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Novel Analytical Methods for and Qualitative Analysis of the Generalized Water Wave Equation

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Abstract

For a significant fluid model and the truncated M-fractional (1 + 1)-dimensional nonlinear generalized water wave equation, distinct types of truncated M-fractional wave solitons are obtained. Ocean waves, tidal waves, weather simulations, river and irrigation flows, tsunami predictions, and more are all explained by this model. We use the improved (G'/G) expansion technique and a modified extended direct algebraic technique to obtain these solutions. Results for trigonometry, hyperbolic, and rational functions are obtained. The impact of the fractional-order derivative is also covered. We use Mathematica software to verify our findings. Furthermore, we use contour graphs in two and three dimensions to illustrate some wave solitons that are obtained. The results obtained have applications in ocean engineering, fluid dynamics, and other fields. The stability analysis of the considered equation is also performed. Moreover, the stationary solutions of the concerning equation are studied through modulation instability. Furthermore, the used methods are useful for other nonlinear fractional partial differential equations in different areas of applied science and engineering.

Keywords: generalized water wave model; fractional derivative; analytical methods; exact solitons; stability analysis; modulation instability analysis

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1. Introduction

Fluid mechanics is an important branch of applied science and engineering. Many models in the field of fluid mechanics have been developed such as the unstable nonlinear Schrödinger model [1], nano-ionic currents model [2], Sharma–Tasso–Olver model [3], Cahn–Allen model [4], simplified modified Camassa–Holm model [5], improved Boussinesq model [6], Boussinesq–Burgers system [7], higher-order Schrödinger equation [8], Whitham–Broer–Kaup equations [9], and perturbed Schrödinger–Hirota equation [10]. Nowadays, fractional models have garnered interest among researchers, such as the generalized Korteweg–de Vries–Zakharov–Kuznetsov model [11], the single-joint robot arm model [12], the Westervelt model [13], the Pareto probability Shannon isotope model [14], the generalized coupled Helmholtz model [15], and the Estevez–Mansfield–Clarkson model [16]. Different methods have been developed to obtain different types of solutions, such as the plane dynamic system method [17], the polynomial complete discriminant system method [18], and the auxiliary function method [19].

We use two simple yet effective methods in this work: the improved (G'/G) expansion and the modified extended direct algebraic method. There are several uses for these techniques. For instance, some new kinds of traveling wave solutions of the Hirota–Ramani model were obtained by using the modified extended direct algebraic technique [20]; distinct wave solutions of the concatenation model were achieved [21]. Many wave solitons of extended shallow water wave equations were obtained by applying the improved (G'/G) expansion method [22], and many exact traveling wave solutions of the Calogero–Bogoyavlenskii-Schiff equation were obtained by utilizing the improved (G'/G) expansion scheme [23].

The nonlinear (1 + 1)-dimensional generalized water wave equation is an important model among the mathematical fluid models. In fluid dynamics, engineering, and other domains, this model is highly significant. This model has been solved using various methods. For instance, the $(\frac{G'}{G'+G+A})$ expansion technique in [24] yielded some new exact wave solutions, while a class $f(G, G')$ expansion schemes in [25] yielded distinct types of exact wave solitons; the modified rational sine–cosine method in [26] provides breathing solutions, kink-singular solutions, and complex-valued solutions.

Our motivation was to study new types of truncated M-fractional exact solitons for the nonlinear (1 + 1)-dimensional generalized shallow water wave equation by using two effective techniques. Both the improved (G'/G) expansion technique and the modified extended direct algebraic technique provide different types of exact soliton solutions. These techniques have not been previously applied for this model. The truncated M-fractional derivative (TMFD) gives solutions closer to the numerical solutions. This fractional derivative satisfies the conditions of both fractional derivatives and integer-order derivatives. The impact of the TMFD on the achieved results is displayed. The truncated M-fractional generalized water wave equation is the more prominent form of the model and helps us to understand the model more clearly. Additionally, stability analysis was used to verify the stability of a relevant equation. Moreover, the stationary solutions of the concerning equation were studied through modulation instability.

This paper consists of several sections: Section 2 explains our techniques, Section 3 discusses the studied equation and its mathematical analysis, Section 4 provides exact solitons, Section 5 provides a graphic representation of some of the results obtained, Section 6 provides the stability analysis, Section 7 explains the modulation instability analysis, Section 8 explains the results and provides a discussion, and Section 9 concludes this paper.

2. Truncated M-Fractional Derivative

Definition 1. *Suppose $g(t) : [0, \infty) \to \Re$; then, the truncated M derivative of g of order α is given as [27]*

$$D_{M,t}^{\alpha,\Upsilon} g(t) = \lim_{\tau \to 0} \frac{g\left(t\, E_\Upsilon(\tau t^{1-\alpha})\right) - g(t)}{\tau}, \quad 0 < \alpha < 1, \;\; \Upsilon > 0,$$

where $E_\Upsilon(.)$ is the truncated Mittag–Leffler function of one parameter, which is defined as [28]

$$E_\Upsilon(z) = \sum_{j-0}^{\infty} \frac{z^j}{\Gamma(\Upsilon j + 1)}, \;\; \Upsilon > 0 \;\; and \;\; z \in \boldsymbol{C}.$$

Characteristics:

Let $\alpha \in (0, 1], \Upsilon > 0, r, s \in \Re$, and g, f are α- differentiable at a point $t > 0$; then, by [27]:

$$\text{(i)}\; D_{M,t}^{\alpha,\Upsilon}(rg(t) + sf(t)) = rD_{M,t}^{\alpha,\Upsilon} g(t) + sD_{M,t}^{\alpha,\Upsilon} f(t).$$

$$\text{(ii) } D_{M,t}^{\alpha,\Upsilon}(g(t).f(t)) = g(t)D_{M,t}^{\alpha,\Upsilon}f(t) + f(t)D_{M,t}^{\alpha,\Upsilon}g(t).$$

$$\text{(iii) } D_{M,t}^{\alpha,\Upsilon}\left(\frac{g(t)}{f(t)}\right) = \frac{f(t)D_{M,t}^{\alpha,\Upsilon}g(t) - g(t)D_{M,t}^{\alpha,\Upsilon}f(t)}{(f(t))^2}.$$

$$\text{(iv) } D_{M,t}^{\alpha,\Upsilon}(A) = 0, \text{ where } A \text{ is a constant.}$$

$$\text{(v) } D_{M,t}^{\alpha,\Upsilon}g(t) = \frac{t^{1-\alpha}}{\Gamma(\Upsilon+1)}\frac{dg(t)}{dt}.$$

3. Description of Strategies

3.1. MEDA Technique

Here, we provide the main steps of the scheme [29]:

Step 1:

Assume a nonlinear partial fractional differential equation:

$$Y(g, D_{M,t}^{\epsilon,\varrho}g^2, g^2g_x, g_{xx}, g_{xt}, \ldots) = 0, \tag{1}$$

where a wave profile is represented by $g = g(x,t)$. Examine the following wave transformations:

$$g(x,t) = G(\Theta), \ \ \Theta = \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + \lambda t^\alpha). \tag{2}$$

Here, λ represents the soliton velocity.

Putting Equation (2) into Equation (1) provides

$$Y(G, G^2G', G'', \ldots) = 0. \tag{3}$$

Step 2:

Assume the solution of Equation (3) is as follows:

$$G(\Theta) = \sum_{-m}^{m} \alpha_s \psi^s(\Theta). \tag{4}$$

Here, $\alpha_s (s = -m, \ldots, 0, 1, 2, 3, \ldots, m)$ are unknowns to be found, where $\alpha_s \neq 0$. A positive integer "m" can be obtained by balancing the highest-order term and the nonlinear term in Equation (3) according to the homogenous balance technique. A function $\psi(\Theta)$ fulfills the ordinary differential equation given as

$$\psi'(\Theta) = \log(d)\Big(p + q\psi(\Theta) + r\psi(\Theta)^2\Big). \tag{5}$$

Here, p, q, and r denote the constants, and $d \neq 0, 1$. Equation (5) provides the answers for the various situations listed below:

Type I: when $q^2 - 4pr < 0$ and r is nonzero, then

$$\psi_1(\Theta) = -\frac{q}{2r} + \frac{\sqrt{-(q^2-4pr)}\tan_B(\frac{1}{2}\sqrt{-(q^2-4pr)}\Theta)}{2r}. \tag{6}$$

$$\psi_2(\Theta) = -\frac{q}{2r} - \frac{\sqrt{4pr-q^2}\cot_B(\frac{1}{2}\sqrt{4pr-q^2}\Theta)}{2r}. \tag{7}$$

$$\psi_3(\Theta) = -\frac{q}{2r} + \frac{\sqrt{4pr-q^2}(\tan_B(\sqrt{4pr-q^2}\Theta) \pm (\sqrt{cf}\ \sec_B(\sqrt{4pr-q^2}\Theta)))}{2r}. \tag{8}$$

$$\psi_4(\Theta) = -\frac{q}{2r} - \frac{\sqrt{4pr-q^2}(\cot_B(\sqrt{4pr-q^2}\Theta) \pm (\sqrt{cf}\ \csc_B(\sqrt{4pr-q^2}\Theta)))}{2r}. \quad (9)$$

$$\psi_5(\Theta) = -\frac{q}{2r} + \frac{\sqrt{4pr-q^2}(\tan_B(\frac{1}{4}\sqrt{4pr-q^2}\Theta) - (\ \cot_B(\frac{1}{4}\sqrt{4pr-q^2}\Theta)))}{2r}. \quad (10)$$

Type II: when $q^2 - 4pr > 0$ and r is nonzero, we have

$$\psi_6(\Theta) = -\frac{q}{2r} - \frac{\sqrt{q^2-4pr}\tanh_B(\frac{1}{2}\sqrt{q^2-4pr}\Theta)}{2r}. \quad (11)$$

$$\psi_7(\Theta) = -\frac{q}{2r} - \frac{\sqrt{q^2-4pr}\coth_B(\frac{1}{2}\sqrt{q^2-4pr}\Theta)}{2r}. \quad (12)$$

$$\psi_8(\Theta) = -\frac{q}{2r} - \frac{\sqrt{q^2-4pr}(\tanh_B(\sqrt{q^2-4pr}\Theta) \pm (\sqrt{cf}\ sech_B(\sqrt{q^2-4pr}\Theta)))}{2r}. \quad (13)$$

$$\psi_9(\Theta) = -\frac{q}{2r} - \frac{\sqrt{q^2-4pr}(\coth_B(\sqrt{q^2-4pr}\Theta) \pm (\sqrt{cf}\ csch_B(\sqrt{q^2-4pr}\Theta)))}{2r}. \quad (14)$$

$$\psi_{10}(\Theta) = -\frac{q}{2r} - \frac{\sqrt{q^2-4pr}(\tanh_B(\frac{1}{4}\sqrt{q^2-4pr}\Theta) - (\ \coth_B(\frac{1}{4}\sqrt{q^2-4pr}\Theta)))}{2r}. \quad (15)$$

Type III: when $pr > 0$ and q is zero, then

$$\psi_{11}(\Theta) = \sqrt{\frac{p}{r}}\tan_B(\sqrt{pr}\Theta). \quad (16)$$

$$\psi_{12}(\Theta) = -\sqrt{\frac{p}{r}}\cot_B(\sqrt{pr}\Theta). \quad (17)$$

$$\psi_{13}(\Theta) = \sqrt{\frac{p}{r}}(\tan_B(2\sqrt{rp}\ \Theta) \pm (\sqrt{fc}\sec_B(2\sqrt{rp}\ \Theta))). \quad (18)$$

$$\psi_{14}(\Theta) = -\sqrt{\frac{p}{r}}(\cot_B(2\sqrt{rp}\ \Theta) \pm (\sqrt{fc}\csc_B(2\sqrt{rp}\ \Theta))). \quad (19)$$

$$\psi_{15}(\Theta) = \frac{1}{2}\sqrt{\frac{p}{r}}(\tan_B(\frac{1}{2}\sqrt{pr}\Theta) - \cot_B(\frac{1}{2}\sqrt{pr}\Theta)). \quad (20)$$

Type IV: when $pr < 0$ and q is zero, then

$$\psi_{16}(\Theta) = -\sqrt{-\frac{p}{r}}\tanh_B(\sqrt{-pr}\Theta). \quad (21)$$

$$\psi_{17}(\Theta) = -\sqrt{-\frac{p}{r}}\coth_B(\sqrt{-pr}\Theta). \quad (22)$$

$$\psi_{18}(\Theta) = -\sqrt{-\frac{p}{r}}(\tanh_B(2\sqrt{-pr}\Theta) \pm \iota\sqrt{cf}\,\mathrm{sech}_B(2\sqrt{-pr}\Theta)). \quad (23)$$

$$\psi_{19}(\Theta) = -\sqrt{-\frac{p}{r}}(\coth_B(2\sqrt{-pr}\Theta) \pm (\sqrt{cf}\,\mathrm{csch}_B(2\sqrt{-pr}\Theta))). \quad (24)$$

$$\psi_{20}(\Theta) = -\frac{1}{2}\sqrt{-\frac{p}{r}}(\tanh_B(\frac{1}{2}\sqrt{-pr}\Theta) + \coth_B(\frac{1}{2}\sqrt{-pr}\Theta)). \quad (25)$$

Type V: when $r = p$ and $q = 0$, we attain

$$\psi_{21}(\Theta) = \tan_B(p\Theta). \quad (26)$$

$$\psi_{22}(\Theta) = -\cot_B(p\Theta). \quad (27)$$

$$\psi_{23}(\Theta) = \tan_B(2p\Theta) \pm \sqrt{cf}\sec_B(2p\Theta). \tag{28}$$

$$\psi_{24}(\Theta) = -\cot_B(2p\Theta) \pm \sqrt{cf}\csc_B(2p\Theta). \tag{29}$$

$$\psi_{25}(\Theta) = \frac{1}{2}\tan_B(\frac{1}{2}p\Theta) - \frac{1}{2}\cot_B(\frac{1}{2}p\Theta). \tag{30}$$

Type VI: when $r = -p$ and q is zero, we have

$$\psi_{26}(\Theta) = -\tanh_B(p\Theta). \tag{31}$$

$$\psi_{27}(\Theta) = -\coth_B(p\Theta). \tag{32}$$

$$\psi_{28}(\Theta) = -\tanh_B(2p\Theta) \pm \iota\sqrt{cf}\,\mathrm{sech}_B(2p\Theta). \tag{33}$$

$$\psi_{29}(\Theta) = -\coth_B(2p\Theta) \pm \sqrt{cf}\,\mathrm{csch}_B(2p\Theta). \tag{34}$$

$$\psi_{30}(\Theta) = -\frac{1}{2}\tanh_B(\frac{1}{2}p\Theta) - \frac{1}{2}\coth_B(\frac{1}{2}p\Theta). \tag{35}$$

Type VII: when $q^2 - 4pr = 0$, then

$$\psi_{31}(\Theta) = -2\frac{p(q\Theta\log(d) + 2)}{q^2\Theta\log(d)}. \tag{36}$$

Type VIII: when $q = \rho$, $p = m\rho$ $(m \neq 0)$ and r is zero, one has

$$\psi_{32}(\Theta) = d^{\rho\Theta} - m. \tag{37}$$

Type IX: if $q = r = 0$, we have

$$\psi_{33}(\Theta) = p\ \Theta\ \log(d). \tag{38}$$

Type X: if $p = q = 0$, we have

$$\psi_{34}(\Theta) = -\frac{1}{\Theta\ \log(d)\ r}. \tag{39}$$

Type XI: when p is zero while q and r are nonzero, we have

$$\psi_{35}(\Theta) = -\frac{cq}{r(\cosh_B(q\Theta) - \sinh_B(q\Theta) + c)}. \tag{40}$$

$$\psi_{36}(\Theta) = -\frac{q(\cosh_B(q\Theta) + \sinh_B(q\Theta))}{r(\cosh_B(q\Theta) + \sinh_B(q\Theta) + f)}. \tag{41}$$

Type XII: when $q = \rho$, $r = m\rho$ $(m \neq 0)$, and p is zero, then

$$\psi_{37}(\Theta) = \frac{cd^{\rho\Theta}}{c - mfd^{\rho\Theta}}. \tag{42}$$

where the deformation parameters are represented by $c, f > 0$.

Step 3:

Enter Equations (4) and (5) into Equation (3) to start this step. The system of equations in $\alpha_s(s = -m, \ldots, 1, 2, 3, \ldots, m)$ and λ can then be obtained by adding the coefficients of every $\psi^s(s = -m, \ldots, 1, 2, 3, \ldots, m)$ term and setting the coefficients of the same power to 0. The results for the unknowns can be obtained by adjusting this collection of equations.

Step 4:

Now that we know the values of $\alpha_s (s = -m, \ldots, 1, 2, 3, \ldots, m)$ and λ, we can insert Equation (4) into Equation (1). The solutions to Equation (1) can be obtained from Equation (3).

The generalized trigonometric and hyperbolic functions are described as follows:

$$\begin{aligned} \cos_B(\Theta) &= \frac{cB^{\iota\Theta} + fB^{-\iota\Theta}}{2}, & \sin_B(\Theta) &= \frac{cB^{\iota\Theta} - fB^{-\iota\Theta}}{2\iota}, \\ \cot_B(\Theta) &= \iota\frac{cB^{\iota\Theta} + fB^{-\iota\Theta}}{cB^{\iota\Theta} - fB^{-\iota\Theta}}, & \tan_B(\Theta) &= -\iota\frac{cB^{\iota\Theta} - fB^{-\iota\Theta}}{cB^{\iota\Theta} + fB^{-\iota\Theta}}, \\ \csc_B(\Theta) &= \frac{2\iota}{cB^{\iota\Theta} - fB^{-\iota\Theta}}, & \sec_B(\Theta) &= \frac{2}{cB^{\iota\Theta} + fB^{-\iota\Theta}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \cosh_B(\Theta) &= \frac{cB^{\Theta} + fB^{-\Theta}}{2}, & \sinh_B(\Theta) &= \frac{cB^{\Theta} - fB^{-\Theta}}{2}, \\ \coth_B(\Theta) &= \frac{cB^{\Theta} + fB^{-\Theta}}{cB^{\Theta} - fB^{-\Theta}}, & \tanh_B(\Theta) &= \frac{cB^{\Theta} - fB^{-\Theta}}{cB^{\Theta} + fB^{-\Theta}}, \\ \operatorname{csch}_B(\Theta) &= \frac{2}{cB^{\Theta} - fB^{-\Theta}}, & \operatorname{sech}_B(\Theta) &= \frac{2}{cB^{\Theta} + fB^{-\Theta}}. \end{aligned}$$

The modified extended direct algebraic method (mEDAM), while powerful for finding the soliton solutions of nonlinear equations, has limitations, particularly when the highest derivative and nonlinear terms are not well balanced, and it might not be applicable to all types of nonlinear equations, especially those with complex structures or non-standard forms.

3.2. The Improved (G'/G) Expansion Method

This subsection briefs the basic steps for the method shown as [30].

Step 1:

Consider a partial fractional nonlinear differential equation:

$$Y(h, D_{M,t}^{\alpha,\Upsilon} h, h^2 h_x, h_y, h_{yy}, h_{xx}, h_{xy}, \ldots) = 0. \tag{43}$$

Step 2:

Assume the following transformation:

$$h(x, y, t) = H(\Theta), \quad \Theta = x - \nu y + \kappa \frac{\Gamma(1+\Upsilon)}{\alpha} t^{\alpha}, \tag{44}$$

Here, ν and κ denote the constants. Using Equation (44) in Equation (43) provides a nonlinear ordinary differential equation:

$$Z(H, H^2 H', H'', \ldots) = 0. \tag{45}$$

Step 3:

Suppose the solution to Equation (45) is given as

$$H(\Theta) = \sum_{j=0}^{m} \alpha_j \left(\frac{G'(\Theta)}{G(\Theta)} \right)^j. \tag{46}$$

In Equation (46), α_0 and $\alpha_j, (j = 1, 2, 3, \ldots, m)$ are to found. By utilizing the homogenous balance method in Equation (45), we obtain m. The profile $G = G(\Theta)$ satisfies the equation below:

$$GG'' - \kappa_1 G^2 - \kappa_2 GG' - \kappa_3 (G')^2 = 0, \tag{47}$$

where κ_1, κ_2, and κ_3 are constants.

Step 4:

Consider a solution to Equation (47) that is given as

Type I: when $\kappa_2 \neq 0$ and $\pi = \kappa_2^2 + 4\kappa_1 - 4\kappa_1\kappa_3 > 0$, we have

$$\frac{G'(\Theta)}{G(\Theta)} = \frac{\kappa_2\sqrt{\pi}\left(C_1 \exp\left(\frac{1}{2}\Theta\sqrt{\pi}\right) + C_2 \exp\left(\frac{1}{2}\Theta(-\sqrt{\pi})\right)\right)}{(2(1-\kappa_3))\left(C_1 \exp\left(\frac{1}{2}\Theta\sqrt{\pi}\right) - C_2 \exp\left(\frac{1}{2}\Theta(-\sqrt{\pi})\right)\right)} + \frac{\kappa_2}{2(1-\kappa_3)}. \tag{48}$$

Type II: if $\kappa_2 \neq 0$ and $\pi = \kappa_2^2 + 4\kappa_1 - 4\kappa_1\kappa_3 < 0$, then

$$\frac{G'(\Theta)}{G(\Theta)} = \frac{\kappa_2\sqrt{-\pi}(C_1\iota\cos(\frac{1}{2}\Theta\sqrt{-\pi}) - C_2\sin(\frac{1}{2}\Theta\sqrt{-\pi}))}{(2(1-\kappa_3))(C_1\iota\sin(\frac{1}{2}\Theta\sqrt{-\pi}) + C_2\cos(\frac{1}{2}\Theta\sqrt{-\pi}))} + \frac{\kappa_2}{2(1-\kappa_3)}. \tag{49}$$

Type III: if $\kappa_2 = 0$ and $\kappa_1 - \kappa_1\kappa_3 \geq 0$, we have

$$\frac{G'(\Theta)}{G(\Theta)} = \frac{\sqrt{\kappa_1 - \kappa_1\kappa_3}(C_2\sin(\Theta\sqrt{\kappa_1 - \kappa_1\kappa_3}) + C_1\cos(\Theta\sqrt{\kappa_1 - \kappa_1\kappa_3}))}{(1-\kappa_3)(C_1\sin(\Theta\sqrt{\kappa_1 - \kappa_1\kappa_3}) - C_2\cos(\Theta\sqrt{\kappa_1 - \kappa_1\kappa_3}))}. \tag{50}$$

Type IV: if $\kappa_2 = 0$ and $\kappa_1 - \kappa_1\kappa_3 < 0$, we have

$$\frac{G'(\Theta)}{G(\Theta)} = \frac{\sqrt{\kappa_1\kappa_3 - \kappa_1}(C_1\iota\cosh(\Theta\sqrt{\kappa_1\kappa_3 - \kappa_1}) - C_2\sinh(\Theta\sqrt{\kappa_1\kappa_3 - \kappa_1}))}{(1-\kappa_3)(C_1\iota\sinh(\Theta\sqrt{\kappa_1\kappa_3 - \kappa_1}) - C_2\cosh(\Theta\sqrt{\kappa_1\kappa_3 - \kappa_1}))}. \tag{51}$$

Here, κ_1, κ_2, κ_3, C_1, and C_2 are constants.

Step 5:

Equations (46) and (47) are combined into Equation (45), and the coefficients of each power of $\left(\frac{G'(\Theta)}{G(\Theta)}\right)$ are obtained. The system of algebraic equations is obtained by setting every coefficient to 0.

Step 6:

By utilizing the Mathematica tool to simplify the system mentioned above, we obtain different solitons to the NLPD Equation (43).

The limitations of the improved (G′/G) expansion method, while offering advantages in finding traveling-wave solutions for nonlinear evolution equations, include the potential for complex algebraic manipulations and the need for specialized software like Maple 17 or Mathematica 11 to handle the resulting equations.

4. Model Description

In 1967, Gardner, Greene, Kruskal, and Miura discovered the generalized shallow water wave equation. The generalized shallow water wave equation is a generalization of the Korteweg–de Vries (KdV) equation, which is a well-known model for water waves. The generalized water wave equation includes higher-order, nonlinear, and dispersive terms, making it a more accurate model for certain types of water waves.

The (1 + 1)-dimensional generalized shallow water wave equation given in [24] is considered by replacing the ordinary derivative with a truncated M-fractional derivative as follows:

$$D_{M,t}^{\alpha,\Upsilon}(D_{M,x}^{3\alpha,\Upsilon}h) + a(D_{M,x}^{\alpha,\Upsilon}h)(D_{M,xt}^{2\alpha,\Upsilon}h) + b(D_{M,t}^{\alpha,\Upsilon}h)(D_{M,x}^{2\alpha,\Upsilon}h) - D_{M,xt}^{2\alpha,\Upsilon}h - D_{M,x}^{2\alpha,\Upsilon}h = 0. \tag{52}$$

Assume the following wave transformations:

$$v = V(\xi); \qquad \xi = \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \lambda t^\alpha). \tag{53}$$

Here, λ is a wave speed. Putting Equation (53) into Equation (52) produces

$$\lambda(a+b)\left(H'\right)^2 + 2\lambda H''' + 2(1-\lambda)H' = 0. \tag{54}$$

We obtain the natural number $m = 1$ with the use of the homogenous balance technique in Equation (54). We now use two distinct approaches to solve Equation (54).

5. Different Types of Wave Solitons

5.1. Through Modified Extended Direct Algebraic Technique

For our case, Equation (4) reduces to this form:

$$H(\xi) = \alpha_{-1}(\psi(\xi))^{-1} + \alpha_0 + \alpha_1\psi(\xi). \tag{55}$$

Using Equations (55) and (5) in Equation (54) provides the given sets:
Set 1:

$$\left\{\alpha_{-1} = \frac{12p\log(d)}{a+b}, \alpha_0 = \alpha_0, \alpha_1 = 0, \lambda = \frac{1}{4p\ r\log^2(d) - q^2\log^2(d) + 1}\right\}. \tag{56}$$

When $q^2 - 4pr < 0$ and r is nonzero, then we have the following by using Equations (6)–(10):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} + \frac{\sqrt{4p\ r-q^2}\tan_B\left(\frac{1}{2}\sqrt{4p\ r-q^2}\xi\right)}{2r}\right)^{-1}. \tag{57}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{4p\ r-q^2}\cot_B\left(\frac{1}{2}\sqrt{4p\ r-q^2}\xi\right)}{2r}\right)^{-1}. \tag{58}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} + \left(\sqrt{4p\ r-q^2}\left(\tan_B\left(\sqrt{4p\ r-q^2}\xi\right)\pm \left(\sqrt{c\ f}\ \sec_B\left(\sqrt{4p\ r-q^2}\xi\right)\right)\right)\right)/(2r)\right)^{-1}. \tag{59}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} - \left(\sqrt{-(q^2-4p\ r)}\left(\cot_B\left(\sqrt{-(q^2-4p\ r)}\xi\right)\pm \left(\sqrt{c\ f}\ \csc_B\left(\sqrt{-(q^2-4p\ r)}\xi\right)\right)\right)\right)/(2r)\right)^{-1}. \tag{60}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} + \left(\sqrt{-(q^2-4p\ r)}\left(\tan_B\left(\frac{1}{4}\sqrt{-(q^2-4p\ r)}\xi\right) - \left(\cot_B\left(\frac{1}{4}\sqrt{-(q^2-4p\ r)}\xi\right)\right)\right)\right)/(2r)\right)^{-1}. \tag{61}$$

When $q^2 - 4pr > 0$ and r is nonzero, we obtain the following solutions by using Equations (11)–(15):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{(q^2-4p\ r)}\tanh_B\left(\frac{1}{2}\sqrt{(q^2-4p\ r)}\xi\right)}{2r}\right)^{-1}. \tag{62}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{(q^2-4p\ r)}\coth_B\left(\frac{1}{2}\sqrt{(q^2-4p\ r)}\xi\right)}{2r}\right)^{-1}. \tag{63}$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{q}{2r} - (\sqrt{q^2-4p\ r}(\tanh_B(\sqrt{q^2-4p\ r}\xi)\pm (\sqrt{c\ f}\ sech_B(\sqrt{q^2-4pr}\xi))))/(2r))^{-1}. \quad (64)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{q}{2r} - (\sqrt{q^2-4p\ r}(\coth_B(\sqrt{q^2-4p\ r}\xi)\pm (\sqrt{c\ f}\ csch_B(\sqrt{q^2-4p\ r}\xi))))/(2r))^{-1}. \quad (65)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{q}{2r} - (\sqrt{q^2-4p\ r}(\tanh_B(\frac{1}{4}\sqrt{q^2-4p\ r}\xi)- (\ \coth_B(\frac{1}{4}\sqrt{q^2-4p\ r}\xi))))/(2r))^{-1}. \quad (66)$$

where $\xi = \frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)-q^2\log^2(d)+1})$.

When $pr > 0$ and q is zero, then we have the following solutions by using Equations (16)–(20):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\sqrt{\frac{p}{r}}\tan_B(\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))^{-1}. \quad (67)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{\frac{p}{r}}\cot_B(\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))^{-1}. \quad (68)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\sqrt{\frac{p}{r}}(\tan_B(2\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))\pm (\sqrt{cf}\sec_B(2\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4pr\log^2(d)+1})))))^{-1}. \quad (69)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{\frac{p}{r}}(\cot_B(2\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))\pm (\sqrt{c\ f}\csc_B(2\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))))^{-1}. \quad (70)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\frac{1}{2}\sqrt{\frac{p}{r}}(\tan_B(\frac{1}{2}\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))- \cot_B(\frac{1}{2}\sqrt{p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))))^{-1}. \quad (71)$$

When $pr < 0$ and q is zero, then we obtain the following solutions by using Equations (21)–(25):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}\tanh_B(\sqrt{-p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))^{-1}. \quad (72)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}\coth_B(\sqrt{-p\ r}\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))^{-1}. \quad (73)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}(\tanh_B(2\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))\pm (\iota\sqrt{cf}sech_B(2\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))))^{-1}. \quad (74)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}(\coth_B(2\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))\pm (\sqrt{cf}csch_B(2\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))))^{-1}. \quad (75)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{1}{2}\sqrt{-\frac{p}{r}}(\tanh_B(\frac{1}{2}\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1}))+ \coth_B(\frac{1}{2}\sqrt{-p\ r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\ r\log^2(d)+1})))))^{-1}. \quad (76)$$

When $r = p$ and $q = 0$, we attain the following solutions by using Equations (26)–(30):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\tan_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1})))^{-1}. \quad (77)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\cot_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1})))^{-1}. \quad (78)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\tan_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))\pm (\sqrt{cf}\sec_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))))^{-1}. \quad (79)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\cot_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))\pm (\sqrt{cf}\csc_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))))^{-1}. \quad (80)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(\frac{1}{2}\tan_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))- \frac{1}{2}\cot_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1})))^{-1}. \quad (81)$$

When $r = -p$ and q is zero, we obtain the following solutions by utilizing Equations (31)–(35):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\tanh_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))^{-1}. \quad (82)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\coth_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))^{-1}. \quad (83)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\tanh_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))\pm (\iota\sqrt{c\ f}sech_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))^{-1}. \quad (84)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\coth_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))\pm (\sqrt{c\ f}csch_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))^{-1}. \quad (85)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{1}{2}\tanh_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))- \frac{1}{2}\coth_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))^{-1}. \quad (86)$$

when $q^2 - 4pr = 0$, then we obtain the following solution by using Equation (36),

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-2\frac{p(q\xi\log(d)+2)}{q^2\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - t^\alpha)\log(d)})^{-1}. \quad (87)$$

When $q = \rho$, $p = m\rho$ $(m \neq 0)$ and r is zero, we obtain the following solution by utilizing Equation (37):

$$h(x,t) = \alpha_0 + \frac{12m\rho\log(d)}{a+b}(d^{\rho\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\rho^2\log^2(d)+1})} - m)^{-1}. \quad (88)$$

if $q = r = 0$, we obtain the following solution by using Equation (38):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(p\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - t^\alpha)\log(d))^{-1}. \quad (89)$$

Set 2:

$$\left\{\alpha_{-1} = 0, \alpha_1 = -\frac{12r\log(d)}{a+b}, \lambda = \frac{1}{4p\ r\log^2(d) - q^2\log^2(d)+1}\right\}. \quad (90)$$

When $q^2 - 4pr < 0$ and r is nonzero, then we have the following by using Equations (6)–(10):

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{q}{2r} + \frac{\sqrt{-(q^2-4p\ r)}\tan_B(\frac{1}{2}\sqrt{-(q^2-4p\ r)}\xi)}{2r}). \quad (91)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{q}{2r} - \frac{\sqrt{-(q^2-4p\ r)}\cot_B(\frac{1}{2}\sqrt{-(q^2-4p\ r)}\xi)}{2r}). \quad (92)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{q}{2r} + \frac{\sqrt{4p\ r-q^2}(\tan_B(\sqrt{4p\ r-q^2}\xi) \pm (\sqrt{c\ f}\ \sec_B(\sqrt{4p\ r-q^2}\xi)))}{2r}). \quad (93)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{q}{2r} - \frac{\sqrt{4p\ r-q^2}(\cot_B(\sqrt{4p\ r-q^2}\xi) \pm (\sqrt{cf}\ \csc_B(\sqrt{4p\ r-q^2}\xi)))}{2r}). \quad (94)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{q}{2r} + \frac{\sqrt{4p\ r-q^2}(\tan_B(\frac{1}{4}\sqrt{4p\ r-q^2}\xi) - (\ \cot_B(\frac{1}{4}\sqrt{4p\ r-q^2}\xi)))}{2r}). \quad (95)$$

When $q^2 - 4pr > 0$ and r is nonzero, we obtain the following solutions by using Equations (11)–(15):

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{(q^2-4p\,r)}\tanh_B\left(\frac{1}{2}\sqrt{(q^2-4p\,r)}\xi\right)}{2r}\right). \quad (96)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{(q^2-4p\,r)}\coth_B\left(\frac{1}{2}\sqrt{(q^2-4p\,r)}\xi\right)}{2r}\right). \quad (97)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{q^2-4p\,r}\left(\tanh_B\left(\sqrt{q^2-4p\,r}\xi\right) \pm \left(\sqrt{c\,f}\ \operatorname{sech}_B\left(\sqrt{q^2-4p\,r}\xi\right)\right)\right)}{2r}\right). \quad (98)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{q^2-4p\,r}\left(\coth_B\left(\sqrt{q^2-4p\,r}\xi\right) \pm \left(\sqrt{c\,f}\ \operatorname{csch}_B\left(\sqrt{q^2-4p\,r}\xi\right)\right)\right)}{2r}\right). \quad (99)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\frac{q}{2r} - \frac{\sqrt{q^2-4p\,r}\left(\tanh_B\left(\frac{1}{4}\sqrt{q^2-4p\,r}\xi\right) - \left(\coth_B\left(\frac{1}{4}\sqrt{q^2-4p\,r}\xi\right)\right)\right)}{2r}\right). \quad (100)$$

where $\xi = \frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d) - q^2\log^2(d) + 1}\right)$.

When $pr > 0$ and q is zero, then we obtain the following solutions by using Equations (16)–(20):

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(\sqrt{\frac{p}{r}}\tan_B\left(\sqrt{p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right). \quad (101)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\sqrt{\frac{p}{r}}\cot_B\left(\sqrt{p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right). \quad (102)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(\sqrt{\frac{p}{r}}\left(\tan_B\left(2\sqrt{p\,r}\frac{1}{\alpha}\Gamma(1+\Upsilon)\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right) \pm \left(\sqrt{c\,f}\sec_B\left(2\sqrt{p\,r}\frac{1}{\alpha}\Gamma(1+\Upsilon)\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right)\right)\right). \quad (103)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\sqrt{\frac{p}{r}}\left(\cot_B\left(2\sqrt{p\,r}\frac{1}{\alpha}\Gamma(1+\Upsilon)\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right) \pm \left(\sqrt{c\,f}\csc_B\left(2\sqrt{p\,r}\frac{1}{\alpha}\Gamma(1+\Upsilon)\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right)\right)\right). \quad (104)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(\frac{1}{2}\sqrt{\frac{p}{r}}\left(\tan_B\left(\frac{1}{2}\sqrt{p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right) - \cot_B\left(\frac{1}{2}\sqrt{p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right)\right). \quad (105)$$

When $pr < 0$ and q is zero, then we obtain the following solutions by using Equations (21)–(25),

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\sqrt{-\frac{p}{r}}\tanh_B\left(\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right). \quad (106)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}\left(-\sqrt{-\frac{p}{r}}\coth_B\left(\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}\right)\right)\right). \quad (107)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}(\tanh_B(2\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}))\pm (\iota\sqrt{c\,f}\,\mathrm{sech}_B(2\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}))))). \quad (108)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\sqrt{-\frac{p}{r}}(\coth_B(2\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}))\pm (\sqrt{c\,f}csch_B(2\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}))))). \quad (109)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{1}{2}\sqrt{-\frac{p}{r}}(\tanh_B(\frac{1}{2}\sqrt{-pr}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p\,r\log^2(d)+1}))+ \coth_B(\frac{1}{2}\sqrt{-p\,r}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4pr\log^2(d)+1})))). \quad (110)$$

When $r = p$ and $q = 0$, we obtain the following solutions by using Equations (26)–(30):

$$h(x,t) = \alpha_0 - \frac{12p\log(d)}{a+b}(\tan_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))). \quad (111)$$

$$h(x,t) = \alpha_0 - \frac{12p\log(d)}{a+b}(-\cot_B(p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))). \quad (112)$$

$$h(x,t) = \alpha_0 - \frac{12p\log(d)}{a+b}(\tan_B(2p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))\pm (\sqrt{c\,f}\sec_B(2p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1})))). \quad (113)$$

$$h(x,t) = \alpha_0 - \frac{12p\log(d)}{a+b}(-\cot_B(2p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))\pm (\sqrt{c\,f}\,\mathrm{csc}_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1})))). \quad (114)$$

$$h(x,t) = \alpha_0 - \frac{12p\log(d)}{a+b}(\frac{1}{2}\tan_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))- \frac{1}{2}\cot_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4p^2\log^2(d)+1}))). \quad (115)$$

When $r = -p$ and q is zero, we achieve the following solutions by utilizing Equations (31)–(35):

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\tanh_B(p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))). \quad (116)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\coth_B(p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))). \quad (117)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\tanh_B(2p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))\pm (\iota\sqrt{c\ f}sech_B(2p\frac{1}{\alpha}\Gamma(1+\Upsilon)(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))). \quad (118)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\coth_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))\pm (\sqrt{c\ f}csch_B(2p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1})))). \quad (119)$$

$$h(x,t) = \alpha_0 + \frac{12p\log(d)}{a+b}(-\frac{1}{2}\tanh_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))- \frac{1}{2}\coth_B(\frac{1}{2}p\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-4p^2\log^2(d)+1}))). \quad (120)$$

when $q^2 - 4pr = 0$, then we obtain the following solution by using Equation (36):

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-2\frac{p(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - t^\alpha)\log(d)+2)}{q^2\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - t^\alpha)\log(d)}). \quad (121)$$

When $p = q = 0$, we obtain the following solution by using the Equation (39),

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-\frac{1}{r\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - t^\alpha)\log(d)}). \quad (122)$$

When p is zero while q and r are nonzero, we obtain the following solutions by using Equations (40)–(41):

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-(cq)/(r(\cosh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1}))- \sinh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1})) + c))). \quad (123)$$

$$h(x,t) = \alpha_0 - \frac{12r\log(d)}{a+b}(-(q(\cosh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1}))+ \sinh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1}))))/(r(\cosh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1})) + \sinh_B(q\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-q^2\log^2(d)+1})) + f))). \quad (124)$$

If $q = \rho$, $r = m\rho$ $(m \neq 0)$ and $p = 0$, then we obtain the following solution by using Equation (42):

$$h(x,t) = \alpha_0 - \frac{12m\rho\log(d)}{a+b}(\frac{cd^{\rho\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\rho^2\log^2(d)+1})}}{c - mfd^{\rho\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\rho^2\log^2(d)+1})}}). \quad (125)$$

5.2. Exact Solitons via Improved (G'/G) Expansion Scheme

For $m = 1$, Equation (46) reduces to the given form:

$$H(\xi) = \alpha_0 + \alpha_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \tag{126}$$

Here, α_0 and α_1 are undetermined.

By putting Equations (126) and (47) in Equation (54), one obtains the given solution set: Set:

$$\left\{ \alpha_1 = -\frac{12(\kappa_3 - 1)}{a + b}, \lambda = \frac{1}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1} \right\}. \tag{127}$$

When $\kappa_2 \neq 0$ and $\pi = \kappa_2^2 + 4\kappa_1 - 4\kappa_1\kappa_3 > 0$, we obtain the following solution by using Equation (48):

$$\begin{aligned} h(x,t) = \alpha_0 - \frac{12(\kappa_3 - 1)}{a + b}((\kappa_2\sqrt{\pi}(C_1 \exp(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1}) \\ \sqrt{\pi}) + C_2 \exp(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})(-\sqrt{\pi})))) / ((2(1 - \kappa_3)) \\ (C_1 \exp(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})\sqrt{\pi}) - C_2 \exp(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha \\ - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})(-\sqrt{\pi}))) + \frac{\kappa_2}{2(1 - \kappa_3)}). \end{aligned} \tag{128}$$

if $\kappa_2 \neq 0$ and $\pi = \kappa_2^2 + 4\kappa_1 - 4\kappa_1\kappa_3 < 0$, then we obtain the following solution by using Equation (49):

$$\begin{aligned} h(x,t) = \alpha_0 - \frac{12(\kappa_3 - 1)}{a + b}((\kappa_2\sqrt{-\pi}(C_1\iota \cos(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1}) \\ \sqrt{-\pi}) - C_2 \sin(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})\sqrt{-\pi}))) / ((2(1 - \kappa_3)) \\ (C_1\iota \sin(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})\sqrt{-\pi}) + C_2 \cos(\frac{1}{2}\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \\ \frac{t^\alpha}{-\kappa_2^2 + 4\kappa_1(\kappa_3 - 1) + 1})\sqrt{-\pi}))) + \frac{\kappa_2}{2(1 - \kappa_3)}). \end{aligned} \tag{129}$$

if $\kappa_2 = 0$ and $\kappa_1 - \kappa_1\kappa_3 \geq 0$, then we achieve the following solution by utilizing Equation (50):

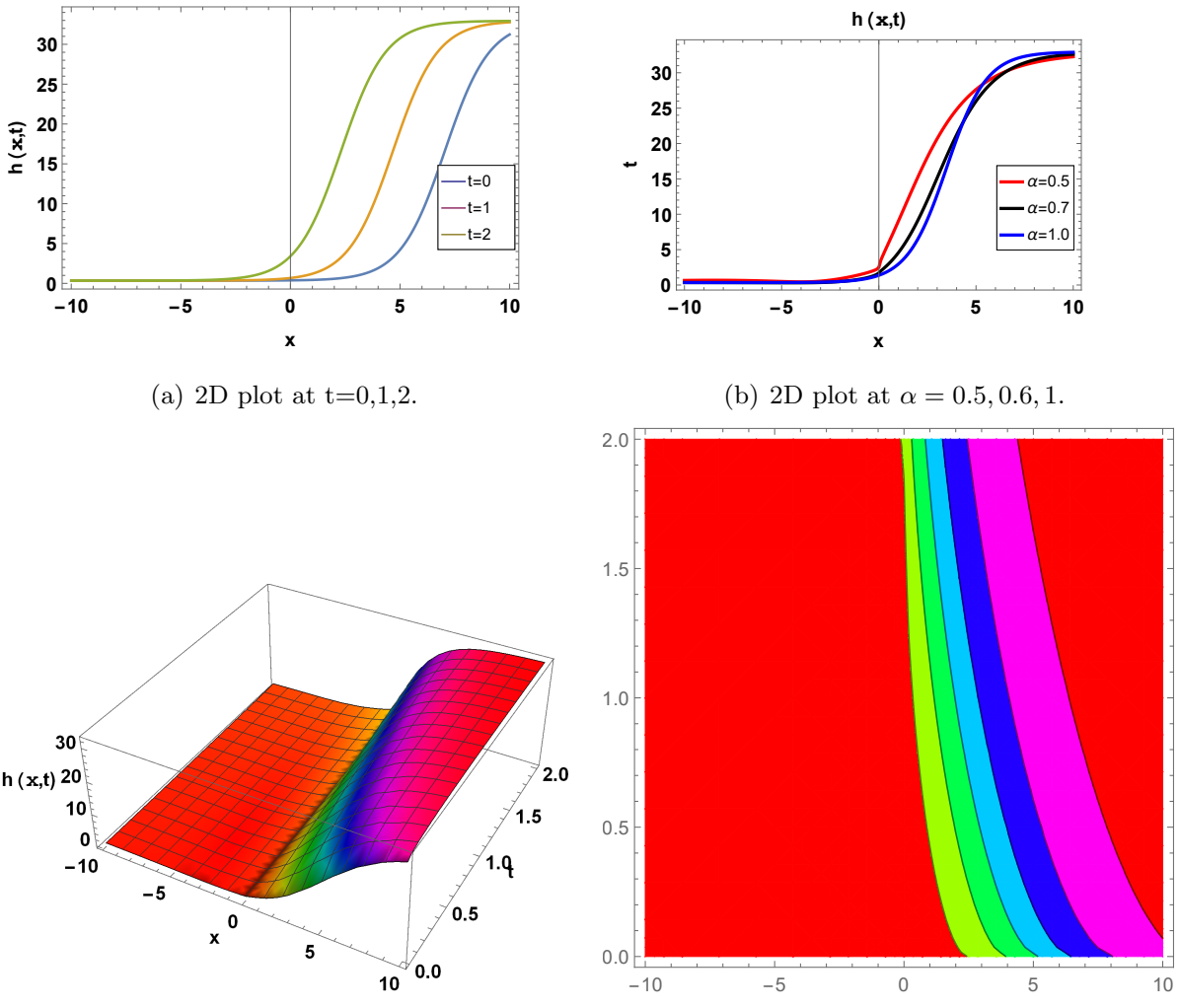
$$\begin{aligned} h(x,t) = \alpha_0 - \frac{12(\kappa_3 - 1)}{a + b}((\sqrt{\kappa_1 - \kappa_1\kappa_3}(C_2 \sin(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3 - 1) + 1}) \\ \sqrt{\kappa_1 - \kappa_1\kappa_3}) + C_1 \cos(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3 - 1) + 1})\sqrt{\kappa_1 - \kappa_1\kappa_3}))) / ((1 - \kappa_3) \\ (C_1 \sin(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3 - 1) + 1})\sqrt{\kappa_1 - \kappa_1\kappa_3}) - C_2 \cos(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha \\ - \frac{t^\alpha}{4\kappa_1(\kappa_3 - 1) + 1})\sqrt{\kappa_1 - \kappa_1\kappa_3})))). \end{aligned} \tag{130}$$

if $\kappa_2 = 0$ and $\kappa_1 - \kappa_1\kappa_3 < 0$, we obtain the following solution by using Equation (51):

$$h(x,t) = \alpha_0 - \frac{12(\kappa_3-1)}{a+b}((\sqrt{\kappa_1\kappa_3-\kappa_1}(C_1\iota\cosh(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3-1)+1})\sqrt{\kappa_1\kappa_3-\kappa_1}) - C_2\sinh(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3-1)+1})\sqrt{\kappa_1\kappa_3-\kappa_1})))/((1-\kappa_3)(C_1\iota\sinh(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3-1)+1})\sqrt{\kappa_1\kappa_3-\kappa_1}) - C_2\cosh(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{4\kappa_1(\kappa_3-1)+1})\sqrt{\kappa_1\kappa_3-\kappa_1})))). \quad (131)$$

6. Graphical Representation

Here, we use contour, 3D, and 2D graphs to illustrate some of our solutions. The graphs for $\alpha = 0.5$, $\alpha = 0.7$, and $\alpha = 1.0$ are also used to illustrate the impact of fractional derivatives in the following Figures 1–6. Graphs are drawn for different values of the involved parameters. Mathematica software 11 was used for drawing the following graphs.



(a) 2D plot at t=0,1,2.

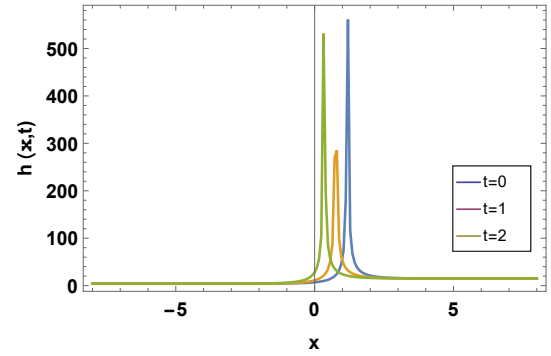
(b) 2D plot at $\alpha = 0.5, 0.6, 1$.

(c) 3-D plot when $\alpha = 0.6$

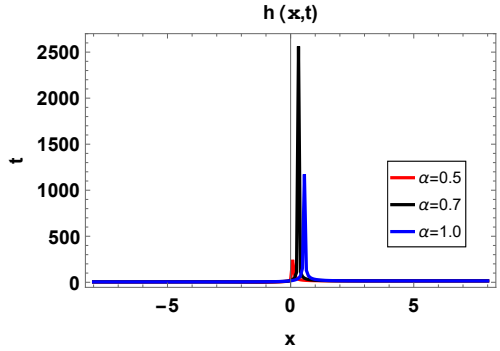
(d) contour plot when $\alpha = 0.6$

Figure 1. Periodic wave solution. A graph of (x,t) in Equation (57) for $\alpha_0 = 0.01$, $a = 1, b = -0.5, p = 0.01, q = -1, r = 1, d = 4$, and $Y = 1$. (**a**) displays a two-dimensional plot for $x \in (-10, 10)$ and $\alpha = 1$, with the blue indicating $t = 0$, the orange indicating $t = 1$, and the green indicating $t = 2$. We can observe that the wave repeats in shape and travels along x over time. The shape is not a single localized hump but shows wave-like oscillations, indicating periodicity.

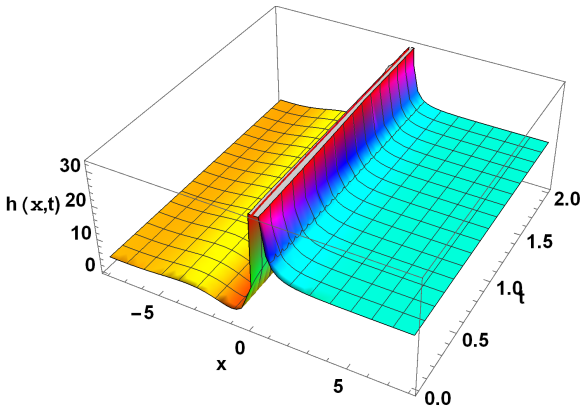
(**b**) displays a two-dimensional plot at $x \in (-10, 10)$ and $t \in (0, 2)$, where the red indicates $\alpha = 0.5$, the black indicates $\alpha = 0.7$, and the blue indicates $\alpha = 1$. It can be observed that the variation in sharpness and spread with increases in α suggests a solution with internal oscillatory behavior. (**c**) shows a 3D plot at $\alpha = 0.6$ for $t \in [0, 2]$. (**d**) shows a contour graph for $\alpha = 0.6$ when $0 < t < 2$. The 3D and contour plots further emphasize a wave repeating in space and time, consistent with a periodic wave solution. Periodic solitons have applications in physics, mathematics, and engineering for explaining water waves, model wave propagation, mechanical systems, etc.



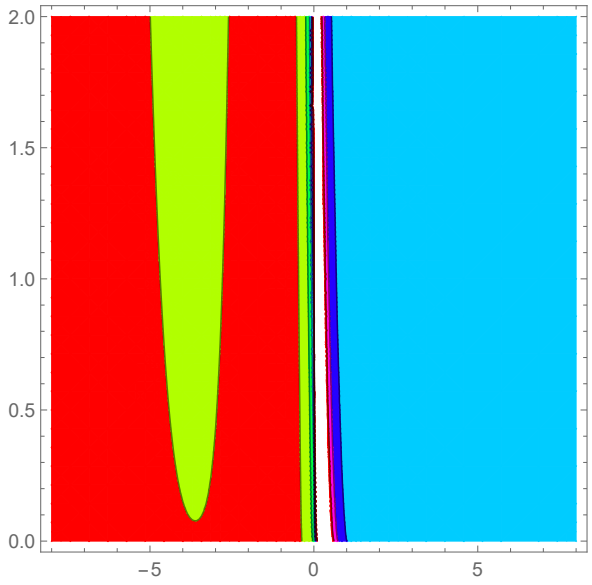
(a) 2D plot for $t = 0, 1, 2$.



(b) 2D plot for $\alpha = 0.5, 0.7, 1$.

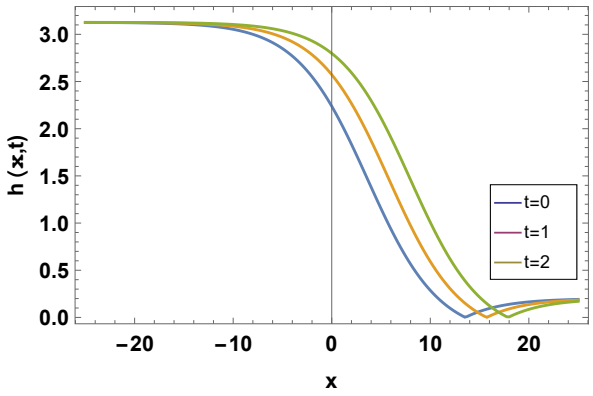


(c) 3-D plot when $\alpha = 0.6$

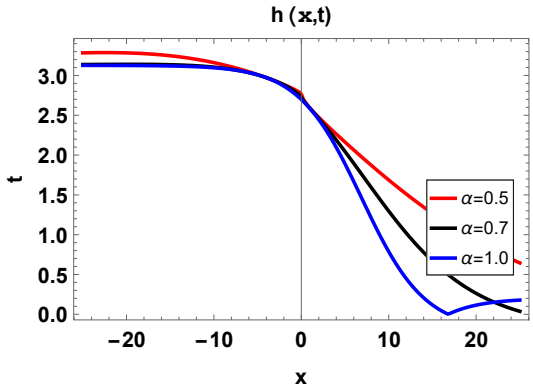


(d) contour graph when $\alpha = 0.6$

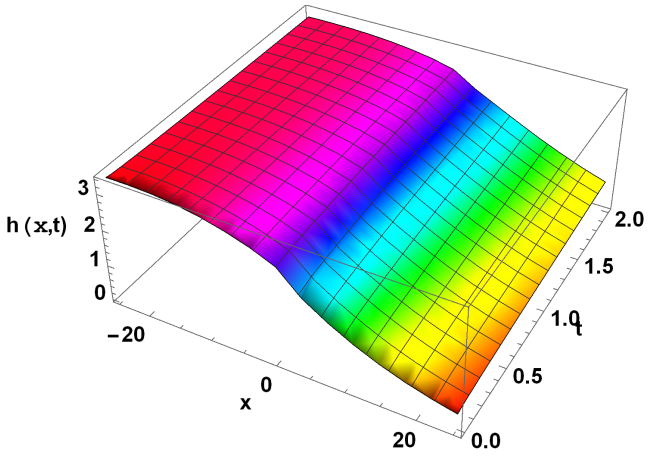
Figure 2. Singular soliton solution. A graph of $h(x, t)$ in Equation (62) is shown for $\alpha_0 = -0.1$, $a = 1$, $b = 0.5$, $p = -0.5$, $q = -1$, $r = 1$, $d = 4$, and $Y = 1$. (**a**) shows a two-dimensional plot for $x \in (-8, 8)$ and $\alpha = 1$, where blue indicates $t = 0$, orange indicates $t = 1$, and green indicates $t = 2$. We can observe that the very sharp peak grows extremely tall over time. This represents a localized energy burst that intensifies over time. (**b**) displays a two-dimensional plot at $x \in (-8, 8)$ and $t \in (0, 2)$, where red indicates $\alpha = 0.5$, black indicates $\alpha = 0.7$, and blue indicates $\alpha = 1$. It can be observed that changing the fractional-order α controls the height and sharpness of the soliton. It also reflects memory effects or nonlocal behavior. (**c**) shows a 3D plot at $\alpha = 0.6$ for $t \in [0, 2]$. The soliton spike is sharply concentrated in space but grows dramatically over time. It shows a localized singular behavior that propagates and intensifies. (**d**) shows a contour graph at $\alpha = 0.6$ when $0 < t < 2$. Extremely tight contours can be seen near the origin, indicating a sudden sharp increase in $h(x, t)$. It confirms local singularities in both x and t. Singular solitons have been studied in various nonlinear evolution equations, including those describing optical fibers, quantum gases, and other physical systems.



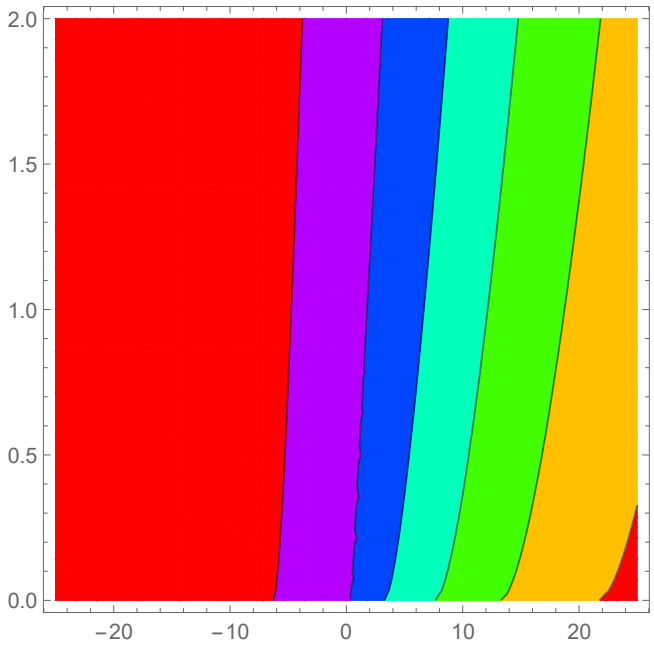
(a) 2D plot for $t = 0, 1, 2$.



(b) 2D plot for $\alpha = 0.5, 0.7, 1$.

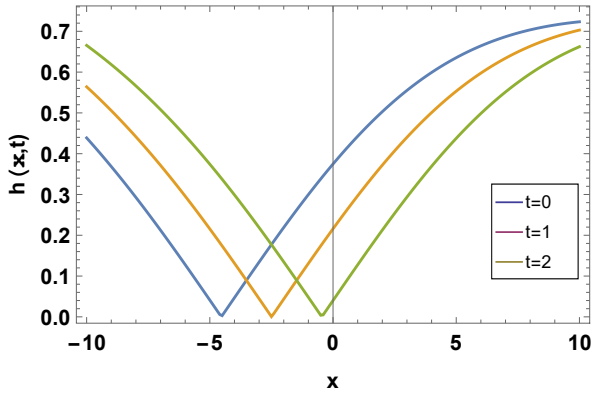


(c) 3-D plot when $\alpha = 0.6$

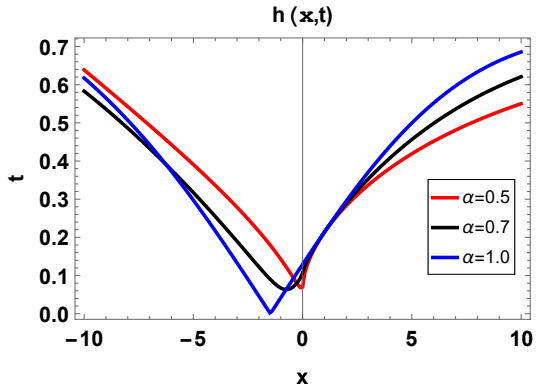


(d) contour plot at $\alpha = 0.6$

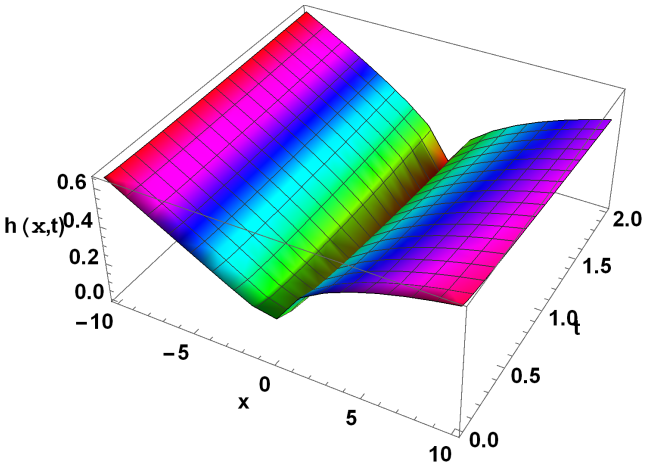
Figure 3. Rational soliton. A graph of $h(x,t)$ in Equation (88) for $\alpha_0 = 0.2, a = 0.5$, $b = 0.5$, $\rho = 0.2$, $m = -5$, $r = 1$, $d = 4$, and $Y = 1$. (**a**) shows a two-dimensional plot for $x \in (-25, 25)$ and $\alpha = 1$, where blue indicates $t = 0$, orange indicates $t = 1$, green indicates $t = 2$. It can be observed that the soliton preserves its structure but shifts in amplitude or position over time. This persistence suggests a stable traveling wave, typical of soliton behavior. (**b**) displays a two-dimensional plot at $x \in (-25, 25)$ and $t \in (0, 2)$, where red indicates $\alpha = 0.5$, black indicates $\alpha = 0.7$, and blue indicates $\alpha = 1$. The parameter α likely controls the nonlinearity or dispersion strength. The variations in shape and height suggest that changing the system parameters can control the wave amplitude or width. (**c**) shows a 3D plot where $\alpha = 0.6$ at $\in [0, 2]$. It provides a comprehensive picture of wave evolution, emphasizing the localized and non-dispersive nature of the rational soliton. (**d**) shows a contour graph at $\alpha = 0.6$ when $0 < t < 2$. The contour pattern illustrates the trajectory of the soliton and how it propagates steadily through space–time. Rational solitons can be found in various fields, including nonlinear optics, water waves, and other areas where nonlinear wave phenomena are studied.



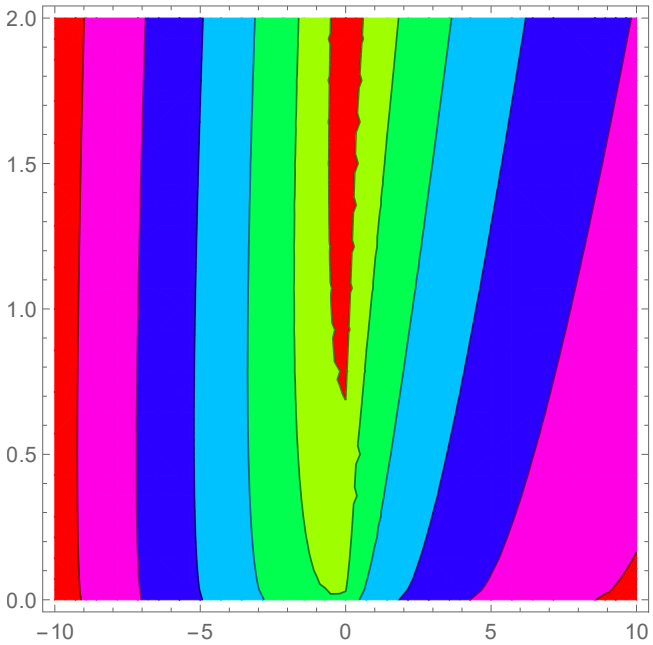
(a) 2D plot when $t = 0, 1, 2$.



(b) 2D plot when $\alpha = 0.5, 0.7, 1$.

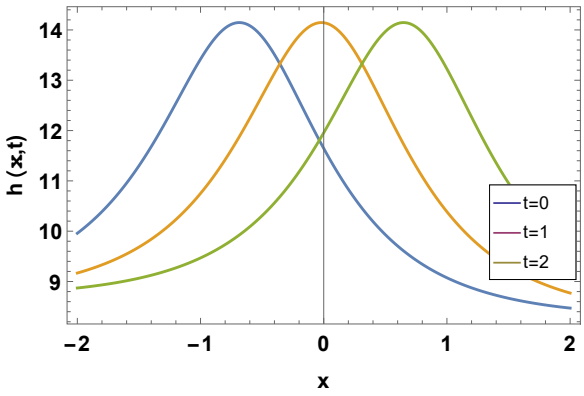


(c) 3-D plot when $\alpha = 0.6$

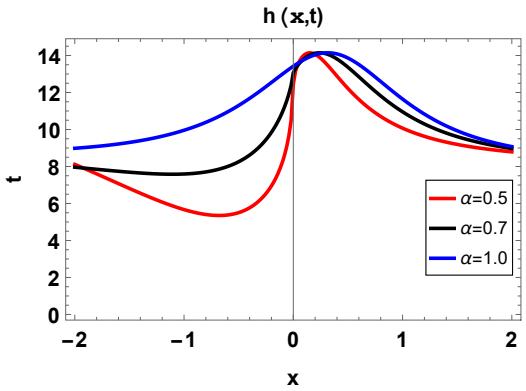


(d) contour plot when $\alpha = 0.6$

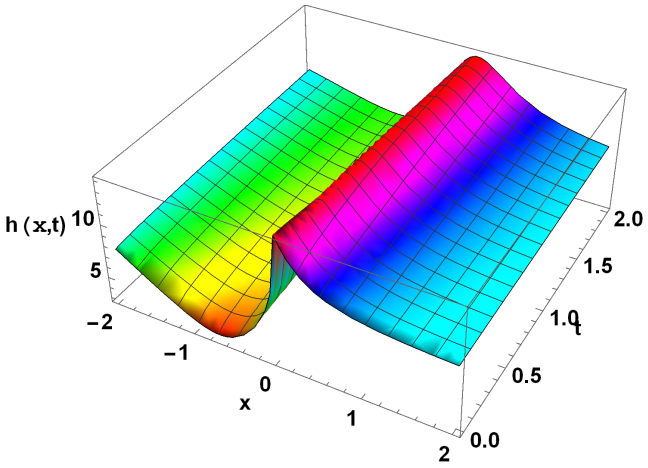
Figure 4. Bright–dark soliton solution. Graph of $h(x,t)$ in Equation (118) for $\alpha_0 = 0.2, a = 0.5$, $b = 0.5$, $c = 0.5$, $f = 0.5$, $p = 0.05$, $d = 4$, and $Y = 1$. (**a**) shows a two-dimensional plot for $x \in [-10, 10]$ and $\alpha = 1$; blue indicates $t = 0$, orange indicates $t = 1$, and green indicates $t = 2$. We can observe that the phases of the wave shift with time. So, the waves are time-dependent. (**b**) displays a two-dimensional plot at $x \in (-10, 10)$ and $t \in (0, 2)$; red indicates $\alpha = 0.5$, black indicates $\alpha = 0.7$, and blue indicates $\alpha = 1$. It can be observed that the phases shift with the increase in the value of α. (**c**) shows a 3D plot at $\alpha = 0.6$, where $t \in [0, 2]$. A sharp transition is shown in both directions. It shows the symmetric property of the wave. (**d**) shows a contour graph at $\alpha = 0.6$ when $0 < t < 2$. Bright–dark solitons are used in the fields of optical communication, optical sensing, and nonlinear optics.



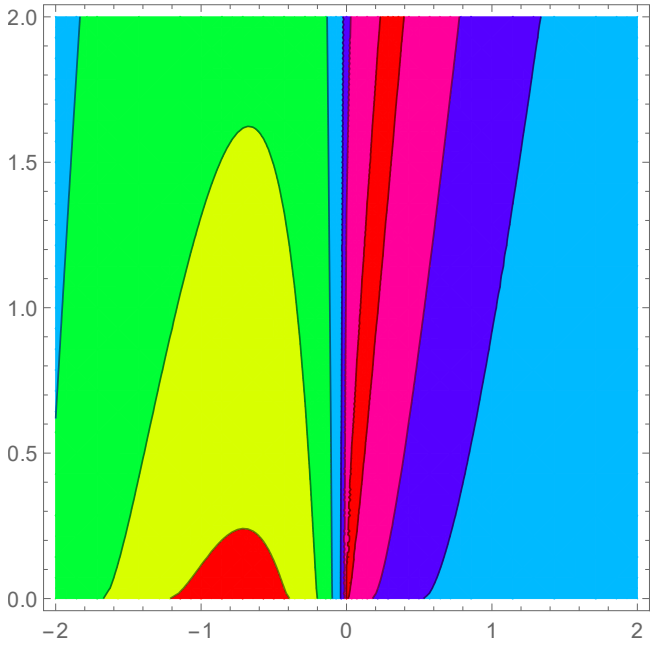
(a) 2D plot when $t = 0, 1, 2$.



(b) 2D plot when $\alpha = 0.5, 0.7, 1$.

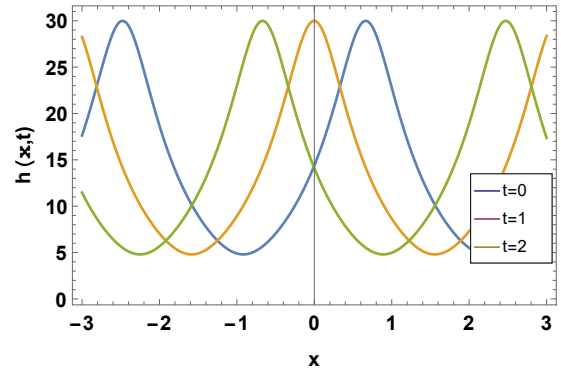


(c) 3-D plot when $\alpha = 0.6$

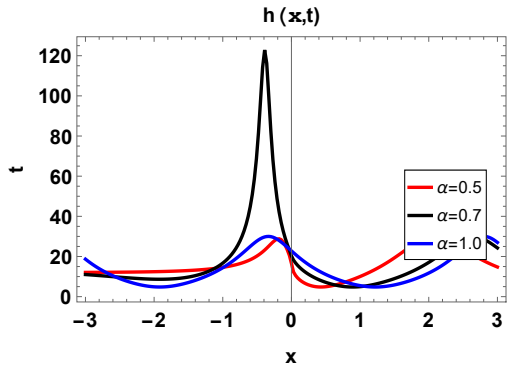


(d) contour plot when $\alpha = 0.6$

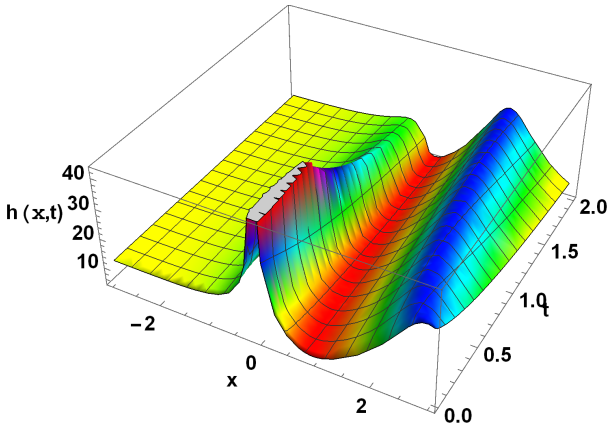
Figure 5. Periodic soliton. A graph of $h(x,t)$ in Equation (130) for $\alpha_0 = -0.2, a = 0.5$, $b = 0.5$, $c_1 = 0.5$, $c_2 = 0.3$, $\kappa_1 = 1$, $\kappa_3 = 1.5$, and $Y = 1$. (**a**) shows a two-dimensional graph, where $x \in [-2,2]$ for $\alpha = 1$; blue indicates $t = 0$, orange indicates $t = 1$, and green indicates $t = 2$. The phases of the waves change with the change in time. It shows the waves are time-dependent. (**b**) displays a two-dimensional plot at $x \in [-2,2]$ and $t \in [0,2]$; red indicates $\alpha = 0.5$, black indicates $\alpha = 0.7$, and blue indicates $\alpha = 1$. The phase of the wave changes with the change in the value of the fractional derivative parameter α. It shows that the phase of the wave is also α-dependent. (**c**) shows a 3D plot at $\alpha = 0.6$ for $t \in [0,2]$. It shows a wave-like pole in both directions. (**d**) shows a contour graph for $\alpha = 0.6$ when $0 < t < 2$.



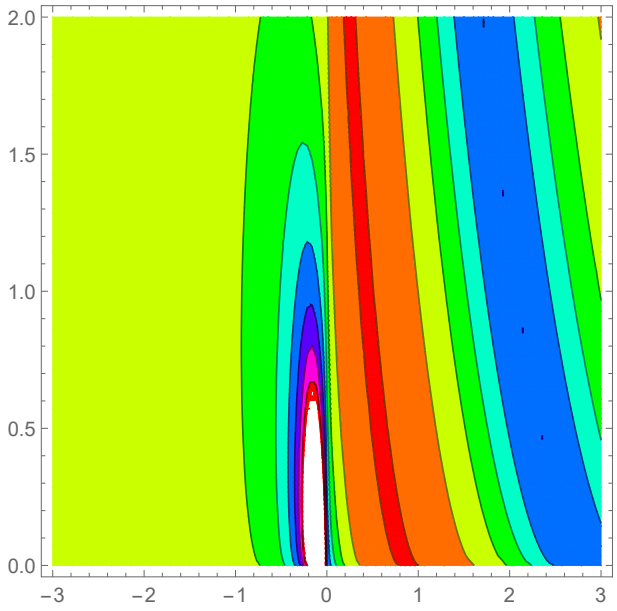
(a) 2D plot when $t = 0, 1, 2$.



(b) 2D plot when $\alpha = 0.5, 0.7, 1$.



(c) 3-D plot when $\alpha = 0.6$



(d) contour plot when $\alpha = 0.6$

Figure 6. Kink-singular soliton. Graph of $h(x,t)$ in Equation (131) for $\alpha_0 = -0.2, a = 0.5,\ b = 0.5,$ $c_1 = -0.5,\ c_2 = -0.2,\ \kappa_1 = -1,\ \kappa_3 = 2$, and $\Upsilon = 1$. (**a**) shows a two-dimensional plot for $x \in (-3,3)$ when $\alpha = 1$; blue indicates $t = 0$, orange indicates $t = 1$, and green indicates $t = 2$. It shows that the phases of the wave shift with time. So, the waves are time-dependent. (**b**) displays a two-dimensional plot at $x \in (-3,3)$ and $t \in (0,2)$; red indicates $\alpha = 0.5$, black indicates $\alpha = 0.7$, and blue indicates $\alpha = 1$. It shows that the phases of the waves also change with the change in α. (**c**) shows a 3D plot for $\alpha = 0.6$ at $t \in [0,2]$. It shows wave-like poles. (**d**) shows a contour graph for $\alpha = 0.6$ when $0 < t < 2$. Kink-singular solitons are used to explain models' phenomena in nerve fibers, liquid crystals, etc.

7. The Stability Analysis (SA)

Stability analysis was employed to describe how the system reacts over time and how it acts in response to outside influences. In order to describe the stability of an equation in applications, SA is carried out by testing some obtained solutions and applying the characteristics of the Hamiltonian system. The stability of various models has been analyzed in the literature, including [31,32]. Let us examine the selected equation. We define the Hamiltonian transformation for the stability of Equation (1) as [33,34]

$$\mathcal{S} = \frac{1}{2}\int_{-\infty}^{\infty} h^2\, dx. \tag{132}$$

where $h(x,t)$ denotes the possibility of power, and $\mathcal{S}$ denotes the momentum factor. We now present a prerequisite for stable soliton solutions.

$$\frac{\partial \mathcal{S}}{\partial \lambda} > 0, \tag{133}$$

where λ indicates the wave speed. Using Equation (88) in Equation (132) provides

$$\mathcal{S} = \frac{1}{2}\int_{-6}^{6}(\alpha_0 + \frac{12m\rho\log(d)}{a+b}(d^{\rho\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \frac{t^\alpha}{-\rho^2\log^2(d)+1})} - m)^{-1})^2 dx, \tag{134}$$

Equation (134) provides the solution:

$$\begin{aligned}\mathcal{S} = (6(\alpha_0^2(a+b)^2 - 2\alpha_0 m(a+b)(d^{\rho\frac{\Gamma(1+Y)}{\alpha}(-\lambda t^\alpha-6)} - d^{\rho\frac{\Gamma(1+Y)}{\alpha}(6-\lambda t^\alpha)} + 12m\rho\log(d))\\ +6m^2\rho\log(d)(24m^2\rho\log(d) - (d^{\rho\frac{\Gamma(1+Y)}{\alpha}(-\lambda t^\alpha-6)} - d^{\rho\frac{\Gamma(1+Y)}{\alpha}(6-\lambda t^\alpha)})(d^{\rho\frac{\Gamma(1+Y)}{\alpha}(-\lambda t^\alpha-6)}\\ +d^{\rho\frac{\Gamma(1+Y)}{\alpha}(6-\lambda t^\alpha)} - 4m))))/((a+b)^2).\end{aligned} \tag{135}$$

where $\frac{1}{-\rho^2\log^2(d)+1} = \lambda$.

By putting the result given in Equation (135) into Equation (133) and using $\lambda = \frac{1}{-\rho^2\log^2(d)+1}$, we obtain the following:

$$\begin{aligned}(6(6m^2\rho\log(d)((\rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}+6)} - \rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}-6)})\\ (d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}-6)} + d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}+6)} - 4m) + (d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}+6)} -\\ d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}-6)})(\rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}-6)} + \rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}+6)}))-\\ 2\alpha_0 m(a+b)(\rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}-6)} - \rho t\log(d)d^{\rho\frac{\Gamma(1+Y)}{\alpha}(\frac{t^\alpha}{\rho^2\log^2(d)-1}+6)})))/((a+b)^2) > 0.\end{aligned} \tag{136}$$

By using the values $Y = 1, \alpha_0 = 0.2, a = 0.5, b = 0.5, m = -5, \rho = 0.2, d = 4, t = 1$ in the above expression (136), we ob tain the solution: $14626.2 > 0$.

Hence, Equation (88) satisfies the given condition. So, Equation (88) is a stable solution. Similarly, we can check the stability of the other solutions.

8. Modulation Instability (MI) Analysis

A steady-state method for solving the generalized shallow water wave equation model is [35,36]

$$h(x,t) = \left(H(x,t) + \sqrt{\tau}\right)e^{\iota\tau t}. \tag{137}$$

where τ is the optical power of normalizing.

Insert Equation (137) into Equation (1). Using linearity, one gets

$$-\iota\tau H_x - H_{xt} - H_{xx} + \iota\tau H_{xxx} + H_{xxxt} = 0. \tag{138}$$

Suppose the solution of Equation (138) is

$$H(x,t) = H_1 e^{\iota(\rho x - \omega t)} + H_2 e^{-\iota(\rho x - \omega t)}. \tag{139}$$

Here, ρ and ω are constants. Equation (139) is substituted into Equation (138). After solving the determinant of the coefficient matrix, we obtain the dispersion relation by adding the coefficients of $e^{\iota(\rho x-\omega t)}$ and $e^{-\iota(\rho x-\omega t)}$.

$$-\rho^6\tau^2 + \rho^6\omega^2 - 2\rho^5\omega - 2\rho^4\tau^2 + 2\rho^4\omega^2 + \rho^4 - 2\rho^3\omega - \rho^2\tau^2 + \rho^2\omega^2 = 0. \tag{140}$$

The dispersion relation can be found in Equation (140) for q, which results in

$$\omega = \pm\tau + \frac{\sqrt{\rho^2}}{\rho^2 + 1}. \tag{141}$$

The steady-state stability is demonstrated by the obtained dispersion relation. The steady-state outcome is not stable when the wave number ω is not real because the perturbation increases exponentially. However, a steady state becomes stable against minor perturbations if ω is not imaginary. The result is an unstable steady state if

$$\rho^2 < 0. \tag{142}$$

It is possible to acquire the MI gain spectrum $G(p)$ as follows:

$$G(\rho) = 2Im(\omega) = \pm\tau + \frac{\sqrt{\rho^2}}{\rho^2 + 1}. \tag{143}$$

9. Findings and Analysis

Now, we compare our solutions with the existing solutions to the considered model. By using the $(\frac{G'}{G'+G+A})$ expansion technique in [24], distinct types of periodic and singular wave solutions are obtained. By applying the class $f(G, G')$ expansion scheme in [25], different kinds of traveling wave solutions are obtained. By utilizing the modified rational sine–cosine method in [26], many explicit wave solutions are obtained. Similar models are discussed in [37,38] through the modified auxiliary equation, (G'/G) expansion, and the improved (G'/G) expansion techniques. We studied the fractional form of the considered equation to understand the phenomena. We utilized effective techniques: the improved (G'/G) expansion technique and the modified extended direct algebraic technique. As a result, we obtained various exact solitons, including periodic, singular, bright–dark, kink-singular, rational solitons, and many others. The impact of the fractional derivative on the obtained solutions is also shown through 2D graphs in Figures 1–6. The validity of the obtained solutions was checked by putting them back into the considered model. By comparing the obtained results with the existing results, we validated our results. Moreover, the stability of the obtained solutions was also checked. Further, the modulation instability is explained with two-dimensional, three-dimensional, and contour graphs as shown in Figure 7. The achieved solutions are useful for future studies of the considered model. Furthermore, the effect of the following terms can be discussed in future studies of the model.

(i) A body force term, representing the applied force or flow;

(ii) A density gradient term, representing the effect of density variations on the wave;

(iii) A viscosity term, representing the effect of friction or dissipation on the wave.

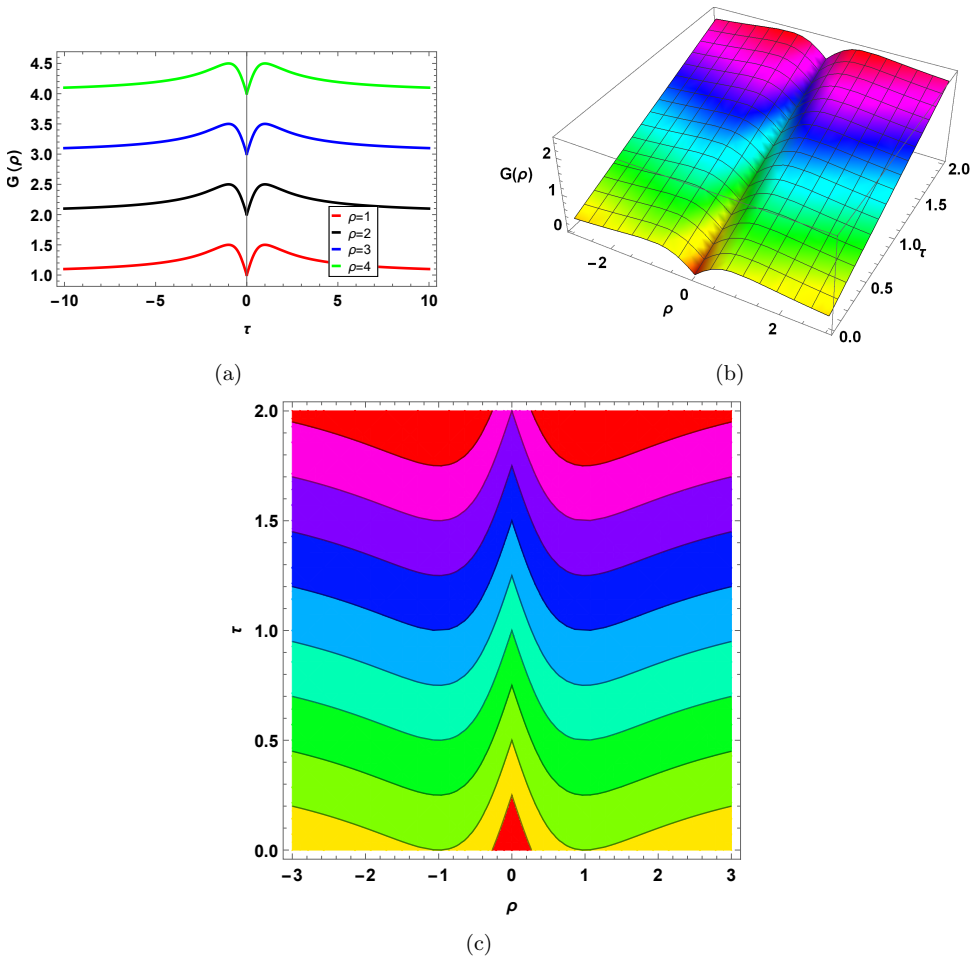


Figure 7. MI gain spectrum. This figure is drawn for the Equation (143). (**a**) represents the two-dimensional graph for $\tau \in (-10, 10)$. (**b**) shows the three-dimensional graph for $\tau \in (0, 2)$ and $\rho \in (-3, 3)$. (**c**) represents the contour graph for $\tau \in (0, 2)$ and $\rho \in (-3, 3)$.

10. Conclusions

This paper successfully obtained diverse exact soliton solutions for a truncated M-fractional nonlinear (1 + 1)-dimensional generalized shallow water wave equation. We employed two efficient techniques: the improved (G'/G) expansion technique and the modified extended direct algebraic technique. The obtained solutions included kink, dark–bright, periodic, and various other wave solitons. We examined the impact of the fractional derivative. Mathematica software was utilized to obtain and validate our results. Additionally, in Figures 1–6, we used 2D, 3D, and contour graphs to visualize some of the obtained wave solitons. Our results can be applied in ocean engineering, fluid dynamics, and other fields. The stability of the governing model was also investigated. In conclusion, these methods are simple and very efficient when applied for solving nonlinear fractional models.

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