, X_s^{t',x'}, \rho(y), z\right)$ converges uniformly on compact sets to $\mathbb{1}_{s\leq\tau_m} f\left(s, X_s^{t',x'}, \rho(y), z\right)$ as $n \to \infty$, and $y_{\tau_m}^{n,l}(\frac{1}{l} \leq y_{\tau_m}^{n,l} \leq m)$ increasingly converges to $\sup_{n\geq 1} y_{\tau_m}^{n,l} = y_{\tau_m}^l$, ref. [7] gives

that there exist processes $y_t^{m,l} := \sup_{n\geq 1} y_t^{m,n,l} = \sup_{n\geq 1} y_{t\wedge\tau_m}^{n,l} = y_{t\wedge\tau_m}^{l}$ and $z_t^{m,l} := \lim_{n\to\infty} z_t^{m,n,l}$ (in $\mathcal{M}^2$), such that $\left(y_t^{m,l}, z_t^{m,l}\right)$ solves the BSDE,

$$y_t^{m,l} = y_{\tau_m}^{l} + \int_t^T \mathbb{1}_{s\leq\tau_m} f\left(s, X_s^{t',x'}, \rho(y_s^{m,l}), z_s^{m,l}\right) ds - \int_t^T z_s^{m,l}\, dW_s,\ t\in[0,\, T].$$

Since $y_{t\wedge\tau_m}^{m+1,l} = y_{t\wedge\tau_m}^{l} = y_t^{m,l}$, $z_t^{m+1,l}\mathbb{1}_{t\leq\tau_m} = z_t^{m,l}$, we define

$$z_t^{l} := z_t^{1,l}\mathbb{1}_{t\leq\tau_1} + \sum_{m\geq 2} z_t^{m,l}\mathbb{1}_{\tau_{m-1}<t\leq\tau_m}.$$

It is clear from [2] that $z^l \in \mathcal{L}^2$. Note that for any $t\in[0,\, T]$, $m,\, l > 1$, $\frac{1}{l} \leq y_{t\wedge\tau_m}^{l} \leq m$; then, we arrive at

$$y_{t\wedge\tau_m}^{l} = y_{\tau_m}^{l} + \int_{t\wedge\tau_m}^{\tau_m} f\left(s, X_s^{t',x'}, y_s^{l}, z_s^{l}\right) ds - \int_{t\wedge\tau_m}^{\tau_m} z_s^{l}\, dW_s,\ t\in[0,\, T].$$

Since $y_t^{m,l}$ is continuous on $[0,\, T]$ and $\tau_m \to T$, y_t^l is continuous on $[0,\, T)$, we know that $\lim_{t\to T} y_t^l = y_T^l$; thus, y^l is continuous on $[0,\, T]$. When $m\to\infty$, we obtain (y^l, z^l), which solves the BSDE (27). Moreover, in view of (26), Remark 1, and Theorem 1, we deduce that $(y^l, z^l) \in \mathcal{S}^p \times \mathcal{M}^p$, $\forall p \geq 1$.

For the PDE part, classical works [29,30] imply that for any $n,\, l > 1$, $u^{n,l}(s,x) := (y_s^{n,l})^{s,x}$, $s\in[0,\, T]$, $x\in\mathbb{R}^n$ is a viscosity solution to the PDE,

$$-\mathscr{L}u(s,\, z) - \partial_s u(s,\, z) - f^n\left(s,\, z,\, u(s,\, z),\, ((\partial_z u)^\top \sigma)(s,\, z)\right) = 0, \quad u(T,\, \cdot) = (\frac{1}{l}) \vee \Psi(\cdot).$$

We know that $\{u^{n,l}\}$ increases pointwise to $u^l(s,x) := (y_s^l)^{s,x} > 0$ as $n\to\infty$, and Dini's theorem ensures uniform convergence on compact sets of $[0,\, T]\times\mathbb{R}^n$. In addition, f^n converges uniformly on compact sets to $0\vee f$. By (A.2) and the stability property of viscosity solutions (see, e.g., Theorem 1.7 in Chapter 5 of [31], or Proposition I.3 in [32]), u^l is a viscosity solution to the PDE

$$-\mathscr{L}u(s,\, z) - \partial_s u(s,\, z) - f\left(s,\, z,\, u(s,\, z),\, ((\partial_z u)^\top \sigma)(s,\, z)\right) = 0, \quad u(T,\, \cdot) = (\frac{1}{l}) \vee \Psi(\cdot).$$

By Theorems 2 and 3, $\{u^l\}$ decreases pointwise to u defined in (22) as $l\to\infty$. Dini's theorem again ensures that $\{u^l\}$ converges to u uniformly on compact sets of $[0,\, T]\times\mathbb{R}^n$. By the stability property of viscosity solutions again, we conclude that u defined in (22) is a viscosity solution to PDE (19). □

We further show the uniqueness of viscosity solutions to (19). Let us consider the following additional assumption.

(A.3) For any given $\tau > 0$, there exists a function $m_\tau : [0,\, +\infty) \to [0,\, +\infty)$ satisfying $\lim_{x\to 0} m_\tau(x) = 0$. Moreover, assume that for any $|x^1|,\, |x^2|,\, |y| \leq \tau$, $z\in\mathbb{R}^{1\times d}$, and $0\leq s\leq T$,

$$|f(s,\, x^1,\, y,\, z) - f(s,\, x^2,\, y,\, z)| \leq m_\tau\Big(|x^1 - x^2|(|z|+1)\Big).$$

Theorem 5 (Uniqueness)**.** *PDE* (19) *has, at most, one viscosity solution that satisfies* (23)*, provided that (A.1), (A.2), and (A.3) are met. In particular, for PDE* (19) *the unique viscosity solution fulfilling* (23) *is u defined in* (22)*.*

Our proof of Theorem 5 is in the spirit of [30]. Let us first present the following lemmas, which utilize the same notation as in the proof of Lemmas 3.7 and 3.8 in [30].

Lemma 4. *Let u be a viscosity subsolution, let κ be a viscosity supersolution of PDE (19), and let both u and κ satisfy (23). Then, $h := u - \kappa$ is a viscosity subsolution of*

$$(-\partial_t h - \mathscr{L}h)h^+ - kh^2 - k|(\partial_x h)^\top \sigma||h| - \delta|(\partial_x h)^\top \sigma|^2 = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \quad (28)$$

where k and δ are constants in (A.2).

Proof. Let $(s_0, w_0) \in [0, T) \times \mathbb{R}^n$ as well as $\varphi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$ satisfy that there is a solid neighborhood of (s_0, w_0) denoted by $U(s_0, w_0)$, such that $0 = h(s_0, w_0) - \varphi(s_0, w_0) > h(s, w) - \varphi(s, w)$ for each $(s, w) \in U(s_0, w_0) \backslash \{(s_0, w_0)\}$. Take a compact set $\mathcal{T} \times B \subset U(s_0, w_0)$, such that $(s_0, w_0) \in \mathcal{T} \times B$. Define a function $\psi_\varepsilon : (0, T) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$\psi_\varepsilon(s, w, z) := u(s, w) - \kappa(s, z) - \frac{|w - z|^2}{2\varepsilon} - \varphi(s, w),$$

in which $\varepsilon > 0$ tends to zero. Owing to the upper semicontinuity of ψ_ε, the maximum of ψ_ε on $\mathcal{T} \times B \times B$ is obtained at a point $(s_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathcal{T} \times B \times B$. It is not hard to derive that

(i) $(s_\varepsilon, w_\varepsilon, z_\varepsilon) \to (s_0, w_0, w_0)$ as $\varepsilon \to 0$;

(ii) $\lim\limits_{\varepsilon \to 0} \dfrac{|w_\varepsilon - z_\varepsilon|^2}{\varepsilon} = 0.$

Indeed, since

$$u(s_\varepsilon, w_\varepsilon) - \kappa(s_\varepsilon, z_\varepsilon) - \frac{|w_\varepsilon - z_\varepsilon|^2}{2\varepsilon} - \varphi(s_\varepsilon, w_\varepsilon) \geq u(s_0, w_0) - \kappa(s_0, w_0) - \varphi(s_0, w_0),$$

it follows that $|w_\varepsilon - z_\varepsilon|^2 \leq 4\varepsilon \max\limits_{(s,w,z) \in \mathcal{T} \times B \times B} |u(s, w) - \kappa(s, z) - \varphi(s, w)|$. Sending $\varepsilon \to 0$, we obtain that w_ε and z_ε converge to the same point. Let us assume $(s_\varepsilon, w_\varepsilon, z_\varepsilon)$ converges to $(\tilde{s}, \tilde{w}, \tilde{w}) \in \mathcal{T} \times B \times B$ as $\varepsilon \to 0$. Then, we have

$$u(\tilde{s}, \tilde{w}) - \kappa(\tilde{s}, \tilde{w}) - \lim_{\varepsilon \to 0} \frac{|w_\varepsilon - z_\varepsilon|^2}{2\varepsilon} - \varphi(\tilde{s}, \tilde{w}) \geq u(s_0, w_0) - \kappa(s_0, w_0) - \varphi(s_0, w_0).$$

Note the fact that $h - \varphi$ achieves its strict maximum in $U(s_0, w_0)$ at (s_0, w_0); the previous inequality implies that both (i) and (ii) hold.

It follows from Theorem 8.3 in [20] that there exist two triplets, $(0, p, X) \in \overline{D}^{2,+}(u - \varphi)(s_\varepsilon, w_\varepsilon)$ and $(0, -p, -Z) \in \overline{D}^{2,+}(-\kappa)(s_\varepsilon, z_\varepsilon)$, such that

(1) $p = \dfrac{w_\varepsilon - z_\varepsilon}{\varepsilon}$;

(2) $$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Z \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

It follows that $(\varphi_s(s_\varepsilon, w_\varepsilon), p + \partial_w \varphi(s_\varepsilon, w_\varepsilon), X + D^2\varphi(s_\varepsilon, w_\varepsilon)) \in \overline{D}^{2,+}u(s_\varepsilon, w_\varepsilon)$ and $(0, p, Z) \in \overline{D}^{2,-}\kappa(s_\varepsilon, z_\varepsilon)$. Since u and κ are, respectively, a viscosity subsolution and a viscosity supersolution of (19) satisfying (23), we deduce that

$$\begin{aligned} &-\varphi_s(s_\varepsilon, w_\varepsilon) - \frac{1}{2} tr\Big(\big(\sigma\sigma^\top\big)(s_\varepsilon, w_\varepsilon)X\Big) - B^\top(s_\varepsilon, w_\varepsilon)p \\ &-\mathscr{L}\varphi(s_\varepsilon, w_\varepsilon) - f\Big(s_\varepsilon, w_\varepsilon, u(s_\varepsilon, w_\varepsilon), (p + \partial_w \varphi(s_\varepsilon, w_\varepsilon))^\top \sigma(s_\varepsilon, w_\varepsilon)\Big) \leq 0 \end{aligned}$$

and

$$-\frac{1}{2}tr\Big(\Big(\sigma\sigma^\top\Big)(s_\varepsilon, z_\varepsilon)Z\Big) - B^\top(s_\varepsilon, z_\varepsilon)p - f\Big(s_\varepsilon, z_\varepsilon, \kappa(s_\varepsilon, z_\varepsilon), p^\top\sigma(s_\varepsilon, z_\varepsilon)\Big) \geq 0.$$

If $(e_1, e_2, ..., e_n)$ is an orthonormal basis of $\mathbb{R}^n$ then we have

$$\begin{aligned}
&tr\Big((\sigma\sigma)^\top(s_\varepsilon, w_\varepsilon)X\Big) - tr\Big(\Big(\sigma\sigma^\top\Big)(s_\varepsilon, z_\varepsilon)Z\Big)\\
&= \sum_{i=1}^{n}\Big[(\sigma(s_\varepsilon, w_\varepsilon)e_i)^\top(X\sigma(s_\varepsilon, w_\varepsilon)e_i) - (\sigma(s_\varepsilon, z_\varepsilon)e_i)^\top(Z\sigma(s_\varepsilon, z_\varepsilon)e_i)\Big] \leq \frac{3J^2}{\varepsilon}|w_\varepsilon - z_\varepsilon|^2.
\end{aligned}$$

It follows from assumptions (A.1) and (A.3) that $p^\top(B(s_\varepsilon, w_\varepsilon) - B(s_\varepsilon, z_\varepsilon)) \leq \frac{J}{\varepsilon}|w_\varepsilon - z_\varepsilon|^2$, and there exists a large enough constant R, such that $|w_\varepsilon|$, $|z_\varepsilon|$, $u(s_\varepsilon, w_\varepsilon)$, $\kappa(s_\varepsilon, z_\varepsilon) \leq R$.

For any fixed $0 < \theta < 1$, subtracting the viscosity inequalities for u and κ yields

$$\begin{aligned}
&-\varphi_s(s_\varepsilon, w_\varepsilon) - \mathscr{L}\varphi(s_\varepsilon, w_\varepsilon)\\
&\leq \frac{1}{2}tr\Big(\Big(\sigma\sigma^\top\Big)(s_\varepsilon, w_\varepsilon)X\Big) - \frac{1}{2}tr\Big(\Big(\sigma\sigma^\top\Big)(s_\varepsilon, z_\varepsilon)Z\Big) + p^\top(B(s_\varepsilon, w_\varepsilon) - B(s_\varepsilon, z_\varepsilon))\\
&+f\Big(s_\varepsilon, w_\varepsilon, u(s_\varepsilon, w_\varepsilon), (p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)\Big)\\
&-f\Big(s_\varepsilon, z_\varepsilon, u(s_\varepsilon, w_\varepsilon), (p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)\Big)\\
&+f\Big(s_\varepsilon, z_\varepsilon, u(s_\varepsilon, w_\varepsilon), (p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)\Big) - f\Big(s_\varepsilon, z_\varepsilon, \kappa(s_\varepsilon, z_\varepsilon), p^\top\sigma(s_\varepsilon, z_\varepsilon)\Big)\\
&\leq \frac{3J^2 + 2J}{2\varepsilon}|w_\varepsilon - z_\varepsilon|^2 + m_R\Big(|w_\varepsilon - z_\varepsilon|\Big(1 + |(p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)|\Big)\Big)\\
&+(1-\theta)f\left(s_\varepsilon, z_\varepsilon, \frac{u(s_\varepsilon, w_\varepsilon) - \theta\kappa(s_\varepsilon, z_\varepsilon)}{1-\theta}, \frac{(p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon) - \theta p^\top\sigma(s_\varepsilon, z_\varepsilon)}{1-\theta}\right).
\end{aligned}$$

Multiplying both sides by $(u(s_\varepsilon, w_\varepsilon) - \theta\kappa(s_\varepsilon, z_\varepsilon))^+$, we have

$$\begin{aligned}
&\Big(-\varphi_s(s_\varepsilon, w_\varepsilon) - \mathscr{L}\varphi(s_\varepsilon, w_\varepsilon)\Big)\Big(u(s_\varepsilon, w_\varepsilon) - \theta\kappa(s_\varepsilon, z_\varepsilon)\Big)^+\\
&\leq \left[\frac{3J^2 + 2J}{2\varepsilon}|w_\varepsilon - z_\varepsilon|^2 + m_R\Big(|w_\varepsilon - z_\varepsilon|\Big(1 + |(p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)|\Big)\Big)\right.\\
&+k(1-\theta)(1 + |z_\varepsilon|^q) + k(u(s_\varepsilon, w_\varepsilon) - \theta\kappa(s_\varepsilon, z_\varepsilon))^+ + k|\Big((\partial_w\varphi)^\top\sigma\Big)(s_\varepsilon, w_\varepsilon)|\\
&\left.+k\frac{|w_\varepsilon - z_\varepsilon|}{\varepsilon}|\sigma(s_\varepsilon, w_\varepsilon) - \theta\sigma(s_\varepsilon, z_\varepsilon)|\right]\Big(u(s_\varepsilon, w_\varepsilon) - \theta\kappa(s_\varepsilon, z_\varepsilon)\Big)^+\\
&+\delta|\Big((\partial_w\varphi)^\top\sigma\Big)(s_\varepsilon, w_\varepsilon)|^2 + \delta\frac{|w_\varepsilon - z_\varepsilon|^2}{\varepsilon^2}|\sigma(s_\varepsilon, w_\varepsilon) - \theta\sigma(s_\varepsilon, z_\varepsilon)|^2.
\end{aligned}$$

By sending $\theta \to 1$, we derive

$$\begin{aligned}
&(-\varphi_s(s_\varepsilon, w_\varepsilon) - \mathscr{L}\varphi(s_\varepsilon, w_\varepsilon))\Big(u(s_\varepsilon, w_\varepsilon) - \kappa(s_\varepsilon, z_\varepsilon)\Big)^+\\
&\leq\left\{\frac{3J^2 + 2J}{2\varepsilon}|w_\varepsilon - z_\varepsilon|^2 + m_R\Big(|w_\varepsilon - z_\varepsilon|\Big(1 + |(p + \partial_w\varphi(s_\varepsilon, w_\varepsilon))^\top\sigma(s_\varepsilon, w_\varepsilon)|\Big)\Big)\right.\\
&+k|u(s_\varepsilon, w_\varepsilon) - \kappa(s_\varepsilon, z_\varepsilon)| + k|\Big((\partial_w\varphi)^\top\sigma\Big)(s_\varepsilon, w_\varepsilon)|\\
&\left.+k\frac{|w_\varepsilon - z_\varepsilon|}{\varepsilon}|\sigma(s_\varepsilon, w_\varepsilon) - \sigma(s_\varepsilon, z_\varepsilon)|\right\}\Big|u(s_\varepsilon, w_\varepsilon) - \kappa(s_\varepsilon, z_\varepsilon)\Big|\\
&+\delta|\Big((\partial_w\varphi)^\top\sigma\Big)(s_\varepsilon, w_\varepsilon)|^2 + \delta\frac{|w_\varepsilon - z_\varepsilon|^2}{\varepsilon^2}|\sigma(s_\varepsilon, w_\varepsilon) - \sigma(s_\varepsilon, z_\varepsilon)|^2.
\end{aligned}$$

By sending $\varepsilon \to 0$, we have

$$\begin{aligned}&(-\varphi_s(s_0, w_0) - \mathscr{L}\varphi(s_0, w_0))(\varphi(s_0, w_0))^+\\ \leq&\Big\{k|\varphi(s_0, w_0)| + k\Big|\Big((\partial_w\varphi)^\top\sigma\Big)(s_0, w_0)|\Big\}|\varphi(s_0, w_0)| + \delta\Big|\Big((\partial_w\varphi)^\top\sigma\Big)(s_0, w_0)|^2.\end{aligned}$$

Therefore, h is a viscosity subsolution of (28). □

The following lemma aims to establish a viscosity supersolution of (28).

Lemma 5. *For any $A > 0$ and sufficiently large C, the function*

$$\mathcal{X}(s, w) := e^{(A+C(T-s))\left(1+|w|^2\right)}, \quad (s, w) \in [s_1, T] \times \mathbb{R}^n,$$

where $s_1 := T - \frac{A}{C}$, satisfies for all $(s, w) \in [s_1, T] \times \mathbb{R}^n$,

$$-\partial_s\mathcal{X}(s, w) - \mathscr{L}\mathcal{X}(s, w) - k\mathcal{X}(s, w) - k\Big|\Big(\sigma^\top\partial_w\mathcal{X}\Big)(s, w)| - \delta\frac{|(\sigma^\top\partial_w\mathcal{X})(s, w)|^2}{\mathcal{X}(s, w)} > 0.$$

Proof. Given $(s, w) \in [s_1, T] \times \mathbb{R}^n$, it is clear that $\partial_s\mathcal{X}(s, w) = -C\mathcal{X}(s, w)(1 + |w|^2)$, $\partial_w\mathcal{X}(s, w) = 2(C(T-s) + A)\mathcal{X}(s, w)w$, and $D^2\mathcal{X}(s, w) = 2(C(T-s) + A)\mathcal{X}(s, w)I + 4(C(T-s) + A)^2\mathcal{X}(s, w)ww^\top$. Consequently,

$$|\partial_w\mathcal{X}(s, w)| \leq 4A\mathcal{X}(s, w)|w| \text{ and } |D^2\mathcal{X}(s, w)| \leq 4A\mathcal{X}(s, w) + 16A^2\mathcal{X}(s, w)|w|^2.$$

Finally, we conclude that

$$\begin{aligned}&-\partial_s\mathcal{X}(s, w) - \mathscr{L}\mathcal{X}(s, w) - k\mathcal{X}(s, w) - k\Big|\Big(\sigma^\top\partial_w\mathcal{X}\Big)(s, w)| - \delta\frac{|(\sigma^\top\partial_w\mathcal{X})(s, w)|^2}{\mathcal{X}(s, w)}\\ \geq&\mathcal{X}(s, w)\Big[C(1 + |w|^2) - J^2\Big(2A + 8A^2|w|^2\Big) - 4AJ(1 + k + |w|)|w| - k - \delta 16A^2J^2|w|^2\Big].\end{aligned}$$

We complete the proof by taking C large enough. □

Proof of Theorem 5. Assume that u, κ are two viscosity solutions to PDE (19) satisfying (23). Define $h := u - \kappa$. In view of Lemma 5 and any fixed $\alpha > 0$, define $N(s, w) := (h(s, w) - \alpha\mathcal{X}(s, w))e^{ks}$, $(s, w) \in [s_1, T] \times \mathbb{R}^n$. Since for some $A > 0$, it holds that for each $s \in [s_1, T]$,

$$\lim_{|w|\to\infty} |h(s, w)|e^{-A|w|^2} = 0,$$

there exists a compact set $E \in \mathbb{R}^n$ satisfying that $N(s, w) < 0, \forall (s, w) \in [s_1, T] \times E^c$. Since N is continuous, $\max_{(s,w)\in[s_1,T]\times E} N(s, w)$ is achieved at some point $(\bar{s}, \bar{w})$. Suppose that $N(\bar{s}, \bar{w}) > 0$. Then, the maximum of N on $[s_1, T] \times \mathbb{R}^n$ is obtained at $(\bar{s}, \bar{w})$ and $\bar{s} < T$. Let us introduce a function,

$$\varphi(s, w) := \alpha\mathcal{X}(s, w) + (h(\bar{s}, \bar{w}) - \alpha\mathcal{X}(\bar{s}, \bar{w}))e^{k(\bar{s}-s)}, (s, w) \in [s_1, T] \times \mathbb{R}^n.$$

Then, $\varphi(\bar{s}, \bar{w}) > 0$ and $(h - \varphi)(s, w) \leq 0 = (h - \varphi)(\bar{s}, \bar{w})$ for all $(s, w) \in [s_1, T] \times \mathbb{R}^n$. According to Lemma 4,

$$((-\partial_s\varphi - \mathscr{L}\varphi)\varphi)(\bar{s}, \bar{w}) - k\varphi^2(\bar{s}, \bar{w}) - k\Big|\Big(\sigma^\top\partial_w\varphi\Big)(\bar{s}, \bar{w})|\varphi(\bar{s}, \bar{w}) - \delta\Big|\Big(\sigma^\top\partial_w\varphi\Big)(\bar{s}, \bar{w})|^2 \leq 0,$$

while it follows from the definition of φ and Lemma 5 that

$$\begin{aligned}&((-\partial_s\varphi-\mathscr{L}\varphi)\varphi)(\bar{s},\bar{w})-k\varphi^2(\bar{s},\bar{w})-k|\left(\sigma^\top\partial_w\varphi\right)(\bar{s},\bar{w})|\varphi(\bar{s},\bar{w})-\delta|\left(\sigma^\top\partial_w\varphi\right)(\bar{s},\bar{w})|^2\\&\geq\alpha h(\bar{s},\bar{w})\left[(-\partial_s\mathcal{X}-\mathscr{L}\mathcal{X}-k\mathcal{X})(\bar{s},\bar{w})-k|\left(\sigma^\top\partial_w\mathcal{X}\right)(\bar{s},\bar{w})|-\delta\frac{|(\sigma^\top\partial_w\mathcal{X})(\bar{s},\bar{w})|^2}{\mathcal{X}(\bar{s},\bar{w})}\right]\\&>0,\end{aligned}$$

which is a contradiction with the previous inequality. Therefore, $h(s,w)\leq\alpha\mathcal{X}(s,w)$, $\forall(s,w)\in[s_1,T]\times\mathbb{R}^n$. Sending $\alpha\to0$, we have that $u\leq\kappa$ on $[s_1,T]\times\mathbb{R}^n$.

Setting $s_2:=\left(s_1-\frac{A}{C}\right)^+$, we obtain analogously that $u\leq\kappa$ on $[s_2,s_1]\times\mathbb{R}^n$. Step by step, we prove that $u\leq\kappa$ on $[0,T]\times\mathbb{R}^n$. Ultimately, interchanging u and κ, we obtain that $\kappa\leq u$ on $[0,T]\times\mathbb{R}^n$. This completes the proof. □

5. Applications in Finance

5.1. Linking Robust Control to Stochastical Differential Utility

Since [33] pioneered the integration of robust control theory from engineering into economics, an increasing number of studies have investigated the impact of model uncertainty on portfolio choices and asset pricing (see [18,19,34–36]). In this paper, we exploit the methodology developed in [18], which showed the connection between a robust control criterion proposed by [34] and a form of the stochastic differential utility (SDU) of [8] without relying on any underlying dynamics. Let us consider a progressively measurable process, $F:[0,T]\times\Omega\times\mathbb{R}\mapsto\mathbb{R}$, a continuous function $\psi:(0,+\infty)\mapsto(0,+\infty)$ together with a terminal value ξ. X represents the set of $N\in\mathcal{L}^2(\mathbb{R}^{d\times1})$ fulfilling that $\mathscr{E}(N^\top\circ W)$ is a $\mathbb{P}$-martingale. For each $N\in X$, E^N denotes the expectation under the probability measure P^N defined by

$$\frac{dP^N}{d\mathbb{P}}\Big|_{\mathcal{F}_t}=\mathscr{E}(N^\top\circ W)_t.$$

We consider the following robust control criterion:

$$\hat{V}_t=\text{ ess inf}\left\{V_t^x:x\in\hat{X}\right\},\tag{29}$$

where

$$V_t^x=E^x\left[\xi+\int_t^T\left(F(s,V_s^x)+\frac{\psi(-V_s^x)}{2}|x_s|^2\right)ds\mid\mathcal{F}_t\right],\tag{30}$$

and

$$\hat{X}:=\{x\in X:\text{ a.s. }P^x,V^x<0\text{ is bounded }\}.$$

In the framework of [18], $\hat{V}$ equals the SDU V that satisfies the BSDE,

$$V_t=\xi+\int_t^T F(r,V_r)-\frac{|M_r|^2}{2\psi(-V_r)}dr-\int_t^T M_rdW_r,\ t\in[0,T],\tag{31}$$

provided that in a properly defined space BSDE (31) has a unique solution, (V,M). The following proposition fills this gap according to our existence and uniqueness results.

Proposition 1. *Assume ξ and the generator of (31) satisfy the following conditions:*

- *ξ is bounded and $\xi<-D$ for some constant $D>0$.*
- *Given any fixed (s,ω), F is concave and continuous and satisfies*

$$\forall y<0,\quad-\alpha_s(\omega)-\beta_s(\omega)\phi(-y)\leq F(s,\omega,y)\leq0,$$

in which α and β are non-negative processes, such that $\int_0^T \alpha_s ds$ and $\int_0^T \beta_s ds$ are bounded, and where ϕ, $\psi : (0, +\infty) \mapsto (0, +\infty)$ satisfying $\phi \in \mathcal{C}^1((0, +\infty))$ is convex, ψ is concave, increasing, and continuous, ϕ/ψ is increasing, and $(\mathbb{1}_{x>0}/\psi(x))_{x\in\mathbb{R}}$ is locally integrable on $\mathbb{R}$.

Then, BSDE (31) admits a unique negative solution (V, M), such that $V \in \mathcal{S}^\infty(\mathbb{R})$ and $M \circ W \in BMO$. With regard to the robust control criterion (29), if $\hat{x} = -\frac{1}{\psi(-V)} M^\top$ then $\hat{x} \in \hat{X}$, and

$$\hat{V} = V^{\hat{x}} = V.$$

Proof. Clearly, (V, M) solves (31) if and only if $(\bar{V}, \bar{M}) := (-V, -M)$ solves

$$\bar{V}_t = -\xi + \int_t^T \left(-F(s, -\bar{V}_s) + \frac{|\bar{M}_s|^2}{2\psi(\bar{V}_s)} \right) ds - \int_t^T \bar{M}_s dW_s,\ t \in [0\, T],$$

which admits a unique solution according to Sections 2 and 3. It remains to prove the last argument.

We first show that for any $x \in \hat{X}$, $V \le V^x$. According to the martingale representation theorem, there exists $M^x \in \mathcal{L}^2$ fulfilling

$$V_t^x = \xi + \int_t^T F(s, V_s^x) + \frac{\psi(-V_s^x)}{2} |x_s|^2 ds - \int_t^T M_s^x dW_s^x. \tag{32}$$

Note that (31) can be reformulated as

$$V_t = \xi + \int_t^T F(s, V_s) - \frac{|M_s|^2}{2\psi(-V_s)} - M_s x_s ds - \int_t^T M_s dW_s^x,\ t \in [0, T].$$

Combining dynamics for V^x and V, we derive from Tanaka's formula that for each fixed $0 < \theta < 1$,

$$\begin{aligned}
&(\theta V_t - V_t^x)^+ \\
=&(\theta - 1)\xi + \int_t^T \mathbb{1}_{\theta V_s > V_s^x} \Big\{ \Big[\theta F(s, V_s) - F(s, V_s^x) \Big] - \frac{\theta\psi(-V_s)}{2} |M_s \frac{1}{\psi(-V_s)} + x_s|^2 \\
&+ \frac{|x_s|^2}{2} \Big(\theta\psi(-V_s) - \psi(-V_s^x) \Big) \Big\} ds - \int_t^T \mathbb{1}_{\theta V_s > V_s^x} (\theta M_s - M_s^x) dW_s^x - \frac{1}{2} \int_t^T dL_s^0 (\theta V - V^x) \\
\le&(\theta - 1)\xi + \int_t^T \mathbb{1}_{\theta V_s > V_s^x} (1 - \theta) \left(\alpha_s + \beta_s \phi(\frac{\theta V_s - V_s^x}{1 - \theta}) \right) ds - \int_t^T \mathbb{1}_{\theta V_s > V_s^x} (\theta M_s - M_s^x) dW_s^x.
\end{aligned}$$

To eliminate $\phi(\frac{\theta V_s - V_s^x}{1-\theta})$, we define

$$G(y) := \int_2^{2+y} \frac{dr}{\phi(r)},\ y > -2.$$

Obviously, $G \in \mathcal{C}^2((-2, +\infty))$ and G is strictly increasing. Applying Itô's formula to

$$P_t := G^{-1} \left(G\left(\frac{(\theta V_t - V_t^x)^+}{1 - \theta} \right) + \int_0^t \left(\frac{\alpha_s}{\phi(1)} + \beta_s \right) ds \right)$$

and denoting

$$Q_t := \frac{\phi(2 + P_t)}{\phi(2 + \frac{(\theta V_t - V_t^x)^+}{1-\theta})} \mathbb{1}_{\theta V_t > V_t^x} \frac{\theta M_t - M_t^x}{1 - \theta}$$

yield that

$$P_t \le G^{-1}\left(G(-\xi) + \int_0^T \left(\frac{\alpha_s}{\phi(1)} + \beta_s\right) ds\right) - \int_t^T Q_s dW_s^x.$$

Provided that V^x, $\int_0^T \alpha_s ds$, and $\int_0^T \beta_s ds$ are bounded, it is apparent from a classical localization technique and the dominated convergence theorem that

$$(\theta V_t - V_t^x)^+ \le (1-\theta) E_t^x \left[G^{-1}\left(G(-\xi) + \int_0^T \left(\frac{\alpha_s}{\phi(1)} + \beta_s\right) ds\right)\right].$$

Sending $\theta \to 1$, we obtain that for any $x \in \hat{X}$, $V \le V^x$. Now, we declare that if $\hat{x} = -\frac{1}{\psi(-V)} M^\top$ then $V^{\hat{x}} \le V$, which can be verified through applying Tanaka's formula to $\left(\theta V^{\hat{x}} - V\right)^+$.

In conclusion, if $\hat{x} \in \hat{X}$ then $\hat{V} = V^{\hat{x}} = V$. Note that $V \in \mathcal{S}^\infty$ and $M \circ W \in BMO$; thus, $\hat{x} \in X$. Because of the boundness of V, ξ, $\int_0^T \alpha_s ds$, and $\int_0^T \beta_s ds$, we have that $V^{\hat{x}}$ is bounded a.s. $P^{\hat{x}}$. This completes the proof. □

Example 1. *As an example, consider the Epstein–Zin-type stochastic differential utility used in [19], which is*

$$F(t, V_t^x) = \frac{1}{1-\rho}\left\{\frac{c_t^{1-\rho}}{((1-\gamma)V_t^x)^{\frac{\gamma-\rho}{1-\gamma}}} - \eta(1-\gamma)V_t^x\right\},$$

where $\rho > 1$, $\gamma > 1$ and $\eta \ge 0$.

Suppose we plug this process back into (30). It is easy to check that Proposition 1 still holds when $\gamma < \rho$, $\psi(x) \le \frac{\gamma-1}{\rho-\gamma} x$, $\forall x > 0$ and $\int_0^T c_t^{1-\rho} dt$ is bounded. This result is a slight extension of that in [19].

5.2. Certainty Equivalent Based on g-Expectation

We take an example to show that the value process of the BSDE considered in this paper can represent a certainty equivalent of the terminal value based on g-expectation.

Example 2. *Suppose that γ is a real-valued non-negative stochastic process, $g(t, z) := \gamma_t |z|$ is a generator, and $\xi < 0$ is a terminal condition. Consider a utility function u defined as*

$$u(y) := e^{2\sqrt{-y}}\left(\frac{1}{2} - \sqrt{-y}\right) - \frac{1}{2}, \quad y < 0.$$

For $y < 0$, $u'(y) = e^{2\sqrt{-y}}$, $u''(y) = \frac{1}{\sqrt{-y}} e^{2\sqrt{-y}}$. If

$$\xi < 0, \mathbb{E}\left[\exp\left\{\frac{p}{p-1}\int_0^T \gamma_s^2 ds\right\}\left(1 + e^{4p\sqrt{-\xi}}(-\xi)^p\right)\right] < +\infty \text{ for some } p > 1$$

then there is only one solution $(\bar{Y}, \bar{Z})$ in $\mathcal{S}^{2p} \times \mathcal{M}^p$ satisfying the BSDE,

$$\bar{Y}_t = u(\xi) + \int_t^T (-\gamma_s |\bar{Z}_s|) ds - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T].$$

We denote the conditional g-expectation of $u(\xi)$ under $\mathcal{F}_t$ by $\varepsilon_t^g[u(\xi)] := \bar{Y}_t$ (see more about g-expectation in [37]). The certainty equivalent of ξ at time $t \in [0, T]$ based on g-expectation is defined as

$$C_t(\xi) := u^{-1}\left(\varepsilon_t^g[u(\xi)]\right).$$

Putting $Z_t = \frac{1}{u'\left(u^{-1}(\bar{Y}_t)\right)}\bar{Z}_t$, *the pair* $(C(\xi),\, Z) \in \mathcal{S}^{2p} \times \mathcal{M}^p$ *is the solution to the BSDE,*

$$Y_t = \xi - \int_t^T \left(\gamma_r |Z_r| + \frac{1}{2}\frac{|Z_r|^2}{\sqrt{-Y_r}}\right) dr - \int_t^T Z_r dW_r, \quad t \in [0,\, T],$$

in which $\frac{1}{\sqrt{-y}}$ *represents the coefficient of absolute risk aversion of* u *at level* y.

6. Discussion

In this article, we establish the existence and comparison theorem for L^p solutions to a class of backward stochastic differential equations (BSDEs) with singular generators, extending the results of [11]. Additionally, we investigate the stability property, derive the Feynman–Kac formula, and provide the uniqueness of viscosity solutions to the associated partial differential equations (PDEs). More importantly, we demonstrate an application in the context of robust control linked to stochastic differential utility, slightly generalizing the results of [18,19]. Additionally, we present an example involving certainty equivalents. In our two applications, the quadratic coefficients in the generators, respectively, quantify ambiguity aversion and absolute risk aversion. Future research will investigate the analysis and applications of singular equations driven by other noise disturbance, such as the well-posedness and stabilities of mean-field equations driven by G-Brownian motion (see, e.g., [38,39]) and stabilization issues in stochastic nonlinear delay systems driven by Lévy processes (see, e.g., [40]).

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References

1. Pardoux, E.; Peng, S. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **1990**, *14*, 55–61. [CrossRef]
2. Briand, P.; Hu, Y. BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Relat. Field* **2006**, *136*, 604–618. [CrossRef]
3. Briand, P.; Hu, Y. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probab. Theory Relat. Field* **2008**, *141*, 543–567. [CrossRef]
4. Tevzadze, R. Solvability of backward stochastic differential equations with quadratic growth. *Stoch. Process. Their Appl.* **2008**, *118*, 503–515. [CrossRef]
5. Barrieu, P.; Karoui, N.E. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.* **2013**, *41*, 1831–1863. [CrossRef]
6. Briand, P.; Richou, A. On the uniqueness of solutions to quadratic BSDEs with non-convex generators. In *Frontiers in Stochastic Analysis–BSDEs, SPDEs and Their Applications*; Cohen, S.N., Gyöngy, I., dos Reis, G., Siska, D., Szpruch, Ł., Eds.; Springer: Cham, Switzerland, 2019; pp. 89–107.
7. Kobylanski, M. Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **2000**, *28*, 558–602. [CrossRef]
8. Duffie, D.; Epstein, L.G. Stochastic differential utility. *Econometrica* **1992**, *60*, 353–394. [CrossRef]

9. Bahlali, K.; Eddahbi, M.; Ouknine, Y. Quadratic BSDE with $\mathbb{L}^2$-terminal data: Krylov's estimate, Itô-Krylov's formula and existence results. *Ann. Probab.* **2017**, *45*, 2377–2397. [CrossRef]
10. Essaky, E.; Hassani, M. General existence results for reflected BSDE and BSDE. *Bull. Sci. Math.* **2011**, *135*, 442–466. [CrossRef]
11. Bahlali, K.; Tangpi, L. BSDEs Driven by $|z|^2/y$ and Applications to PDEs and Decision Theory. Available online: https://arxiv.org/pdf/1810.05664.pdf (accessed on 1 June 2021).
12. Laeven, R.J.; Gianin, E.R.; Zullino, M. Geometric BSDEs. Available online: https://arxiv.org/pdf/2405.09260 (accessed on 1 February 2025).
13. Zheng, S. Well-Posedness of Quadratic RBSDEs and BSDEs with One-Sided Growth Restrictions. Available online: https://arxiv.org/pdf/2412.21172 (accessed on 1 February 2025).
14. Vázquez, J.L. *The Porous Medium Equation: Mathematical Theory*; Oxford University Press: New York, NY, USA, 2007; ISBN 978-0-19-856903-9.
15. Giachetti, D.; Petitta, F.; De León, S.S. Elliptic equations having a singular quadratic gradient term and a changing sign datum. *Commun. Pure Appl. Anal.* **2012**, *11*, 1875–1895. [CrossRef]
16. Shamarova, E. Solutions with positive components to quasilinear parabolic systems. *J. Math. Anal. Appl.* **2024**, *536*, 128243. [CrossRef]
17. Ma, H.; Tian, D. Generalized entropic risk measures and related BSDEs. *Stat. Probab. Lett.* **2021**, *174*, 109–110. [CrossRef]
18. Skiadas, C. Robust control and recursive utility. *Financ. Stoch.* **2003**, *7*, 475–489. [CrossRef]
19. Maenhout, P.J. Robust portfolio rules and asset pricing. *Rev. Financ. Stud.* **2004**, *17*, 951–983. [CrossRef]
20. Crandall, M.; Ishii, H.; Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* **1992**, *27*, 1–67. [CrossRef]
21. Bahlali, K.; Eddahbi, M.; Ouknine, Y. Solvability of some quadratic BSDEs without exponential moments. *C. R. Math.* **2013**, *351*, 229–233. [CrossRef]
22. Bahlali, K. Solving Unbounded Quadratic BSDEs by a Domination Method. Available online: https://arxiv.org/pdf/1903.11325 (accessed on 1 June 2021).
23. Bahlali, K.; Essaky, E.; Hassani, M. Existence and uniqueness of multidimensional BSDEs and of systems of degenerate PDEs with superlinear growth generator. *SIAM J. Math. Anal.* **2015**, *47*, 4251–4288. [CrossRef]
24. Yang, H. Lp Solutions of Quadratic BSDEs. Available online: https://arxiv.org/pdf/1506.08146v1.pdf (accessed on 1 June 2021).
25. Bri, P.; Delyon, B.; Hu, Y.; Pardoux, E.; Stoica, L. L^p solutions of backward stochastic differential equations. *Stoch. Process. Appl.* **2003**, *108*, 109–129.
26. Fan, S.; Hu, Y. Well-posedness of scalar BSDEs with sub-quadratic generators and related PDEs. *Stoch. Process. Appl.* **2021**, *131*, 21–50. [CrossRef]
27. Delbaen, F.; Hu, Y.; Richou, A. On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **2011**, *47*, 559–574. [CrossRef]
28. Lepeltier, J.P.; San Martin, J. Backward stochastic differential equations with continuous coefficient. *Stat. Probab. Lett.* **1997**, *32*, 425–430. [CrossRef]
29. El Karoui, N.; Peng, S.; Quenez, M.C. Backward stochastic differential equations in finance. *Math. Financ.* **1997**, *7*, 1–71. [CrossRef]
30. Barles, G.; Buckdahn, R.; Pardoux, E. Backward stochastic differential equations and integral-partial differential equations. *Stochastics* **1997**, *60*, 57–83. [CrossRef]
31. Bardi, M.; Dolcetta, I.C. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*; Birkhäuser: Boston, MA, USA, 1997; Volume 12, ISBN 978-0-8176-4754-4.
32. Lions, P.L. Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations part 2: Viscosity solutions and uniqueness. *Comm. Partial Differ. Equ.* **1983**, *8*, 1229–1276. [CrossRef]
33. Hansen, L.P.; Sargent, T.J. Discounted linear exponential quadratic gaussian control. *IEEE Trans. Autom. Control.* **1995**, *40*, 968–971. [CrossRef]
34. Hansen, L.P.; Sargent, T.J.; Turmuhambetova, G.; Williams, N. Robustness and Uncertainty Aversion. Available online: https://pages.stern.nyu.edu/~dbackus/Exotic/1Robustness/HSTW%20robust%20Apr%2002.pdf (accessed on 1 June 2021).
35. Yang, Z.; Liang, G.; Zhou, C. Constrained portfolio-consumption strategies with uncertain parameters and borrowing costs. *Math. Financ. Econ.* **2019**,*13*, 393–427. [CrossRef]
36. Pu, J.; Zhang, Q. Robust consumption portfolio optimization with stochastic differential utility. *Automatica* **2021**, *133*, 109835. [CrossRef]
37. Peng, S. Backward SDE and related g-expectation. In *Backward Stochastic Differential Equations*; Pitman Research Notes in Mathematics Series, No. 364; Karoui, N.E., Mazliak, L., Eds.; Longman: Essex, UK, 1997; pp. 141–159.
38. Yuan, H.; Zhu, Q. The well-posedness and stabilities of mean-field stochastic differential equations driven by G-Brownian motion. *SIAM J. Control Optim.* **2025**, *63*, 596–624. [CrossRef]

39. Bollweg, K.W.G.; Meyer-Brandis, T. Mean-field stochastic differential equations driven by G-Brownian motion. *Probab. Uncertain. Quant. Risk* **2025**, *10*, 241–264. [CrossRef]
40. Zhu, Q. Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Lévy processes. *IEEE Trans. Autom. Control.* **2025**, *70*, 1176–1183. [CrossRef]

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