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Controllability of Bilinear Systems: Lie Theory Approach and Control Sets on Projective Spaces

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Abstract

Bilinear systems can be developed from the point of view of time-varying linear differential equations or from the symmetry of Lie theory, in particular Lie algebras, Lie groups, and Lie semigroups. For bilinear control systems with bounded control range, we analyze when a unique control set (i.e., a maximal set of approximate controllability) with nonvoid interior exists, for the induced system on projective space. We use the system semigroup by considering piecewise constant controls and use spectral properties to extend the result to bilinear systems in $\mathbb{R}^d$. The contribution of this paper highlights the relationship between all the existent control sets. We show that the controllability property of a bilinear system is equivalent to the existence and uniqueness of a control set of the projective system.

Keywords: bilinear control systems; Lie theory; control sets; homogeneous spaces; Lyapunov exponents; Floquet spectrum; linear control semigroups

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1. Introduction

Bilinear systems are a special type of nonlinear systems: they are control systems whose dynamics are jointly linear in the state and control variables. The study of bilinear systems originated from their emergence in research on the dynamics of nuclear reactors carried out in the 1960s. Within the specialized literature on bilinear systems, we have the classic Mohler [1] monograph, in which a theoretical and computational basis for bilinear control systems is established. The author presents some sufficient conditions for complete controllability and optimal regulation of the bilinear systems with bounded control. In addition, applications are exhibited in nuclear and thermal control processes, ecological and physiological control, and socioeconomic systems. Moreover, the book examines applications in nuclear and thermal control processes, where reactor dynamics and heat transfer are modeled using bilinear systems. In the ecological and physiological realms, biological processes are described by compartmental models whose flow rates between compartments are defined as bilinear functions. Finally, in socioeconomic systems, urban and economic dynamics models are presented in which investment multipliers, public spending, or migration rates act bilinearly on variables such as population, production, or capital.

Furthermore, bilinear mathematical models are invaluable for capturing the dynamic behavior of a variety of important processes in the real world, since they can represent

systems in physics, mathematics, chemistry, biology, and social systems. In [2], applications as diverse as deterministic and stochastic bilinear population models, dynamic structures of immune response, body temperature regulation, CO_2 concentration in blood, blood flow, and water balance through parametric control via pulmonary ventilation rate, vascular dilation, and membrane permeability are described, as well as the modeling of vehicle breaking and certain aeronautical dynamics.

Similarly, refs. [3,4] present applications in power systems, ref. [5] illustrates case studies in modern epidemiology using compartmental disease models such as COVID-19, and ref. [6] examines applications in quantum computing.

Bilinear systems can be developed from the point of view of time-varying linear differential equations or from the symmetry of Lie theory, in particular Lie algebras, Lie groups, and Lie semigroups [7–10].

An exposition from a matrix Lie group point of view is given by Elliot's [11] monograph, which also discusses connections of bilinear systems with other system classes, such as switching systems and spin control in quantum physics. In addition, interesting applications are shown with the use of software for bilinear control problems, and the above shows us the interdisciplinary breadth of these models. Among the literature on controllability of bilinear control systems, based on the theory of semigroups in Lie groups, we have, for example, Boothby and Wilson [12], Jurdjevic and Kupka [13], Jurdjevic and Sallet [14], and San Martin [15].

Mathematically, they are also applicable to the analysis of complicated nonlinear systems, e.g., in the vicinity of fixed operating points. Moreover, bilinear control systems exhibit additional structure that allows for a more detailed study of the extremal Lyapunov exponents, i.e., the supremum and infimum of the Lyapunov spectrum. For these systems, the semigroup that describes the behavior forward in time is a linear semigroup. Consequently, the extremal exponents can always be approximated by Floquet exponents. Hence, the suprema (and infima) of the Floquet spectrum, the Lyapunov spectrum, and the Morse spectrum always coincide for bilinear systems; compare ([16], Chapter 7).

To exhibit part of the beauty that bilinear systems contain, we consider nonlinear control systems at a singular point y^0, i.e., y^0 is a common fixed point of the uncontrolled system and of all control vector fields. Because we are interested only in local questions, we study systems in $\mathbb{R}^d$.

We start considering the following class of control affine systems:

$$\dot{y}(t) = f(y(t), u(t)) = f_0(y(t)) + \sum_{i=1}^{m} u_i(t) f_i(y(t)) \text{ in } \mathbb{R}^d \tag{1}$$

$$u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m,\ u(t) \in U \text{ for all } t \in \mathbb{R},\ \text{locally integrable}\}$$

with a compact and convex set $U \in \mathbb{R}^m$ containing 0. We assume that $f_0, \ldots, f_m$ are C^1 vector fields with Lipschitz continuous first derivatives. Furthermore, suppose that for all $(u, y) \in \mathcal{U} \times \mathbb{R}^d$, (1) has a unique solution $\phi(t, y, u)$, $t \in \mathbb{R}$, with $\phi(0, y, u) = 0$.

Let $y^0 \in \mathbb{R}^d$ be a singular point of (1), i.e.,

$$f_i(y^0) = 0 \text{ for all } i = 0, \ldots, m. \tag{2}$$

Then, the system linearized at y^0 has the form

$$\dot{x}(t) = \left(A_0 + \sum_{i=1}^{m} u_i(t) A_i \right) x(t) = A(u(t))x(t),\ t \in \mathbb{R},\ x(t) \in \mathbb{R}^d\ u \in \mathcal{U}, \tag{3}$$

where $A_i = \frac{\partial}{\partial y} f_i(y)|_{y=y^0}$ denote the Jacobians at y^0 and $A_0, A_1, \ldots, A_m \in \mathrm{gl}(d, \mathbb{R})$ (the $d \times d$ matrices with real entries).

Thus, the linearization in y^0 yields a standard bilinear system in $\mathbb{R}^d$. This was studied in Chapter 7 of [16] along with a stable manifold theorem (see Theorem 7.4.1 of [16]), which links the linearization to the local behavior of the nonlinear system around y^0. Hence, locally, the bilinear control system (3) represents the behavior of the nonlinear system (1) in a neighborhood of y^0.

This paper is based mainly on the work of Colonius and Kliemann [17], in which they study linear control semigroups acting on projective spaces (or on spheres) that describe situations where the system is not completely controllable. This work characterizes dynamical and control behavior of such semigroups, defining maximal approximate controllability subsets (called *control sets)* in the projective space $\mathbb{P}^{d-1}$, using eigenvalue perturbation theory and (generalized) eigenspaces of matrices in the associated semigroup. The main result of [17] is Theorem 3.10, which fully describes k control sets, $1 \leq k \leq d$, whose interior consists of the eigenspaces of matrices in the interior of the system semigroup. These sets can be linearly ordered according to the accessibility structure of the bilinear system projection onto the projective space $\mathbb{P}^{d-1}$. Undoubtedly, article [17] is a great contribution to control theory, since it reduces the study of the controllability of bilinear systems in $\mathbb{P}^{d-1}$ (or in $\mathbb{S}^{d-1}$) to the study of eigenvalues and eigenspaces of matrices of the semigroup $\mathcal{S}$(defined later) associated with the system.

When there exists just one control set of the system in the projective space $\mathbb{P}^{d-1}$, it is well known that Theorem 3.10 in [17] gives a sufficient condition for the controllability of the original bilinear control systems on $\mathbb{R}^d$. According to our search, we could not find in the literature a demonstration of this fact. As a main contribution of this article, we write down a proof for it and we show that the condition is also necessary.

Then, we consider systems of the form (3), where the solutions of (3) are denoted by $\varphi(t, x, u)$ for the initial value $\varphi(0, x, u) = x \in \mathbb{R}^d$.

For fixed control $u \in \mathcal{U}$, (3) is a nonautonomous linear differential equation. Denote by $\Phi_u(t, s) \in \mathrm{gl}(d, \mathbb{R})$ the principal matrix solution, i.e., the solution of

$$\frac{d}{dt}\Phi_u(t, s) = A(u(t))\Phi_u(t, s), \qquad \Phi_u(s, s) = I. \tag{4}$$

Thus, the solutions of (3) are

$$\varphi(t, x_0, u) = \Phi_u(t, 0)x_0 \qquad t \in \mathbb{R}.$$

Following the approach of [17], we use piecewise constant controls to associate system (3) with the semigroup of the system:

$$\mathcal{S} := \{\exp(t_n B_n) \cdots \exp(t_1 B_1) : t_j \geq 0, B_j = A(u_j), u_j \in U, \\ j = 1, \ldots, n, n \in \mathbb{N}\} \subset \mathrm{GL}(d, \mathbb{R}).$$

where $\mathrm{GL}(d, \mathbb{R})$ is the set of invertible $d \times d$ matrices with real entries.

We will assume the rank condition for the Lie algebra for the induced system in the projective space $\mathbb{P}^{d-1}$ and focus our attention on the action of $\mathcal{S}$ in that space.

This allows us to determine a key fact: if there exists a unique control set in $\mathbb{P}^{d-1}$, then the projected system is completely controllable in the projective space $\mathbb{P}^{d-1}$ (Proposition 4). We present necessary and sufficient conditions that describe when this situation occurs (Theorem 3 and corollaries).

Note that the methodology proposed in this paper does not directly apply to questions of controllability of nonlinear systems. However, bilinear systems as presented here de-

scribe the linearization of a nonlinear system at an equilibrium point. Hence, together with the invariant manifold theorem from Theorem 5.6.1 in [16], one obtains local information about nonlinear situations. Another potential limitation of the use of spectral methods in control systems is always the dimensionality: Lyapunov spectra are notoriously hard to compute. But references [3,4] show that systems with a few dozen dimensions may be within the reach of the presented methodology. Finally, as pointed out, e.g., in [16,17] from a dynamics point of view, perturbations and control play similar roles in systems theory. $H\infty$ theory gives specific approaches to robust design for linear systems from an output feedback point of view—see, e.g., [18]—while the Lie-based approach presented here deals with perturbations only in the sense that extremal Lyapunov exponents indicate the stability (and controllability) gap for a system.

The structure of this paper is as follows: In Section 2, we collect preliminary results and introduce some notations. Section 3 presents a collection of results on control sets in homogeneous spaces, and in Section 4, we present the main results of this work and some applications.

2. Preliminaries

In Section 2.1, notation and some basic properties of control systems are recalled, and results on control sets for nonlinear control systems are reviewed. In Section 2.2, we present a specific description for bilinear control systems.

2.1. Basic Properties of Nonlinear Control Systems

System (3) has associated an angular system: we denote the projection of $\mathbb{R}^d$ onto the Euclidean unit sphere $\mathbb{S}^{d-1}$ by $\mathbb{S}$ and the projection onto real projective space $\mathbb{P}^{d-1}$(obtained by identifying opposite points on the sphere) by $\mathbb{P}$. For a trajectory $\varphi(t)$ of (3), define

$$s(t) := \mathbb{S}(\varphi(t)) = \frac{\varphi(t)}{\|\varphi(t)\|}, \qquad t \in \mathbb{R}.$$

Then, system (3) has the associated *angular system*; it is defined by the projection of (3) onto the Euclidean unit sphere $\mathbb{S}^{d-1}$,

$$\dot{s}(t) = h(u(t), s(t)) = h_0(s(t)) + \sum_{i=1}^{m} u_i(t) h_i(s(t)), \tag{5}$$

$h_0(s) = (A_0 - s^T A_0 s \cdot I)s$, $h_i(s) = (A_i - s^T A_i s \cdot I)s$ for $i = 1, \ldots, m$, where ^{T} denotes the transpose of a matrix and I is the identity matrix. One also obtains an induced control system on projective space $\mathbb{P}^{d-1}$ with vector fields $\mathbb{P}h(u, \cdot)$ since $h_i(s) = -h_i(-s)$ for all $0 \leq i \leq m$. With a slight abuse of notation, we have the projection of (3) onto the projective space $\mathbb{P}^{d-1}$.

$$\dot{s} = h_0(s) + \sum_{i=1}^{m} u_i(t) h_i(s) =: h(u, s), \text{ on } \mathbb{P}^{d-1} \tag{6}$$

where $h_j(s) = (A_j - q_j(s) \cdot I) \cdot s$, $q_i(s) = s^T A_j s$ for $j = 0, \ldots, m$. We also write

$$q(u, s) = q_0(s) + \sum_{i=1}^{m} u_i q_i(s). \tag{7}$$

Also, we consider the time-reversed systems corresponding to (3) and (6)

$$\dot{x}^*(t) = -\left(A_0 + \sum_{i=1}^{m} u_i(t) A_i\right) x^*(t), \; u \in \mathcal{U}. \tag{8}$$

and

$$\dot{s}^*(t) = -h(u(t), s^*(t)), \tag{9}$$

respectively.

For a nonlinear control system on the state space M, the set of points reachable from $x \in M$ and controllable to $x \in M$ up to time $T > 0$ are defined by

$$\mathcal{O}^+_{\leq T}(x) := \{y \in M, \text{ there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } \varphi(t, x, u) = y\}$$

and

$$\mathcal{O}^-_{\leq T}(x) := \{y \in M, \text{ there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } \varphi(t, y, u) = x\}$$

respectively. Furthermore, the reachable set (or "positive orbit") from x and controllable set (or "negative orbit") to x are

$$\mathcal{O}^+(x) = \bigcup_{T \geq 0} \mathcal{O}^+_{\leq T}(x) \qquad \text{and} \qquad \mathcal{O}^-(x) = \bigcup_{T \geq 0} \mathcal{O}^-_{\leq T}(x),$$

respectively. A control system is (completely) controllable on M, if for every point $x \in M$, we have $\mathcal{O}^+(x) = \mathcal{O}^-(x) = M$. The control system is said to be *accessible* from $x \in M$ if $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$ have nonvoid interior in M. It is *locally accessible* from $x \in M$ if the orbits up to time T have nonvoid interior for all $T > 0$.

The Lie algebra of (3) is given by $\mathcal{LA}\{A_0 + \sum_{i=1}^m u_i A_i,\ u \in U\} \subset \mathrm{gl}(d, \mathbb{R})$. In general, for each $x \in \mathbb{R}^d$, we have

$$\mathcal{LA}\left\{A_0 + \sum_{i=1}^m u_i A_i,\ u \in U\right\}(x) \subset T_x \mathbb{R}^d.$$

Definition 1. *The bilinear control system (3) satisfies the Lie algebra rank condition at $x \in \mathbb{R}^d$ if*

$$\dim\left(\mathcal{LA}\left\{A_0 + \sum_{i=1}^m u_i A_i,\ u \in U\right\}(x)\right) = \dim \mathbb{R}^d = d. \tag{10}$$

If (10) holds for all $x \in \mathbb{R}^d \setminus \{0\}$, we say that the bilinear system satisfies the Lie algebra rank condition (10).

Note that bilinear control systems, as well as the induced angular systems on the projective space and Euclidean unit sphere, are real analytic systems, with corresponding accessibility rank conditions on $\mathbb{S}^{d-1}$ and $\mathbb{P}^{d-1}$

$$\dim \mathcal{LA}\{h(u, \cdot), u \in U\}(s) = d - 1 \text{ for all } s \in \mathbb{S}^{d-1} \tag{11}$$

$$\dim \mathcal{LA}\{h(u, \cdot), u \in U\}(p) = d - 1 \text{ for all } p \in \mathbb{P}^{d-1}. \tag{12}$$

The following result shows that controllability is invariant under a change of base. To do this, let us consider the transformation $\mathcal{P}(A) = Q^{-1}AQ$ associated with base change $x = Qy$, $Q \in \mathrm{GL}(d, \mathbb{R})$, then we have the following:

Lemma 1. *Consider the bilinear control system (3) and the transformation $\mathcal{P}(A) = Q^{-1}AQ$ of A. Let*

$$\dot{y}(t) = \left(Q^{-1}A_0Q + \sum_{i=1}^m u_i(t) Q^{-1}A_iQ\right) y(t) \tag{13}$$

denote system (3) transformed by $\mathcal{P}$. Then, $\mathcal{P}$ preserves controllability of the system in $\mathbb{R}^d \setminus \{0\}$.

Proof. We assume that the bilinear control system (3) is controllable in $\mathbb{R}^d \setminus \{0\}$. Let $y_1, y_2 \in \mathbb{R}^d \setminus \{0\}$, then there exist $x_1, x_2 \in \mathbb{R}^d \setminus \{0\}$, such that

$$y_1 = Q^{-1}x_1 \text{ and } y_2 = Q^{-1}x_2.$$

Since system (3) is controllable, there are $t > 0$ and $u \in \mathcal{U}$ such that $\varphi(t, x_1, u(t)) = x_2$. Applying the transformation $\mathcal{P}$ and using the linearity of the right hand side of (3), we get

$$y_2 = Q^{-1}x_2 = Q^{-1}\varphi(t, x_1, u(t)) = \varphi(t, Q^{-1}x_1, u(t)) = \varphi(t, y_1, u(t)).$$

Therefore, (13) is controllable. □

However, there is not always complete controllability, but under certain conditions, it is possible to find subsets of complete approximate controllability (see Definition 2.3 of [17]).

Definition 2. *A nonempty set D in a state space M is called a control set of a control system if it has the following properties:*

(i) For all $x \in D$, there is a control function $u \in \mathcal{U}$, such that $\varphi(t, x, u) \in D$ for all $t \geq 0$;
(ii) For all $x \in D$, one has $D \subset \text{cl } \mathcal{O}^+(x)$;
(iii) D is maximal with these properties; that is, if $D' \supset D$ satisfies conditions (i) and (ii), then $D' = D$.

A control set $D \subset M$ is called an invariant control set if $\text{cl } D = \text{cl } \mathcal{O}^+(x)$ for all $x \in D$. All other control sets are called variant. Also, in this article a control set with nonvoid interior is called a main control set.

Note that in the following, we denote by ${}_{\mathbb{P}}D$ and ${}_{\mathbb{S}}D$ the main control sets in $\mathbb{P}^{d-1}$ and $\mathbb{S}^{d-1}$, respectively.

We present Proposition 3.2.3 of Chapter 3 of [16], which shows that one cannot return to a control set.

Lemma 2. *Let D be a main control set of a nonlinear control system (1) in a state space M. Suppose that for an element $x \in D$, there are $T > 0$ and $u \in \mathcal{U}$, such that $\varphi(T, x, u) \in D$. Then, $\varphi(t, x, u) \in D$ for all $0 \leq t \leq T$.*

2.2. Specific Description for Bilinear Control Systems

We denote by N the constant right-hand side of (3), i.e.,

$$N = \left\{ A_0 + \sum_{i=1}^{m} u_i A_i, u \in U \right\} \subset \text{gl}(d, \mathbb{R}) \tag{14}$$

Then, the systems' group $\mathcal{G}$ and semigroup $\mathcal{S}$ are given as

$$\mathcal{G} = \{\exp(t_n B_n) \cdots \exp(t_1 B_1), t_j \in \mathbb{R}, B_j \in N, j = 1, \ldots, n \in \mathbb{N}\}.$$

$$\mathcal{S} = \{\exp(t_n B_n) \cdots \exp(t_1 B_1), t_j \geq 0, B_j \in N, j = 1, \ldots, n \in \mathbb{N}\}.$$

They correspond to the solutions of (3) with piecewise constant controls. The set $\mathcal{S}_{\leq t}$ is the subset of $\mathcal{S}$ with $\sum_{j=1}^{n} t_j \leq t$. Also, $S_0 = I$, where I is the identity matrix.

Note the following relationships between system (3) and the time-reversed system (8): $N^* = -N$, $\mathcal{G}^* = \mathcal{G}$, $\mathcal{S}^* = \mathcal{S}^{-1}$, where * refers to the system in reverse time correspondent.

The Lie group $\mathcal{G}$ induces a Lie group $\mathbb{P}\mathcal{G}$ obtained via a projection onto the projective space $\mathbb{P}^{d-1}$, identifying g_1 and g_2 if $g_1 = \alpha g_2$ for some $\alpha \neq 0$. This group and the associated

semigroup $\mathbb{P}\mathcal{S}$ correspond to the control system (6). The Lie algebra rank condition (12) implies that $\mathbb{P}\mathcal{G}$ acts transitively on $\mathbb{P}^{d-1}$ (Theorem 3, p. 44 [19]), i.e., for all $p \in \mathbb{P}^{d-1}$, one has $\{gs : g \in \mathcal{G}\} = \mathbb{P}^{d-1}$. Furthermore, (12) implies that int $\mathbb{P}\mathcal{S}_{\leq t}$ in $\mathbb{P}\mathcal{G}$ is nonvoid for every $t > 0$. In the following, we write, with a slight abuse of notation, $g \in \text{int } \mathcal{S}_{\leq t}$, if we mean elements, $g \in \mathcal{S}$ with $\mathbb{P}g \in \text{int } \mathbb{P}\mathcal{S}_{\leq t}$. Also, in the following, we do not distinguish notationally between $\mathbb{P}\mathcal{G}$ and $\mathcal{G}$, $\mathbb{P}\mathcal{S}$ and $\mathcal{S}$, etc.

Throughout this section, we assume that the accessibility condition (12) is satisfied.

The following lemma discusses the structure of the semigroup $\mathcal{S}$ (for more details, see Lemmas 7.3.1 and 7.3.2 in [16]).

Lemma 3.

(i) *For all $t > 0$, one has* cl int $\mathcal{S}_{\leq t}$ = cl $\mathcal{S}_{\leq t}$, *where the interior is taken with respect to the topology of the Lie group $\mathcal{G}$. In particular,* int $\mathcal{S}_{\leq t} \neq \emptyset$ *for all $t > 0$.*

(ii) *If $g \in$ int $\mathcal{S}_{\leq t}$, then $gx \in$ int $\mathcal{O}^+_{\leq t}(x)$ and $x \in$ int $\mathcal{O}^-_{\leq t}(gx)$ for all $x \in \mathbb{P}^{d-1}$.*

(iii) *For all $t > 0$ and all $x \in \mathbb{P}^{d-1}$,* cl int $\mathcal{O}^+_{\leq t}(x) =$ cl $\mathcal{O}^+_{\leq t}(x)$.

(iv) *Let $g_i \in$ int $\mathcal{S}_{\leq t_i}$ for $i = 0, 1$. Then, there is a continuous path in* int $\mathcal{S}_{\leq t_0+t_1}$, *connecting g_0 and g_1.*

(v) *For some $t > 0$, let $g \in \mathcal{S}_t$. Then, there exist a decreasing sequence $t_n \searrow t$ and $g_n \in \mathcal{S}_t \cap$ int $\mathcal{S}_{\leq t_n}$ with $\lim_{n\to\infty} g_n = g$. In particular, for all $g \in \mathcal{S}_t$, there are $g_n \in \mathcal{S}_t \cap$ int $\mathcal{S}_{\leq t+1}$ with $g_n \to g$.*

Recall that elements g of the system semigroup correspond to piecewise constant periodic control functions in the following way: every

$$g = \exp(A(u_n)t_n) \dots \exp(A(u_1)t_1) \in \mathcal{S}_T \tag{15}$$

with $u_i \in U$, $t_i > 0$, $i = 1, \dots, n$, $\sum_i t_i = T$, corresponds to $u_g \in \mathcal{U}$ defined by

$$u_g(t) = u_{j+1} \text{ for } t \in \left[\sum_{i=0}^{j} t_i, \sum_{i=0}^{j+1} t_i\right), \qquad j = 0, 1, \dots, n-1 \tag{16}$$

with $t_0 = 0$ and extended T-periodically to $\mathbb{R}$.

Conversely, every piecewise constant T-periodic control function u defines an element g_u of $\mathcal{S}_T$.

In the following, the set of eigenvalues of a matrix $g \in \text{GL}(d, \mathbb{R})$ is called spec g, and the real eigenspace for an eigenvalue $\lambda \in \mathbb{C}$ is $E(\lambda)$ (for more details, see Chapter 1 of [20]).

We present the concept of Lyapunov exponents as on page 266 of [16].

Definition 3. *Let*

$$\lambda(u, x) := \limsup_{t\to\infty} \frac{1}{t} \log |\varphi(t, x, u)|, \tag{17}$$

the Lyapunov exponent associated with the solution $\varphi(t, x, u)$ of (3), with $\varphi(0, x, u) = x \neq 0$, and the Lyapunov spectrum of (3) consist of all Lyapunov exponents, i.e.,

$$\Sigma_{Ly} = \{\lambda(u, x); (u, x) \in \mathcal{U} \times \mathbb{R}^d, x \neq 0\}. \tag{18}$$

Furthermore, for $\mathbb{P}^{d-1}$, using (7) and the chain rule, we obtain

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(u(\tau), \mathbb{P}\varphi(\tau, \mathbb{P}x_0, u))d\tau, \tag{19}$$

with $q(u, s) = q_0(s) + \sum_{i=1}^m u_i(\cdot)q_i(s)$.

Note that $\lambda(u,s) = \lambda(u,\alpha s)$ for all $\alpha \in \mathbb{R}, \alpha \neq 0$, then (19) describes all Lyapunov exponents of $A(u(t))$ for any initial value $x \neq 0$. Observe that the Lyapunov exponents are constant on lines through the origin.

A subset of the Lyapunov spectrum is the Floquet spectrum. As we have seen, the periodic trajectories are related to the semigroup $\mathcal{S}$ of the system. The importance of the Floquet spectrum is that we can describe the periodic trajectories of the projected control system in the following way: we consider a periodic trajectory $\mathbb{P}\varphi(\cdot, \mathbb{P}x, u(\cdot))$, then there exist $T \geq 0$ and $\alpha \in \mathbb{R}$ with

$$\varphi(T, x, u(\cdot)) = \alpha x$$

for an $x \in \mathbb{R}^d$. Thus, x is an eigenvector for a real eigenvalue α of a T-periodic solution.

Next, we define the Floquet spectrum for a main control set in $\mathbb{P}^{d-1}$ (see Definition 4.3 in [21]) and $\mathbb{S}^{d-1}$ (see Definition 3.13 in [22]).

Definition 4.

(i) Let ${}_{\mathbb{P}}D$ be a main control set of (6). The Floquet spectrum of the family (3) over ${}_{\mathbb{P}}D$ is defined as

$$\Sigma_{Fl}({}_{\mathbb{P}}D) = \left\{ \lambda(u,p) \,\middle|\, \begin{array}{c} (u,p) \in \mathcal{U} \times \text{int}\,({}_{\mathbb{P}}D),\ u \text{ is piecewise constant, periodic} \\ \text{with period } \tau \geq 0 \text{ such that } \mathbb{P}\varphi(\tau,p,u) = p \end{array} \right\},$$

with initial condition $\mathbb{P}\varphi(0,p,u) = p$. The Floquet spectrum of the family (3) is

$$\Sigma_{Fl} = \bigcup_{i=1}^{k} \Sigma_{Fl}({}_{\mathbb{P}}D_i),$$

where ${}_{\mathbb{P}}D_i$ are the main control sets of (6).

(ii) For a main control set ${}_{\mathbb{S}}D$ in $\mathbb{S}^{d-1}$, the Floquet spectrum is given by

$$\Sigma_{Fl}({}_{\mathbb{S}}D) = \left\{ \lambda(u,x) \,\middle|\, \begin{array}{c} x \in \text{int}\,({}_{\mathbb{S}}D) \text{ and } u \text{ is piecewise constant} \\ \tau - \text{periodic for some } \tau \geq 0 \text{ with } s(\tau,x,u) = x \end{array} \right\}.$$

The following result establishes the relationship between the Floquet spectrum in $\mathbb{P}^{d-1}$ and in $\mathbb{S}^{d-1}$.

Proposition 1. *If ${}_{\mathbb{S}}D$ is a control set with nonvoid interior on $\mathbb{S}^{d-1}$ that projects to a control set ${}_{\mathbb{P}}D$ in $\mathbb{P}^{d-1}$, then*

$$\Sigma_{Fl}({}_{\mathbb{S}}D) = \Sigma_{Fl}({}_{\mathbb{P}}D).$$

Consider the differential equation with T-periodic coefficients

$$\dot{x}(t) = A(u_g(t))x(t) \tag{20}$$

with u_g given by (16). By the Floquet theory (see e.g, ref. [20], Section 7.2), the Floquet multipliers $\rho_j \in \mathbb{C}$ are defined as the eigenvalues of $\Phi_{u_g}(T,0)$ (with Φ_{u_g} as in (4)) and the Floquet exponents are $\lambda_j := \frac{1}{T}\log|\rho_j|$. They coincide with the Lyapunov exponents (Theorem 7.2.9 of [20]).

The Floquet spectrum can be characterized using the system semigroups of the systems on $\mathbb{R}^d \setminus \{0\}$ and on $\mathbb{P}^{d-1}$. Suppose that the accessibility rank condition in $\mathbb{P}^{d-1}$ holds, then Corollary 7.3.18 in [16] implies that the Floquet spectrum of a control set ${}_{\mathbb{P}}D$ consists of the Floquet exponents $\lambda_j := \frac{1}{T}\log|\rho_j|$, where ρ_j is a real eigenvalue of some Φ_{u_g}.

To complete the preliminaries, we mention the following result of [22] (Theorem 3.14): the minimal and maximal Lyapunov exponent of system (3) are defined by

$$\kappa^* = \inf_{u\in\mathcal{U}} \inf_{x\neq 0} \lambda(u,x) \qquad \text{and} \qquad \kappa = \sup_{u\in\mathcal{U}} \sup_{x\neq 0} \lambda(u,x) \tag{21}$$

respectively. Extremal Lyapunov exponents are those exponential growth rates that are defined globally as supremum and infima over the initial values and the controls. These rates generalize the minimal and maximal real parts of the eigenvalues of the matrix A in the linear, autonomous equation $\dot{x} = Ax$.

Remark 1. *If the system is completely controllable on $\mathbb{P}^{d-1}$, then κ and κ^* can be realized for all $x_0 \neq 0$. Furthermore, by Corollary 7.3.23 of [16], the system has exactly one interval of the Floquet spectrum and it satisfies*

$$[\kappa^*, \kappa] = \text{cl}\ \Sigma_{Fl} = \Sigma_{Ly}$$

3. Control Sets for Bilinear Control Systems

We develop this section from [17,22] to present results about control sets for bilinear control systems and their projections.

The following results connect the eigenspaces for $g \in \text{int}\ \mathcal{S}_{\leq t}$ with the interior of the control sets; for more details, see Propositions 3.7 and 3.8 of [17].

Proposition 2. *Assume (12); then, the following results hold:*

(i) *Let $g \in \text{int}\ \mathcal{S}_{\leq t}$ for some $t > 0$, and suppose that $\lambda_{i_1}, \ldots, \lambda_{i_j} \in \text{spec}\ g$ are such that $|\lambda_{i_1}| = \cdots = |\lambda_{i_j}|$. Then, there exists a control set ${}_{\mathbb{P}}D$ such that the corresponding generalized eigenspaces satisfy*

$$\mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j})) \subset \text{int}\ {}_{\mathbb{P}}D.$$

(ii) *Let ${}_{\mathbb{P}}D$ be a main control set. Then, for every $x \in \text{int}\ {}_{\mathbb{P}}D$, there are $t > 0$, $g \in \mathcal{S}_t \cap \text{int}\ \mathcal{S}_{\leq t+1}$, and $\lambda \in \text{spec}\ g \cap \mathbb{R}$ with $\mathbb{P}E(\lambda) \subset \text{int}\ {}_{\mathbb{P}}D$.*

The following theorem (Theorem 3.10 of [17]) characterizes the structure of the main control sets in $\mathbb{P}^{d-1}$.

Theorem 1. *Consider the bilinear control system (3) with its projected system (6). Assume that accessibility condition (12) holds. Then, the following assertions hold:*

(i) *There are k control sets ${}_{\mathbb{P}}D_i$ with nonvoid interior in $\mathbb{P}^{d-1}$ and $1 \leq k \leq d$.*

(ii) *The main control sets are linearly ordered, where the order is defined by*

$${}_{\mathbb{P}}D_i \prec {}_{\mathbb{P}}D_j \text{ if there exist } x_i \in {}_{\mathbb{P}}D_i,\ x_j \in {}_{\mathbb{P}}D_j,\ t \geq 0$$

$$\text{and } g \in \mathcal{S}_t \text{ with } gx_i = x_j$$

We enumerate the control sets such that ${}_{\mathbb{P}}D_1 \prec {}_{\mathbb{P}}D_2 \prec \cdots \prec {}_{\mathbb{P}}D_k$.

(iii) *For every $t > 0$, every $g \in \text{int}\ \mathcal{S}_{\leq t}$, and every $\lambda \in \text{spec}\ g$, there is a main control set ${}_{\mathbb{P}}D_i$, such that the generalized eigenspace $E(\lambda)$ satisfies*

$$\mathbb{P}(E(\lambda)) \subset \text{int}\ {}_{\mathbb{P}}D_i$$

The interior of the main control sets consists exactly of those elements $x \in \mathbb{P}^{d-1}$ that are eigenvectors for a real eigenvalue of some $g \in \mathcal{S}_t \cap \text{int}\ \mathcal{S}_{\leq t+1}$ for some $t > 0$.

(iv) *For every $g \in \mathcal{S}$ and every $\lambda \in \operatorname{spec} g$, there is some main control set ${}_{\mathbb{P}}D_i$ with $\mathbb{P}E(\lambda) \cap \operatorname{cl} {}_{\mathbb{P}}D_i \neq \emptyset$; for every main control set ${}_{\mathbb{P}}D_i$ and every $g \in \mathcal{S}$, there is $\lambda \in \operatorname{spec} g$ with $\mathbb{P}E(\lambda) \cap \operatorname{cl} {}_{\mathbb{P}}D_i \neq \emptyset$.*

(v) *The control set $C := {}_{\mathbb{P}}D_k$ is closed and invariant and $C = \cap_{x \in \mathbb{P}^{d-1}} \operatorname{cl} \mathcal{O}^+(x)$; the control set $C^- := \operatorname{int} C^* = {}_{\mathbb{P}}D_1$ is open and $\operatorname{cl} C^- = \cap_{x \in \mathbb{P}^{d-1}} \operatorname{cl} \mathcal{O}^-(x)$; all other main control sets are neither open nor closed. In particular, the invariant control set is unique.*

Remark 2. *Using the classification of transitive Lie groups on projective space and the theory of control sets for semigroups in semisimple Lie groups, Barros and San Martin [23] were able, in certain cases, to give sharper estimates on the number of main control sets.*

In the following proposition, we present some additional results about control sets in $\mathbb{P}^{d-1}$; for more details, see Theorem 3.13, Proposition 3.17, and the proof of Theorem 3.10 of [17].

Proposition 3. *Assume that (12) is satisfied. Then,*

(i) *For each $g \in \operatorname{int} \mathcal{S}_{\leq t}$, $t > 0$, and each main control set ${}_{\mathbb{P}}D_i$, there exists $\lambda \in \operatorname{spec} g$ such that $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D_i$.*

(ii) *${}_{\mathbb{P}}D_i \prec {}_{\mathbb{P}}D_j$ if and only if, for each $g \in \operatorname{int} \mathcal{S}$, one has $\lambda, \lambda' \in \operatorname{spec} g$ with $\operatorname{Re} \lambda < \operatorname{Re} \lambda'$ and $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D_i$, $\mathbb{P}E(\lambda') \subset \operatorname{int} {}_{\mathbb{P}}D_j$. Furthermore, one has ${}_{\mathbb{P}}D_i \prec {}_{\mathbb{P}}D_j$ if there exists $x \in {}_{\mathbb{P}}D_i$ with $\mathcal{O}^+(x) \cap {}_{\mathbb{P}}D_j \neq \emptyset$.*

(iii) *The time-reversed system (8) has the same number k of main control sets as the forward system (3). The order among the main control sets is reversed, and for the corresponding main control sets, one has*

$$\operatorname{int} D_i^* = \operatorname{int} D_{k+1-i} \text{ for } i = 1, \ldots, k$$

where D^* denotes main control sets of (8).

Proof. (i) By part (i) of Proposition 2, there exists at least one main control set. Part (ii) of Theorem 2 helps us understand that it suffices to prove that there is $h \in \mathcal{S}_T \cap \operatorname{int} \mathcal{S}_{\leq T+1}$ for some $T > 0$, such that for all main control sets ${}_{\mathbb{P}}D$, there is $\lambda \in \operatorname{spec} h$ with $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D$. Let $g_j \in \operatorname{int} \mathcal{S}_{\leq t_j}$, $t_j > 0, j = 0, 1$, then by part (iv) of Lemma 3, there exists a continuous path $g : [0, t_0 + t_1] \to \operatorname{int} \mathcal{S}_{\leq t_0 + t_1}$ with $g(0) = g_0$ and $g(t_1 + t_2) = g_1$. Along this path, the eigenvalues vary continuously and the projections of the corresponding eigenspaces are contained in the appropriate main control sets. For more details, see page 508 of [17].

(ii) Assume that ${}_{\mathbb{P}}D_i \prec {}_{\mathbb{P}}D_j$, by (ii), there exists $x \in \operatorname{int} {}_{\mathbb{P}}D_i, y \in \operatorname{int} {}_{\mathbb{P}}D_j, \tau > 0$ and $g \in \mathcal{S}_\tau \cap \operatorname{int} \mathcal{S}_{\leq \tau+1}$ with $gx = y$. By part (i), there exists $\lambda, \lambda_j \in \operatorname{spec} g$ such that $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D_i$, $\mathbb{P}E(\lambda_j) \subset \operatorname{int} {}_{\mathbb{P}}D_j$. Then, $x \notin \mathbb{P}E(\lambda)$ and $gx = y \in \mathbb{P}(\oplus_{\lambda_j} E(\lambda_j))$, where the sum is taken over all $\lambda \in \operatorname{spec} g$ with $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D_j$. Then, by the dynamics of the system, $\operatorname{Re} \lambda < \operatorname{Re} \lambda_j$.

Conversely, let $g \in \operatorname{int} \mathcal{S}$ with $\lambda, \lambda' \in \operatorname{spec} g$ and $\mathbb{P}E(\lambda) \subset \operatorname{int} {}_{\mathbb{P}}D_i$, $\mathbb{P}E(\lambda') \subset \operatorname{int} {}_{\mathbb{P}}D_j$, and let $x \in \operatorname{int} {}_{\mathbb{P}}D_i$. Since $\mathbb{R}^d = \bigoplus_{\lambda \in \operatorname{spec} g} E(\lambda)$, x can be represented as $x = x_1 + \cdots + x_k$, with $x_j \in \bigoplus E(\lambda)$, where the sum is taken over all λ such that $\mathbb{P}(\bigoplus E(\lambda))$ is contained in the interior of one of the control sets ${}_{\mathbb{P}}D$. In particular, we have a component in $\mathbb{P}E(\lambda')$, and by the dynamics of the system, solutions converge to $\mathbb{P}E(\lambda') \subset \operatorname{int} {}_{\mathbb{P}}D_j$.

(iii) See Proposition 3.17 of [17]. □

The following theorem, Theorem 3.2 of [22], characterizes conditions under which a control set in $\mathbb{S}^{d-1}$ generates a control set in $\mathbb{R}^d$.

Theorem 2. *Let ${}_{\mathbb{S}}D$ be a main control set for system (5) on the unit sphere $\mathbb{S}^{d-1}$ and suppose that*

(i) *Every point in* int $({}_{\mathbb{S}}D)$ *is locally accessible;*
(ii) *There are* $\alpha_0^+ > 1$ $\delta_0 > 0$, *and* $\alpha^- \in (0,1)$ *such that for all* $\alpha^+ \in (\alpha_0^+, \alpha_0^+ + \delta_0)$, *there are points* $s^+, s^- \in$ int $({}_{\mathbb{S}}D)$, *controls* $u^+, u^- \in \mathcal{U}$, *and times* $\sigma^+, \sigma^- > 0$ *with*

$$\varphi(\sigma^+, s^+, u^+) = \alpha^+ s^+ \qquad \varphi(\sigma^-, s^-, u^-) = \alpha^- s^-. \tag{22}$$

Then, the cone $\{\alpha s \in \mathbb{R}^d | \alpha > 0, s \in {}_{\mathbb{S}}D\}$ *is a main control set in* $\mathbb{R}^d$.

Condition (ii) of this theorem is satisfied, e.g., if Observation 3.6 of [22] holds:

Remark 3. *Suppose that for a control set* ${}_{\mathbb{S}}D$ *on the unit sphere, every point in the interior is locally accessible and there are control values* $u^\pm \in$ int (U) *such that* $A(u^+)$ *has an eigenvalue* $\lambda^+ > 0$ *and* $A(u^-)$ *has an eigenvalue* $\lambda^- < 0$, *with eigenvalues satisfying* $E(\lambda^\pm) \cap$ int $({}_{\mathbb{S}}D) \neq \emptyset$. *Then, Assumption (ii) of Theorem 2 holds.*

The following result establishes that the connection between controllability in $\mathbb{R}^d$ and in its homogeneous spaces $\mathbb{S}^{d-1}$ and $\mathbb{P}^{d-1}$ is given by the Lyapunov exponents (see Corollary 3.21 in [22]).

Corollary 1. *Assume that the bilinear control system (3) satisfies the accessibility rank condition (12) on* $\mathbb{P}^{d-1}$. *Then, it is a controllable in* $\mathbb{R}^d \setminus \{0\}$ *if and only if the induced system on* $\mathbb{P}^{d-1}$ *is controllable and* $\kappa^* < 0 < \kappa$.

In Corollary 1, κ and κ^* are defined as in (21).

4. Controllability of Bilinear Control Systems

In this section, our goal is to characterize complete controllability of bilinear control systems; to do this, we combine ideas presented in [17] with the following result.

Proposition 4. *Let us consider the projected system (6) in* $\mathbb{P}^{d-1}$ *and assume that (12) holds. Then, the angular system (6) in* $\mathbb{P}^{d-1}$ *is completely controllable if and only if it has exactly one main control set.*

Proof. Assume that the projected system has a single control set ${}_{\mathbb{P}}D$, and furthermore ${}_{\mathbb{P}}D \neq \mathbb{P}^{d-1}$. Then, there exists $p \in \mathbb{P}^{d-1} \setminus {}_{\mathbb{P}}D$. Next, by part (v) of Theorem 1, ${}_{\mathbb{P}}D$ is invariant and ${}_{\mathbb{P}}D \subset$ cl $\mathcal{O}^+(p)$ for all $p \in \mathbb{P}^{d-1}$. Hence, there are $\tau > 0$ and $h \in \mathcal{S}_\tau$ such that $hp = q'$, with $q' \in$ int ${}_{\mathbb{P}}D$, and there exists a neighborhood V of q' such that $V \subset$ int ${}_{\mathbb{P}}D$. By part (v) of Lemma 3, there are $h_n \in \mathcal{S}_\tau \cap$ int $\mathcal{S}_{\leq \tau+1}$ with $h_n \to h$. Thus, by continuity, we can find $n \in \mathbb{N}$ such that $h_n p = q \in V$. Hence, we can find $g = h_n \in$ int $\mathcal{S}_{\leq t}$ with $gp = q \in$ int ${}_{\mathbb{P}}D$ for some $t > 0$. Now, consider the associated time-reversed system. Since $g \in$ int $\mathcal{S}_{\leq t}$ by part (ii) of Lemma 3, it follows that $gp \in$ int $\mathcal{O}^+_{\leq t}(p)$ and hence, $q \in$ int $\mathcal{O}^{*-}_{\leq t}(p)$. Then, by the same argument used in the previous paragraph, there exists $g^* \in$ int $\mathcal{S}^*_{\leq t}$ such that $p = g^* q$.

By Theorem 1, ${}_{\mathbb{P}}D^*$ is open and by part (iii) of Proposition 3, ${}_{\mathbb{P}}D^* =$ int ${}_{\mathbb{P}}D$; therefore, cl ${}_{\mathbb{P}}D^* \subset {}_{\mathbb{P}}D$, and hence, $p \notin$ cl ${}_{\mathbb{P}}D^*$.

Now, consider $(g^*)^m q$ with $m \to \infty$, by Lemma 2, $(g^*)^m q \notin {}_{\mathbb{P}}D^*$, there exists a main control set ${}_{\mathbb{P}}D'^* \neq {}_{\mathbb{P}}D^*$, such that $(g^*)^m q \in$ int ${}_{\mathbb{P}}D'^*$. Thus, we have at least two main control sets; this contradicts Proposition 3 (iii).

The reverse follows immediately. □

Next, we present the main result of this section.

Theorem 3. *Assume that (12) is satisfied and let $h \in \mathcal{S}$.*

Consider $\lambda_{1_1}, \ldots, \lambda_{1_i}, \lambda_{k_1}, \ldots, \lambda_{k_j} \in \operatorname{spec} h$, such that $\operatorname{Re} \lambda_{1_1} = \cdots = \operatorname{Re} \lambda_{1_i} = \mu_1$, $\operatorname{Re} \lambda_{k_1} = \cdots = \operatorname{Re} \lambda_{k_j} = \mu_k$, where $\mu_1 = \min\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} h\}$, and $\mu_k = \max\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} h\}$. Then, system (6), is completely controllable in $\mathbb{P}^{d-1}$ if and only if there exists $v(t) \in \mathcal{U}$ such that $\mathbb{P}\varphi(\tau, x, v(t)) = y$, with $\mathbb{P}\varphi(0, x, v(t)) = x$, $\tau > 0$, $x \in \mathbb{P}(E(\lambda_{k_1}) \oplus \cdots \oplus E(\lambda_{k_j}))$, and $y \in \mathbb{P}(E(\lambda_{1_1}) \oplus \cdots \oplus E(\lambda_{1_i}))$.

Proof. By Theorem 1, $\mathbb{P}(E(\lambda_{k_1}) \oplus \cdots \oplus E(\lambda_{k_j})) \cap {}_{\mathbb{P}}D_k \neq \emptyset$, and $\mathbb{P}(E(\lambda_{1_1}) \oplus \cdots \oplus E(\lambda_{1_i})) \cap \operatorname{cl} {}_{\mathbb{P}}D_1 \neq \emptyset$ holds. Next, by assumption, one has $v(t) \in \mathcal{U}$, with $\mathbb{P}\varphi(\tau, x, v(t)) = y$, where $x \in \mathbb{P}(E(\lambda_{k_1}) \oplus \cdots \oplus E(\lambda_{k_j})) \cap {}_{\mathbb{P}}D_k$ and $y \in \mathbb{P}(E(\lambda_{1_1}) \oplus \cdots \oplus E(\lambda_{1_i})) \cap \operatorname{cl} {}_{\mathbb{P}}D_1$. If $y \in \operatorname{int} {}_{\mathbb{P}}D_1 = {}_{\mathbb{P}}D_1$, then we have ${}_{\mathbb{P}}D_k \preceq {}_{\mathbb{P}}D_1$.

If $y \in \partial_{\mathbb{P}}D_1$, by Theorem 1 and Proposition 3 (iii), we have $\operatorname{cl} {}_{\mathbb{P}}D_1 = {}_{\mathbb{P}}D_k^*$, which for the time-reversed system, is invariant. This means $y \in {}_{\mathbb{P}}D_1^*$, and by exact controllability, there exists $g'^* \in \mathcal{S}_\sigma^*$, such that $g'^* z = y$, with $z \in \operatorname{int} {}_{\mathbb{P}}D_1^*$. Consequently, for the control $v(t) \in \mathcal{U}$, there exists $g_v \in \mathcal{S}_\tau$, such that $g_v x = y$. Concatenating g_v and $g' \in \mathcal{S}_\sigma$, we have $g' g_v x = z$, with $z \in {}_{\mathbb{P}}D_1$ and $g' g_v \in \mathcal{S}_{\tau+\sigma}$. Thus, ${}_{\mathbb{P}}D_k \preceq {}_{\mathbb{P}}D_1$, which implies that the system has only one main control set and by Proposition 4, system (6) is completely controllable in $\mathbb{P}^{d-1}$. □

Corollary 2. *Assume that (12) is satisfied. Let ${}_{\mathbb{P}}D$ be a main control set of system (6), and let λ_g, λ_h be eigenvalues corresponding to the largest and smallest real part of $g, h \in \operatorname{int} \mathcal{S}$, respectively. If $\mathbb{P}(E(\lambda_g)) \subset \operatorname{int} {}_{\mathbb{P}}D$ and $\mathbb{P}(E(\lambda_h)) \subset \operatorname{int} {}_{\mathbb{P}}D$, then ${}_{\mathbb{P}}D$ is the only main control set of system (6).*

Proof. Assume that there is another main control set ${}_{\mathbb{P}}D'$. Then there is $\lambda_g' \in \operatorname{spec} g$ such that $\mathbb{P}(E(\lambda_g')) \subset \operatorname{int} {}_{\mathbb{P}}D'$ and $\operatorname{Re} \lambda_g' < \operatorname{Re} \lambda_g$. By Proposition 3 (ii), one has ${}_{\mathbb{P}}D' \prec {}_{\mathbb{P}}D$. In the same way, there exists $\lambda_h' \in \operatorname{spec} h$ such that $\mathbb{P}(E(\lambda_h')) \subset \operatorname{int} {}_{\mathbb{P}}D'$ and $\operatorname{Re} \lambda_h < \operatorname{Re} \lambda_h'$, hence ${}_{\mathbb{P}}D \prec {}_{\mathbb{P}}D'$. This is a contradiction, since the order between the main control sets does not depend on the choice of $g \in \operatorname{int} \mathcal{S}$. □

Corollary 3. *Let $A(u) \in \mathfrak{gl}(d, \mathbb{R})$, with $u \in U$ constant. We consider $\lambda_{1_1}, \ldots, \lambda_{1_i}, \lambda_{k_1}, \ldots, \lambda_{k_j} \in \operatorname{spec} A(u)$, such that $\operatorname{Re} \lambda_{1_1} = \cdots = \operatorname{Re} \lambda_{1_i} = \mu_1$, $\operatorname{Re} \lambda_{k_1} = \cdots = \operatorname{Re} \lambda_{k_j} = \mu_k$, where $\mu_1 = \min\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} A(u)\}$, and $\mu_k = \max\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} A(u)\}$. Then, system (6) is completely controllable in $\mathbb{P}^{d-1}$ if and only if there exist $v(t) \in \mathcal{U}$, such that $\mathbb{P}\varphi(\tau, x, v(t)) = y$, with $\mathbb{P}\varphi(0, x, v(t)) = x$, $\tau > 0$, and $x \in \mathbb{P}(E(\lambda_{k_1}) \oplus \cdots \oplus E(\lambda_{k_j}))$, $y \in \mathbb{P}(E(\lambda_{1_1}) \oplus \cdots \oplus E(\lambda_{1_i}))$.*

Proof. By Proposition 1.3.2. of [20], the eigenspaces associated with $A(u)$ and g_u are the same. □

To discuss applicability of our results, we present numerical examples of bilinear systems.

Example 1. *Let us consider the bilinear control system in $\mathbb{R}^3$ given by*

$$\dot{x} = \left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + u(t) \begin{bmatrix} 5 & -3 & -1 \\ -3 & 5 & -1 \\ -1 & -1 & 3 \end{bmatrix} \right) x$$

with $u(t) \in [-1, 1]$.

For $u \equiv -1$, the eigenvalues with respective eigenvectors are

$$\lambda_1 = -7 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -1 \Rightarrow \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \lambda_3 = 2 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $u \equiv 1$, the eigenvalues with respective eigenvectors are

$$\lambda_1 = 9 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 7 \Rightarrow \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \lambda_3 = 4 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

By Corollary 3, the system is completely controllable in $\mathbb{P}^2$. Furthermore, by Remark 1, the system has exactly one interval of the Floquet spectrum. This example has positive and negative eigenvalues for $u(t)$ constant, hence $k^* < 0 < k$. Hence, by Corollary 1, the system is controllable in $\mathbb{R}^3 \setminus \{0\}$.

Example 2. *Let us consider the bilinear control system in $\mathbb{R}^4$ given by*

$$\dot{x} = \left(\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} + u(t) \begin{bmatrix} 2 & -1 & -1 & 3 \\ 1 & -2 & 1 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix} \right) x$$

with $u(t) \in [-1, 1]$.

For $u \equiv -1$, the eigenvalues with respective eigenvectors are

$$\lambda_1 = 8 \Rightarrow \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 6 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_3 = 4 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_4 = 2 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For $u \equiv 1$, the eigenvalues with respective eigenvectors are

$$\lambda_1 = 2 \Rightarrow \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 4 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_3 = 6 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_4 = 8 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, by Corollary 3, the system is completely controllable in $\mathbb{P}^3$ and arguing in a way similar to the previous example, the system is also completely controllable in $\mathbb{R}^4 \setminus \{0\}$.

Example 3. *Let us consider the bilinear control system in $\mathbb{R}^4$ given by*

$$\dot{x} = \left(\begin{bmatrix} 21 & 0 & 0 & -1 \\ -1 & 20 & 0 & 1 \\ -1 & -2 & 22 & 1 \\ -1 & 0 & 0 & 21 \end{bmatrix} + u(t) \begin{bmatrix} 7 & 14 & -2 & -7 \\ 5 & -2 & 2 & 7 \\ 5 & 0 & 0 & 7 \\ 5 & 10 & 2 & -5 \end{bmatrix} \right) x$$

with $u(t) \in [-1, 1]$.

For $u \equiv -1$, the eigenvalues with respective eigenvectors are:

$$\lambda_1 = 8 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 20 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_3 = 24 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_4 = 32 \Rightarrow \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

For $u \equiv 1$, the eigenvalues with respective eigenvectors are

$$\lambda_1 = 32 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 24 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_3 = 20 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda_4 = 8 \Rightarrow \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, by Corollary 3, the system is completely controllable in $\mathbb{P}^3$ and arguing in a way similar to the previous examples, the system is also completely controllable in $\mathbb{R}^4 \setminus \{0\}$.

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