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On the Arithmetic Average of the First n Primes

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Abstract

The arithmetic average of the first n primes, $\bar{p}_n = \frac{1}{n}\sum_{i=1}^{n} p_i$, exhibits very many interesting and subtle properties. Since the transformation from $p_n \to \bar{p}_n$ is extremely easy to invert, $p_n = n\bar{p}_n - (n-1)\bar{p}_{n-1}$, it is clear that these two sequences $p_n \longleftrightarrow \bar{p}_n$ must ultimately carry exactly the same information. But the averaged sequence $\bar{p}_n$, while very closely correlated with the primes, ($\bar{p}_n \sim \frac{1}{2}p_n$), is much "smoother" and much better behaved. Using extensions of various standard results, I shall demonstrate that the prime-averaged sequence $\bar{p}_n$ satisfies prime-averaged analogues of the Cramer, Andrica, Legendre, Oppermann, Brocard, Fourges, Firoozbakht, Nicholson, and Farhadian conjectures. (So these prime-averaged analogues are not conjectures; they are theorems). The crucial key to enabling this pleasant behaviour is the "smoothing" process inherent in averaging. While the asymptotic behaviour of the two sequences is very closely correlated, the local fluctuations are quite different.

Keywords: the n-th prime p_n; average of the first n primes; prime-averaged conjectures; Cramer; Andrica; Legendre; Oppermann; Brocard; Fourges; Firoozbakht; Nicholson; Farhadian

MSC: 11A41; 11N05; 11N80

1. Introduction and Background

Consider the average of the first n primes

$$\bar{p}_n = \frac{1}{n}\sum_{i=1}^{n} p_i. \tag{1}$$

Explicitly

$$\bar{p}_n \in \left\{2, 2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, 5\tfrac{3}{5}, 6\tfrac{5}{6}, 8\tfrac{2}{7}, 9\tfrac{5}{8}, 11\tfrac{1}{9}, 12\tfrac{9}{10}, \ldots\right\} \quad \text{for} \quad n \in \{1, 2, 3, \ldots\}. \tag{2}$$

Quite a lot is already known about this sequence [1–9]. For instance the asymptotic behaviour is known to be (OEIS A034387 [1])

$$\bar{p}_n \sim \frac{p_n}{2} \sim \frac{n \ln n}{2}. \tag{3}$$

More precisely (Dusart [2])

$$\bar{p}_n = \frac{1}{2}\left(n \ln(n \ln n) - \frac{3n}{2} + o(1)\right). \tag{4}$$

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Several higher-order terms in the asymptotic expansion are also known [2–6] but are not of direct relevance to the present discussion.

Various other explicit upper and lower bounds are also known (Mandl [7], OEIS A351914 [8], Hassani [3], Dusart [2])

$$\bar{p}_n < \tfrac{p_n}{2}; \qquad (p_n > 19; n \geq 9); \tag{5}$$

$$\bar{p}_n < \tfrac{p_n}{2} - \tfrac{n}{14}; \qquad (n \geq 10); \tag{6}$$

$$\bar{p}_n > p_{\lfloor n/2 \rfloor}; \qquad (n \geq 2). \tag{7}$$

Herein, I wish to take the discussion in a rather different direction.

Since $\bar{p}_n \sim \frac{p_n}{2}$, these two sequences are very closely correlated. Furthermore, since the relation $p_n \longleftrightarrow \bar{p}_n$ is easily invertible to $p_n = n\bar{p}_n - (n-1)\bar{p}_{n-1}$, these two sequences ultimately carry completely identical information. Nevertheless, as discussed below, the two sequences also exhibit profound differences.

Specifically, I shall demonstrate below that the prime-averaged sequence $\bar{p}_n$ *satisfies* suitably formulated prime-averaged analogues of the usual Cramer, Andrica, Legendre, Oppermann, Brocard, Fourges, Firoozbakht, Nicholson, and Farhadian conjectures. (So these prime-averaged analogues are not conjectures; they are theorems). The crucial observation enabling this pleasant behaviour is the "smoothing" process inherent in averaging. We shall soon see that the prime-averaged gaps are in a suitable sense extremely small. On the other hand, while the asymptotic behaviour of the two sequences is very tightly correlated, we shall see that the fluctuations are quite different.

2. Standard and Easy Results

In this section I will introduce a few basic tools (Rosser [10], Rosser–Schoenfeld [11]):

$$p_n > n \ln n; \qquad (n \geq 1); \tag{8}$$

$$p_n < n \ln(n \ln n); \qquad (n \geq 6). \tag{9}$$

It is sometimes useful to note

$$\ln(n \ln n) = \ln n + \ln \ln n = \ln n \left(1 + \frac{\ln \ln n}{\ln n}\right) \leq (1 + e^{-1})\ \ln n. \tag{10}$$

Consequently

$$p_n < (1 + e^{-1})\ n \ln n; \qquad (n \geq 4). \tag{11}$$

Lemma 1. *Still easy (but perhaps somewhat less well known) is the first slightly non-trivial (post Bertrand–Chebyshev) bound on the n-th prime gap:*

$$g_n = p_{n+1} - p_n < n; \qquad \textit{so that} \qquad p_{n+1} < p_n + n; \qquad (n \geq 1). \tag{12}$$

Proof. Rosser and Schoenfeld [11] give $|\vartheta(x) - x| < \frac{1}{2}\ \frac{x}{\ln x}$ for $x \geq 563 = p_{103}$. Thence, (checking small values of x by explicit computation), $|\vartheta(x) - x| < \frac{1}{2}\ \pi(x)$ for $x \geq 347 = p_{69}$. But then $|\vartheta(p_n) - p_n| < \frac{1}{2} n$ for $n \geq 26$. By evaluating $|\vartheta(x) - x|$ at $x = p_{n+1}^{-}$, just below the $(n+1)^{th}$ prime, we also see $|\vartheta(p_n) - p_{n+1}| < \frac{1}{2} n$ for $n \geq 69$. Thence by the triangle inequality $p_{n+1} - p_n < n$, certainly for all integers $n \geq 69$. Checking lower integers by explicit computation, $p_{n+1} - p_n < n$ for $n \geq 1$. □

Note that this in turn implies

$$p_{n+1} < p_n + n < n[\ln(n \ln n) + 1]; \qquad (n \geq 5). \tag{13}$$

For another lower bound on $\bar{p}_n$ note

$$\bar{p}_n > \frac{1}{n}\sum_{i=1}^{n} i \ln i = \frac{1}{n}\sum_{i=2}^{n} i \ln i. \tag{14}$$

We now bound this sum by the integral

$$\bar{p}_n > \frac{1}{n}\int_1^n u \ln u \, du = \frac{n \ln n}{2} - \frac{n}{4} + \frac{1}{4n} = \frac{n}{2}\left(\ln n - \frac{1}{2}\right) + \frac{1}{4n}, \tag{15}$$

with this final inequality holding for $n \geq 1$.

Finally, we shall also have occasion to use (Dusart [12])

$$\pi(x) > \frac{x}{\ln x - 1}; \qquad (x > 5393 = p_{711}), \tag{16}$$

and (Dusart [12])

$$\pi(x) < \frac{x}{\ln x - \frac{139}{125}}; \qquad (x > e^{139/125}). \tag{17}$$

3. Counting the Averaged Primes

Define

$$\bar{\pi}(x) := \#\{i : \bar{p}_i \leq x\}. \tag{18}$$

This counts the averaged-primes $\bar{p}_n$. (I emphasize that here we are counting the averaged-primes, not averaging the count of primes. That would instead be something akin to $\pi_{\text{average}}(x) = \frac{1}{\pi(x)}\sum_{i=1}^{\pi(x)} \pi(i)$, a rather different quantity).

- From $\bar{p}_n < \frac{p_n}{2}$ we see $\{i : \bar{p}_i \leq x\} \supseteq \{i : \frac{p_i}{2} \leq x\} = \{i : p_i \leq 2x\}$, and so for $x \geq 19$ we have $\#\{i : \bar{p}_i \leq x\} \geq \#\{i : p_i \leq 2x\}$. In terms of the average-prime counting function, this implies

$$\bar{\pi}(x) \geq \pi(2x); \qquad (x > 19 = p_8). \tag{19}$$

  Explicitly checking smaller values of x this saturates the domain of validity.
- From $\bar{p}_n > p_{[n/2]}$ we see $\{i : \bar{p}_i \leq x\} \subseteq \{i : p_{[i/2]} \leq x\}$ whence for $n \geq 2$ (implying $x \geq 3$) we have the following:

$$\bar{\pi}(x) \leq \#\{i : p_{[i/2]} \leq x\} \leq 2\,\#\{i : p_i \leq x\} \leq 2\,\pi(x). \tag{20}$$

  Explicitly checking smaller values of x this saturates the domain of validity.

$$\bar{\pi}(x) \leq 2\pi(x); \qquad (x \geq 3 = p_2). \tag{21}$$

Combining, we see that we have reasonably tight upper and lower bounds on the count of averaged primes:

$$(x > 19 = p_8) \qquad \pi(2x) \leq \bar{\pi}(x) \leq 2\pi(x) \qquad (x \geq 3 = p_2). \tag{22}$$

4. Bounding the Gaps in the Averaged Primes

First note that

$$\bar{p}_{n+1} = \frac{\sum_{i=1}^{n+1} p_i}{n+1} = \frac{\sum_{i=1}^{n} p_i + p_{n+1}}{n+1} = \frac{n\bar{p}_n + p_{n+1}}{n+1} = \bar{p}_n + \frac{p_{n+1} - \bar{p}_n}{n+1}. \tag{23}$$

In view of Mandl's inequality ($\bar{p}_n < p_n/2$ for $n \geq 9$, see [7,8]), that last term is definitely positive, and the sequence $\bar{p}_n$ is monotone increasing. Let us define the gap in the averages as

$$\bar{g}_n = \bar{p}_{n+1} - \bar{p}_n. \tag{24}$$

While the average primes $\bar{p}_n$ are monotone increasing, the gaps in the averages $\bar{g}_n$ have no nice monotonicity properties.

(I emphasise, $\bar{g}_n$ is the gap in the averages, not the averages of the gaps — that would instead be

$$(g_n)_{\text{average}} = \frac{1}{n}\sum_{i=1}^{n} g_i = \frac{1}{n}\sum_{i=1}^{n}(p_{i+1}-p_i) = \frac{1}{n}(p_{n+1}-p_1) = \frac{1}{n}(p_{n+1}-2), \tag{25}$$

a quantity interesting in its own right, but for totally different reasons).

For the gap in the averages we have

$$\bar{g}_n = \bar{p}_{n+1} - \bar{p}_n = \frac{p_{n+1}-\bar{p}_n}{n+1} < \frac{n[\ln(n\ln n)+1] - \frac{n}{2}\left(\ln n - \frac{1}{2}\right)}{n+1}, \tag{26}$$

with the inequality holding for $n \geq 5$. Rearranging

$$\bar{g}_n < \ln n\left\{\frac{n}{n+1}\ \left[\left(1+\frac{\ln\ln n}{n}+\frac{1}{\ln n}\right) - \frac{1}{2}\left(1-\frac{1}{2\ln n}\right)\right]\right\}, \tag{27}$$

that is

$$\bar{g}_n < \frac{\ln n}{2}\left\{\frac{n}{n+1}\ \left[1+\frac{2\ln\ln n}{n}+\frac{5}{2\ln n}\right]\right\}, \qquad (n \geq 5). \tag{28}$$

Within the domain of validity of this inequality, the quantity in braces is monotone decreasing (but always exceeds unity) and, searching for a value of n such that quantity in braces drops below 2, we certainly have $\bar{g}_n < \ln n$ for $n \geq 439$. Explicitly checking the smaller integers, one finds

$$\bar{g}_n < \ln n, \qquad (n \geq 3). \tag{29}$$

In the other direction

$$\bar{g}_n = \frac{p_{n+1}-\bar{p}_n}{n+1} > \frac{p_n+2-\frac{1}{2}p_n}{n+1} = \frac{1}{2}\frac{p_n+4}{n+1} > \frac{1}{2}\frac{n\ln n+4}{n+1} = \frac{\ln n}{2}\left\{\frac{1+\frac{4}{\ln n}}{1+\frac{1}{n}}\right\}. \tag{30}$$

Thence, since the quantity in braces always exceeds unity, we see

$$\bar{g}_n > \frac{\ln n}{2}, \qquad (n \geq 1). \tag{31}$$

Overall we have rather good upper and lower bounds

$$(n \geq 1), \qquad \frac{\ln n}{2} < \bar{g}_n < \ln n, \qquad (n \geq 3). \tag{32}$$

Indeed, from this we see $\bar{g}_n \sim \ln n$ and consequently $\frac{\bar{g}_n}{\bar{p}_n} \sim \frac{2}{n}$. This extremely rapid falloff in the size of the relative gaps in the averaged primes is ultimately the key observation underlying the computations below.

To be more explicit, for $n \geq 3$ we certainly have

$$\frac{\bar{g}_n}{\bar{p}_n} < \frac{\ln n}{\frac{n}{2}(\ln n - 1/2) + 1/4} < \frac{2.871142034}{n}. \tag{33}$$

By somewhat rearranging and tightening the discussion, from (23) we have

$$n\,\frac{\bar{g}_n}{\bar{p}_n} = \frac{n}{n+1}\left(\frac{p_{n+1}}{\bar{p}_n}-1\right) < \frac{n}{n+1}\left(\frac{[n+1]\ln([n+1]\ln[n+1])}{\frac{n}{2}\left(\ln n - \frac{1}{2}\right)+\frac{1}{4}} - 1\right). \tag{34}$$

Now this last quantity is less than 2 for $n \geq 100$. Thence certainly

$$\frac{\bar{g}_n}{\bar{p}_n} < \frac{2}{n}; \qquad (n > 100). \tag{35}$$

Explicitly checking smaller integers we have the stronger result

$$\frac{\bar{g}_n}{\bar{p}_n} < \frac{2}{n}; \qquad (n \geq 4). \tag{36}$$

In the other direction, for $n \geq 6$ we have

$$\frac{\bar{g}_n}{\bar{p}_n} > \frac{\frac{1}{2}\ln n}{\frac{1}{2}p_n} > \frac{\ln n}{n\ln(n\ln n)} = \frac{1}{n(1+\frac{\ln\ln n}{\ln n})} \tag{37}$$

which is bounded below by

$$\frac{1}{n(1+e^{-1})} > \frac{0.7310585785}{n}; \qquad (n \geq 6). \tag{38}$$

Explicitly checking smaller integers

$$\frac{\bar{g}_n}{\bar{p}_n} > \frac{1}{n(1+e^{-1})} > \frac{0.7310585785}{n}; \qquad (n \geq 3). \tag{39}$$

A slightly tidier summary of these results comes from noting

$$(n \geq 4) \qquad \frac{7}{10n} < \frac{\bar{g}_n}{\bar{p}_n} < \frac{2}{n}; \qquad (n \geq 4). \tag{40}$$

We shall re-use these bounds several times in the discussion below.

5. Prime-Average Analogues of Some Standard Conjectures

Given the rather tight bound on the prime-average gaps derived above, it is now in very many cases relatively easy to formulate *provable* prime-average analogues of various standard conjectures. (Often the only major difficulty lies in designing and formulating a suitable analogue). Below I present a few examples.

5.1. Prime-Average Analogue of Cramer

The ordinary Cramer conjecture is the hypothesis that the ordinary prime gaps g_n satisfy $g_n = \mathcal{O}([\ln p_n]^2)$. The prime-average analogue would be $\bar{g}_n = \mathcal{O}([\ln \bar{p}_n]^2)$. But since $\bar{p}_n = \mathcal{O}(n\ln n)$ we have $\ln \bar{p}_n = \mathcal{O}(\ln n)$ and $\mathcal{O}([\ln \bar{p}_n]^2) = \mathcal{O}([\ln n]^2)$. So the prime-average analogue of Cramer would be tantamount to making the claim $\bar{g}_n = \mathcal{O}([\ln n]^2)$. But since we have already proved the very much stronger result that $\bar{g}_n = \mathcal{O}(\ln n)$, this is a triviality.

5.2. Prime-Average Analogue of Andrica

The ordinary Andrica conjecture is the hypothesis that the ordinary primes p_n satisfy $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$. The prime-average analogue would be that the averaged primes $\bar{p}_n$ satisfy $\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} < 1$. But this is easily checked to be true and in fact much more can be said.

Using our previous results we compute (for $n > 1$)

$$\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} = \frac{\bar{p}_{n+1} - \bar{p}_n}{\sqrt{\bar{p}_{n+1}} + \sqrt{\bar{p}_n}} < \frac{\ln n}{2\sqrt{\bar{p}_n}} < \frac{\ln n}{2\sqrt{\frac{n}{2}(\ln n - \frac{1}{2})}}. \tag{41}$$

That is

$$\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} < \sqrt{\frac{\ln n}{n}} \left\{ 2\left(1 - \frac{1}{2\ln n}\right) \right\}^{-1/2}. \tag{42}$$

But for $n > e$ (implying $n \geq 3$) the quantity in braces lies in the range $(1, 2)$, and so in this range

$$\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} < \sqrt{\frac{\ln n}{n}}; \qquad (n \geq 3). \tag{43}$$

Explicitly checking $n \in \{1, 2\}$, in fact we see

$$\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} < \sqrt{\frac{\ln n}{n}}; \qquad (n \geq 2). \tag{44}$$

Note this is asymptotically much stronger than just a constant bound, and we could replace it with a considerably weaker statement with a slightly greater range of validity

$$\sqrt{\bar{p}_{n+1}} - \sqrt{\bar{p}_n} \leq \sqrt{\bar{p}_3} - \sqrt{\bar{p}_2} < \frac{1}{4}; \qquad (n \geq 1). \tag{45}$$

So the prime-average analogue of Andrica is unassailably true.

5.3. Prime-Average Analogue of Legendre

The ordinary Legendre conjecture is the hypothesis that the ordinary primes p_n satisfy $\pi([m+1]^2) > \pi(m^2)$ for integer $m \geq 1$. The most naive prime-average analogue would be that the averaged primes $\bar{p}_n$ satisfy $\bar{\pi}([m+1]^2) > \bar{\pi}(m^2)$ for integer $m \geq 1$. But this is easily checked to be true and in fact much more can be said.

Recall

$$\pi(2x) \leq \bar{\pi}(x) \leq 2\pi(x). \tag{46}$$

Let $\bar{\pi}(m^2) = n$, then by definition $\bar{p}_n$ is the largest prime-average below m^2, and $\bar{p}_{n+1}$ is the smallest prime-average above m^2. But then, using our previous results, for integer $n \geq 3$ corresponding to $m \geq \sqrt{5}$, we have

$$\bar{p}_{n+1} = \bar{p}_n + \bar{g}_n < m^2 + \ln n = m^2 + \ln \bar{\pi}(m^2) < m^2 + \ln[2\pi(m^2)] < m^2 + \ln 2 + \ln \pi(m^2). \tag{47}$$

Thence, since $\pi(m^2)$ is certainly (very much) less than m^2, and $\ln 2 < 1$, we certainly have the (extremely crude) bound

$$\bar{p}_{n+1} < m^2 + \ln 2 + \ln(m^2) < m^2 + 1 + 2\ln m = (m+1)^2 - 2[m - \ln m] < (m+1)^2. \tag{48}$$

(This crude bound could also be derived directly from the analogue Andrica discussion above). This bound is more than sufficient to prove the prime-average version of Legendre for $m \geq \sqrt{5}$, though it is clear that the argument can be very considerably tightened. In particular, it is easy to check that

$$\bar{p}_n^2 \in \left\{ 4, 6\tfrac{1}{4}, 11\tfrac{1}{9}, 19\tfrac{1}{16}, 31\tfrac{9}{25}, 46\tfrac{25}{36}, 68\tfrac{32}{49}, 92\tfrac{41}{64} \ldots \right\} \text{ for } n \in \{1, 2, 3, \ldots\}. \tag{49}$$

Consequently, comparing with

$$\bar{p}_n \in \left\{ 2, 2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, 5\tfrac{3}{5}, 6\tfrac{5}{6}, 8\tfrac{2}{7}, 9\tfrac{5}{8}, 11\tfrac{1}{9}, 12\tfrac{9}{10}, \ldots \right\} \text{ for } n \in \{1, 2, 3, \ldots\}. \tag{50}$$

we explicitly see that

$$\bar{\pi}(m^2) \in \{0, 3, 5, 7, 11, 16, \ldots\} \text{ for } m \in \{1, 2, 3, \ldots\}. \tag{51}$$

That is, the prime-average version of Legendre actually holds for integer $m \geq 1$.

Furthermore, the argument above nowhere explicitly uses the fact that m need be an integer — so the prime-average version of Legendre actually holds for real $m \geq \sqrt{5}$. Finally to check real range $m < \sqrt{5}$ use a suitable finite truncation of the sum

$$\bar{\pi}(x) = \sum_{i=1}^{\infty} \text{Heaviside}(x - \bar{p}_i); \qquad \text{Heaviside}(0) = 1. \tag{52}$$

Thereby one verifies that the prime-average version of Legendre certainly holds for real $m \geq 1$. (Actually $m \geq \sqrt{\bar{p}_2} - 1 = \sqrt{5/2} - 1 = 0.581138830$ is sufficient).

For another way of extending the argument above, consider this: If we were to define $\bar{\pi}([m+1]^2) = n_*$, then the average-prime gaps *below* p_{n_*} would be *at most* of size $\ln n_*$ and so

$$\bar{\pi}([m+1]^2) - \bar{\pi}(m^2) \geq \frac{2m+1}{\ln n_*} = \frac{2m+1}{\ln \bar{\pi}([m+1]^2)} \geq \frac{2m+1}{\ln\{2\pi([m+1]^2)\}}. \tag{53}$$

Thence for $[m+1]^2 > \exp(139/125)$, certainly for $m \geq 1$, we have

$$\bar{\pi}([m+1]^2) - \bar{\pi}(m^2) \geq \frac{2m+1}{\ln\{2[m+1]^2/(\ln([m+1]^2) - \frac{139}{125})\}}. \tag{54}$$

Expanding the logarithm

$$\bar{\pi}([m+1]^2) - \bar{\pi}(m^2) \geq \frac{2m+1}{\{2\ln[m+1] + \ln 2 - \ln(2\ln[m+1] - \frac{139}{125})\}}. \tag{55}$$

Ultimately, for $m \geq 4$ we certainly have

$$\bar{\pi}([m+1]^2) - \bar{\pi}(m^2) \geq \frac{2m+1}{2\ln[m+1]}. \tag{56}$$

Sacrificing a little precision in the interests of a slightly wider range of validity, for $m \geq 1$ we have

$$\bar{\pi}([m+1]^2) - \bar{\pi}(m^2) \geq \frac{2m-1}{2\ln[m+1]}. \tag{57}$$

So there will be *many* average-primes $\bar{p}_n$ between consecutive squares.

5.4. Prime-Average Analogue of Oppermann

The ordinary Oppermann conjecture is the hypothesis that the ordinary primes p_n satisfy $\pi(m[m+1]) > \pi(m^2) > \pi(m[m-1])$ for integer $m \geq 2$. This is completely equivalent to demanding that $\pi([m+\frac{1}{2}]^2) > \pi(m^2) > \pi([m-\frac{1}{2}]^2)$.

Then the most compelling prime-average analogue of Oppermann would be that the averaged primes $\bar{p}_n$ satisfy $\bar{\pi}([m+\frac{1}{2}]^2) > \bar{\pi}(m^2) > \bar{\pi}([m-\frac{1}{2}]^2)$. But (adapting the argument presented above for Legendre) this is easily checked to be true, and in fact much more can be said.

Let $\bar{\pi}(m^2) = n$, then for integer $n \geq 3$ corresponding to $m \geq \sqrt{5}$ we still have

$$m^2 < \bar{p}_{n+1} < m^2 + \ln 2 + 2\ln m. \tag{58}$$

Thence

$$m^2 < \bar{p}_{n+1} < \left(m + \frac{1}{2}\right)^2 - \left[m - 2\ln m + \frac{1}{4} - \ln 2\right] < \left(m + \frac{1}{2}\right)^2. \tag{59}$$

It is easy to check that for small integers m this continues to hold for $m \geq 1$. (For real values of m the domain of validity is $m \geq \sqrt{\bar{p}_1} - \frac{1}{2} = \sqrt{2} - \frac{1}{2} > 0.91421356237$).

Similarly let $\bar{\pi}([m-\frac{1}{2}]^2) = n_*$, then through an entirely analogous argument

$$\left(m-\frac{1}{2}\right)^2 < \bar{p}_{n_*+1} < \left(m-\frac{1}{2}\right)^2 + \ln 2 + 2\ln\left(m-\frac{1}{2}\right). \tag{60}$$

Thence

$$\left(m-\frac{1}{2}\right)^2 < \bar{p}_{n_*+1} < m^2 - \left[m - 2\ln\left(m-\frac{1}{2}\right) + \frac{1}{4} - \ln 2\right] < m^2. \tag{61}$$

It is easy to check that for small integers m this continues to hold for $m \geq 2$. (For real values of m the domain of validity is $m \geq \sqrt{\bar{p}_1} = \sqrt{2} = 1.414213562$).

So the prime-average analogue of Opperman is unassailably true for real $m \geq 2$.

5.5. Prime-Average Analogue of Brocard

The ordinary Brocard conjecture is the hypothesis that the ordinary primes p_n satisfy $\pi(p_{n+1}^2) - \pi(p_n^2) \geq 4$ for $n \geq 2$. Note that the ordinary Brocard conjecture is implied by the ordinary Oppermann conjecture. Note that $p_{n+1} \geq p_n + 2$ and that ordinary Oppermann implies the existence of at least one prime in each of the four regions

$$(p_n^2, [p_n+\tfrac{1}{2}]^2), \ ([p_n+\tfrac{1}{2}]^2, [p_n+1]^2), \ ([p_n+1]^2, [p_n+\tfrac{3}{2}]^2), \ ([p_n+\tfrac{3}{2}]^2, [p_n+2]^2). \tag{62}$$

Generalizing this, the ordinary Oppermann conjecture implies a generalization of the ordinary Brocard conjecture

$$\pi(p_{n+1}^2) - \pi(p_n^2) \geq 2g_n; \qquad (n \geq 2). \tag{63}$$

Then the most compelling prime-average analogue of Brocard would be that the averaged primes $\bar{p}_n$ satisfy $\bar{\pi}(\bar{p}_{n+1}^2) - \bar{\pi}(\bar{p}_n^2) \geq 4$ for suitable values of n. But (adapting the argument presented above for Legendre) this is easily checked to be true, and in fact much more can be said.

Write $\bar{p}_{n+1} = \bar{p}_n + \bar{g}_n$ and use prime-average analogue of Oppermann. Then

$$\bar{\pi}(\bar{p}_{n+1}^2) - \bar{\pi}(\bar{p}_n^2) = \bar{\pi}([\bar{p}_n + \bar{g}_n]^2) - \bar{\pi}(\bar{p}_n^2) > 2\bar{g}_n; \qquad (n \geq 1). \tag{64}$$

But $\bar{g}_n > \frac{\ln n}{2}$ for $n \geq 1$ so

$$\bar{\pi}(\bar{p}_{n+1}^2) - \bar{\pi}(\bar{p}_n^2) > \ln n; \qquad (n \geq 1). \tag{65}$$

So certainly

$$\bar{\pi}(\bar{p}_{n+1}^2) - \bar{\pi}(\bar{p}_n^2) > 4; \qquad (n \geq e^4). \tag{66}$$

Noting that $e^4 = 54.59815003$, to complete the analysis it suffices to explicitly check the first 55 average primes. Then

$$\bar{\pi}(\bar{p}_{n+1}^2) - \bar{\pi}(\bar{p}_n^2) > 4; \qquad (n \geq 4; \ \bar{p}_n \geq 4\tfrac{1}{4}). \tag{67}$$

So certainly the prime-average analogue of Brocard holds over a suitable range.

5.6. Prime-Average Analogue of Firoozbakht

The ordinary Firoozbakht conjecture is the hypothesis that the quantity $(p_n)^{1/n}$ is monotone decreasing for the ordinary primes p_n. The prime-average analogue of Firoozbakht would be the hypothesis that $(\bar{p}_n)^{1/n}$ is monotone decreasing for the averaged primes $\bar{p}_n$. But this is easily checked to be true (see for instance reference [9]) and in fact much more can be said.

Note

$$(\bar{p}_{n+1})^{1/(n+1)} < (\bar{p}_n)^{1/n} \quad \Longleftrightarrow \quad n \ln \bar{p}_{n+1} < (n+1) \ln \bar{p}_n. \tag{68}$$

So let us compute

$$Q := n \ln \bar{p}_{n+1} - (n+1) \ln \bar{p}_n = n \ln(\bar{p}_n + \bar{g}_n) - (n+1) \ln \bar{p}_n = n \ln(1 + \bar{g}_n / \bar{p}_n) - \ln \bar{p}_n. \tag{69}$$

This quantity is certainly less than

$$Q < \frac{n \bar{g}_n}{\bar{p}_n} - \ln \bar{p}_n < \frac{n \ln n}{\frac{1}{2} n \ln n \left[1 + \frac{\ln \ln n}{\ln n}\right]} - \ln \bar{p}_n < 2 - \ln \bar{p}_n, \tag{70}$$

which clearly becomes negative for $\bar{p}_n > e^2$, that is, for $n \geq 12$. Checking $n \in [1, 11]$ by direct computation shows that $Q < 0$ for all $n \geq 1$, so the prime-average analogue of Firoozbakht is unassailably true for all $n \geq 1$.

5.7. Prime-Average Analogue of Fourges

The ordinary Fourges conjecture is typically presented in terms of first rearranging the ordinary Firoozbakht conjecture into the form

$$\left(\frac{\ln p_{n+1}}{\ln p_n}\right)^n < \left(1 + \frac{1}{n}\right)^n; \qquad (n \geq 1), \tag{71}$$

and then *weakening* it by making the *less restrictive* demand that

$$\left(\frac{\ln p_{n+1}}{\ln p_n}\right)^n < e; \qquad (n \geq 1), \tag{72}$$

To obtain a prime-average analogue of the Fourges conjecture we first rearrange the prime-average analogue Firoozbakht conjecture into the form

$$\left(\frac{\ln \bar{p}_{n+1}}{\ln \bar{p}_n}\right)^n < \left(1 + \frac{1}{n}\right)^n; \qquad (n \geq 1), \tag{73}$$

and then *weaken* it by making the *less restrictive* demand that

$$\left(\frac{\ln \bar{p}_{n+1}}{\ln \bar{p}_n}\right)^n < e; \qquad (n \geq 1). \tag{74}$$

Since this is a weakening, the prime-average analogue of Fourges is unassailably true for all $n \geq 1$.

5.8. Prime-Average Analogue of Nicholson

The ordinary Nicholson conjecture is typically presented in terms of first rearranging the ordinary Firoozbakht conjecture into the form

$$\left(\frac{p_{n+1}}{p_n}\right)^n < p_n; \qquad (n \geq 1), \tag{75}$$

and then (slightly) *strengthening* it by making the *more restrictive* demand that

$$\left(\frac{p_{n+1}}{p_n}\right)^n < n \ln n; \qquad (n \geq 1). \tag{76}$$

To obtain a prime-average analogue of the Nicholson conjecture we first rearrange the prime-average analogue Firoozbakht conjecture into the form

$$\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right)^n < \bar{p}_n; \qquad (n \geq 1), \tag{77}$$

and then, noting $\bar{p}_n > p_{[n/2]} > (n/2)\ln(n/2)$, with the final inequality holding for $n \geq 20$, (slightly) *strengthen* it by making the *more restrictive* demand that

$$\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right)^n < (n/2)\ln(n/2); \qquad (n \geq n_0), \tag{78}$$

for some $n_0 \leq 20$ yet to be determined. That is, our proposed form of the prime-average analogue of Nicholson is equivalent to

$$n\ln\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right) < \ln((n/2)\ln(n/2)); \qquad (n \geq n_0). \tag{79}$$

Since

$$n\ln\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right) = n\ln\left(1+\frac{\bar{g}_n}{\bar{p}_n}\right) < n\ln\left(1+\frac{2}{n}\right) < 2, \tag{80}$$

with the inequalities holding for $n \geq 4$, we see that our proposed version of the prime-average analogue of the Nicholson conjecture will plausibly hold whenever

$$\ln[(n/2)\ln(n/2)] > 2, \tag{81}$$

that is, whenever $n \geq 5$, and so will certainly hold for all $n \geq 20$. Finally, explicitly checking all integers below 20 we have an explicit prime-average analogue of Nicholson:

$$\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right)^n < (n/2)\ln(n/2); \qquad (n \geq 6). \tag{82}$$

5.9. Prime-Average Analogue of Farhadian

The ordinary Farhadian conjecture is typically presented in terms of (slightly) *strengthening* the ordinary Nicholson conjecture by making the even *more restrictive* demand that

$$\left(\frac{p_{n+1}}{p_n}\right)^n < \frac{p_n \ln n}{\ln p_n}; \qquad (n \geq 1). \tag{83}$$

To obtain a prime-average analogue of the Farhadian conjecture we again (slightly) *strengthen* the prime-average analogue of Nicholson conjecture by making *some* (slightly) more restrictive demand. There is a potential infinity of stronger demands that one might make, but we shall try to keep close to the spirit of the original Farhadian conjecture by demanding

$$\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right)^n < \frac{p_n}{2\ln p_n}\ln(n/2); \qquad (n \geq n_0). \tag{84}$$

Since $p_n/\ln p_n < n$ for $n \geq 4$ this certainly strengthens the prime-average analogue of Nicholson.

This would be equivalent to

$$n\ln\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right) < \ln\left(\frac{p_n}{2\ln p_n}\ln(n/2)\right); \qquad (n \geq n_0). \tag{85}$$

As previously noted

$$n\ln\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right) = n\ln\left(1+\frac{\bar{g}_n}{\bar{p}_n}\right) < n\ln\left(1+\frac{2}{n}\right) < 2, \tag{86}$$

with these inequalities holding for $n \geq 4$.

This now suggests we consider the inequality

$$\ln\left(\frac{p_n}{2\ln p_n}\,\ln(n/2)\right) > 2 \tag{87}$$

which holds for $n \geq 5$ (though the original weakening to get to prime-average analogue Nicholson holds only for $n \geq 20$). Explicitly checking small integers

$$n\ln\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right) < \ln\left(\frac{p_n}{2\ln p_n}\,\ln(n/2)\right); \qquad (n \geq 7). \tag{88}$$

Equivalently

$$\left(\frac{\bar{p}_{n+1}}{\bar{p}_n}\right)^n < \frac{p_n}{2\ln p_n}\,\ln(n/2); \qquad (n \geq 7). \tag{89}$$

As expected, because the prime-average analogue of Farhadian is slightly stronger than the prime-average analogue of Nicholson, it is valid only on a slightly smaller domain.

6. Discussion

We have explicitly seen above that the averaged primes $\bar{p}_n = \frac{1}{n}\sum_{i=1}^{n} p_i$ satisfy suitably defined averaged-prime analogues of the Cramer, Andrica, Legendre, Oppermann, Fourges, Firoozbakht, Nicholson, and Farhadian hypotheses. The key step in all cases was the extremely tight bound in the averaged-prime gaps: $\bar{g}_n = \bar{p}_{n+1} - \bar{p}_n < \ln n$ for $n \geq 3$. The only (relatively minor) potential difficulty lies in the design and formulation of truly compelling prime-average analogues of these hypotheses.

On the other hand, what does this tell us about the ordinary primes p_n?
Not as much as one might hope. We certainly have the identity

$$g_n = [(n+1)\bar{p}_{n+1} - n\bar{p}_n] - [n\bar{p}_n - (n-1)\bar{p}_{n-1}] = [\bar{g}_n + \bar{g}_{n-1}] + n[\bar{g}_n - \bar{g}_{n-1}]. \tag{90}$$

Unfortunately a naive application of this equality merely leads to $g_n = \mathcal{O}(n\ln n)$— bounds too weak to be useful. So even though the numerical evidence is quite compelling [13,14], we must conclude that significant progress on any of the usual Cramer, Andrica, Legendre, Oppermann, Fourges, Firoozbakht, Nicholson, and Farhadian conjectures for ordinary primes would require considerably more subtle arguments.

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