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# Relation-Theoretic Boyd–Wong Contractions of Pant Type with an Application to Boundary Value Problems

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**Abstract**

Non-unique fixed-point theorems play a pivotal role in the mathematical modeling to solve certain typical equations, which admit more than one solution. In such situations, traditional outcomes fail due to uniqueness of fixed points. The primary aim of the present article is to investigate a non-unique fixed-point theorem in the framework of a metric space endowed with a local class of transitive binary relations. To obtain our main objective, we introduce a new nonlinear contraction-inequality that subsumes the ideas involved in four noted contraction conditions, namely: almost contraction, Boyd–Wong contraction, Pant contraction and relational contraction. We also establish the corresponding uniqueness theorem for the proposed contraction under some additional hypotheses. Several examples are furnished to illustrate the legitimacy of our newly proved results. In particular, we deduce a fixed-point theorem for almost Boyd–Wong contractions in the setting of abstract metric space. Our results generalize, enhance, expand, consolidate and develop a number of known results existing in the literature. The practical relevance of the theoretical findings is demonstrated by applying to study the existence and uniqueness of solution of a specific periodic boundary value problem.

**Keywords:** fixed point; binary relation; control functions

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## 1. Introduction

The BCP is the most vital and conventional methodology. In addition to ensuring that there is a uniqueness fixed point, the BCP provides a useful technique as a means to pick out the fixed point. The BCP was made accessible from an application perspective due to its simplicity. To portray the existence of solutions for matrix equations, BVP, integral equations, etc., numerous authors have employed contraction maps. Numerous generalizations of this intriguing finding have been thoroughly examined in the field of study. The Boyd–Wong Theorem [1] is one of the generalizations of the BCP that has earned a great deal of attention during the past fifty years. In fact, Boyd and Wong [1] upgraded the contraction-inequality by the replacement of a control function for the Lipschitz constant $a \in (0,1)$, adopting the following class of functions:

$$\Phi = \Big\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi(t) < t, \text{ for all } t \in \mathbb{R}^+ \setminus \{0\} \text{ and } \limsup_{r \to t^+} \varphi(r) < t, \text{ for all } t \in \mathbb{R}^+ \setminus \{0\}\Big\}.$$

**Theorem 1** ([1]). *Let $\mathcal{S}$ be a self-map on a CMS $(\mathbf{Q}, \delta)$, which verifies for some $\varphi \in \Phi$ that*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p, q)), \quad \forall\, p, q \in \mathbf{Q}.$$

*Then $\mathcal{S}$ enjoys a unique fixed point.*

The above contraction-inequality is referred to as a nonlinear contraction or $\varphi$-contraction. Under restriction $\varphi(t) = a \cdot t$, $0 < a < 1$, $\varphi$-contraction is transformed into usual contraction and the Theorem 1 is transformed into the BCP.

Pant [2] recently enhanced BCP by looking into the following non-unique fixed-point outcome.

**Theorem 2.** *Let $\mathcal{S}$ be a self-map on a CMS $(\mathbf{Q}, \delta)$, which verifies for some $a \in [0, 1)$ that*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p, q), \quad \forall\, p, q \in \mathbf{V} \text{ with } [p \neq \mathcal{S}(p) \text{ or } q \neq \mathcal{S}(q)].$$

*Then, $\mathcal{S}$ admits a fixed point.*

Subsequently, Pant [3] proved a generalization of Theorem 2 for $\Phi$-contraction.

Berinde [4] proposed an innovative generalization of BCP in 2004 which is commonly termed "almost contraction."

**Definition 1** ([4]). *A self-map $\mathcal{S}$ on a MS $(\mathbf{Q}, \delta)$ is referred as almost contraction if there exists $a \in (0, 1)$ and $l \in \mathbb{R}^+$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p, q) + l \cdot \delta(p, \mathcal{S}q), \quad \text{for all } p, q \in \mathbf{Q}.$$

The last contraction-inequality condition is alike the following one because of the symmetric feature of $\delta$.

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p, q) + l \cdot \delta(q, \mathcal{S}p), \quad \text{for all } p, q \in \mathbf{Q}.$$

**Theorem 3** ([4]). *An almost contraction map on a CMS enjoys a fixed point.*

Many researchers have explored the idea of almost contraction; for scenarios, see [5–9]. An almost contraction need not possess a unique fixed; however, the sequence of iterates of such a map converges to a fixed point. Babu et al. [7] constructed a significantly stronger class of almost contraction conditions to validate a theorem of uniqueness.

**Definition 2** ([7]). *A self-map $\mathcal{S}$ on a MS $(\mathbf{Q}, \delta)$ is referred to as a strict almost contraction if there are constants $a \in (0, 1)$ and $l \in \mathbb{R}^+$ verifying*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p, q) + l \cdot \min\{\delta(p, \mathcal{S}p), \delta(q, \mathcal{S}q), \delta(p, \mathcal{S}q), \delta(q, \mathcal{S}p)\}, \quad \forall\, p, q \in \mathbf{Q}.$$

A strict almost contraction is obviously an almost contraction. However, the reverse is usually not true, as demonstrated by Example 2.6 [7].

**Theorem 4** ([7]). *A strict almost contraction map on a CMS possesses a unique fixed point.*

In this continuation, Turinici [8] (subsequently Alfuraidan et al. [9]) developed a nonlinear model of almost contraction replacing the Lipschitz constant $l$ with a test function $\theta$. Indeed, Alfuraidan et al. [9] initiated the following of functions.

$$\Omega = \{\theta : \mathbb{R}^+ \to \mathbb{R}^+ : \lim_{t \to 0^+} \theta(t) = \theta(0) = 0\}.$$

Evidence of the fixed-point findings in relational MS has turned into a prominent subject for investigations in metric fixed-point theory in the past decade. Instead of being satisfied for all elements, the contraction entailed in such outcomes must be satisfied for solely comparative elements in connection to the given BR. In the meantime, relational contractions generalize the typical contractions. Alam and Imdad [10] pioneered this trend in 2015 when they discovered a variety of the BCP of the context of relational MS. As such, an assortment of findings are established in this direction. We resort to [11–23] along with others to cite some of them.

In this work, we obtain an analogue of the recent fixed-point findings of Alshaban et al. [19]. Indeed the resultant contraction-inequality subsumes the earlier contraction conditions: Boyd–Wong contraction, almost contraction, relational contraction and Pant contraction. Several examples are furnished, which confirm the efficacy of our findings. For illustration of our outcomes, we constructed two instances. We deduce a number of classical fixed-point assessments, especially owing to Boyd and Wong [1], Pant [3], Babu et al. [7], Berinde [4], Turinici [8], Alam and Imdad [12], Alharbi and Khan [17], and similar others.

Likewise, for relation-theoretic contraction principles [10], in order to prove their relation-theoretic formulations, a few generalized contractions require an arbitrary BR for the existence of fixed points of such a map. Apart from this, in the context of nonlinear contractions, the transitivity of the underlying relation is additionally required. But the transitivity requirement is very restrictive. In order to employ an optimal condition of transitivity, we visited various types of weaker transitive BR. In this regards, $\mathcal{S}$-transitivity and local transitivity, which were introduced by Roldán-López-de-Hierro et al. [24] and Turinici [25], respectively, are two noted variants of transitivity. But both are independent. To unify them, Alam and Imdad [12] investigated the concept of local $\mathcal{S}$-transitivity. Thus far, we adopted a locally $\mathcal{S}$-transitive BR, which remains more general than what is utilized in similar outcomes utilized by earlier authors. Nevertheless, a further premise is needed to achieve the uniqueness outcome (i.e., the image given must be $\varrho$-directed).

As already pointed out, a contraction condition is used, which is comparatively weaker than those encountered in current studies. Because of their limited character, the results demonstrated here and similar results in future works can be used in situations where it is not feasible to utilize classical fixed-point outcomes such as nonlinear integral equations, nonlinear cantilever beams, nonlinear elliptic problems, fractional differential equations, nonlinear matrix equations, delayed hematopoiesis models and specific kinds of BVP, wherein classical fixed-point theorems cannot be applied. As a precaution, we applied our outcomes to figure out the unique solution of certain BVP satisfying certain additional hypotheses in the presence of a lower solution.

## 2. Preliminaries

As usual, $\mathbf{Q}^2$ will denote the Cartesian product of two copies of the nonempty set $\mathbf{Q}$, i.e., $\mathbf{Q}^2 := \mathbf{Q} \times \mathbf{Q}$. Thus far, any element of $\mathbf{Q}^2$ is an ordered pair $(p, q)$, where $p, q \in \mathbf{Q}$. Recall that a subset of $\mathbf{Q}^2$ is said to be a BR on the nonempty set $\mathbf{Q}$. In what follows, we assume that $\mathbf{Q}$ is a nonempty set, $\delta$ is a metric on $\mathbf{Q}$, $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map, and $\varrho$ is a BR on $\mathbf{Q}$. We say the following:

**Definition 3** ([10]). *The elements $p, q \in \mathbf{Q}$ are $\varrho$-comparative if $(p, q) \in \varrho$ or $(q, p) \in \varrho$. Such a couple is indicated by $[p, q] \in \varrho$.*

**Definition 4** ([26]). *The BR* $\varrho^{-1} := \{(p,q) \in \mathbf{Q}^2 : (q,p) \in \varrho\}$ *is inverse of* $\varrho$. *Also, the BR* $\varrho^s := \varrho \cup \varrho^{-1}$ *is symmetric closure of* $\varrho$.

**Remark 1** ([10]). $(p,q) \in \varrho^s \iff [p,q] \in \varrho$.

**Definition 5** ([10]). *A sequence* $\{p_n\} \subset \mathbf{Q}$ *satisfying* $(p_n, p_{n+1}) \in \varrho$, $\forall\, n \in \mathbb{N}$, *is* $\varrho$*-preserving.*

**Definition 6** ([27]). *For a subset* $\mathbf{R} \subseteq \mathbf{Q}$, *the BR*

$$\varrho|_{\mathbf{R}} := \varrho \cap \mathbf{R}^2,$$

*(on* $\mathbf{R}$*), is the restriction of* $\varrho$ *on* $\mathbf{R}$.

**Definition 7** ([12]). $\varrho$ *is locally* $\mathcal{S}$*-transitive BR if for every* $\varrho$*-preserving sequence* $\{p_n\} \subset \mathcal{S}(\mathbf{Q})$ *(with range* $\mathbf{R} = \{p_n : n \in \mathbb{N}\}$*),* $\varrho|_{\mathbf{R}}$ *remains transitive.*

**Definition 8** ([10]). $\varrho$ *is* $\mathcal{S}$*-closed BR if for every* $(p,q) \in \varrho$, *we attain*

$$(\mathcal{S}p, \mathcal{S}q) \in \varrho.$$

**Example 1.** *Consider* $\mathbf{Q} = [0,1]$ *equipped with a BR* $\varrho :=<$ *(usual strict order). Define the map* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *by*

$$\mathcal{S}(p) = p^2.$$

*Since* $p < q \Longrightarrow \mathcal{S}(p) = p^2 < q^2 = \mathcal{S}(q)$, *therefore* $\varrho$ *is* $\mathcal{S}$*-closed.*

**Example 2.** *Consider* $\mathbf{Q} = [-1,1]$ *equipped with a BR* $\varrho :=<$ *(usual strict order). Define the map* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *by*

$$\mathcal{S}(p) = p^2.$$

*Here,* $\varrho$ *is not* $\mathcal{S}$*-closed as* $-1 < \frac{1}{2}$ *but* $\mathbf{Q}(-1) = 1 > \frac{1}{4} = \mathbf{Q}(\frac{1}{2})$.

**Definition 9** ([11]). *The map* $\mathcal{S}$ *is* $\varrho$*-continuous if for all* $p \in \mathbf{Q}$ *and for every* $\varrho$*-preserving sequence* $\{p_n\} \subset \mathbf{Q}$ *with* $p_n \xrightarrow{\delta} p$,

$$\mathcal{S}(p_n) \xrightarrow{\delta} \mathcal{S}(p).$$

**Example 3.** *Consider* $\mathbf{Q} = (0,1]$ *equipped with standard metric* $\varrho$. *On* $\mathbf{Q}$, *endow a BR* $\varrho = \{(p,q) : \frac{1}{4} \leq p \leq q \leq \frac{1}{3}$ or $\frac{1}{2} \leq p \leq q \leq 1\}$. *If* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *is a map given by*

$$\mathcal{S}(p) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq p < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq p \leq 1. \end{cases}.$$

*Then,* $\mathcal{S}$ *is* $\varrho$*-continuous; however, it is not continuous.*

**Definition 10** ([10]). $\varrho$ *is* $\delta$*-self-closed if every* $\varrho$*-preserving convergent sequence in* $\mathbf{Q}$ *has a subsequence, whose terms are* $\varrho$*-comparative to the convergence limit.*

**Definition 11** ([11]). *The MS* $(\mathbf{Q}, \delta)$ *is* $\varrho$*-complete if each* $\varrho$*-preserving Cauchy sequence in* $\mathbf{Q}$ *is convergent.*

Clearly, each complete MS is $\varrho$-complete but the converse is not true. In Example 3, $(\mathbf{Q}, \delta)$ is $\varrho$-complete; however, it is not complete.

**Definition 12** ([28]). *A subset* $\mathbf{R} \subseteq \mathbf{Q}$ *is* $\varrho$*-directed if for every pair* $p, q \in \mathbf{Q}$, $\exists\ u \in \mathbf{Q}$ *satisfying* $(p, u) \in \varrho$ *and* $(q, u) \in \varrho$.

**Definition 13** ([27]). *Given* $p, q \in \mathbf{Q}$, *a finite sequence* $\{r_0, r_1, r_2, ..., r_k\} \subset \mathbf{Q}$ *is a path of length k (where k is a natural number) between* $p$ *and* $q$ *that satisfies the following:*

$$r_0 = p,\ r_k = q \text{ and } (r_i, r_{i+1}) \in \varrho, \text{ for each } i\ (0 \leq i \leq k-1).$$

**Definition 14** ([11]). *A subset* $\mathbf{R}$ *of* $\mathbf{Q}$ *is called* $\varrho$*-connected if there exists a path in* $\varrho$ *between each pair of elements of* $\mathbf{R}$.

**Remark 2.** *Every* $\varrho$*-directed set is* $\varrho$*-connected.*

**Definition 15** ([8]). *A sequence* $\{p_n\}$ *in a MS* $(\mathbf{Q}, \delta)$ *is semi-Cauchy if*

$$\lim_{n\to\infty} \delta(p_n, p_{n+1}) = 0.$$

Any Cauchy sequence is obviously semi-Cauchy, but not either way around.

**Proposition 1** ([12]). *If* $\varrho$ *is* $\mathcal{S}$*-closed, then* $\varrho$ *is* $\mathcal{S}^n$*-closed, for each* $n \in \mathbb{N}_0$.

**Lemma 1** ([29]). *If* $\{p_n\}$*is a sequence in a MS* $(\mathbf{Q}, \delta)$*, which is not a Cauchy, then there exists* $\epsilon_0 > 0$ *and subsequences* $\{p_{n_k}\}$ *and* $\{p_{l_k}\}$ *of* $\{p_n\}$ *with*

(i) $k \leq l_k < n_k,\ \ \forall\ k \in \mathbb{N}$,
(ii) $\delta(p_{l_k}, p_{n_k}) > \epsilon_0,\ \ \forall\ k \in \mathbb{N}$,
(iii) $\delta(p_{l_k}, p_{n_{k-1}}) \leq \epsilon_0,\ \ \forall\ k \in \mathbb{N}$.

*Moreover, if* $\{p_n\}$ *is semi-Cauchy then*

(iv) $\lim_{k\to\infty} \delta(p_{l_k}, p_{n_k}) = \epsilon_0$,
(v) $\lim_{k\to\infty} \delta(p_{l_k}, p_{n_k+1}) = \epsilon_0$,
(vi) $\lim_{k\to\infty} \delta(p_{l_k+1}, p_{n_k}) = \epsilon_0$,
(vii) $\lim_{k\to\infty} \delta(p_{l_k+1}, p_{n_k+1}) = \epsilon_0$.

Using the symmetric property of the metric $\delta$, the following conclusion is immediate.

**Proposition 2.** *Given* $\varphi \in \Phi$ *and* $\theta \in \Omega$, *the following conditions are equivalent:*

(I) $\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p, q)) + \min\{\theta(\delta(p, \mathcal{S}p)), \theta(\delta(q, \mathcal{S}q)), \theta(\delta(p, \mathcal{S}q)), \theta(\delta(q, \mathcal{S}p))\}$,
$\forall\ (p, q) \in \varrho$ *with* $[p \neq \mathcal{S}(p)$ *or* $q \neq \mathcal{S}(q)]$.
(II) $\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p, q)) + \min\{\theta(\delta(p, \mathcal{S}p)), \theta(\delta(q, \mathcal{S}q)), \theta(\delta(p, \mathcal{S}q)), \theta(\delta(q, \mathcal{S}p))\}$,
$\forall\ [p, q] \in \varrho$ *with* $[p \neq \mathcal{S}(p)$ *or* $q \neq \mathcal{S}(q)]$.

## 3. Main Results

We now proceed with the following finding on the existence of fixed points for relational almost Boyd–Wong contraction of Pant type.

**Theorem 5.** *Assuming that* $(\mathbf{Q}, \delta)$ *is a MS,* $\varrho$ *is a BR on* $\mathbf{Q}$ *and* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *is a map. Also,*

*(a)* $(\mathbf{Q}, \delta)$ *is* $\varrho$*-complete,*
*(b)* $\exists\ p_0 \in \mathbf{Q}$ *with* $(p_0, \mathcal{S}p_0) \in \varrho$,
*(c)* $\varrho$ *is locally* $\mathcal{S}$*-transitive and* $\mathcal{S}$*-closed,*
*(d)* $\mathbf{Q}$ *is* $\varrho$*-continuous, or* $\varrho$ *is* $\delta$*-self-closed,*

*(e)* $\exists\ \varphi \in \Phi$ *and* $\theta \in \Omega$ *with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p,q)) + \min\{\theta(\delta(p,\mathcal{S}p)), \theta(\delta(q,\mathcal{S}q)), \theta(\delta(p,\mathcal{S}q)), \theta(\delta(q,\mathcal{S}p))\},$$
$$\forall\ (p,q) \in \varrho \text{ with } [p \neq \mathcal{S}(p) \text{ or } q \neq \mathcal{S}(q)].$$

*Then,* $\mathcal{S}$ *owns a fixed point.*

**Proof.** We will achieve this finding in a number of steps.
**Step-1.** Define Picard sequence $\{p_n\} \subset \mathbf{Q}$ beginning with $p_0 \in \mathbf{Q}$ so that

$$p_n := \mathcal{S}^n(p_0) = \mathcal{S}(p_{n-1}), \quad \forall\ n \in \mathbb{N}. \tag{1}$$

**Step-2.** We prove that $\{p_n\}$ is $\varrho$-preserving sequence. Using hypothesis $(b)$, $\mathcal{S}$-closedness of $\varrho$ and Proposition 1, we conclude

$$(\mathcal{S}^n p_0, \mathcal{S}^{n+1} p_0) \in \varrho,$$

which utilizing (1) reduces to

$$(p_n, p_{n+1}) \in \varrho, \quad \forall\ n \in \mathbb{N}_0. \tag{2}$$

**Step-3.** Denote $\delta_n := \delta(p_n, p_{n+1})$. If $\exists\ n_0 \in \mathbb{N}_0$ with $\delta_{n_0} = 0$, then by (1), we attain $p_{n_0} = p_{n_0+1} = \mathcal{S}(p_{n_0})$; so, $p_{n_0}$ serves as a fixed point of $\mathcal{S}$ and hence, our task is through. Otherwise, we have $\delta_n > 0$, $\forall\ n \in \mathbb{N}_0$ so that we move on the Step-4.
**Step-4.** We prove that the sequence $\{p_n\}$ is semi-Cauchy, i.e., $\lim\limits_{n\to\infty} \delta(p_n, p_{n+1}) = 0$. Using condition $(e)$, (1) and (2), we obtain

$$\begin{aligned} \delta_n &= \delta(p_n, p_{n+1}) = \delta(\mathcal{S}p_{n-1}, \mathcal{S}p_n) \\ &\leq \varphi(\delta(p_{n-1}, p_n)) + \min\{\theta(\delta(p_{n-1}, p_n)), \theta(\delta(p_n, p_{n+1})), \theta(\delta(p_{n-1}, p_{n+1})), \theta(0)\}, \end{aligned}$$

which, by employing the property of $\theta$, deduces

$$\delta_n \leq \varphi(\delta_{n-1}), \quad \forall\ n \in \mathbb{N}_0. \tag{3}$$

Utilizing the axiom of $\varphi$ in (3), we obtain

$$\delta_n \leq \varphi(\delta_{n-1}) < \delta_{n-1}, \quad \forall\ n \in \mathbb{N}.$$

Therefore, $\{\delta_n\}$ is a decreasing sequence in $\mathbb{R}^+$. Since $\{\delta_n\}$ is bounded below, therefore $\exists$ $l \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = l. \tag{4}$$

Now, we prove that $l = 0$. If $l > 0$ then proceeding upper limit in (3) and employing (4) and the axiom of $\varphi$, we attain

$$l = \limsup_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \varphi(\delta_{n-1}) = \limsup_{\delta_n \to l^+} \varphi(\delta_{n-1}) < l,$$

which is contradictory, resulting $l = 0$. Hence, we attain

$$\lim_{n\to\infty} \delta_n = 0. \tag{5}$$

**Step-5.** We prove that $\{p_n\}$ is a Cauchy sequence. On contrary, if $\{p_n\}$ is not Cauchy, then using Lemma 1, there exist $\epsilon_0 > 0$ and the subsequences $\{p_{n_k}\}$ and $\{p_{l_k}\}$ of $\{p_n\}$ verifying

$$k \leq l_k < n_k,\ \delta(p_{l_k}, p_{n_k}) > \epsilon_0 \geq \delta(p_{l_k}, p_{n_{k-1}}), \quad \forall\, k \in \mathbb{N}.$$

Define $\mu_k := \delta(p_{l_k}, p_{n_k})$.Due to (1) , (2) and locally $\mathcal{S}$-transitivity of $\varrho$, we conclude $(p_{l_k}, p_{n_k}) \in \varrho$. Thus, using hypothesis $(e)$, we attain

$$\begin{aligned}\delta(p_{l_k+1}, p_{n_k+1}) &= \delta(\mathcal{S}p_{l_k}, \mathcal{S}p_{n_k}) \leq \varphi(\delta(p_{l_k}, p_{n_k})) \\ &\quad + \min\{\theta(\delta(p_{l_k}, \mathcal{S}p_{l_k})), \theta(\delta(p_{n_k}, \mathcal{S}p_{n_k})), \theta(\delta(p_{l_k}, \mathcal{S}p_{n_k})), \theta(\delta(p_{n_k}, \mathcal{S}p_{l_k}))\}\end{aligned}$$

so that

$$\delta(p_{l_k+1}, p_{n_k+1}) \leq \varphi(\delta_k) + \min\{\theta(\delta_{l_k}), \theta(\delta_{n_k}), \theta(\delta(p_{l_k}, p_{n_k+1})), \theta(\delta(p_{n_k}, p_{l_k+1}))\}. \tag{6}$$

Employing (5) and the property of $\varphi$, we obtain

$$\lim_{k\to\infty} \theta(\delta_{l_k}) = \lim_{k\to\infty} \theta(\delta_{n_k}) = \lim_{\mathsf{t}\to 0^+} \theta(\mathsf{t}) = 0. \tag{7}$$

Taking the upper limit in (6) and utilizing (7) and the axiom of $\varphi$, we obtain

$$\epsilon_0 = \limsup_{k\to\infty} \delta(p_{l_k+1}, p_{n_k+1}) \leq \limsup_{k\to\infty} \varphi(\mu_k) + 0 = \limsup_{s\to\epsilon_0^+} \varphi(s) < \epsilon_0,$$

which is contradictory, resulting in $\{p_n\}$ being Cauchy. Consequently, $\{p_n\}$ being $\varrho$-preserving and Cauchy and $\mathbf{Q}$ being $\varrho$-complete guarantee the existence of $\bar{p} \in \mathbf{Q}$ with $p_n \xrightarrow{\delta} \bar{p}$.

**Step-6.** We prove that $\bar{p}$ is the fixed point of $\mathcal{S}$ by the hypothesis $(d)$. Assuming that $\mathcal{S}$ is $\varrho$-continuous, then $p_{n+1} = \mathcal{S}(p_n) \xrightarrow{\delta} \mathcal{S}(\bar{p})$. Thus, we conclude $\mathcal{S}(\bar{p}) = \bar{p}$.

If $\varrho$ is $\delta$-self-closed, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with $[p_{n_k}, \bar{p}] \in \varrho$, $\forall\, k \in \mathbb{N}$. Employing hypothesis $(e)$, Proposition 2, $[p_{n_k}, \bar{p}] \in \varrho$ and property of $\theta$, we find

$$\begin{aligned}\delta(p_{n_k+1}, \mathcal{S}\bar{p}) &= \delta(\mathcal{S}p_{n_k}, \mathcal{S}\bar{p}) \\ &\leq \varphi(\delta(p_{n_k}, \bar{p})) + \min\{\theta(\delta(p_{n_k}, p_{n_k+1})), \theta(0), \theta(\delta(p_{n_k}, \bar{p})), \theta(\delta(\bar{p}, p_{n_k+1}))\} \\ &= \varphi(\delta(p_{n_k}, \bar{p})).\end{aligned}$$

We claim that

$$\delta(p_{n_k+1}, \mathcal{S}\bar{p}) \leq \delta(p_{n_k}, \bar{p}), \quad \forall\, k \in \mathbb{N}. \tag{8}$$

If $\delta(p_{n_{k_0}}, \bar{p}) = 0$ for some $k_0 \in \mathbb{N}$, then we attain $\delta(\mathcal{S}p_{n_{k_0}}, \mathcal{S}\bar{p}) = 0$; so $\delta(p_{n_{k_0}+1}, \mathcal{S}\bar{p}) = 0$ and, hence, (8) holds for such $k_0 \in \mathbb{N}$. Regardless of the scenario, we attain $\delta(p_{n_k}, \bar{p}) > 0$, $\forall\, k \in \mathbb{N}$. Utilizing the axiom of $\varphi$, we attain $\delta(p_{n_k+1}, \mathcal{S}\bar{p}) \leq \varphi(\delta(p_{n_k}, \bar{p})) < \delta(p_{n_k}, \bar{p})$, $\forall\, k \in \mathbb{N}$. Hence (8) holds for every $k \in \mathbb{N}$. Taking limit of (8) and by $p_{n_k} \xrightarrow{\delta} \bar{p}$, we attain $p_{n_k+1} \xrightarrow{\delta} \mathcal{S}(\bar{p})$; so $\mathcal{S}(\bar{p}) = \bar{p}$. Thus, $\bar{p}$ is a fixed point of $\mathcal{S}$. □

Now, we are equipped to prove the following uniqueness outcome.

**Theorem 6.** *Along with the assumptions of Theorem 5, if $\mathcal{S}(\mathbf{Q})$ is $\varrho$-directed, then $\mathcal{S}$ owns a unique fixed point.*

**Proof.** With an allusion to Theorem 5, if $\bar{p}, \bar{q} \in \mathbf{Q}$ are two fixed points of $\mathcal{S}$, then

$$\mathcal{S}(\bar{p}) = \bar{p} \text{ and } \mathcal{S}(\bar{q}) = \bar{q}. \tag{9}$$

As $\bar{p}, \bar{q} \in \mathcal{S}(\mathbf{Q})$, using our assumption, $\exists\, u \in \mathbf{Q}$ satisfying

$$(\bar{p}, u) \in \varrho \quad \text{and} \quad (\bar{q}, u) \in \varrho. \tag{10}$$

Denote $\varrho_n := \delta(\bar{p}, \mathcal{S}^n u)$. By (9), (10) and condition $(e)$, we obtain

$$\begin{aligned} \varrho_n = \delta(\bar{p}, \mathcal{S}^n u) &= \delta(\mathcal{S}\bar{p}, \mathcal{S}(\mathcal{S}^{n-1}u)) \leq \varphi(\delta(\bar{p}, \mathcal{S}^{n-1}u)) \\ &\quad + \min\{\theta(0), \theta(\delta(\mathcal{S}^{n-1}u, \mathcal{S}^n u)), \theta(\delta(\bar{p}, \mathcal{S}^n u)), \theta(\delta(\mathcal{S}^{n-1}u, \bar{p}))\} \\ &= \varphi(\varrho_{n-1}) \end{aligned}$$

so that

$$\varrho_n \leq \varphi(\varrho_{n-1}). \tag{11}$$

If $\exists\, n_0 \in \mathbb{N}$ such that $\varrho_{n_0} = 0$, then we have $\varrho_{n_0} \leq \varrho_{n_0-1}$. Regardless of the scenario, we have $\varrho_n > 0$, for all $n \in \mathbb{N}$, using the definition of $\varphi$, (11) reduces to $\varrho_n < \varrho_{n-1}$. Hence, in both cases, we have

$$\varrho_n \leq \varrho_{n-1}.$$

Proceeding the arguments as utilized in Theorem 5, the foregoing inequality implies that

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \delta(\bar{p}, \mathcal{S}^n u) = 0. \tag{12}$$

Similarly, one can find

$$\lim_{n\to\infty} \delta(\bar{p}, \mathcal{S}^n u) = 0. \tag{13}$$

By (12), (13) and the triangular inequality, we obtain

$$\delta(\bar{p}, \bar{q}) = \delta(\bar{p}, \mathcal{S}^n u) + \delta(\mathcal{S}^n u, \bar{q}) \to 0, \quad \text{as} \quad n \to \infty.$$

This concludes that $\bar{p} = \bar{q}$; so $\mathcal{S}$ has a unique fixed point. □

## 4. Illustrative Examples

We employ the following instances to showcase Theorems 5 and 6.

**Example 4.** *Let* $\mathbf{Q} = \mathbb{R}^+$ *with Euclidean metric* $\delta$ *and the BR* $\varrho := \{(p, q) \in \mathbf{Q}^2 : p - q > 0\}$. *Consider the map* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *defined by* $\mathcal{S}(p) = \frac{p}{p+1}$. *Then the BR* $\varrho$ *is locally* $\mathcal{S}$*-transitive and* $\mathcal{S}$*-closed. Also, the MS* $(\mathbf{Q}, \delta)$ *is* $\varrho$*-complete and the map* $\mathcal{S}$ *is* $\varrho$*-continuous. Define the auxiliary functions* $\varphi(t) = \frac{t}{t+1}$ *and* $\theta$ *as arbitrarily. Then for all* $(p, q) \in \varrho$, *we conclude*

$$\begin{aligned} \delta(\mathcal{S}p, \mathcal{S}q) &= \left|\frac{q}{p+1} - \frac{q}{q+1}\right| = \left|\frac{p-q}{1+p+q+pq}\right| \\ &\leq \frac{p-q}{1+(p-q)} = \frac{\delta(p,q)}{1+\delta(p,q)} \\ &\leq \phi(\delta(p,q)) + \min\{\theta(\delta(p, \mathcal{S}p)), \theta(\delta(q, \mathcal{S}q)), \theta(\delta(p, \mathcal{S}q)), \theta(\delta(q, \mathcal{S}p))\}. \end{aligned}$$

*Thus far, the inequality* $(e)$ *of Theorem 5 holds. Similarly, the left over of the presumptions of Theorems 5 and 6 are met. It follows that* $\mathcal{S}$ *owns a unique fixed point* ($\bar{p} = 0$).

**Example 5.** *Let* $\mathbf{Q} = [0, 2]$ *with Euclidean metric* $\delta$ *and the BR* $\varrho :=\leq$. *Consider the map* $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ *defined by*

$$\mathcal{S}(p) = \begin{cases} p^2, & \text{if } 0 \leq p < 1/2 \\ 0, & \text{if } 1/2 \leq p \leq 2. \end{cases}$$

*Clearly, $\varrho$ is locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed BR on $\mathbf{Q}$. Moreover, the MS $(\mathbf{Q}, \delta)$ is $\varrho$-complete. $\mathcal{S}$ however is not $\varrho$-continuous, yet $\varrho$ is $\delta$-self-closed. The inequality $(e)$ of Theorem 5 are also met for the auxiliary functions $\varphi(t) = t/2$ and $\theta(t) = t$. Similarly, the left over presumptions of Theorems 5 and 6 are met. It follows that $\mathcal{S}$ owns a unique fixed point ($\bar{p} = 0$).*

**Example 6.** *Let $\mathbf{Q} = [0,1]$ with Euclidean metric $\delta$ and the BR $\varrho := [0,1] \times (\mathbb{Q} \cap [0,1])$. Consider $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ as identity map. Then the BR $\varrho$ is locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed. Also, the MS $(\mathbf{Q}, \delta)$ is $\varrho$-complete and the map $\mathcal{S}$ is $\varrho$-continuous. With any fixed $\mu \in [0,1)$, define auxiliary functions $\varphi(t) = \mu \cdot t$ and $\theta(t) = t - \mu \cdot t$. The inequality $(e)$ of Theorem 5 are also met. Similarly, the left over presumptions of Theorem 5 are also met.*

*Herein, $\mathcal{S}(\mathbf{Q})$ is not $\varrho$-directed; consequently, Theorem 6 is not applicable for this example. Each point of domain serves as a fixed point of $\mathcal{S}$.*

## 5. Consequences

Using of our findings, we derive some known fixed-point outcomes of the existing literature. Under restriction $\varrho = \mathbf{Q}^2$, the universal relation, Theorem 6 produces the following fixed-point finding under nonlinear almost Boyd–Wong contraction of Pant type.

**Corollary 1.** *Assuming that $(\mathbf{Q}, \delta)$ is a CMS and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map. If $\exists\ \varphi \in \Phi$ and $\theta \in \Omega$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p,q)) + \min\{\theta(\delta(p, \mathcal{S}p)), \theta(\delta(q, \mathcal{S}q)), \theta(\delta(p, \mathcal{S}q)), \theta(\delta(q, \mathcal{S}p))\},$$
$$\text{for all } p, q \in \mathbf{Q} \text{ with } [p \neq \mathcal{S}(p) \text{ or } q \neq \mathcal{S}(q)],$$

*then $\mathcal{S}$ enjoys a unique fixed point.*

If we take $\varphi(t) = a \cdot t$ (where $a \in (0,1)$) in Corollary 1, then we derive the following outcome of Turinici [8].

**Corollary 2** ([8])**.** *Assuming that $(\mathbf{Q}, \delta)$ is a CMS and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map. If $\exists\ a \in (0,1)$ and $\theta \in \Omega$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p,q) + \min\{\theta(\delta(p, \mathcal{S}p)), \theta(\delta(q, \mathcal{S}q)), \theta(\delta(p, \mathcal{S}q)), \theta(\delta(q, \mathcal{S}p))\}, \text{ for all } p, q \in \mathbf{Q},$$

*then $\mathcal{S}$ enjoys a unique fixed point.*

The following outcome of Babu et al. [7] (also proved by Berinde [4]) can be deduced from Corollary 1 for the auxiliary functions $\varphi(t) = a \cdot t$ (where $a \in (0,1)$) and $\theta(t) = l \cdot t$ (where $l \in \mathbb{R}^+$).

**Corollary 3** ([4,7])**.** *Assuming that $(\mathbf{Q}, \delta)$ is a CMS and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map. If $\exists\ a \in (0,1)$ and $l \in \mathbb{R}^+$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq a \cdot \delta(p,q) + l \cdot \min\{\delta(p, \mathcal{S}p), \delta(q, \mathcal{S}q), \delta(p, \mathcal{S}q), \delta(q, \mathcal{S}p)\}, \text{ for all } p, q \in \mathbf{Q},$$

*then $\mathcal{S}$ enjoys a unique fixed point.*

On setting for $\theta(t) = 0$, Corollary 1 reduces to the outcome of Pant [3], which runs as follows.

**Corollary 4** ([3])**.** *Assuming that $(\mathbf{Q}, \delta)$ is a CMS and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a mapping. If $\exists\ \varphi \in \Phi$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p,q)), \text{ for all } p, q \in \mathbf{Q} \text{ with } [p \neq \mathcal{S}(p) \text{ or } q \neq \mathcal{S}(q)],$$

*then $\mathcal{S}$ enjoys a unique fixed point.*

Particularly for $\theta(t) = 0$, Theorem 6 deduces the following result of Alam and Imdad [12].

**Corollary 5** ([12]). *Assuming that $(\mathbf{Q}, \delta)$ is a MS, $\varrho$ is a BR on $\mathbf{Q}$ and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map. Also,*

*(a)* *$(\mathbf{Q}, \delta)$ is $\varrho$-complete,*
*(b)* *$\exists\, p_0 \in \mathbf{Q}$ with $(p_0, \mathcal{S}p_0) \in \varrho$,*
*(c)* *$\varrho$ is locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed,*
*(d)* *$\mathbf{Q}$ is $\varrho$-continuous, or $\varrho$ is $\delta$-self-closed,*
*(e)* *$\exists\, \varphi \in \Phi$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p, q)) \text{ for all } (p, q) \in \varrho.$$

*Then, $\mathcal{S}$ enjoys a fixed point. Moreover, if $\mathcal{S}(\mathbf{Q})$ is $\varrho$-directed, then $\mathcal{S}$ possesses a unique fixed point.*

In particular, for $\theta(t) = l \cdot t$ (where $l \in \mathbb{R}^+$), Theorem 6 deduces the following outcome of Alharbi and Khan [17].

**Corollary 6** ([17]). *Assuming that $(\mathbf{Q}, \delta)$ is MS, $\varrho$ is a BR on $\mathbf{Q}$ and $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ is a map. Also,*

*(a)* *$(\mathbf{Q}, \delta)$ is $\varrho$-complete,*
*(b)* *$\exists\, p_0 \in \mathbf{Q}$ with $(p_0, \mathcal{S}p_0) \in \varrho$,*
*(c)* *$\varrho$ is locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed,*
*(d)* *$\mathbf{Q}$ is $\varrho$-continuous, or $\varrho$ is $\delta$-self-closed,*
*(e)* *$\exists\, \varphi \in \Phi$ and $l \in \mathbb{R}^+$ with*

$$\delta(\mathcal{S}p, \mathcal{S}q) \leq \varphi(\delta(p, q)) + l \cdot \min\{\delta(p, \mathcal{S}p), \delta(q, \mathcal{S}q), \delta(p, \mathcal{S}q), \delta(q, \mathcal{S}p)\},$$

*for all $(p, q) \in \varrho$.*

*Then, $\mathcal{S}$ enjoys a fixed point. Moreover, if $\mathcal{S}(\mathbf{Q})$ is $\varrho$-directed, then $\mathcal{S}$ possesses a unique fixed point.*

## 6. Applications to BVP

Let us look at the BVP that follows:

$$\begin{cases} \vartheta'(r) = \digamma(r, \vartheta(r)), & r \in [0, L] \\ \vartheta(0) = \vartheta(L) \end{cases} \tag{14}$$

where $\digamma : [0, L] \times \mathbb{R} \to \mathbb{R}$ remains a continuous function.

In the sequel, $\Psi$ will denote the class of increasing and continuous functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) < t$, for all $t > 0$. Obviously, $\Psi \subset \Phi$.

Following [30], we say that a function $\tilde{\vartheta} \in \mathcal{C}'[0, L]$ is a lower solution of (14) if

$$\begin{cases} \tilde{\vartheta}'(r) \leq \digamma(r, \tilde{\vartheta}(r)), & r \in [0, L] \\ \tilde{\vartheta}(0) \leq \tilde{\vartheta}(L). \end{cases}$$

The primary outcome of this section is as outlined below:

**Theorem 7.** *In addition to Problem (14), if there exists $\tau > 0$, and $\varphi \in \Psi$ verifying*

$$0 \leq [\digamma(r, b) + \tau b] - [\digamma(r, a) + \tau a] \leq \tau\varphi(b - a), \quad \forall\, r \in [0, L] \text{ and } a, b \in \mathbb{R} \text{ with } a \leq b, \tag{15}$$

*then the Problem* (14) *enjoys a unique solution when it owns a lower solution.*

**Proof.** Equation (14) can be reformulated as

$$\begin{cases} \vartheta'(r) + \tau\vartheta(r) = \digamma(r, \vartheta(r)) + \tau\vartheta(r), \quad \forall\, r \in [0, L] \\ \vartheta(0) = \vartheta(L). \end{cases}$$

The above BVP is equivalent to the Fredholm integral equation:

$$\vartheta(r) = \int_0^L M(r, \xi)[\digamma(\xi, \vartheta(\xi)) + \tau\vartheta(\xi)]d\xi, \tag{16}$$

where $M(r, \xi)$ being the Green function is described by

$$M(r, \xi) = \begin{cases} \frac{e^{\tau(L+\xi-r)}}{e^{\tau L}-1}, & 0 \le \xi < r \le L \\ \frac{e^{\tau(\xi-r)}}{e^{\tau L}-1}, & 0 \le r < \xi \le L. \end{cases}$$

Denote $\mathbf{Q} := \mathcal{C}[0, L]$. Define a map $\mathcal{S} : \mathbf{Q} \to \mathbf{Q}$ by

$$(\mathcal{S}\vartheta)(r) = \int_0^L M(r, \xi)[\digamma(\xi, \vartheta(\xi)) + \tau\vartheta(\xi)]d\xi, \quad \text{for all } r \in [0, L]. \tag{17}$$

Take a BR $\varrho$ on $\mathbf{Q}$ by

$$\varrho = \{(\vartheta, \omega) \in \mathbf{Q} \times \mathbf{Q} : \vartheta(r) \le \omega(r), \text{ for all } r \in [0, L]\}. \tag{18}$$

Next, define a metric $\delta$ on $\mathbf{Q}$ by

$$\delta(\vartheta, \omega) = \sup_{r \in [0,L]} |\vartheta(r) - \omega(r)|, \quad \text{for all } \vartheta, \omega \in \mathbf{Q}. \tag{19}$$

Now, we verify all the presumptions of Theorems 5 and 6.
*(a)* The MS $(\mathbf{Q}, \delta)$ being complete is $\varrho$-complete.
*(b)* if $\tilde{\vartheta} \in \mathcal{C}'[0, L]$ forms a lower solution of (14), then we attain

$$\tilde{\vartheta}'(r) + \tau\tilde{\vartheta}(r) \le \digamma(r, \tilde{\vartheta}(r)) + \tau\tilde{\vartheta}(r), \quad \text{for all } r \in [0, L].$$

Taking multiplication with $e^{\tau r}$, we attain

$$(\tilde{\vartheta}(r)e^{\tau r})' \le [\digamma(r, \tilde{\vartheta}(r)) + \tau\tilde{\vartheta}(r)]e^{\tau r}, \quad \text{for all } r \in [0, L],$$

so that

$$\tilde{\vartheta}(r)e^{\tau r} \le \tilde{\vartheta}(0) + \int_0^r [\digamma(\xi, \tilde{\vartheta}(\xi)) + \tau\tilde{\vartheta}(\xi)]e^{\tau\xi}d\xi, \quad \text{for all } r \in [0, L]. \tag{20}$$

As $\tilde{\vartheta}(0) \le \tilde{\vartheta}(L)$, we conclude

$$\tilde{\vartheta}(0)e^{\tau L} \le \tilde{\vartheta}(L)e^{\tau L} \le \tilde{\vartheta}(0) + \int_0^L [\digamma(\xi, \tilde{\vartheta}(\xi)) + \tau\tilde{\vartheta}(\xi)]e^{\tau\xi}d\xi,$$

thereby yielding

$$\tilde{\vartheta}(0) \le \int_0^L \frac{e^{\tau\xi}}{e^{\tau L} - 1}[\digamma(\xi, \tilde{\vartheta}(\xi)) + \tau\tilde{\vartheta}(\xi)]d\xi. \tag{21}$$

From (20) and (21), we get

$$\begin{aligned}\tilde{\vartheta}(r)e^{\tau r} &\leq \int_0^L \frac{e^{\tau\xi}}{e^{\tau L}-1}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi+\int_0^r e^{\tau\xi}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi\\ &= \int_0^r \frac{e^{\tau(L+\xi)}}{e^{\tau L}-1}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi+\int_r^L \frac{e^{\tau\xi}}{e^{\tau L}-1}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi,\end{aligned}$$

thereby yielding

$$\begin{aligned}\tilde{\vartheta}(r) &\leq \int_0^r \frac{e^{\tau(L+\xi-r)}}{e^{\tau L}-1}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi+\int_r^L \frac{e^{\tau(\xi-r)}}{e^{\tau L}-1}[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi\\ &= \int_0^L M(r,\xi)[F(\xi,\tilde{\vartheta}(\xi))+\tau\tilde{\vartheta}(\xi)]d\xi\\ &= (\mathcal{S}\tilde{\vartheta})(r),\quad \forall\, r\in[0,L],\end{aligned}$$

so that $(\tilde{\vartheta},\mathcal{S}\tilde{\vartheta})\in\varrho$.

*(c)* The BR $\varrho$ being transitive is locally $\mathcal{S}$-transitive. To verify the $\mathcal{S}$-closedness of $\varrho$, take $\vartheta,\omega\in\mathbf{Q}$ with $(\vartheta,\omega)\in\varrho$. Using (15), we conclude

$$F(r,\vartheta(r))+\tau\vartheta(r)\leq F(r,\omega(r))+\tau\omega(r),\quad \text{for all } r\in[0,L]. \tag{22}$$

Using (17), (22) and $M(r,\xi)>0$, for all $r,\xi\in[0,L]$, we get

$$\begin{aligned}(\mathcal{S}\vartheta)(r) &= \int_0^L M(r,\xi)[F(\xi,\vartheta(\xi))+\tau\vartheta(\xi)]d\xi\\ &\leq \int_0^L M(r,\xi)[F(\xi,\omega(\xi))+\tau\omega(\xi)]d\xi\\ &= (\mathcal{S}\omega)(r),\quad \text{for all } r\in[0,L],\end{aligned}$$

which with the help of (18) demonstrate that $(\mathcal{S}\vartheta,\mathcal{S}\omega)\in\varrho$. Thus, $\varrho$ is $\mathcal{S}$-closed.

*(d)* If $\{\vartheta_n\}\subset\mathbf{Q}$ is an $\varrho$-preserving sequence, that converges to $\vartheta\in\mathbf{Q}$, then, we conclude that $v_n(r)\leq\vartheta(r)$, $\forall\, n\in\mathbb{N}$ and $\forall\, r\in[0,L]$. Using (18), we find $(v_n,\vartheta)\in\varrho$, $\forall\, n\in\mathbb{N}$. Hence, $\varrho$ is $\delta$-self-closed.

*(e)* Take $\vartheta,\omega\in\mathbf{Q}$ with $(\vartheta,\omega)\in\varrho$. Using (15), (17) and (19), we get

$$\begin{aligned}\delta(\mathcal{S}\vartheta,\mathcal{S}\omega) &= \sup_{r\in[0,L]}|(\mathcal{S}\vartheta)(r)-(\mathcal{S}\omega)(r)| = \sup_{r\in[0,L]}\big((\mathcal{S}\omega)(r)-(\mathcal{S}\vartheta)(r)\big)\\ &\leq \sup_{r\in[0,L]}\int_0^L M(r,\xi)[F(\xi,\omega(\xi))+\tau\omega(\xi)-F(\xi,\vartheta(\xi))-\tau\vartheta(\xi)]d\xi\\ &\leq \sup_{r\in[0,L]}\int_0^L M(r,\xi)\tau\varphi(\omega(\xi)-\vartheta(\xi))d\xi.\end{aligned} \tag{23}$$

As $0\leq\omega(\xi)-\vartheta(\xi)\leq\delta(\vartheta,\omega)$, monotonicity of $\varphi$ yields that

$$\varphi(\omega(\xi)-\vartheta(\xi))\leq\varphi(\delta(\vartheta,\omega)).$$

By last inequality, (23) becomes

$$\begin{aligned}\delta(\mathcal{S}\vartheta,\mathcal{S}\omega) &\leq \tau\varphi(\delta(\vartheta,\omega))\sup_{r\in[0,L]}\int_0^L M(r,\xi)d\xi\\ &= \tau\varphi(\delta(\vartheta,\omega))\sup_{r\in[0,L]}\frac{1}{e^{\tau L}-1}\left[\frac{1}{\tau}e^{\tau(L+\xi-r)}\Big|_0^r+\frac{1}{\tau}e^{\tau(\xi-r)}\Big|_r^L\right]\end{aligned}$$

$$\begin{aligned} &= \tau\varphi(\delta(\vartheta,\omega))\frac{1}{\tau(e^{\tau L}-1)}(e^{\tau L}-1) \\ &= \varphi(\delta(\vartheta,\omega)) \end{aligned}$$

thereby yielding

$$\delta(\mathcal{S}\vartheta,\mathcal{S}\omega) \leq \varphi(\delta(\vartheta,\omega)) + \min\{\theta(\delta(\vartheta,\mathcal{S}\vartheta)),\theta(\delta(\omega,\mathcal{S}\omega)),\theta(\delta(\vartheta,\mathcal{S}\omega)),\theta(\delta(\omega,\mathcal{S}\vartheta))\},$$
$$\text{for all } \vartheta,\omega \in \mathbf{Q} \text{ satisfying } (\vartheta,\omega) \in \varrho$$

where $\theta \in \Omega$ is arbitrary.

Therefore, the presumptions $(a)$–$(e)$ of Theorem 5 are satisfied. Consequently, $\mathcal{S}$ has a fixed point.

Take arbitrary $\vartheta,\omega \in \mathbf{Q}$ so that $\mathcal{S}(\vartheta),\mathcal{S}(\omega) \in \mathcal{S}(\mathbf{Q})$. Set $\varpi := \max\{\mathcal{S}\vartheta,\mathcal{S}\omega\}$, thereby implying $(\mathcal{S}\vartheta,\varpi) \in \varrho$ and $(\mathcal{S}\omega,\varpi) \in \varrho$. Thus, the set $\mathcal{S}(\mathbf{Q})$ is $\varrho$-directed. Consequently, by Theorem 6, $\mathcal{S}$ possesses a unique fixed point, which indeed serves the desired unique solution of Problem (14). □

We evaluate the following numerical example for the purpose to demonstrate Theorem 7.

**Example 7.** *Let* $\digamma(r,\vartheta(r)) = \cos r$ *for* $0 \leq r \leq \pi$, *then* $\hbar$ *is a continuous function. Herein,* $\underline{\vartheta} = 0$ *serves as a lower solution to* $\vartheta'(r) = \cos r$. *So far, Theorem 7 can be employed for the given problem; consequently,* $\vartheta(r) = \sin r$ *remains the unique solution.*

## 7. Conclusions

We explored the fixed-point findings in this work through a locally $\mathcal{S}$-transitive BR for a nonlinear almost $\varphi$-contraction in the spirit of Boyd and Wong [1]. Additionally, we derived an analogous finding in abstract MS that generalizes the primary findings of Boyd and Wong [1], Pant [3], and Babu et al. [7]. Our findings complement, build upon and develop a number of recent findings, particularly those attributed to Berinde [4], Turinici [8], Alam and Imdad [12], Alharbi and Khan [17] and similar others. This emphasizes the advantages of our findings over a few recognized findings from a review of the contemporary literary works. Sometimes, certain real-world problems of nonlinear analysis admit non-unique solutions. In such situations, the earlier classical findings (such as, Boyd–Wong Theorem) can not be utilized; instead, our non-unique fixed-point theorem can be applied.

To explain our outcomes, we formed three different examples. Examples 4 and 5 demonstrated Theorem 6 which, in turn, confirms two different alternate hypotheses (either $\mathbf{Q}$ is $\varrho$-continuous or $\varrho$ is $\delta$-self-closed). In contrast, Example 6 fulfills only the premise of existence outcome (i.e., Theorem 5) in lacking uniqueness.

As further investigation to the findings of Alam et al. [13], similar versions of Theorems 5 and 6 for locally finitely $\mathcal{S}$-transitive BR under nonlinear almost $\varphi$-contraction can be proven. We demonstrated the existence and uniqueness finding for BVP in the circumstance of a lower solution through using our findings. Similar findings can also be illustrated when an upper solution is available. In recent years, many problems of fractional differential equations are solved by numerical methods, e.g., [31,32]. Our outcomes can also be applied to solve certain types of fractional differential equations.

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## Notations and Abbreviations

The following acronyms and symbols are utilized in this assessment:

| | |
|---|---|
| $\mathbb{R}^+$ | the set of nonnegative real numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{N}$ | the set of natural numbers |
| BR | binary relation |
| BCP | Banach contraction principle |
| BVP | boundary value problems |
| MS | metric space |
| CMS | complete metric space |
| RHS | right hand side |
| iff | if and only if |
| $\mathcal{C}(D)$ | the family of all real valued continuous functions on a set $D$ |
| $\mathcal{C}'(D)$ | the family of all real valued differentiable continuous functions on a set $D$. |

## References

1. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
2. Pant, R.P. Extended contraction mappings. *Filomat* **2024**, *38*, 1987–1990. [CrossRef]
3. Pant, R.P. Extended Φ-contraction mappings. *J. Anal.* **2024**, *32*, 1661–1670. [CrossRef]
4. Berinde, V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **2004**, *9*, 43–53.
5. Berinde, V.; Păcurar, M. Fixed points and continuity of almost contractions. *Fixed Point Theory* **2008**, *9*, 23–34.
6. Berinde, V.; Takens, F. *Iterative Approximation of Fixed Points*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 1912.
7. Babu, G.V.R.; Sandhy, M.L.; Kameshwari, M.V.R. A note on a fixed point theorem of Berinde on weak contractions. *Carpathian J. Math.* **2008**, *24*, 8–12.
8. Turinici, M. Weakly contractive maps in altering metric spaces. *ROMAI J.* **2013**, *9*, 175–183.
9. Alfuraidan, M.R.; Bachar, M.; Khamsi, M.A. Almost monotone contractions on weighted graphs. *J. Nonlinear Sci. Appl.* **2016**, *9*, 5189–5195. [CrossRef]
10. Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [CrossRef]
11. Alam, A.; Imdad, M. Relation-theoretic metrical coincidence theorems. *Filomat* **2017**, *31*, 4421–4439. [CrossRef]
12. Alam, A.; Imdad, M. Nonlinear contractions in metric spaces under locally $T$-transitive binary relations. *Fixed Point Theory* **2018**, *19*, 13–24. [CrossRef]
13. Alam, A.; Arif, M.; Imdad, M. Metrical fixed point theorems via locally finitely $T$-transitive binary relations under certain control functions. *Miskolc Math. Notes* **2019**, *20*, 59–73. [CrossRef]
14. Arif, M.; Imdad, M.; Alam, A. Fixed point theorems under locally $T$-transitive binary relations employing Matkowski contractions. *Miskolc Math. Notes* **2022**, *23*, 71–83. [CrossRef]
15. Alam, A.; George, R.; Imdad, M. Refinements to relation-theoretic contraction principle. *Axioms* **2022**, *11*, 316. [CrossRef]
16. Khan, F. A. Almost contractions under binary relations. *Axioms* **2022**, *11*, 441. [CrossRef]
17. Alharbi, A.F.; Khan, F.A. Almost Boyd-Wong type contractions under binary relations with applications to boundary value problems. *Axioms* **2023**, *12*, 896. [CrossRef]
18. Algehyne, E.A.; Altaweel, N.H.; Areshi, M.; Khan, F.A. Relation-theoretic almost $\phi$-contractions with an application to elastic beam equations. *AIMS Math.* **2023**, *8*, 18919–18929. [CrossRef]
19. Alshaban, E.; Alatawi, A.; Alamrani, F.M.; Alamer, A.; Alrshidi, N.N.; Khan, F.A. Nonlinear almost contractions of Pant type under binary relations with an application to boundary value problems. *Mathematics* **2025**, *13*, 906. [CrossRef]
20. Sawangsup, K.; Sintunavarat, W.; de Hierro, A.F.R.L. Fixed point theorems for $F_{\Re}$-contractions with applications to solution of nonlinear matrix equations. *J. Fixed Point Theory Appl.* **2017**, *19*, 1711–1725. [CrossRef]

21. Senapati, T.; Dey, L.K. Relation-theoretic metrical fixed-point results via $w$-distance with applications. *J. Fixed Point Theory Appl.* **2017**, *19*, 2945–2961. [CrossRef]
22. Tomar, A.; Joshi, M.; Padaliya, S.K.; Joshi, B.; Diwedi, A. Fixed point under set-valued relation-theoretic nonlinear contractions and application. *Filomat* **2019**, *33*, 4655–4664. [CrossRef]
23. Tomar, A.; Joshi, M. Relation-theoretic nonlinear contractions in an $\mathcal{F}$-metric space and applications. *Rend. Circ. Mat. Palermo Ser.* **2021**, *70*, 835–852. [CrossRef]
24. Roldán-López-de-Hierro, A.F.; Karapinar, E.; de la Sen, M. Coincidence point theorems in quasi-metric spaces without assuming the mixed monotone property and consequences in $G$-metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 184. [CrossRef]
25. Turinici, M. Contractive maps in locally transitive relational metric spaces. *Sci. World J.* **2014**, *2014*, 169358. [CrossRef]
26. Lipschutz, S. *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*; McGraw-Hill: New York, NY, USA, 1964.
27. Kolman, B.; Busby, R.C.; Ross, S. *Discrete Mathematical Structures*, 6th ed.; Pearson/Prentice Hall: Hoboken, NJ, USA, 2009.
28. Samet, B.; Turinici, M. Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications. *Commun. Math. Anal.* **2012**, *13*, 82–97.
29. Jleli, M.; Rajic, V.C.; Samet, B.; Vetro, C. Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations. *J. Fixed Point Theory Appl.* **2012**, *12*, 175–192. [CrossRef]
30. Nieto, J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
31. Rahimkhani, P.; Ordokhani, Y.; Babolian, E. Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet. *J. Comput. Appl. Math.* **2017**, *309*, 493–510. [CrossRef]
32. Rahimkhani, P.; Ordokhani, Y. A numerical scheme based on Bernoulli wavelets and collocation method for solving fractional partial differential equations with Dirichlet boundary conditions. *Numer. Methods Partial Differ. Equ.* **2019**, *35*, 34–59. [CrossRef]


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