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# Some Centrally Extended Derivations on Banach Algebras

**Ick-Soon Chang [1] and Jaiok Roh [2,*]**

[1] Department of Mathematics, Chungnam National University, Daejeon 34134, Republic of Korea; ischang@cnu.ac.kr
[2] Ilsong Liberal Art Schools (Mathematics), Hallym University, Chuncheon 24252, Republic of Korea
* Correspondence: joroh@hallym.ac.kr

**Abstract**

In this article, we consider centrally extended derivations on Banach algebras. We first establish the stability of centrally extended derivations on Banach algebras. Then, we prove that under some conditions, approximate centrally extended derivations on Banach algebras are linear CE derivations.

**Keywords:** Banach algebra; CE derivation; derivation; stability; superstability

**MSC:** 16N60; 16W80; 39B72; 39B82; 46H40

## 1. Introduction

Let $\mathcal{R}$ be a ring with center $Z(\mathcal{R})$. For all $x, y \in \mathcal{R}$, the symbol $[x, y]$ will denote the commutator $xy - yx$. A ring $\mathcal{R}$ is called *prime* if $IJ \neq 0$ for any two nonzero ideals $I, J \subseteq \mathcal{R}$, and *semiprime* if it contains no nonzero ideal whose square is zero. The following result was investigated by R.N Ferreira and B.L.M. Ferreira [1], who demonstrated the following:

*Let $\mathcal{R}$ be a 3-torsion free alternative ring. $\mathcal{R}$ is a prime ring if and only if $x\mathcal{R} \cdot y = 0$ (or $x \cdot \mathcal{R}y = 0$) implies $x = 0$ or $y = 0$ for $x, y \in \mathcal{R}$.*

In this paper, since we are working in an associative ring, we use the definition of a prime ring in [1]. One should note that the ring $\mathcal{R}$ in this paper is an associated ring. Recall that $\mathtt{ann}(\mathcal{R}) = \{x \in \mathcal{R} : \mathcal{R}x = 0\}$ denotes the two-sided annihilator of $\mathcal{R}$. An additive mapping $\delta : \mathcal{R} \to \mathcal{R}$ is said to be a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in \mathcal{R}$. A *centrally extended derivation* (*CE derivation*) $\delta : \mathcal{R} \to \mathcal{R}$ is a mapping that is centrally additive and centrally multiplicative; i.e.,

$$\begin{aligned} &\delta(x+y) - \delta(x) - \delta(y) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}, \\ &\delta(xy) - \delta(x)y - x\delta(y) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \end{aligned}$$

The concept of CE derivation was introduced by Bell and Daif [2]. From the definition, it is easy to see that all derivations are CE derivations. Furthermore, if $\mathcal{R}$ is commutative, every mapping $\delta : \mathcal{R} \to \mathcal{R}$ is a CE derivation. Recently, the study of centrally extended mappings in ring under various conditions has been investigated (for example, see [3]). CE derivations in ring theory and Banach algebra are important because they generalize the concepts of traditional derivation, providing a broader perspective on structure and behavior. On the other hand, the study of characterization is not limited to associative structures but also extends to nonassociative structures such as alternative rings and Jordan algebras (cf. [1,4,5]).

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Academic Editor: Victor Orlov

Received: 4 May 2025
Revised: 26 June 2025
Accepted: 28 June 2025
Published: 15 July 2025

**Citation:** Chang, I.-S.; Roh, J. Some Centrally Extended Derivations on Banach Algebras. *Mathematics* **2025**, *13*, 2271. https://doi.org/10.3390/math13142271

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A type of superstability of functional equations was studied by Baker, Lawrence and Zorzitto [6]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. Then, the exponential functional equation is said to be *superstable*. It was the first result concerning the *superstability phenomenon* of functional equations. The following year, this famous result was generalized with a simplified proof by Baker [7], who demonstrated the following:

*Let $f$ map a semigroup $S$ into complex field $\mathbb{C}$. If $f$ is an approximately exponential function, i.e., there exists a nonnegative number $\varepsilon$ such that*

$$\|f(x \cdot y) - f(x)f(y)\| \leq \varepsilon \text{ for all } x, y \in S,$$

*then $f$ is either bounded or exponential.*

Later, the superstability for derivations between operator algebras was investigated by Šemrl [8]. Badora [9] presented the stability of derivation in Banach algebra. The study of the stability of functional equation has its origin in the famous talk of Ulam [10]. Hyers [11] had answered affirmatively the question of Ulam for Banach spaces. Since then, many authors have generalized Hyers' results in [12–17]. A great amount of subsequent studies of stability to various functional equations involving derivations are still being conducted.

CE derivations on Banach algebra are the generalizations of concepts for derivations. The study concerning the stability of CE derivations on Banach algebras helps to understand the behavior of approximate CE derivations and their relation to actual CE derivations. So, studying the stability problem of CE derivations is necessary and important. On the other hand, extending the stability of derivations to the stability of CE derivations contributes to the broader understanding of derivations and mappings of derivation type in Banach algebra theory. The main objective of the present article is to investigate the stability of CE derivations on Banach algebras. Then, we are going to prove that under certain conditions, approximate CE derivations on Banach algebras can be linear.

## 2. Stability of CE Derivations

Let us begin with the following lemma as a starting point for our investigations. Now, let $\mathcal{A}$ be an algebra, and we will write the unit element (or identity element) of $\mathcal{A}$ by $e$. The set of invertible elements of algebra $\mathcal{A}$ will be denoted by $\mathrm{Inv}(\mathcal{A})$. A normed algebra is unital if it has a unit element and $\|e\| = 1$. A Banach algebra or complete normed algebra is a normed algebra such that the normed vector space with a norm is complete (i.e., every Cauchy sequence converges) (cf. [18]). As example of a centrally extended derivation in algebra, we can consider the following:

**Example 1.** *Let* $\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ *be an algebra. We define a map* $\delta : \mathcal{A} \to \mathcal{A}$ *by*

$$\delta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & c-a \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}$$

*for all* $a, b, c \in \mathbb{C}$. *Then, we have that* $Z(\mathcal{A}) = \left\{ \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$. *It is easy to show that* $\delta$ *is a CE derivation but not a derivation.*

**Lemma 1.** *Let $\mathcal{A}$ be a unital Banach algebra. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping such that for some $\varepsilon, \theta \geq 0$,*

$$\left\| [\delta(x+y) - \delta(x) - \delta(y), z] \right\| \leq \varepsilon \quad (x, y, z \in \mathcal{A}), \tag{1}$$
$$\left\| [\delta(xy) - \delta(x)y - x\delta(y), z] \right\| \leq \theta \quad (x, y, z \in \mathcal{A}). \tag{2}$$

*If there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ such that*

$$\left[ \frac{\delta(2^n x)}{2^n}, x_0 \right] \in \mathrm{Inv}(\mathcal{A}) \qquad \text{for all } x \in \mathcal{A} \text{ and all } n \geq n_0, \tag{3}$$

*then there exists a CE derivation $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\left\| [\mathcal{L}(x) - \delta(x), z] \right\| \leq \varepsilon \text{ for all } x, z \in \mathcal{A}. \tag{4}$$

**Proof.** Letting $y = x$ in (1), we arrive at

$$\left\| \left[ \frac{\delta(2x)}{2} - \delta(x), z \right] \right\| \leq \frac{\varepsilon}{2} \text{ for all } x, z \in \mathcal{A}. \tag{5}$$

Considering $x := 2^{n-1}x$ in (5), we obtain

$$\left\| \left[ \frac{\delta(2^n x)}{2^n} - \frac{\delta(2^{n-1}x)}{2^{n-1}}, z \right] \right\| \leq \frac{\varepsilon}{2^n} \text{ for all } x, z \in \mathcal{A}. \tag{6}$$

Then, we have by (5) and (6) that

$$\begin{aligned} \left\| \left[ \frac{\delta(2^n x)}{2^n} - \delta(x), z \right] \right\| &\leq \left\| \left[ \frac{\delta(2^n x)}{2^n} - \frac{\delta(2^{n-1}x)}{2^{n-1}}, z \right] \right\| + \left\| \left[ \frac{\delta(2^{n-1}x)}{2^{n-1}} - \frac{\delta(2^{n-2}x)}{2^{n-2}}, z \right] \right\| \\ &\quad \cdots + \left\| \left[ \frac{\delta(2^3 x)}{2^3} - \frac{\delta(2^2 x)}{2^2}, z \right] \right\| + \left\| \left[ \frac{\delta(2x)}{2} - \delta(2x), z \right] \right\| \\ &\leq \left( 1 - \frac{1}{2^n} \right) \varepsilon \end{aligned} \tag{7}$$

for all $x, z \in \mathcal{A}$ and all $n \in \mathbb{N}$. It follows from (7) that

$$\begin{aligned} \left\| \left[ \frac{\delta(2^m x)}{2^m} - \frac{\delta(2^n x)}{2^n}, z \right] \right\| &\leq \left\| \left[ \frac{\delta(2^{m-n+n} x)}{2^{m-n+n}} - \frac{\delta(2^n x)}{2^n}, z \right] \right\| \\ &\leq \frac{1}{2^n} \left\| \left[ \frac{\delta(2^{m-n} \cdot 2^n x)}{2^{m-n}} - \delta(2^n x), z \right] \right\| \\ &\leq \left( \frac{1}{2^n} - \frac{1}{2^m} \right) \varepsilon \end{aligned}$$

for all $x, z \in \mathcal{A}$ and all $m, n \in \mathbb{N}$. This means that $\left\{ \left[ \frac{\delta(2^n x)}{2^n}, z \right] \right\}$ is a Cauchy sequence. Hence, the sequence $\left\{ \left[ \frac{\delta(2^n x)}{2^n}, z \right] \right\}$ converges for all $x, z \in \mathcal{A}$. Considering $z := \frac{\delta(2^n x)}{2^n} x_0$ in the last expression, we see that

$$\lim_{n \to \infty} \frac{\delta(2^n x)}{2^n} \left[ \frac{\delta(2^n x)}{2^n}, x_0 \right] = \lim_{n \to \infty} \left[ \frac{\delta(2^n x)}{2^n}, \frac{\delta(2^n x)}{2^n} x_0 \right]$$

exists. Based on (3), since the map $s \mapsto s^{-1}$ of $\mathrm{Inv}(\mathcal{A}) \to \mathrm{Inv}(\mathcal{A})$ is continuous (for example, see [18,19]), we then have that $\lim_{n \to \infty} \left[ \frac{\delta(2^n x)}{2^n}, x_0 \right]^{-1}$ exists and

$$\lim_{n \to \infty} \left[ \frac{\delta(2^n x)}{2^n}, x_0 \right]^{-1} = \left( \lim_{n \to \infty} \left[ \frac{\delta(2^n x)}{2^n}, x_0 \right] \right)^{-1}.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}\frac{\delta(2^n x)}{2^n} &= \lim_{n\to\infty}\frac{\delta(2^n x)}{2^n}\left[\frac{\delta(2^n x)}{2^n}, x_0\right]\cdot\left[\frac{\delta(2^n x)}{2^n}, x_0\right]^{-1}\\ &= \lim_{n\to\infty}\frac{\delta(2^n x)}{2^n}\left[\frac{\delta(2^n x)}{2^n}, x_0\right]\cdot\lim_{n\to\infty}\left[\frac{\delta(2^n x)}{2^n}, x_0\right]^{-1}\end{aligned}$$

exists. Therefore, we can define a mapping $\mathcal{L}: \mathcal{A}\to\mathcal{A}$ by

$$\mathcal{L}(x) := \lim_{n\to\infty}\frac{\delta(2^n x)}{2^n} \quad \text{for all } x\in\mathcal{A}. \tag{8}$$

We now have from (1) that

$$\begin{aligned}\|[\mathcal{L}(x+y)-\mathcal{L}(x)-\mathcal{L}(y), z]\| &= \lim_{n\to\infty}\frac{1}{2^n}\|[\delta(2^n(x+y))-\delta(2^n x)-\delta(2^n y), z]\|\\ &\le \lim_{n\to\infty}\frac{\varepsilon}{2^n} = 0\end{aligned}$$

for all $x, y, z\in\mathcal{A}$, which gives that $[\mathcal{L}(x+y)-\mathcal{L}(x)-\mathcal{L}(y), z] = 0$. That is, $\mathcal{L}$ is centrally additive. Also, it follows from (7) and (8) that the mapping $\mathcal{L}$ satisfies (4).

On the other hand, we have by the condition (2)

$$\begin{aligned}\|[\mathcal{L}(xy)-\mathcal{L}(x)y-x\mathcal{L}(y), z]\| &= \lim_{n\to\infty}\frac{1}{2^{2n}}\|[\delta(2^n x\cdot 2^n y)-\delta(2^n x)y-2^n x\delta(2^n y), z]\|\\ &\le \lim_{n\to\infty}\frac{\theta}{2^{2n}} = 0,\end{aligned}$$

which implies that

$$[\mathcal{L}(xy)-\mathcal{L}(x)y-x\mathcal{L}(y), z] = 0 \text{ for all } x, y, z\in\mathcal{A}. \tag{9}$$

Hence, $\mathcal{L}$ is centrally multiplicative. Therefore, $\mathcal{L}$ is a CE derivation. The proof of the lemma is complete. □

**Theorem 1.** *Let $\mathcal{A}$ be a semiprime unital Banach algebra. Suppose that $\delta: \mathcal{A}\to\mathcal{A}$ is a mapping satisfying the inequalities (1) and (2). Assume that there exists an element $x_0\in\mathcal{A}$ and an integer $n_0\in\mathbb{N}$ with (3). Then, $\delta$ is a CE derivation.*

**Proof.** By Lemma 1, there exists a CE derivation $\mathcal{L}: \mathcal{A}\to\mathcal{A}$ satisfying the relation (4). In this case, the mapping $\mathcal{L}$ is defined as (8).

On the other hand, by virtue of (2), we see that

$$\|[\mathcal{L}(xy)-\mathcal{L}(x)y-x\delta(y), z]\| = \lim_{n\to\infty}\frac{1}{2^n}\|[\delta(2^n x\cdot y)-\delta(2^n x)y-2^n x\delta(y), z]\| \le \lim_{n\to\infty}\frac{\theta}{2^n} = 0,$$

which implies that

$$[\mathcal{L}(xy)-\mathcal{L}(x)y-x\delta(y), z] = 0 \text{ for all } x, y, z\in\mathcal{A}. \tag{10}$$

Also, since $\mathcal{L}$ is centrally multiplicative, the mapping $\mathcal{L}$ satisfies the expresstion (9). Subtract (10) from (9) to obtain

$$[x(\mathcal{L}(y)-\delta(y)), z] = 0 \text{ for all } x, y, z\in\mathcal{A}. \tag{11}$$

The identity (11) can be represented as

$$[x, z]\cdot(\mathcal{L}(y)-\delta(y)) + x[\mathcal{L}(y)-\delta(y), z] = 0. \tag{12}$$

Replacing $z$ by $\mathcal{L}(y) - \delta(y)$ in (12), we have

$$[\mathcal{L}(y) - \delta(y), x] \cdot (\mathcal{L}(y) - \delta(y)) = 0. \tag{13}$$

Substitute $xz$ instead of $x$ in (13) and then use (13) to obtain the following equation

$$[\mathcal{L}(y) - \delta(y), x]z(\mathcal{L}(y) - \delta(y)) = 0. \tag{14}$$

Using the relation obtained by multiplying $x$ in the right-hand side of (14) and the expression attained by substituting $zx$ instead of $z$ in (14), it can be derived as follows.

$$[\mathcal{L}(y) - \delta(y), x]z[\mathcal{L}(y) - \delta(y), x] = 0$$

for all $x, y, z \in \mathcal{A}$. So, by the semiprimeness of $\mathcal{A}$, we have $[\mathcal{L}(y) - \delta(y), x] = 0$. This means that

$$[\mathcal{L}(y), x] = [\delta(y), x] \text{ for all } x, y \in \mathcal{A}. \tag{15}$$

The expression (2) implies that

$$\|[\mathcal{L}(xy) - \delta(x)y - x\mathcal{L}(y), z]\| = \lim_{n\to\infty} \frac{1}{2^n} \|[\delta(x \cdot 2^n y) - 2^n \delta(x)y - x\delta(2^n y), z]\| \leq \lim_{n\to\infty} \frac{\theta}{2^n} = 0.$$

Then, we are forced to conclude that

$$[\mathcal{L}(xy) - \delta(x)y - x\mathcal{L}(y), z] = 0 \text{ for all } x, y, z \in \mathcal{A}. \tag{16}$$

Comparing (11) and (15) in (16), we obtain $[\delta(xy) - \delta(x)y - x\delta(y), z] = 0$, that is,

$$\delta(xy) - \delta(x)y - x\delta(y) \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}. \tag{17}$$

Now, since $\mathcal{L}$ is centrally additive, one obtains that $[\mathcal{L}(x + y) - \mathcal{L}(x) - \mathcal{L}(y), z] = 0$. Then, we have by (15) that $[\delta(x + y) - \delta(x) - \delta(y), z] = 0$. Hence, we see that

$$\delta(x + y) - \delta(x) - \delta(y) \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}. \tag{18}$$

Therefore, by (17) and (18), $\delta$ is a CE derivation, which concludes the proof. □

Recall that a central ideal $I$ of a ring $\mathcal{R}$ is an ideal of $\mathcal{R}$ such that $xr = rx$ for all $r \in \mathcal{R}$ and $x \in I$. Note that if $\mathcal{R}$ is a semiprime ring with no nonzero central ideals, then every CE derivation on $\mathcal{R}$ is a derivation (refer to [2] (Theorem 2.5)). Due to this fact and Theorem 1, we obtain the following result.

**Corollary 1.** *Let $\mathcal{A}$ be a semiprime unital Banach algebra with no nonzero central ideals. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying (1) and (2). Assume that there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ with (3). Then, $\delta$ is a CE derivation. Moreover, it can be a derivation.*

It is pointed out that the same result for Theorem 1 can be also obtained in prime noncommutative unital Banach algebra by applying a similar argument to the proof of Theorem 1. In particular, considering the proof of [2] (Theorem 2.5), we can see that every CE derivation on a noncommutative prime ring is a derivation. So, we arrive at the next conclusions.

**Corollary 2.** *Let $\mathcal{A}$ be a noncommutative prime unital Banach algebra. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying (1) and (2). Assume that there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ with (3). Then, $\delta$ is a CE derivation. Moreover, it can be a derivation.*

**Proof.** According to Lemma 1, there exists a CE derivation $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying the relation (4), where $\mathcal{L}$ is a mapping defined by (8).

In particular, note that in this case, the CE derivation $\mathcal{L}$ is unique. Indeed, to show the uniqueness of $\mathcal{L}$, let us assume that $T : \mathcal{A} \to \mathcal{A}$ is another CE derivation satisfying (4). Then, we have by (4)

$$\begin{aligned}\big\|\big[\mathcal{L}(x) - T(x), z\big]\big\| &= \lim_{n\to\infty} \frac{1}{2^n}\big\|\big[\mathcal{L}(2^n x) - T(2^n x), z\big]\big\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\big\{\big\|\big[\mathcal{L}(2^n x) - \delta(2^n x), z\big]\big\| + \big\|\big[\delta(2^n x) - T(2^n x), z\big]\big\|\big\} \\ &\leq \lim_{n\to\infty} \frac{\varepsilon}{2^{n-1}} = 0\end{aligned}$$

for all $x, z \in \mathcal{A}$, which means that $\big[\mathcal{L}(x) - T(x), z\big] = 0$, so that, $\big[\mathcal{L}(x) - T(x), x\big] = 0$. Consider a mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ defined by $\mathcal{D}(x) := \mathcal{L}(x) - T(x)$ for all $x \in \mathcal{A}$. Then, a simple calculation can prove that the mapping $\mathcal{D}$ is a CE derivation. Based on the facts mentioned above, $\mathcal{D}$ is a derivation with $\big[\mathcal{D}(x), x\big] = 0$ for all $x \in \mathcal{A}$. So, by Posner's second theorem [20] (Theorem 2), we see that $\mathcal{D} = 0$; i.e., $\mathcal{L} = T$. Therefore, $\mathcal{L}$ is unique.

The remainder of this proof can be proved using the same technique as Theorem 1. The proof of the theorem is complete. □

**Theorem 2.** *Let $\mathcal{A}$ be a noncommutative prime unital Banach algebra. Suppose that a mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfies (1) and for some $\theta \geq 0$,*

$$\big\|\big[(\delta(xy) - \delta(x)y - x\delta(y))a, z\big]\big\| \leq \theta \quad (x, y \in \mathcal{A}), \tag{19}$$

*where $0 \neq a \in Z(\mathcal{A})$. Assume that there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ with (3). Then, $\delta$ is a CE derivation. Moreover, it can be a derivation.*

**Proof.** As in the proof of Lemma 1, it follows that there exists a centrally additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ defined by (8) such that

$$[\mathcal{L}(y)a, x] = [\delta(y)a, x],\ \big(\delta(xy) - \delta(x)y - x\delta(y)\big)a \in Z(\mathcal{A}) \tag{20}$$

for all $x, y \in \mathcal{A}$. Then, since $0 \neq a \in Z(\mathcal{A})$, considering the primeness of $\mathcal{A}$ and reference [21], we have (17).

On the other hand, due to the fact that $\mathcal{L}$ is centrally additive, we yield that

$$\big[\big(\mathcal{L}(x + y) - \mathcal{L}(x) - \mathcal{L}(y)\big)a, z\big] = 0.$$

In view of (20), one can obtain that

$$\big[\big(\delta(x + y) - \delta(x) - \delta(y)\big)a, z\big] = 0,$$

so that $\big(\delta(x + y) - \delta(x) - \delta(y)\big)a \in Z(\mathcal{A})$. Since $\mathcal{A}$ is prime and $0 \neq a \in Z(\mathcal{A})$, we figure out (18). Therefore, by (17) and (18), we conclude that $\delta$ is a CE derivation. So, it follows from [2] (Theorem 2.5) that $\delta$ is a derivation. □

In [22], Tammam El-Sayiad et al. gave the notion of a *multiplicative centrally extended derivation* (*multiplicative CE derivation*) of ring $\mathcal{R}$ to be a mapping $\delta$ of $\mathcal{R}$ satisfying $\delta(xy) -$

$\delta(x)y - x\delta(y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{A}$. For an example of a multiplicative centrally extended derivation in algebra, we have the following:

**Example 2.** *Let* $\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ *be an algebra. Define a map* $\delta : \mathcal{A} \to \mathcal{A}$ *given by*

$$\delta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & ac & b \\ 0 & 0 & ac \\ 0 & 0 & 0 \end{bmatrix}$$

*for all* $a, b, c \in \mathbb{C}$. *Then, we see that* $Z(\mathcal{A}) = \left\{ \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$. *It is easy to check that* $\delta$ *is a multiplicative CE derivation.*

Next, we will address the superstability associated with multiplicative CE derivation.

**Theorem 3.** *Let* $\mathcal{A}$ *be a normed unital algebra. Suppose that* $\delta : \mathcal{A} \to \mathcal{A}$ *is a mapping subjected to the inequality* (2). *Assume that*

$$\lim_{n\to\infty} \left(\frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n}x - \delta(x)\right) \cdot [y, z] = 0 \text{ for all } x, y, z \in \mathcal{A}. \tag{21}$$

*Then,* $\delta$ *is a multiplicative CE derivation.*

**Proof.** It follows from (2) that

$$\left\| \left[\frac{\delta(nx \cdot y)}{n} - \frac{\delta(nx)}{n}y - x\delta(y), z\right] \right\| \leq \frac{\theta}{n} \quad (x, y, z \in \mathcal{A},\ n \in \mathbb{N}),$$

which leads to

$$\lim_{n\to\infty} \left[\frac{\delta(nx \cdot y)}{n} - \frac{\delta(nx)}{n}y - x\delta(y), z\right] = 0 \tag{22}$$

for all $x, y, z \in \mathcal{A}$. Moreover, we have by (2) that

$$\left\| \left[\frac{\delta(x \cdot ny)}{n} - \delta(x)y - x\frac{\delta(ny)}{n}, z\right] \right\| \leq \frac{\theta}{n} \quad (x, y, z \in \mathcal{A},\ n \in \mathbb{N}),$$

which implies that

$$\lim_{n\to\infty} \left[\frac{\delta(x \cdot ny)}{n} - \delta(x)y - x\frac{\delta(ny)}{n}, z\right] = 0 \tag{23}$$

for all $x, y, z \in \mathcal{A}$. In particular, considering $x = e$ and $y = x$ in (22), we see that

$$\lim_{n\to\infty} \left[\frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n}x - \delta(x), z\right] = 0 \tag{24}$$

for all $x, z \in \mathcal{A}$. Note that

$$\begin{aligned} &\left\| \left[\left(\frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n}x - \delta(x)\right)y, z\right] \right\| \\ &\leq \left\| \left[\frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n}x - \delta(x), z\right] \right\| \|y\| + \left\| \left(\frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n}x - \delta(x)\right)[y, z] \right\| \end{aligned} \tag{25}$$

for all $x, y, z \in \mathcal{A}$ and all $n \in \mathbb{N}$. However, we have

$$\lim_{n\to\infty} \Big[ \Big( \frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n} x - \delta(x) \Big) y, z \Big] = 0 \tag{26}$$

for all $x, y, z \in \mathcal{A}$. Indeed, if the condition (21) is true, then, with aid of (24) and (25), we are forced to conclude that the mapping $\delta$ satisfies the relation (26).

On the other hand, we obtain that

$$\begin{aligned}
&\| [\delta(xy) - \delta(x)y - x\delta(y), z] \| \\
&= \Big\| \Big[ \delta(xy) - \frac{\delta(ne \cdot xy)}{n} + \frac{\delta(ne)}{n} xy - \delta(x)y + \frac{\delta(x \cdot ny)}{n} - x\frac{\delta(ny)}{n} - x\delta(y) + \frac{\delta(nx \cdot y)}{n} \\
&\quad - \frac{\delta(nx)}{n} y - \frac{\delta(ne)}{n} xy + x\frac{\delta(ny)}{n} - \frac{\delta(nx \cdot y)}{n} + \frac{\delta(nx)}{n} y - \delta(x)y + \delta(x)y, z \Big] \Big\| \\
&\le \Big\| \Big[ \delta(xy) - \frac{\delta(ne \cdot xy)}{n} + \frac{\delta(ne)}{n} xy, z \Big] \Big\| + \Big\| \Big[ -\delta(x)y + \frac{\delta(x \cdot ny)}{n} - x\frac{\delta(ny)}{n}, z \Big] \Big\| \\
&\quad + \Big\| \Big[ -x\delta(y) + \frac{\delta(nx \cdot y)}{n} - \frac{\delta(nx)}{n} y, z \Big] \Big\| + \Big\| \Big[ x\frac{\delta(ny)}{n} - \frac{\delta(nx \cdot y)}{n} + \delta(x)y, z \Big] \Big\| \\
&\quad + \Big\| \Big[ \Big( \frac{\delta(ne \cdot x)}{n} - \frac{\delta(ne)}{n} x - \delta(x) \Big) y, z \Big] \Big\|
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$ and all $n \in \mathbb{N}$. Applying (22), (23) and (26), we find that the right-hand side of the last inequality tends to zero when $n \to \infty$. Then, we arrive at the result $[\delta(xy) - \delta(x)y - x\delta(y), z] = 0$, say,

$$\delta(xy) - \delta(x)y - x\delta(y) \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

Therefore, $\delta$ is a multiplicative CE derivation, which completes the proof. □

## 3. Linearity of CE Derivations

Derivations, CE derivations and similar mappings of derivation types in Banach algebras are important because they provide a way to study the structure and properties of these algebras. They are essential tools for understanding various aspects of Banach algebras, including their structure, stability, properties and connections to other areas of mathematics like functional analysis, operator theory and representation theory. In particular, it is more meaningful and useful only when most of the derivations, CE derivations and similar mappings of derivation types are linearity in terms of functional analysis and Banach algebras.

In [2] (Theorem 2.4), it is proved that if $\mathcal{A}$ is an algebra with no nonzero central ideals, then every CE derivation is additive. So, in order to establish the linearity, it is sufficient to show that $\delta(tx) = t\delta(x)$ holds for all $x \in \mathcal{A}$ and $t \in \mathbb{C}$.

**Lemma 2.** *Let $\mathcal{A}$ be an algebra with no nonzero central ideals. Suppose that a CE derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\delta(tx) - t\delta(x) \in Z(\mathcal{A}) \ \ (x \in \mathcal{A},\ t \in \mathbb{C}). \tag{27}$$

*Then, the CE derivation $\delta$ is linear.*

**Proof.** First, for all $x, y \in \mathcal{A}$ and all $t \in \mathbb{C}$, we let

$$\delta(ty) - t\delta(y) = c, \tag{28}$$
$$\delta(txy) - t\delta(xy) = c_1, \tag{29}$$
$$\delta(xy) - \delta(x)y - x\delta(y) = c_2, \tag{30}$$
$$\delta(x(ty)) - \delta(x)(ty) - x\delta(ty) = c_3 \tag{31}$$

be fulfilled, where $c, c_1, c_2, c_3 \in Z(\mathcal{A})$.

Then, we have from (28) and (31) that

$$\begin{aligned} \delta(x(ty)) &= t\delta(x)y + x\delta(ty) + c_3 \\ &= t\delta(x)y + x(t\delta(y) + c) + c_3 \\ &= t(\delta(x)y + x\delta(y)) + xc + c_3. \end{aligned} \tag{32}$$

On the other hand, if we calculate in a different way, then in view of (29) and (30), we obtain

$$\begin{aligned} \delta(x(ty)) &= \delta(txy) = t\delta(xy) + c_1 \\ &= t(\delta(x)y + x\delta(y) + c_2) + c_1 \\ &= t(\delta(x)y + x\delta(y)) + tc_2 + c_1. \end{aligned} \tag{33}$$

Combining (32) and (33) gives $xc = c_1 + tc_2 - c_3$ and $tc_2 \in Z(\mathcal{A})$. Hence, $\mathcal{A}c$ is a central ideal and therefore, $\mathcal{A}c = \{0\}$. Thus, if $\mathtt{ann}(\mathcal{A})$ is the two-sided annihilator of $\mathcal{A}$, we obtain $c \in \mathtt{ann}(\mathcal{A})$. But $\mathtt{ann}(\mathcal{A})$ is always a central ideal; hence, our assumption forces to $\mathtt{ann}(\mathcal{A}) = \{0\}$, and so $c = 0$. Therefore, by (28), $\delta(ty) = t\delta(y)$ for all $y \in \mathcal{A}$ and all $t \in \mathbb{C}$; that is, $\delta$ is linear. This proves the theorem completely. □

To demonstrate the following theorem, we will use the well-known results involving semisimple Banach algebra contained in [18,23]: *Note that semisimple algebras are semiprime and that any linear derivation on a semisimple Banach algebra is continuous.* An algebra is said to be semisimple if its (Jacobson) radical contains only the zero element. The next conclusion can be derived from Corollary 1 and Lemma 2.

**Theorem 4.** *Let $\mathcal{A}$ be a semisimple unital Banach algebra with no nonzero central ideals. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping such that for some $\sigma \geq 0$,*

$$\|[\delta(tx) - t\delta(x), z]\| \leq \sigma \ \ (x, z \in \mathcal{A},\ t \in \mathbb{C}) \tag{34}$$

*together with (1) and (2). Assume that there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ with (3). Then, $\delta$ is a linear CE derivation. Moreover, it can be a continuous linear derivation.*

**Proof.** It follows from (8) and (34) that

$$\left\| \left[\mathcal{L}(tx) - t\mathcal{L}(x), z\right] \right\| = \lim_{n\to\infty} \frac{1}{2^n} \left\| \left[\delta(2^n tx) - t\delta(2^n x), z\right] \right\| \leq \lim_{n\to\infty} \frac{\sigma}{2^n} = 0.$$

Hence, $\left[\mathcal{L}(tx) - t\mathcal{L}(x), z\right] = 0$. We then have by (15)

$$[\delta(tx) - t\delta(x), z] = 0 \ \ (x \in \mathcal{A},\ t \in \mathbb{C}),$$

so that $\delta$ satisfies (27). Therefore, by Corollary 1 and Lemma 2, $\delta$ is a CE derivation and $\delta$ is also linear derivation. The semisimplicity ensures that $\delta$ is continuous. This completes the proof of the corollary. □

In the next theorem, we will see the linearity of CE derivation in noncommutative algebra.

**Theorem 5.** *Let $\mathcal{A}$ be a noncommutative prime unital Banach algebra. Suppose that a mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfies (1) and (2). Assume that there exists an element $x_0 \in \mathcal{A}$ and an integer $n_0 \in \mathbb{N}$ with (3). In addition, if $\delta(te) = t\delta(e)$ holds for all $t \in \mathbb{C}$, then $\delta$ is a linear CE derivation.*

**Proof.** It follows from Corollary 2 that $\delta$ is a CE derivation. Then, $\delta$ satisfies

$$[\delta(xy) - \delta(x)y - x\delta(y), z] = 0 \text{ for all } x, y, z \in \mathcal{A}. \tag{35}$$

Let $x = y = e$ in (35), and then to obtain

$$[\delta(e), z] = 0 \text{ for all } z \in \mathcal{A}. \tag{36}$$

Putting $y = e$ in (35), we have $[x\delta(e), z] = 0$ and thus, by (36), $[x, z]\delta(e) = 0$. Replacing $x$ by $xy$ in the last expression, we are lead to $[x, z]y\delta(e) = 0$. Since $\mathcal{A}$ is noncommutative prime, we have $\delta(e) = 0$.

Considering $y = te$ in (35), by the assumption and the last relation, we find that

$$[\delta(tx) - t\delta(x), z] = 0 \ \ (x, z \in \mathcal{A}, \ t \in \mathbb{C}).$$

Hence, $\delta$ satisfies the relation (27). Therefore, by Lemma 2, the CE derivation $\delta$ is linear. The corollary is completely established. □

Next, for an example of Theorems 4 and 5, we can consider the following example.

**Example 3.** *Let $\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be an algebra satisfying the conditions of Theorems 4 and 5. Define a map $\delta : \mathcal{A} \to \mathcal{A}$ given by*

$$\delta(X) = XB - BX + N, \quad \text{for} \quad X \in \mathcal{A},$$

*where $B$ is a suitable matrix in $\mathcal{A}$ and $N = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now, if we can find a matrix $C \in \mathcal{A}$ such that $det([\delta(X), C]) \neq 0$, then our example satisfies the conditions (1), (2) and (3). As a result, we can say that $\delta$ is a linear CE derivation.*

**Conclusions:** We presented some theorems concerning the stability of CE derivation in a unital Banach algebra. In addition, we designed our result regarding the superstability of multiplicative CE derivation in a normed unital algebra. On the other hand, we established certain conditions in which approximate CE derivation can be a linear mapping. In particular, some of these results can be applied to $C^*$ algebra. However, there is an important question that we could leave as an open problem: Are the main results still valid for a more general algebraic structure such as nonassociative algebras, namely, alternative algebras?

**Author Contributions:** Writing—review & editing, I.-S.C. and J.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** Jaiok Roh was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2021R1A2C109489611).

**Data Availability Statement:** The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

**Acknowledgments:** We would like to thank reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us improve the quality of the manuscript.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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