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Spherical Harmonics and Gravity Intensity Modeling Related to a Special Class of Triaxial Ellipsoids

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Abstract

The G-modified Helmholtz equation is a partial differential equation that allows gravity intensity g to be expressed as a series of spherical harmonics, with the radial distance r raised to irrational powers. In this study, we consider a non-rotating triaxial ellipsoid parameterized by the geodetic latitude φ and geodetic longitude λ, and eccentricities e_e, e_x, e_y. On its surface, the value of gravity potential has a constant value, defining a level triaxial ellipsoid. In addition, the gravity intensity is known on the surface, which allows us to formulate a Dirichlet boundary value problem for determining the gravity intensity as a series of spherical harmonics. This expression for gravity intensity is presented here for the first time, filling a gap in the study of triaxial ellipsoids and spheroids. Given that the triaxial ellipsoid has very small eccentricities, a first order approximation can be made by retaining only the terms containing ${e_e}^2$ and ${e_x}^2$. The resulting expression in spherical harmonics contains even degree and even order harmonic coefficients, along with the associated Legendre functions. The maximum degree and order that occurs is four. Finally, as a special case, we present the geometrical degeneration of an oblate spheroid.

Keywords: gravity; triaxial ellipsoid; Dirichlet problem; spherical harmonics

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1. Introduction

Various surfaces are used as geometric models of the Earth. As a first approximation, Earth can be modeled as a sphere with a radius R = 6371 km. Since the polar semi axis [1] is shorter than the equatorial semi axis by approximately 20 km, a more accurate second approximation is to model the Earth as a biaxial ellipsoid (oblate spheroid). These surfaces are chosen to approximate the physical shape of the Earth, known as the geoid. In this section, we will briefly address this topic, as it is relevant to our investigation.

According to Bektaş [2], geodetic studies require an accepted model of the Earth. The reference surface should be chosen to reflect the Earth's actual shape, and calculations must be easily linked to this model. The appropriate reference surface is the geoid, which is [3] an equipotential surface of the Earth's gravity field. This surface is approximated by the global Mean Sea Level with high accuracy in sea areas, assuming the absence of other influences such us wind, tides, and ocean currents.

The geoid serves as a reference surface for orthometric heights, as all its points have a zero orthometric height. Its determination is a fundamental problem in Geodesy. The

determination of the geoid using the Laplace equation involves solving an inverse boundary value problem (Stokes' case), where the known gravity anomalies define the geoid as the unknown boundary surface that must be determined. To solve this problem, we approximate the surface of the geoid with that of a rotational ellipsoid (oblate spheroid). Thus, gravity anomalies are defined as the difference between the Earth's gravity intensity g on the geoid (which [3] depends on the mass and density distribution beneath the Earth's surface) and the gravity intensity γ on the surface of the ellipsoid. Due to the lack of accurate data on mass and density distribution [4,5], the determination of the geoid is performed point by point. The selection of an appropriate rotational ellipsoid is crucial, as the geoid is defined as a set of points with known distances from the surface of the ellipsoid. The distances are measured along the vertical line from the point on the geoid to the ellipsoid. This distance is referred to as the geoid undulation N.

The geoid undulation depends on [6] the Earth's disturbing potential T and the gravity intensity γ, which is related to the gravity field generated by the chosen ellipsoid. This implies that the selected rotational ellipsoid should not only be geometrically close to the surface of the geoid (the physical surface of the Earth) but also physically close. The predominant [3] geometric model for the geoid is an ellipsoid that minimizes the mean square of the geometric heights (the distance measured along the vertical line from a point of interest on the Earth's physical surface to the ellipsoid). The issue with this ellipsoid is that it introduces locally significant deviations in regions such as coastlines, or areas with submarine faults. To overcome this problem [3], a triaxial ellipsoid is introduced, which provides more accurate geometric heights values. This improvement is particularly important in regions of special geodetic interest.

Traditionally, rotational ellipsoids have been used for geodetic computations due to the simplicity they offer. However, with modern computing capabilities [1,2,7,8], it is no longer necessary to rely on simplifying assumptions for the sake of computational ease. Although the difference between the Earth's equatorial semimajor and semiminor axes is small, it is not negligible. Therefore, triaxial ellipsoids are well suited for various geodetic calculations, including the solution of geodetic transformation problems.

Many celestial bodies (such as asteroids, comets, natural satellites) have shapes that closely approximate triaxial ellipsoids [9–12]. In addition, the theory of ellipsoidal spline functions enables the interpolation and smoothing of the gravity data on a triaxial ellipsoid. Smoothing gravity data is essential for selecting the appropriate method for determining the geoid undulation in a specific region (for example, using the Stokes' problem or an alternative approach). Triaxial ellipsoids are also preferable [13] for modeling strongly magnetic, compact ore bodies, as ellipsoids are the only bodies for which self-demagnetization can be treated exactly and analytically.

The gravity anomaly, defined as the difference between the Earth's gravity g on the geoid and the ellipsoidal gravity γ on the surface of a rotational ellipsoid, is a key quantity in determining the gravimetric geoid through the solution of Stokes problem. The variation in gravity anomaly is directly related to the variation in geoid undulation. Since the 19th century [14], scientists have recognized that a triaxial ellipsoid provides a better geometric model of the Earth's physical shape. Pizzetti solved the problem of determining the geoid in the case of a triaxial ellipsoid and derived expressions for the components of the gravity force Γ_x, Γ_y, and Γ_z with respect to the coordinate axles. Building on Pizzetti's solution, Mineo [15], in 1928, was the first to provide a closed form expression for gravity γ (or gravity intensity γ) on a triaxial ellipsoid, whose surface is an equipotential surface. The gravity intensity γ is expressed as a function of geodetic longitude and geodetic latitude (further details are provided in the following paragraph). The usefulness of the triaxial ellipsoid is now widely recognized across numerous applications. Additionally, the

choice of coordinate system plays a significant role in gravity field modeling, particularly in expressing gravitational potential. An insightful paper [16] provides an analytical treatment of this topic.

In this work, we determine the gravity intensity γ outside a triaxial ellipsoid with low eccentricities. Our results serve as a desired complement—long sought for decades—to the work of Mineo, since the G-modified Helmholtz equation enables us to express gravity intensity as a series of spherical harmonics in three-dimensional space. This constitutes the central novelty of our solution, which offers significant advantages including: (a) the formula for gravity γ is simpler than the expression obtained from the root of the sum of the squared components γ_x, γ_y, and γ_z; (b) the formula is expressed as a series of spherical harmonics something that has not existed until now.

2. Preliminaries

The following table will be helpful to understand all formulae that will be used in order to express gravity intensity γ as a series of spherical harmonics.

Table 1 includes all the necessary variables which are used in the text. The triaxial ellipsoid [2] has three distinct semi axles, i.e., a_x, a_y, and b. Its standard equation, when centered at the origin of a Cartesian system (x, y, z) and aligned with the axes, is

$$\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} + \frac{z^2}{b^2} = 1. \tag{1}$$

A triaxial ellipsoid has three eccentricities, the first and second polar eccentricity and the first equatorial eccentricity. Their relations are

$$e_x^2 = \frac{a_x^2 - b^2}{a_x^2}, \tag{2}$$

$$e_y^2 = \frac{a_y^2 - b^2}{a_y^2}, \tag{3}$$

$$e_e^2 = \frac{a_x^2 - a_y^2}{a_x^2}. \tag{4}$$

A parameterization of a triaxial ellipsoid, using geodetic φ and λ as parameters, is [17]

$$x(\phi.\lambda) = \frac{a_x}{\sqrt{1 - e_x^2 \sin^2\phi - e_e^2 \cos^2\phi \sin^2\lambda}} \cos\phi \cos\lambda, \tag{5}$$

$$y(\phi.\lambda) = \frac{a_x(1 - e_e^2)}{\sqrt{1 - e_x^2 \sin^2\phi - e_e^2 \cos^2\phi \sin^2\lambda}} \cos\phi \sin\lambda, \tag{6}$$

$$z(\phi.\lambda) = \frac{a_x(1 - e_x^2)}{\sqrt{1 - e_x^2 \sin^2\phi - e_e^2 \cos^2\phi \sin^2\lambda}} \sin\phi. \tag{7}$$

The symbols a_x, a_y, and b stand for major equatorial semiaxis, minor equatorial semiaxis and polar semiaxis, respectively. In our case it holds that

$$a_x > a_y > b. \tag{8}$$

Table 1. Variables that are used in the text.

Symbol	Description
γ	Gravity intensity on and above the triaxial ellipsoid
γ_s	Gravity intensity on and above the oblate spheroid
a_x	Semi Major semi axis of the triaxial ellipsoid
a_y	Semi Intermediate axis of the triaxial ellipsoid
b	Semi Minor axis of the triaxial ellipsoid
e_x^2	First polar eccentricity of the triaxial ellipsoid
e_y^2	Second polar eccentricity of the triaxial ellipsoid
e_e^2	First equatorial eccentricity of the triaxial ellipsoid
e^2	Eccentricity of the oblate spheroid $a_x = a_y = a$, b
ϕ	Geodetic latitude on the triaxial ellipsoid
λ	Geodetic longitude on the triaxial ellipsoid
θ'	Spherical latitude
λ_c	Spherical longitude
r	Radial distance
r_e	Radial distance on the surface of the triaxial ellpsoid
r_s	Radial distance on the surface of the oblate spheroid $a_x = a_y = a$, b
γ_{a_x}	Gravity intensity value at point (a_x, 0, 0)
$\delta\gamma_x$	Difference between the gravity values at points (a_x, 0, 0) and (0, a_y, 0)
$\delta\gamma_z$	Difference between the gravity values at points (a_x, 0, 0) and (0, 0, b)
$P'_{nm}(\sin\theta')$	Derivative of the Legendre function with respect to the argument $\sin\theta'$
$P'_n(\sin\theta')$	Derivative of the Legendre polynomial with respect to the argument $\sin\theta'$

The geometric interpretation of geodetic and spherical coordinates is shown in Figure 1. The geodetic latitude φ of point P on the surface of a triaxial ellipsoid is the angle between the vertical line passing through P and its projection onto xy-plane.

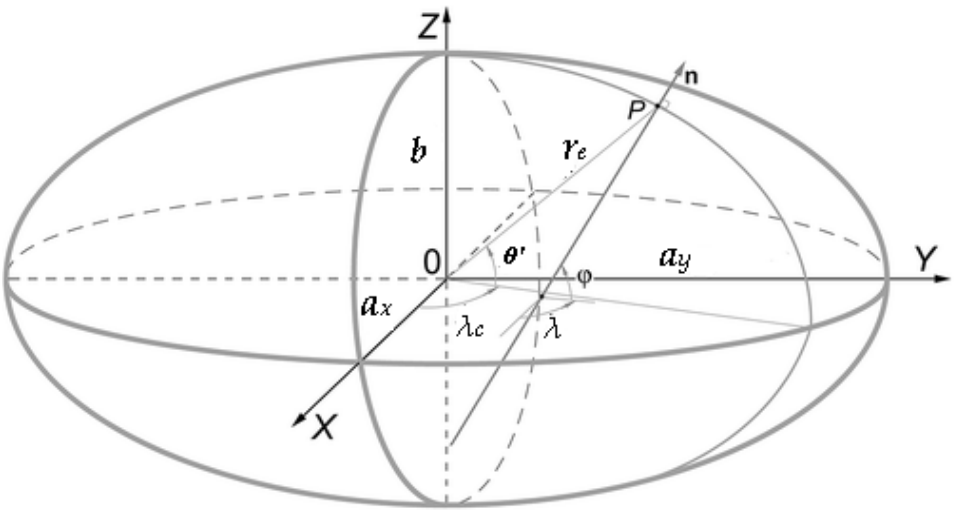


Figure 1. Triaxial ellipsoid, geodetic, and spherical coordinates.

The geodetic longitude λ of a point P on the surface of a triaxial ellipsoid is the angle between the projection of the vertical line through P onto the xy-plane and the line obtained by parallel transporting the x-axis to the trace of the vertical line on the xy-plane.

The normal vector $\boldsymbol{n}$ (see Figure 1) at a point P on the surface of a triaxial ellipsoid is the vector perpendicular to the tangent plane at point P.

The gravity intensity vector at a point P on the surface of a triaxial ellipsoid lies along the vertical line through point P, pointing to the opposite direction to the normal vector.

Relationships (2)–(4) define the first eccentricities of the three principal ellipses, which are mutually perpendicular. The gravity force on the surface of a triaxial level ellipsoid is given as [14]

$$\gamma(\phi,\lambda)=\frac{a_x\gamma_{a_x}\cos^2\phi\cos^2\lambda+a_y\gamma_{a_y}\cos^2\phi\sin^2\lambda+b\gamma_b\sin^2\phi}{\sqrt{(a_x^2\cos^2\lambda+a_y^2\sin^2\lambda)\cos^2\phi+b^2\sin^2\phi}}. \tag{9}$$

In the above relation, the gamma symbols on the right hand side represent the values of the gravity force at the end of the corresponding semiaxes. The geodetic longitude is defined as the angle between the x-axis and the projection of the ellipsoid's normal vector on the xy -plane. If λ_c denotes the spherical longitude, then it holds that

$$\sin^2\lambda_c=\frac{y^2}{x^2+y^2}=\frac{(1-e_e^2)^2\cos^2\phi\sin^2\lambda}{\cos^2\phi\cos^2\lambda+(1-e_e^2)^2\cos^2\phi\sin^2\lambda}=\frac{(1-e_e^2)^2\sin^2\lambda}{\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda}. \tag{10}$$

From the above relations we have that

$$\sin^2\lambda=\frac{\sin^2\lambda_c}{\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c}, \tag{11}$$

$$\cos^2\lambda=\frac{(1-e_e^2)^2\cos^2\lambda_c}{(1-e_e^2)^2\cos^2\lambda_c+\sin^2\lambda_c}. \tag{12}$$

If θ' is the spherical latitude, then

$$\cos^2\theta'=\frac{z^2}{r^2}=\frac{(1-e_x^2)^2\sin^2\phi}{\cos^2\phi\cos^2\lambda+(1-e_e^2)^2\cos^2\phi\sin^2\lambda+(1-e_x^2)^2\sin^2\phi}. \tag{13}$$

After some manipulations we obtain

$$\sin^2\phi=\frac{(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}, \tag{14}$$

$$\cos^2\phi=\frac{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}. \tag{15}$$

For $e_e=0$, the above relations hold for an ellipsoid of revolution ($a_x=a_y=a$) with semi axes a, a, b, and first eccentricity e, hence, we have

$$\sin^2\phi=\frac{\sin^2\theta'}{\sin^2\theta'+(1-e^2)\cos^2\theta'}, \tag{16}$$

$$\cos^2\phi=\frac{(1-e^2)^2\cos^2\theta'}{\sin^2\theta'+(1-e^2)\cos^2\theta'}. \tag{17}$$

To determine gravity intensity γ on and above the surface of the level triaxial ellispoid S, we must solve a Dirichlet boundary value problem, which involves the G—modified Helmholtz equation. This reads

$$\nabla^2\gamma - \frac{2}{r^2}\gamma = 0, \tag{18}$$

$$\gamma|_S = \gamma(\phi,\lambda) = \frac{(a_x\gamma_{a_x}\cos^2\lambda + a_y\gamma_{a_y}\sin^2\lambda)\cos^2\phi + b\gamma_b\sin^2\phi}{\sqrt{(a_x^2\cos^2\lambda + a_y^2\sin^2\lambda)\cos^2\phi + b^2\sin^2\phi}}. \tag{19}$$

The boundary condition for the Dirichlet boundary value problem, as described above, is expressed in geodetic coordinates φ and λ. To derive an expression for gravity intensity as a series of spherical harmonics, we must first rewrite the boundary condition in spherical coordinates θ' and λ_c. This transformation is quite complex and involves several manipulations, which will be demonstrated in the sequel.

The procedure consists of two steps: In the first step we express the boundary condition in geodetic coordinates φ and λ, retaining only the ${e_x}^2$ and ${e_e}^2$ terms. In the second step we substitute the geodetic coordinates φ and λ by the spherical coordinates θ' and λ_c, maintaining the same level of approximation for the ${e_x}^2$ and ${e_e}^2$ terms. Using Equations (2)–(4), the boundary condition (19) becomes

$$\gamma(\phi,\lambda) = \frac{(a_x\gamma_{a_x}\cos^2\lambda + a_x\sqrt{1-e_e^2}\gamma_{a_y}\sin^2\lambda)\cos^2\phi + a_x\sqrt{1-e_x^2}\gamma_b\sin^2\phi}{\sqrt{(a_x^2\cos^2\lambda + a_x^2(1-e_e^2)\sin^2\lambda)\cos^2\phi + a_x^2(1-e_x^2)\sin^2\phi}}. \tag{20}$$

In addition, we set

$$\gamma_{a_y} = \gamma_{a_x} + \delta\gamma_x, \tag{21}$$

$$\gamma_b = \gamma_{a_x} + \delta\gamma_z. \tag{22}$$

Thus,

$$\begin{aligned}&\gamma(\phi,\lambda) = \\ &= \frac{a_x[(\gamma_{a_x}\cos^2\lambda + (\gamma_{a_x}+\delta\gamma_x)\sqrt{1-e_e^2}\sin^2\lambda)\cos^2\phi + (\gamma_{a_x}+\delta\gamma_z)\sqrt{1-e_x^2}\sin^2\phi]}{\sqrt{(a_x^2\cos^2\lambda + a_x^2(1-e_e^2)\sin^2\lambda)\cos^2\phi + a_x^2(1-e_x^2)\sin^2\phi}} = \\ &= \frac{\gamma_{a_x}\left\{\left[\cos^2\lambda + \left(1+\frac{\delta\gamma_x}{\gamma_{a_x}}\right)\sqrt{1-e_e^2}\sin^2\lambda\right]\cos^2\phi + \left(1+\frac{\delta\gamma_z}{\gamma_{a_x}}\right)\sqrt{1-e_x^2}\sin^2\phi\right\}}{\sqrt{(\cos^2\lambda + (1-e_e^2)\sin^2\lambda)\cos^2\phi + (1-e_x^2)\sin^2\phi}}\end{aligned}. \tag{23}$$

The boundary condition given in (20) will be approximated carefully. Due to low eccentricities, e_e and e_x, the quantities $\delta\gamma_\chi$ and $\delta\gamma_z$ are small. Therefore, Equation (20) can now be simplified. Using Newton's dyonym for the eccentricities, we obtain the following

$$\begin{aligned}&\gamma(\phi,\lambda) = \\ &= \frac{\gamma_{a_x}\left\{\left[\cos^2\lambda + \left(1+\frac{\delta\gamma_x}{\gamma_{a_x}}\right)\left(1-\frac{e_e^2}{2}\right)\sin^2\lambda\right]\cos^2\phi + \left(1+\frac{\delta\gamma_z}{\gamma_{a_x}}\right)\left(1-\frac{e_x^2}{2}\right)\sin^2\phi\right\}}{\sqrt{1-e_e^2\cos^2\phi\sin^2\lambda - e_x^2\sin^2\phi}}\end{aligned}. \tag{24}$$

In addition,

$$(1 - e_e^2\cos^2\phi\sin^2\lambda - e_x^2\sin^2\phi)^{-\frac{1}{2}} = 1 + \frac{e_e^2\cos^2\phi\sin^2\lambda}{2} + \frac{e_x^2\sin^2\phi}{2}, \tag{25}$$

$$\begin{aligned}&\left[\cos^2\lambda + \left(1+\frac{\delta\gamma_x}{\gamma_{a_x}}\right)\left(1-\frac{e_e^2}{2}\right)\sin^2\lambda\right]\cos^2\phi + \left(1+\frac{\delta\gamma_z}{\gamma_{a_x}}\right)\left(1-\frac{e_x^2}{2}\right)\sin^2\phi = \\ &= \cos^2\phi + \left(\frac{\delta\gamma_x}{\gamma_{a_x}} - \frac{e_e^2}{2} - \frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda + \sin^2\phi + \left(\frac{\delta\gamma_z}{\gamma_{a_x}} - \frac{e_x^2}{2} - \frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi = \\ &= 1 + \left(\frac{\delta\gamma_x}{\gamma_{a_x}} - \frac{e_e^2}{2} - \frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda + \left(\frac{\delta\gamma_z}{\gamma_{a_x}} - \frac{e_x^2}{2} - \frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi\end{aligned}, \tag{26}$$

$$\begin{aligned}
&\left[1+\left(\frac{\delta\gamma_x}{\gamma_{a_x}}-\frac{e_e^2}{2}-\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda+\left(\frac{\delta\gamma_z}{\gamma_{a_x}}-\frac{e_x^2}{2}-\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi\right]\cdot\\
&\left(1+\frac{e_e^2\cos^2\phi\sin^2\lambda}{2}+\frac{e_x^2\sin^2\phi}{2}\right)=\\
&=1+\left(\frac{\delta\gamma_x}{\gamma_{a_x}}-\frac{e_e^2}{2}-\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda+\left(\frac{\delta\gamma_z}{\gamma_{a_x}}-\frac{e_x^2}{2}-\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi+\\
&+\frac{e_e^2\cos^2\phi\sin^2\lambda}{2}+\frac{e_x^2\sin^2\phi}{2}+\frac{\delta\gamma_x}{\gamma_{a_x}}\frac{e_e^2\cos^4\phi\sin^4\lambda}{2}+\frac{\delta\gamma_x}{\gamma_{a_x}}\frac{e_x^2\sin^2\phi\cos^2\phi\sin^2\lambda}{2}+\\
&+\frac{\delta\gamma_z}{\gamma_{a_x}}\frac{e_e^2\sin^2\phi\cos^2\phi\sin^2\lambda}{2}+\frac{\delta\gamma_z}{\gamma_{a_x}}\frac{e_x^2\sin^4\phi}{2}=\\
&=1+\left(\frac{\delta\gamma_x}{\gamma_{a_x}}-\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda+\left(\frac{\delta\gamma_z}{\gamma_{a_x}}-\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi+\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\cos^4\phi\sin^4\lambda+\\
&+\frac{\delta\gamma_x e_x^2}{2\gamma_{a_x}}\sin^2\phi\cos^2\phi\sin^2\lambda+\frac{\delta\gamma_z e_e^2}{2\gamma_{a_x}}\sin^2\phi\cos^2\phi\sin^2\lambda+\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\sin^4\phi
\end{aligned} \tag{27}$$

Hence, the boundary condition is written as

$$\begin{aligned}
&\gamma(\phi,\lambda)=\\
&=\gamma_{a_x}\left\{1+\left(\frac{\delta\gamma_x}{\gamma_{a_x}}-\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\right)\cos^2\phi\sin^2\lambda+\left(\frac{\delta\gamma_z}{\gamma_{a_x}}-\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\right)\sin^2\phi+\frac{\delta\gamma_x e_e^2}{2\gamma_{a_x}}\cos^4\phi\sin^4\lambda+\right.\\
&\left.+\frac{\delta\gamma_x e_x^2}{2\gamma_{a_x}}\sin^2\phi\cos^2\phi\sin^2\lambda+\frac{\delta\gamma_z e_e^2}{2\gamma_{a_x}}\sin^2\phi\cos^2\phi\sin^2\lambda+\frac{\delta\gamma_z e_x^2}{2\gamma_{a_x}}\sin^4\phi\right\}
\end{aligned} \tag{28}$$

Formula (28) provides the desired expression for gravity intensity γ in geodetic coordinates φ and λ, retaining only the ${e_x}^2$ and ${e_e}^2$ terms. Now, we move to the second step where we express the above boundary condition in spherical coordinates. This is presented in Appendix A. The desired result is as follows

$$\begin{aligned}
&\gamma(\theta',\lambda_c)=\gamma_{a_x}+\\
&+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+\\
&+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+\\
&+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+\\
&+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\\
&+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'
\end{aligned} \tag{29}$$

Formula (29) represents the desired form of the boundary condition (see Equation (19)) in spherical coordinates. With this we can proceed to solve the Dirichlet boundary value problem.

The procedure of deriving the series of spherical harmonics for the gravity intensity γ is carried on in the following section.

3. Main Results

In this paragraph, we will need the associated Legendre functions of the first kind in order to derive the solution for gravity intensity γ as a series of spherical harmonics. These functions are written as P_{nm} (where n is the degree and m is the order) and as variable we use either $\sin\theta'$ or $\cos\theta'$. In our case we will use $\sin\theta'$.

There are two significant challenges: (i) the radial distance r on the surface of the triaxial ellispoid is a function of gedetic coordinates φ and λ, and (ii) the functions $P_{nm}(\sin\varphi)\cos m\lambda$, $P_{nm}(\sin\varphi)\sin m\lambda$ are not suitable for constructing the desired series for the gravity intensity γ. Therefore, we need to express these functions in spherical coordinates θ' and λ_c, while retaining only the ${e_x}^2$ and ${e_e}^2$ terms. These manipulations are quite com-

plex. We begin by addressing the radial distance r. The square of the radial distance of an arbitrary point on the surface of the triaxial ellispoid is given by

$$\begin{aligned} r_e^2 &= \frac{a_x^2}{1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda}\cos^2\phi\cos^2\lambda+ \\ &+\frac{a_x^2(1-e_e^2)^2}{1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda}\cos\phi^2\sin^2\lambda+ \\ &+\frac{a_x^2(1-e_x^2)^2}{1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda}\sin^2\phi = \\ &= \frac{a_x^2\cos^2\phi\cos^2\lambda+a_x^2(1-e_e^2)^2\cos\phi^2\sin^2\lambda+a_x^2(1-e_x^2)^2\sin^2\phi}{1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda} \end{aligned} . \quad (30)$$

We replace φ anf λ with spherical coordinates θ' and λ_c. The procedure is made step by step, using Equations (11), (12), (14) and (15). To this end, we calculate

$$\begin{aligned} &1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda = \\ &= 1-e_x^2\frac{(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}- \\ &-e_e^2\frac{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}}\frac{\sin^2\lambda_c}{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]} = \end{aligned} . \quad (31)$$

Hence, the denominator in relation (30) is equal to

$$\begin{aligned} &1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda = \\ &= \frac{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'-e_x^2(1-e_e^2)^2\sin^2\theta'}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}}- \\ &-\frac{e_e^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'} = \\ &= \frac{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_x^2)(1-e_e^2)^2\sin^2\theta'-e_e^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}} \end{aligned} . \quad (32)$$

Now, we are going to express the nominator of relation (30) in spherical coordinates θ' and λ_c. It holds that

$$\begin{aligned} &a_x^2\cos^2\phi\cos^2\lambda+a_x^2(1-e_e^2)^2\cos\phi^2\sin^2\lambda = \\ &= a_x^2\cos^2\phi[\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda] \end{aligned} , \quad (33)$$

$$\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda = \frac{(1-e_e^2)^2}{\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c}, \quad (34)$$

$$a_x^2\cos^2\phi = \frac{a_x^2[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}, \quad (35)$$

$$a_x^2(1-e_x^2)^2\sin^2\phi = \frac{a_x^2(1-e_x^2)^2(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}. \quad (36)$$

Consequently,

$$\begin{aligned} &a_x^2\cos^2\phi[\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda] = \\ &= \frac{a_x^2(1-e_e^2)^2[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]} \end{aligned} , \quad (37)$$

$$a_x^2\cos^2\phi[\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda]+a_x^2(1-e_x^2)^2\sin^2\phi=$$
$$=\frac{a_x^2(1-e_e^2)^2[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]}+$$
$$+\frac{[a_x^2(1-e_x^2)^2(1-e_e^2)^2\sin^2\theta'][\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]}= \quad (38)$$
$$=\frac{a_x^2(1-e_e^2)^2(1-e_x^2)^2\cos^2\theta'+a_x^2(1-e_x^2)^2(1-e_e^2)^2\sin^2\theta'}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}}$$

Thus,

$$a_x^2\cos^2\phi[\cos^2\lambda+(1-e_e^2)^2\sin^2\lambda]+a_x^2(1-e_x^2)^2\sin^2\phi=$$
$$=\frac{a_x^2(1-e_e^2)^2(1-e_x^2)^2}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}} \quad (39)$$

Combining relationships (A13), (29) and (34), the radial distance on the surface of the triaxial ellipsoid is equal to

$$r_e=\frac{a_x(1-e_e^2)(1-e_x^2)}{\sqrt{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_x^2)(1-e_e^2)^2\sin^2\theta'-e_e^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c}} \quad (40)$$

or

$$r_e=\frac{a_x(1-e_e^2)\sqrt{1-e_x^2}}{\sqrt{(1-e_x^2)\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'-e_e^2(1-e_x^2)\cos^2\theta'\sin^2\lambda_c}}=$$
$$=\frac{b(1-e_e^2)}{\sqrt{(1-e_x^2)\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'-e_e^2(1-e_x^2)\cos^2\theta'\sin^2\lambda_c}} \quad (41)$$

For $e_e = 0$ and $a_x = a$, $e_x = e$, the above formula [18] gives the radial distance r_s of a point on the surface of an oblate spheroid with semi axles a, a, b, that is

$$r_s=\frac{a\sqrt{1-e^2}}{\sqrt{(1-e^2)\cos^2\theta'+\sin^2\theta'}}\Rightarrow r_s=\frac{a\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2\theta'}}=\frac{b}{\sqrt{1-e^2\cos^2\theta'}}. \quad (42)$$

We will return to Equation (41) shortly, as it is not in a convenient form to later integration. The transformation of the associated Legendre functions $P_{nm}(\sin\varphi)$ in spherical coordinates is necessary. The Dirichlet problem at hand is given as the G—modified Helmoltz equation (see (18)) and the boundary condition (29). For the Legendre functions, we have the following

$$P_{nm}(\sin\phi)=P_{nm}\left(\frac{(1-e_e^2)\sin\theta'}{\sqrt{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}}\right). \quad (43)$$

But, bearing in mind relations (A3) and (A4), we obtain

$$(1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c)^{-\frac{1}{2}}=$$
$$=1+e_x^2\cos^2\theta'+e_e^2\sin^2\theta'+e_e^2\cos^2\theta'\cos^2\lambda_c \quad (44)$$

Therefore,

$$\sin\phi=(1-e_e^2)\sin\theta'(1+e_x^2\cos^2\theta'+e_e^2\sin^2\theta'+e_e^2\cos^2\theta'\cos^2\lambda_c)=$$
$$=\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'+e_e^2\sin\theta'\cos^2\theta'\cos^2\lambda_c-e_e^2\sin\theta'= \quad (45)$$
$$=\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c$$

Formula (43) is written as

$$P_{nm}(\sin\phi) = P_{nm}(\sin\theta' + e_x^2 \sin\theta' \cos^2\theta' + e_e^2 \sin^3\theta' - e_e^2 \sin\theta' \cos^2\theta' \sin^2\lambda_c). \quad (46)$$

Relationship (46) represents the desired relation for the associated Legendre functions. For $\sin m\lambda$ and $\cos m\lambda$ we have the following

$$\begin{array}{l} \sin m\lambda = \frac{\sin m\lambda_c}{\sqrt{\sin^2 m\lambda_c + (1-e_e^2)^2 \cos^2 m\lambda_c}} = \sin m\lambda_c (1 + e_e^2 \cos^2 m\lambda_c) = \\ = \sin m\lambda_c (1 + e_e^2 - e_e^2 \sin^2 m\lambda_c) = \sin m\lambda_c + e_e^2 \sin m\lambda_c - e_e^2 \sin^3 m\lambda_c \end{array}, \quad (47)$$

$$\begin{array}{l} \cos m\lambda = \frac{(1-e_e^2)\cos m\lambda_c}{\sqrt{\sin^2 \lambda_c + (1-e_e^2)^2 \cos^2 m\lambda_c}} = \\ = (\cos m\lambda_c - e_e^2 \cos m\lambda_c)(1 + e_e^2 \cos^2 m\lambda_c) = \cos m\lambda_c - e_e^2 \cos m\lambda_c + e_e^2 \cos^3 m\lambda_c \end{array}. \quad (48)$$

The general expression for the gravity intensity γ as a series of spherical harmonics is given as

$$\begin{array}{l} \gamma(r, \theta', \lambda_c) = \sum\limits_{n=0}^{+\infty} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \\ \{ \sum\limits_{m=0}^{n} [a_{nm} P_{nm}(\sin\theta' + e_x^2 \sin\theta' \cos^2\theta' + e_e^2 \sin^3\theta' - e_e^2 \sin\theta' \cos^2\theta' \sin^2\lambda_c) \\ \quad (\cos m\lambda_c - e_e^2 \cos m\lambda_c + e_e^2 \cos^3 m\lambda_c) + \\ \quad + b_{nm} P_{nm}(\sin\theta' + e_x^2 \sin\theta' \cos^2\theta' + e_e^2 \sin^3\theta' - e_e^2 \sin\theta' \cos^2\theta' \sin^2\lambda_c) \\ \quad (\sin m\lambda_c + e_e^2 \sin m\lambda_c - e_e^2 \sin^3 m\lambda_c)] \} \end{array}. \quad (49)$$

We need to modify Equation (41), holding that

$$b(1 - e_e^2) = b - e_e^2 b, \quad (50)$$

$$\begin{array}{l} (1-e_x^2)\cos^2\theta'[\sin^2\lambda_c + (1-e_e^2)^2 \cos^2\lambda_c] + (1-e_e^2)^2 \sin^2\theta' - \\ -e_e^2(1-e_x^2)\cos^2\theta' \sin^2\lambda_c = (\cos^2\theta' - e_x^2 \cos^2\theta')(1 - 2e_e^2 \cos^2\lambda_c) + \\ + \sin^2\theta' - 2e_e^2 \sin^2\theta' - e_e^2 \cos^2\theta' \sin^2\lambda_c = \\ = \cos^2\theta' - 2e_e^2 \cos^2\theta' \cos^2\lambda_c - e_x^2 \cos^2\theta' + \sin^2\theta' - 2e_e^2 \sin^2\theta' - e_e^2 \cos^2\theta' \sin^2\lambda_c = \\ = 1 - 2e_e^2 \cos^2\theta' \cos^2\lambda_c - e_x^2 \cos^2\theta' - 2e_e^2 \sin^2\theta' - e_e^2 \cos^2\theta' \sin^2\lambda_c \end{array}, \quad (51)$$

$$\begin{array}{l} (1 - 2e_e^2 \cos^2\theta' \cos^2\lambda_c - e_x^2 \cos^2\theta' - 2e_e^2 \sin^2\theta' - e_e^2 \cos^2\theta' \sin^2\lambda_c)^{-\frac{1}{2}} = \\ = 1 + e_e^2 \cos^2\theta' \cos^2\lambda_c + \frac{e_x^2}{2} \cos^2\theta' + e_e^2 \sin^2\theta' + \frac{e_e^2}{2} \cos^2\theta' \sin^2\lambda_c \end{array}. \quad (52)$$

Therefore,

$$\begin{array}{l} r_e = (b - e_e^2 b) \\ \left(1 + e_e^2 \cos^2\theta' \cos^2\lambda_c + \frac{e_x^2}{2} \cos^2\theta' + e_e^2 \sin^2\theta' + \frac{e_e^2}{2} \cos^2\theta' \sin^2\lambda_c\right) \end{array} \quad (53)$$

or

$$r_e = b - e_e^2 b + e_e^2 b \cos^2\theta' \cos^2\lambda_c + \frac{e_x^2}{2} b \cos^2\theta' + \frac{e_e^2}{2} b \sin^2\theta' + \frac{e_e^2}{2} b \cos^2\theta' \sin^2\lambda_c. \quad (54)$$

For $e_e = 0$, $e_x = 0$ the above equation gives the radial distance of an oblate spheroid with semi axles a, a, b, that is

$$r = \frac{b}{\sqrt{1 - e^2 \cos^2\theta'}} = b + \frac{e^2}{2} b \cos^2\theta'. \quad (55)$$

Having evaluated all necessary expressions, we need to determine the coefficients a_{nm} and b_{nm} of relation (64). On the surface of the triaxial ellipsoid, it holds that (see Equations (29), (46)–(48) and (54))

$$\begin{aligned}
&\gamma_{a_x}+\\
&+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+\\
&+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+\\
&+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+\\
&+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\\
&+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'=\\
&=\sum_{n=0}^{+\infty}(b-e_e^2b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\frac{e_x^2}{2}b\cos^2\theta'+\frac{e_e^2}{2}b\sin^2\theta'+\frac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&\{\sum_{m=0}^{n}[a_{nm}P_{nm}(\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)[(\cos m\lambda_c(1-e_e^2+e_e^2\cos^2 m\lambda_c)]+\\
&\quad+b_{nm}P_{nm}(\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)(\sin m\lambda_c+e_e^2\sin m\lambda_c-e_e^2\sin^3 m\lambda_c)]\}
\end{aligned} \quad . \tag{56}$$

We aim to split the above series into two parts. The first part will consist of a series of spherical harmonics containing the associated Legendre functions $P_{nm}(\sin\theta')$, while the second part will include a singular component which contains the ${e_x}^2$ and ${e_e}^2$ terms. Setting

$$\mu_0=\sin\theta', \tag{57}$$

$$\mu=e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c, \tag{58}$$

we make the following substitution (see relation (46)):

$$P_{nm}(\mu_0+\mu)=P_{nm}(\mu_0)+{P'}_{nm}(\mu_0)(\mu_0+\mu-\mu_0)=P_{nm}(\mu_0)+\mu{P'}_{nm}(\mu_0). \tag{59}$$

The prime symbol denotes differentiation with respect to the argument $\sin\theta'$. Hence,

$$\begin{aligned}
&P_{nm}(\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)=\\
&=P_{nm}(\sin\theta')+(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c){P'}_{nm}(\sin\theta')
\end{aligned} \quad . \tag{60}$$

Therefore, boundary condition (56) on the surface of the triaxial ellipsoid becomes

$$
\begin{aligned}
&\gamma_{a_x}+\\
&+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+\\
&+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+\\
&+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+\\
&+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\\
&+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'=\\
&=\sum_{n=0}^{+\infty}(b-e_e^2b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\frac{e_x^2}{2}b\cos^2\theta'+\frac{e_e^2}{2}b\sin^2\theta'+\frac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&\{\sum_{m=0}^{n}\{a_{nm}[P_{nm}(\sin\theta')(\cos m\lambda_c-e_e^2\cos m\lambda_c+e_e^2\cos^3 m\lambda_c)+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)(\cos m\lambda_c-e_e^2\cos m\lambda_c+e_e^2\cos^3 m\lambda_c)]\}+\\
&+b_{nm}\{[P_{nm}(\sin\theta')(\sin m\lambda_c+e_e^2\sin m\lambda_c-e_e^2\sin^3 m\lambda_c)+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)(\sin m\lambda_c+e_e^2\sin m\lambda_c-e_e^2\sin^3 m\lambda_c)]\}\}
\end{aligned}
\tag{61}
$$

The terms that involve the derivatives of the associated Legendre functions $P_{nm}(\sin\theta')$ represent the singular part of our series. By retaining only ${e_x}^2$ and ${e_e}^2$ terms, the above boundary condition becomes

$$
\begin{aligned}
&\gamma_{a_x}+\\
&+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+\\
&+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+\\
&+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+\\
&+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\\
&+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'=\\
&=\sum_{n=0}^{+\infty}(b-e_e^2b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\frac{e_x^2}{2}b\cos^2\theta'+\frac{e_e^2}{2}b\sin^2\theta'+\frac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&\{\sum_{m=0}^{n}\{a_{nm}[P_{nm}(\sin\theta')\cos m\lambda_c-e_e^2P_{nm}(\sin\theta')\cos m\lambda_c+e_e^2P_{nm}(\sin\theta')\cos^3 m\lambda_c+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\cos m\lambda_c]\}+\\
&+b_{nm}\{[P_{nm}(\sin\theta')\sin m\lambda_c+e_e^2P_{nm}(\sin\theta')\sin m\lambda_c-e_e^2P_{nm}(\sin\theta')\sin^3 m\lambda_c+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\sin m\lambda_c]\}\}
\end{aligned}
\tag{62}
$$

The right hand side of (62) can be further modified, since

$$
\begin{aligned}
&b-e_e^2b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\tfrac{e_x^2}{2}b\cos^2\theta'+\tfrac{e_e^2}{2}b\sin^2\theta'+\tfrac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c=\\
&=b-e_e^2b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\tfrac{e_x^2}{2}b\cos^2\theta'+\tfrac{e_e^2}{2}b-\tfrac{e_e^2}{2}b\cos^2\theta'+\tfrac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c=\\
&=b-\tfrac{e_e^2}{2}b+e_e^2b\cos^2\theta'\cos^2\lambda_c+\tfrac{e_x^2}{2}b\cos^2\theta'-\tfrac{e_e^2}{2}b\cos^2\theta'+\tfrac{e_e^2}{2}b\cos^2\theta'\sin^2\lambda_c=\\
&=b\left(1-\tfrac{e_e^2}{2}\right)\left(1+\tfrac{2e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c+\tfrac{e_x^2-e_e^2}{2-e_e^2}\cos^2\theta'-\tfrac{e_e^2}{2-e_e^2}\cos^2\theta'\sin^2\lambda_c\right)=\\
&=b\left(1-\tfrac{e_e^2}{2}\right)\left(1+\tfrac{e_e^2}{2-e_e^2}\cos^2\theta'+\tfrac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c+\tfrac{e_x^2-e_e^2}{2-e_e^2}\cos^2\theta'\right)=\\
&=b\left(1-\tfrac{e_e^2}{2}\right)\left(1+\tfrac{e_x^2}{2-e_e^2}\cos^2\theta'+\tfrac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c\right)
\end{aligned}
\tag{63}
$$

We substitute Formula (63) into (62) and multiply both sides with $P_{jk}(\cos\theta')\cos m\lambda_c$. By integrating both sides, we can determine the coefficients a_{nm}. This is expressed in the following relation:

$$\begin{aligned}
&\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\gamma_{a_x}+\\
&+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+\\
&+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+\\
&+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+\\
&+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\\
&+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'\Big]P_{jk}(\sin\theta')\cos m\lambda_c\cos\theta' d\theta' d\lambda_c=\\
&=\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{n=0}^{+\infty}b^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\left(1-\frac{e_e^2}{2}\right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\left(1+\frac{e_x^2}{2-e_e^2}\cos^2\theta'+\frac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c\right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&\{\sum_{m=0}^{n}\{a_{nm}[P_{nm}(\sin\theta')\cos m\lambda_c-e_e^2P_{nm}(\sin\theta')\cos m\lambda_c+e_e^2P_{nm}(\sin\theta')\cos^3 m\lambda_c+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\cos m\lambda_c]\}+\\
&+b_{nm}\{[P_{nm}(\sin\theta')\sin m\lambda_c+e_e^2P_{nm}(\sin\theta')\sin m\lambda_c-e_e^2P_{nm}(\sin\theta')\sin^3 m\lambda_c+\\
&+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\sin m\lambda_c]\}\}P_{jk}(\sin\theta')\cos m\lambda_c\cos\theta' d\theta' d\lambda_c
\end{aligned} \tag{64}$$

The summation over the index n will be carried out up to a degree n_0. On the left hand side of relationship (64), the function $\sin^2\lambda_c\sin m\lambda_c$ is always expressed in odd powers. Therefore, all the integrals involving this function are equal to zero for every m. The same applies to the function $\sin^4\lambda_c\sin m\lambda_c$. This conclusion simplifies the process, as there will be no coefficients b_{nm} in the final formula for the gravity intensity γ (b_{nm} = 0 for every n, m).

The degree n_0 is equal to four. Due to the structure of the left hand side of Formula (64), the non-zero coefficients are a_{00}, a_{20}, a_{22}, a_{40}, a_{42}, and a_{44}. Additionally, only the following associated Legendre functions appear, $P_{00}(\sin\theta')$, $P_{20}(\sin\theta')$, $P_{22}(\sin\theta')$, $P_{40}(\sin\theta')$, $P_{42}(\sin\theta')$ and $P_{44}(\sin\theta')$. In Appendix A we present this result. On the other hand, Appendix B shows that integrals involving derivatives of Legendre functions are also non-zero. This indicates that the singular terms contribute significantly to determining the values of the coefficients a_{2n2m}. The final formula will be derived from relation (56).

The series of spherical harmonics for the gravity intensity has the following form:

$$\begin{aligned}
&\gamma(r,\theta',\lambda_c)=\sum_{n=0}^{4}\sum_{m=0}^{n}a_{nm}r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&P_{nm}(\sin\theta'+e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\cdot\\
&(\cos m\lambda_c-e_e^2\cos m\lambda_c+e_e^2\cos^3 m\lambda_c)
\end{aligned} \tag{65}$$

Using (60), the relation (65) becomes

$$\begin{aligned}
&\gamma(r,\theta',\lambda_c)=\sum_{n=0}^{4}\sum_{m=0}^{n}a_{nm}r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\
&[P_{nm}(\sin\theta')+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta'+e_e^2\sin^3\theta'-e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)]\cdot\\
&(\cos m\lambda_c-e_e^2\cos m\lambda_c+e_e^2\cos^3 m\lambda_c)
\end{aligned} \tag{66}$$

or

$$\gamma(r,\theta',\lambda_c) = \sum_{n=0}^{4}\sum_{m=0}^{n} a_{nm} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \\ [P_{nm}(\sin\theta') + P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta' + e_e^2\sin^3\theta' - e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)]\cdot \\ \cos m\lambda_c(1-e_e^2\sin^2 m\lambda_c) \tag{67}$$

We do not employ a recursion formula for $P'_{nm}(\sin\theta')$ to avoid singularities. Finally, Equation (67) can be written as

$$\boxed{\begin{array}{l}\gamma(r,\theta',\lambda_c) = \sum\limits_{n=0}^{2}\sum\limits_{m=0}^{n} a_{2n2m} r^{-\frac{1+\sqrt{9+8n(2n+1)}}{2}} \\ [P_{2n2m}(\sin\theta') + e_x^2 P'_{2n2m}(\sin\theta')\sin\theta'\cos^2\theta' + \\ \quad + e_e^2 P'_{2n2m}(\sin\theta')(\sin^3\theta' - \sin\theta'\cos^2\theta'\sin^2\lambda_c)](1-e_e^2\sin^2 2m\lambda_c)\cos 2m\lambda_c\end{array}} \tag{68}$$

Relationship (68) represents the desired expression for the gravity intensity γ on and outside of the triaxial level ellipsoid. It contains only the coefficients a_{2n2m} and associated Legendre functions of even degree and order. A special case of this relation occurs when the triaxial ellipsoid becomes an ellipsoid of revolution (oblate spheroid) with semi axles a, a, b, i.e., when $e_e = 0$ and $e_x \equiv e$. Due to rotational symmetry of its gravity field [19], it follows that $m = 0$ and (68) simplifies to

$$\boxed{\begin{array}{l}\gamma_s(r,\theta') = \sum\limits_{n=0}^{2} a_{2n} r^{-\frac{1+\sqrt{9+8n(2n+1)}}{2}}[P_{2n}(\sin\theta') + e^2 P'_{2n}(\sin\theta')\sin\theta'\cos^2\theta'] = \\ \sum\limits_{n=0}^{2} a_{2n} r^{-\frac{1+\sqrt{9+8n(2n+1)}}{2}} \cdot \{P_{2n}(\sin\theta') + e^2(2n+1)\sin\theta'[P_{2n}(\sin\theta')\sin\theta' - P_{2n+1}(\sin\theta')]\}\end{array}} \tag{69}$$

The determination of the leading term a_{00} and its normalized form A_{00} (see Appendix C) is as follows:

(a) For the case of a triaxial ellipsoid

$$a_{00} = \frac{b^2\left(1-\frac{e_e^2}{2}\right)^2\left[4\gamma_{a_x} + \frac{4}{3}\delta\gamma_x + \frac{4}{3}\delta\gamma_z + \left(-\frac{6}{15}e_x^2 + \frac{59}{30}e_e^2\right)\delta\gamma_x + \left(\frac{12}{15}e_x^2 - \frac{6}{15}e_e^2\right)\delta\gamma_z\right]}{\left[4 - \frac{16e_x^2}{3(2-e_e^2)} - \frac{4e_e^2}{3(2-e_e^2)}\right]}, \tag{70}$$

$$A_{00} = \frac{\left[4\gamma_{a_x} + \frac{4}{3}\delta\gamma_x + \frac{4}{3}\delta\gamma_z + \left(-\frac{6}{15}e_x^2 + \frac{59}{30}e_e^2\right)\delta\gamma_x + \left(\frac{12}{15}e_x^2 - \frac{6}{15}e_e^2\right)\delta\gamma_z\right]}{\left[4 - \frac{16e_x^2}{3(2-e_e^2)} - \frac{4e_e^2}{3(2-e_e^2)}\right]}. \tag{71}$$

(b) For the case of an oblate spheroid ($a_x = a_y = a$, b, $e_e = 0$, $e_x = e$)

$$a_{00} = \frac{b^2\left(\gamma_{a_x} + \frac{1}{3}\delta\gamma_z + \frac{1}{5}e^2\delta\gamma_z\right)}{\left(1-\frac{2e^2}{3}\right)}, \tag{72}$$

$$A_{00} = \frac{\left(\gamma_{a_x} + \frac{1}{3}\delta\gamma_z + \frac{1}{5}e^2\delta\gamma_z\right)}{\left(1-\frac{2e^2}{3}\right)}. \tag{73}$$

(c) For the case of a sphere of radius $R = a_x$ ($a_x = a_y = b$, $e_e = e_x = 0$)

$$a_{00} = GM, \tag{74}$$

$$A_{00} = \frac{GM}{R^2} \tag{75}$$

where G is the gravitational constant and M is the Earth's mass. The formulae expressing the normalized leading term A_{00} are very encouraging for the validity of our results, as in all cases the value of A_{00} is close to—or equal to—the reference gravity value.

4. Conclusions

This study investigates the gravity field generated by a triaxial ellipsoid with low eccentricities and that of an oblate spheroid as a special case. The gravity intensity was determined by solving a Dirichlet boundary value problem associated with the G—modified Helmholtz equation. This novel and original approach enables the expression of gravity intensity as a series of spherical harmonics.

The significance of this novel series is significantly important for slightly flattened triaxial ellipsoids and spheroids. Determining the magnitude of gradient of the gravity potential in ellipsoidal coordinates results in an extremely complicated formula for gravity intensity. This complicated formula is not suitable for gravity field studies. In contrast the presented series substantially simplifies the study of gravity fields, making it possible to reveal physical and geometrical properties of ellipsoidal physical bodies. It is impossible to perform the latter using the gravity potential gradient.

This resulting series consists of two parts: the first includes the associated Legendre functions, while the second represents the singular component, which involves the derivatives of the associated Legendre functions with respect to the argument $\sin\theta'$. The use of spherical harmonics allows for a straightforward and efficient determination of gravity intensity, as the radial and angular components are separated.

From the last two relations in the previous paragraph, it is evident that the differences between the gravity intensity of a triaxial ellipsoid and that of an ellipsoid of revolution (oblate spheroid) are small. Both expressions require only the coefficients a_{nm} with even values of n and neither contains any b_{nm} coefficients. Additionally, in both cases, a maximum degree $n = 2$ is sufficient to describe the gravity intensity (within a specific level of approximation).

The expression for gravity intensity offers a significant advantage, as it enables the study of gravity intensity without requiring prior determination of the gravity potential. Gravity intensity plays a crucial role in geodetic and geophysical studies: low degree harmonics represent global phenomena, (such us tectonic plates motion), while high degree terms capture localized anomalies. Therefore, this approach provides an opportunity to refine gravity anomalies and gravity disturbances within the Earth's gravity field. The gravity intensity of a triaxial ellipsoid can be used to obtain an improved value for the normal vertical gradient of gravity. This refinement is essential not only for geodetic applications but also for a wide range of fields, including planetary and cometary gravity studies, volcano monitoring, geophysical exploration, detection of subsurface structures, mass distribution variations, marine gravity field research, and even archaeology: detection of subsurface cavities (air filled cavities, water filled cavities), crypts, cellars, and tunnels beneath churches and castles.

Improving the normal vertical gradient of gravity will enable the development of a refined vertical gravity anomaly gradient, which in turn will enhance our understanding of the structural morphology of the Earth's interior and the configuration of the seafloor.

Future work could involve the independent development of new types of models, such as Global Normal Gravity Models. In the case of the Earth, the construction of Global Gravity Intensity Models for gravity intensity g would be particularly valuable. Unlike gravity potential, gravity intensity is a directly measurable quantity and can be

expressed as a series of spherical harmonics by solving appropriate Dirichlet boundary value problems. The representation of the Earth's gravity intensity, g, as a series of spherical harmonics, combined with the development of new models, will be highly valuable for constructing improved gravity anomaly and gravity disturbance models. The numerical evaluation of these new models along with the numerical evaluation of the proposed series for gravity intensity will be highly valuable for geophysical and geodetic studies. This, as a consequence, will contribute to a more accurate and comprehensive understanding of the Earth's gravity field.

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Appendix A

Using Equations (11), (12), (14) and (15) we are led to

$$\cos^2\phi\sin^2\lambda = \frac{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}\cdot \cdot\frac{\sin^2\lambda_c}{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]} = = \frac{(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'} \tag{A1}$$

$$(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c = \cos^2\theta'\sin^2\lambda_c - 2e_x^2\cos^2\theta'\sin^2\lambda_c \tag{A2}$$

$$\begin{aligned}&(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta' = \\ &= (1-2e_x^2)\cos^2\theta'[\sin^2\lambda_c+(1-2e_e^2)\cos^2\lambda_c]+(1-2e_e^2)\sin^2\theta' = \\ &= (1-2e_x^2)\cos^2\theta'(1-2e_e^2\cos^2\lambda_c)+(1-2e_e^2)\sin^2\theta' = \\ &= 1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c\end{aligned} \tag{A3}$$

$$\begin{aligned}&(1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c)^{-1} = \\ &= 1+2e_x^2\cos^2\theta'+2e_e^2\sin^2\theta'+2e_e^2\cos^2\theta'\cos^2\lambda_c\end{aligned} \tag{A4}$$

Thus,

$$\begin{aligned}&\cos^2\phi\sin^2\lambda = (\cos^2\theta'\sin^2\lambda_c-2e_x^2\cos^2\theta'\sin^2\lambda_c)\cdot \\ &\quad(1+2e_x^2\cos^2\theta'+2e_e^2\sin^2\theta'+2e_e^2\cos^2\theta'\cos^2\lambda_c) = \\ &= \cos^2\theta'\sin^2\lambda_c-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+ \\ &+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c\end{aligned} \tag{A5}$$

$$\begin{aligned}&\sin^2\phi = \frac{(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'} = \\ &= (1-2e_e^2)\sin^2\theta'(1+2e_x^2\cos^2\theta'+2e_e^2\sin^2\theta'+2e_e^2\cos^2\theta'\cos^2\lambda_c) = \\ &= \sin^2\theta'+2e_x^2\sin^2\theta'\cos^2\theta'+2e_e^2\sin^4\theta'+2e_e^2\sin^2\theta'\cos^2\theta'\cos^2\lambda_c-2e_e^2\sin^2\theta' = \\ &= \sin^2\theta'+2e_x^2\sin^2\theta'\cos^2\theta'+2e_e^2\sin^2\theta'\cos^2\theta'\cos^2\lambda_c-2e_e^2\sin^2\theta'\cos^2\theta' = \\ &= \sin^2\theta'+2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c\end{aligned} \tag{A6}$$

$$\cos^4\phi\sin^4\lambda = \frac{\{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c](1-e_x^2)^2\cos^2\theta'\}^2}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}^2}\cdot$$
$$\frac{\sin^4\lambda_c}{[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]^2} = \tag{A7}$$
$$= \frac{(1-e_x^2)^4\cos^4\theta'\sin^4\lambda_c}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}^2}$$

$$(1-e_x^2)^4\cos^4\theta'\sin^4\lambda_c = \cos^4\theta'\sin^4\lambda_c - 4e_x^2\cos^4\theta'\sin^4\lambda_c \tag{A8}$$

$$\begin{aligned}&\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}^2 =\\ &= (1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c)^2 =\\ &= (1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c)\cdot\\ &(1-2e_x^2\cos^2\theta'-2e_e^2\sin^2\theta'-2e_e^2\cos^2\theta'\cos^2\lambda_c) =\\ &= 1-4e_x^2\cos^2\theta'-4e_e^2\sin^2\theta'-4e_e^2\cos^2\theta'\cos^2\lambda_c\end{aligned} \tag{A9}$$

$$\begin{aligned}&(1-4e_x^2\cos^2\theta'-4e_e^2\sin^2\theta'-4e_e^2\cos^2\theta'\cos^2\lambda_c)^{-1} =\\ &= 1+4e_x^2\cos^2\theta'+4e_e^2\sin^2\theta'+4e_e^2\cos^2\theta'\cos^2\lambda_c\end{aligned} \tag{A10}$$

Hence,

$$\begin{aligned}\cos^4\phi\sin^4\lambda &= (\cos^4\theta'\sin^4\lambda_c - 4e_x^2\cos^4\theta'\sin^4\lambda_c)\cdot\\ &\quad(1+4e_x^2\cos^2\theta'+4e_e^2\sin^2\theta'+4e_e^2\cos^2\theta'\cos^2\lambda_c) =\\ &= \cos^4\theta'\sin^4\lambda_c - 4e_x^2\cos^4\theta'\sin^4\lambda_c + 4e_x^2\cos^6\theta'\sin^4\lambda_c +\\ &+4e_e^2\sin^2\theta'\cos^4\theta'\sin^4\lambda_c + 4e_e^2\cos^6\theta'\sin^4\lambda_c\cos^2\lambda_c\end{aligned} \tag{A11}$$

With the aid of (29) and (34), we have that

$$\sin^2\phi\cos^2\phi\sin^2\lambda = \frac{(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'}\cdot$$
$$\frac{(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'} = \tag{A12}$$
$$= \frac{(1-e_e^2)^2(1-e_x^2)^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c}{\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\}^2}$$

$$\begin{aligned}&(1-e_e^2)^2(1-e_x^2)^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c = (1-2e_e^2)(1-2e_x^2)\sin^2\theta'\cos^2\theta'\sin^2\lambda_c =\\ &= \sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_x^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c\end{aligned} \tag{A13}$$

Therefore, according to relations (37) and (38), we arrive at

$$\begin{aligned}&\sin^2\phi\cos^2\phi\sin^2\lambda =\\ &(\sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_x^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)\cdot\\ &(1+4e_x^2\cos^2\theta'+4e_e^2\sin^2\theta'+4e_e^2\cos^2\theta'\cos^2\lambda_c) =\\ &= \sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c - 2e_x^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c +\\ &+4e_x^2\sin^2\theta'\cos^4\theta'\sin^2\lambda_c + 4e_e^2\sin^4\theta'\cos^2\theta'\sin^2\lambda_c +\\ &+4e_e^2\sin^2\theta'\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c\end{aligned} \tag{A14}$$

Finally,

$$\begin{aligned}\sin^4\phi &= \frac{(1-e_e^2)^4\sin^4\theta'}{\left\{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c+(1-e_e^2)^2\cos^2\lambda_c]+(1-e_e^2)^2\sin^2\theta'\right\}^2} = \\ &= (1-4e_e^2)\sin^4\theta'(1+4e_x^2\cos^2\theta'+4e_e^2\sin^2\theta'+4e_e^2\cos^2\theta'\cos^2\lambda_c) = \\ &= \sin^4\theta'+4e_x^2\sin^4\theta'\cos^2\theta'+4e_e^2\sin^6\theta'+4e_e^2\sin^4\theta'\cos^2\theta'\cos^2\lambda_c-4e_e^2\sin^4\theta'\end{aligned} \tag{A15}$$

Using the above relationships (33), (34), (39), (42), (43), the boundary condition (28) is rewritten as

$$\begin{aligned}&\gamma(\theta',\lambda_c) = \gamma_{a_x}+ \\ &+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+ \\ &+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\tfrac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+ \\ &+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\tfrac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+ \\ &+\tfrac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\tfrac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+ \\ &+\tfrac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\tfrac{\delta\gamma_z e_x^2}{2}\sin^4\theta'\end{aligned} \tag{A16}$$

We determine the following integrals for m = 0, 1, 2, 3, 4. We use tables of integrals which can be found at [18].

For m = 0, cos$m\lambda_c$ = 1. Therefore,

$$\int_0^{2\pi}\cos^2\lambda_c d\lambda_c = \frac{1}{2}\left[\lambda_c+\frac{\sin 2\lambda_c}{2}\right]_0^{2\pi} = \pi \tag{A17}$$

$$\int_0^{2\pi}\sin^2\lambda_c d\lambda_c = \frac{1}{2}\left[\lambda_c-\frac{\sin 2\lambda_c}{2}\right]_0^{2\pi} = \pi \tag{A18}$$

For m = 1,

$$\int_0^{2\pi}\cos^2\lambda_c\cos\lambda_c d\lambda_c = \left[\frac{1}{12}\sin 2\lambda_c+\frac{3}{4}\sin 3\lambda_c\right]_0^{2\pi} = 0 \tag{A19}$$

$$\int_0^{2\pi}\sin^2\lambda_c\cos\lambda_c d\lambda_c = \left[\frac{\sin^3\lambda_c}{3}\right]_0^{2\pi} = 0 \tag{A20}$$

For m = 2,

$$\int_0^{2\pi}\cos^2\lambda_c\cos 2\lambda_c d\lambda_c = \left[\frac{\sin 4\lambda_c+(4\sin 2\lambda_c+4x)}{16}\right]_0^{2\pi} = \frac{\pi}{2} \tag{A21}$$

$$\int_0^{2\pi}\sin^2\lambda_c\cos 2\lambda_c d\lambda_c = \left[-\frac{\sin 4\lambda_c-4\sin 2\lambda_c+4x}{16}\right]_0^{2\pi} = -\frac{\pi}{2} \tag{A22}$$

On the other hand, the integrals of the form

$$\begin{aligned}&\int_0^{2\pi}\sin^2\lambda_c\cos m\lambda_c d\lambda_c = \\ &= \frac{(m^2\sin^2\lambda_c-2)\sin m\lambda_c+2m\sin\lambda_c\cos\lambda_c\cos m\lambda_c}{m^3-3m} = 0, \quad m = 3,4,5,\ldots\end{aligned} \tag{A23}$$

are equal to zero.

For $m = 0$, $\cos m\lambda_c = 1$, thus

$$\int_0^{2\pi} \cos^4 \lambda_c d\lambda_c = \frac{1}{2}\left[\frac{\sin 4\lambda_c + 8\sin 2\lambda_c + 12\lambda_c}{32}\right]_0^{2\pi} = \frac{3\pi}{4} \tag{A24}$$

For $m = 1$,

$$\int_0^{2\pi} \cos^4 \lambda_c \cos \lambda_c d\lambda_c = \left[\frac{1}{5}\sin^5 \lambda_c - \frac{2}{3}\sin^3 \lambda_c + \sin \lambda_c\right]_0^{2\pi} = 0 \tag{A25}$$

For $m = 2$,

$$\begin{aligned} &\int_0^{2\pi} \cos^4 \lambda_c \cos 2\lambda_c d\lambda_c = \\ &= \left[\frac{\sin 6\lambda_c + 6\sin 4\lambda_c + 21\sin 2\lambda_c + 24x)}{96}\right]_0^{2\pi} = \frac{\pi}{2} \end{aligned} \tag{A26}$$

For $m = 3$,

$$\begin{aligned} &\int_0^{2\pi} \cos^4 \lambda_c \cos 3\lambda_c d\lambda_c = \\ &= \left[-\frac{4}{7}\sin^7 \lambda_c + \frac{9}{5}\sin^5 \lambda_c - 2\sin^3 \lambda_c + \sin \lambda_c\right]_0^{2\pi} = 0 \end{aligned} \tag{A27}$$

For $m = 4$,

$$\begin{aligned} &\int_0^{2\pi} \cos^4 \lambda_c \cos 4\lambda_c d\lambda_c = \\ &= \left[\frac{3\sin 8\lambda_c + 16\sin 6\lambda_c + 36\sin 4\lambda_c + 48\sin 2\lambda_c + 24\lambda_c}{384}\right]_0^{2\pi} = \frac{\pi}{8} \end{aligned} \tag{A28}$$

For $m = 0$,

$$\int_0^{2\pi} \sin^4 \lambda_c d\lambda_c = \left[\frac{\sin 4\lambda_c}{32} - \frac{\sin 2\lambda_c}{4} + \frac{12\lambda_c}{32}\right]_0^{2\pi} = \frac{3\pi}{4} \tag{A29}$$

For $m = 1$,

$$\int_0^{2\pi} \sin^4 \lambda_c \cos \lambda_c d\lambda_c = \left[\frac{\sin^5 \lambda_c}{5}\right]_0^{2\pi} = 0 \tag{A30}$$

For $m = 2$,

$$\begin{aligned} &\int_0^{2\pi} \sin^4 \lambda_c \cos^2 \lambda_c d\lambda_c = \\ &= \left[\frac{\sin 6\lambda_c - 6\sin 4\lambda_c + 21\sin 2\lambda_c - 24\lambda_c}{96}\right]_0^{2\pi} = -\frac{\pi}{2} \end{aligned} \tag{A31}$$

For $m = 3$,

$$\int_0^{2\pi} \sin^4 \lambda_c \cos^3 \lambda_c d\lambda_c = \left[\frac{\sin^5 \lambda_c}{5} - \frac{4\sin^7 \lambda_c}{7}\right]_0^{2\pi} = 0 \tag{A32}$$

For $m = 4$,

$$\begin{aligned}&\int_0^{2\pi} \sin^4\lambda_c \cos 4\lambda_c d\lambda_c = \\ &= \left[\frac{3\sin 8\lambda_c - 16\sin 6\lambda_c + 36\sin 4\lambda_c - 48\sin 2\lambda_c + 24\lambda_c}{384}\right]_0^{2\pi} = \frac{\pi}{8}\end{aligned} \tag{A33}$$

while,

$$\begin{aligned}&\int_0^{2\pi} \sin^4\lambda_c \cos m\lambda_c d\lambda_c = \frac{\sin((m+4)\lambda_c)}{16(m+4)} - \frac{\sin((m+2)\lambda_c)}{4(m+2)} + \\ &+\frac{3\sin m\lambda_c}{8m} - \frac{\sin((m-2)\lambda_c)}{4(m-2)} + \frac{\sin((m-4)\lambda_c)}{16(m-4)} = 0\ , \qquad m = 5,6,7,\ldots\end{aligned} \tag{A34}$$

For $m = 0$,

$$\begin{aligned}&\int_0^{2\pi} \gamma_{a_x}(\cos^2\lambda_c + \sin^2\lambda_c) d\lambda_c = \gamma_{a_x}\left[\frac{\lambda_c}{2} + \frac{\sin 2\lambda_c}{2} + \frac{\lambda_c}{2} - \frac{\sin 2\lambda_c}{4}\right]_0^{2\pi} = \\ &= 2\pi\gamma_{a_x}\end{aligned} \tag{A35}$$

For $m = 1$,

$$\int_0^{2\pi} \gamma_{a_x}(\cos^2\lambda_c + \sin^2\lambda_c)\cos\lambda_c d\lambda_c = 0 \tag{A36}$$

For $m = 2$,

$$\begin{aligned}&\int_0^{2\pi} \gamma_{a_x}(\cos^2\lambda_c + \sin^2\lambda_c)\cos 2\lambda_c d\lambda_c = \\ &= \int_0^{2\pi} \gamma_{a_x}\cos^2\lambda_c \cos 2\lambda_c d\lambda_c + \int_0^{2\pi} \gamma_{a_x}\sin^2\lambda_c \cos 2\lambda_c d\lambda_c = \\ &= \gamma_{a_x}\left[\frac{\sin 4\lambda_c + 4(\sin 2\lambda_c + \lambda_c) - \sin 4\lambda_c + 4\sin 2\lambda_c - 4\lambda_c}{16}\right]_0^{2\pi} = 0\end{aligned} \tag{A37}$$

For $m = 0$, $\sin\lambda_c = 0$. Therein, for $m = 1$,

$$\int_0^{2\pi} \sin^3\lambda_c d\lambda_c = \left[\frac{\cos^3\lambda_c}{3} - \cos\lambda_c\right]_0^{2\pi} = 0 \tag{A38}$$

For $m = 2$,

$$\int_0^{2\pi} \sin^2\lambda_c \sin 2\lambda_c d\lambda_c = \left[\frac{\sin^4\lambda_c}{2}\right]_0^{2\pi} = 0 \tag{A39}$$

For $m = 1$,

$$\int_0^{2\pi} \sin^5\lambda_c d\lambda_c = \left[-\frac{\cos^5\lambda_c}{5} + \frac{2\cos^3\lambda_c}{3} - \cos\lambda_c\right]_0^{2\pi} = 0 \tag{A40}$$

For $m = 2$,

$$\int_0^{2\pi} \sin^4\lambda_c \sin 2\lambda_c d\lambda_c = \left[\frac{\sin^6\lambda_c}{3}\right]_0^{2\pi} = 0 \tag{A41}$$

For $m = 0$,

$$\int_0^{2\pi} \sin^2 \lambda_c \cos^2 \lambda_c d\lambda_c = \left[-\frac{\sin 4\lambda_c}{32} + \frac{4}{32}\lambda_c \right]_0^{2\pi} = \frac{\pi}{4} \tag{A42}$$

For $m = 1$,

$$\int_0^{2\pi} \sin^2 \lambda_c \cos^3 \lambda_c d\lambda_c = \left[\frac{\sin^3 \lambda_c}{3} - \frac{\sin^5 \lambda_c}{5} \right]_0^{2\pi} = 0 \tag{A43}$$

For $m = 2$,

$$\int_0^{2\pi} \sin^2 \lambda_c \cos^2 \lambda_c \cos 2\lambda_c d\lambda_c = \left[\frac{\sin^3 \lambda_c}{24} \right]_0^{2\pi} = 0 \tag{A44}$$

$$\begin{aligned} &\int_0^{2\pi} \sin^2 \lambda_c \cos^2 \lambda_c \cos m\lambda_c d\lambda_c = \\ &= -\frac{\sin((m+4)\lambda_c)}{16(m+4)} + \frac{\sin m\lambda_c}{8m} - \frac{\sin((m-4)\lambda_c)}{16(m-4)} = 0\,, \quad m = 3, 4, \ldots \end{aligned} \tag{A45}$$

For $m = 1$,

$$\int_0^{2\pi} \sin^3 \lambda_c \cos^2 \lambda_c d\lambda_c = \left[-\frac{\cos^3 \lambda_c}{3} + \frac{\cos^5 \lambda_c}{5} \right]_0^{2\pi} = 0 \tag{A46}$$

For $m = 2$,

$$\int_0^{2\pi} \sin^2 \lambda_c \cos^2 \lambda_c \sin 2\lambda_c d\lambda_c = \left[\frac{\cos 6\lambda_c - 9\cos 2\lambda_c}{96} \right]_0^{2\pi} = 0 \tag{A47}$$

$$\begin{aligned} &\int_0^{2\pi} \sin^2 \lambda_c \cos^2 \lambda_c \sin m\lambda_c d\lambda_c = \\ &= \frac{\cos((m+4)\lambda_c)}{16(m+4)} - \frac{\cos m\lambda_c}{8m} + \frac{\cos((m-4)\lambda_c)}{16(m-4)} = 0\,, \quad m = 3, 4, \ldots \end{aligned} \tag{A48}$$

$$\begin{aligned} \int_0^{2\pi} \sin^2 \lambda_c \sin m\lambda_c d\lambda_c = \frac{2m \sin m\lambda_c \sin \lambda_c \cos \lambda_c}{m^3 - 3m} + \\ + \frac{+(2 - m^2 \sin^2 \lambda_c) \cos m\lambda_c}{m^3 - 3m} = 0\,, \quad m = 1, 2, 3, \ldots \end{aligned} \tag{A49}$$

$$\begin{aligned} &\int_0^{2\pi} \sin^4 \lambda_c \sin m\lambda_c d\lambda_c = \frac{-\cos((m+4)\lambda_c)}{16(m+4)} + \frac{\cos((m+2)\lambda_c)}{4(m+2)} - \\ &- \frac{3\sin m\lambda_c}{8m} + \frac{\cos((m-2)\lambda_c)}{4(m-2)} - \frac{\cos((m-4)\lambda_c)}{16(m-4)} = 0\,, \quad m = 1, 2, 3, \ldots \end{aligned} \tag{A50}$$

In addition,

$$\int_0^{2\pi} \cos^2 m\lambda_c d\lambda_c = \left[\frac{\lambda_c}{2} + \frac{\sin 2m\lambda_c}{4m} \right]_0^{2\pi} = \pi \tag{A51}$$

$$\int_0^{2\pi} \cos^4 m\lambda_c d\lambda_c = \left[\frac{3\lambda_c}{8} + \frac{\sin 2m\lambda_c}{4m} + \frac{\sin 4m\lambda_c}{32m} \right]_0^{2\pi} = \frac{3\pi}{4} \tag{A52}$$

$$\int_0^{2\pi} \sin m\lambda_c \cos m\lambda_c d\lambda_c = \left[\frac{\sin^2 m\lambda_c}{2m} \right]_0^{2\pi} = 0 \tag{A53}$$

$$\int_0^{2\pi} \sin^3 m\lambda_c \cos m\lambda_c d\lambda_c = \left[\frac{\sin^4 m\lambda_c}{4m}\right]_0^{2\pi} = 0 \tag{A54}$$

$$\int_0^{2\pi} \sin^2 m\lambda_c d\lambda_c = \left[\frac{\lambda_c}{2} - \frac{\sin 2m\lambda_c}{4m}\right]_0^{2\pi} = \pi \ , \ m = 1, 2, 3, \ldots \tag{A55}$$

$$\int_0^{2\pi} \sin^4 m\lambda_c d\lambda_c = \left[\frac{12\lambda_c}{32} \frac{\sin 4m\lambda_c - 8\sin 2m\lambda_c}{32m}\right]_0^{2\pi} = \frac{3\pi}{4} \ , \ m = 1, 2, 3, \ldots \tag{A56}$$

$$\int_0^{2\pi} \sin m\lambda_c \cos m\lambda_c d\lambda_c = \left[\frac{\sin^2 m\lambda_c}{2m}\right]_0^{2\pi} = 0 \ , \ m = 1, 2, 3, \ldots \tag{A57}$$

$$\int_0^{2\pi} \sin m\lambda_c \cos^3 m\lambda_c d\lambda_c = \left[-\frac{\cos^4 m\lambda_c}{2m}\right]_0^{2\pi} = 0 \ , \ m = 1, 2, 3, \ldots \tag{A58}$$

The integrals above for $m = 0$ are equal to zero. Since all integrals for $m = 1, 3$ and for $m > 4$ are also zero, the non-zero corefficients are a_{00}, a_{20}, a_{22}, a_{40}, a_{42} and a_{44}. Therefore, for our level of approximation $n = 0, 2, 4$.

Appendix B

$$\int_0^{2\pi} \sin m\lambda_c = \left[-\frac{1}{m}\cos m\lambda_c\right]_0^{2\pi} = 0 \ , \ m \neq 0 \tag{A59}$$

$$\int_0^{2\pi} \sin^3 m\lambda_c d\lambda_c = \left[\frac{1}{3m}\cos^3 m\lambda_c - \frac{\cos m\lambda_c}{m}\right]_0^{2\pi} = 0 \ , \ m \neq 0 \tag{A60}$$

$$\int_0^{2\pi} \sin^3 \lambda_c d\lambda_c = \left[\frac{1}{3}\cos^3 \lambda_c - \cos \lambda_c\right]_0^{2\pi} = 0 \tag{A61}$$

$$\begin{aligned} &\int_0^{2\pi} \cos m\lambda \cos^2 \lambda_c d\lambda_c = \\ &= \left[\frac{(m^2\cos^2 \lambda_c - 2)\sin m\lambda_c - 2m\cos \lambda_c \sin \lambda_c \cos m\lambda_c}{m^3 - 4m}\right]_0^{2\pi} = 0 \ , \ m \neq 0 \end{aligned} \tag{A62}$$

$$\begin{aligned} &\int_0^{2\pi} \cos^3 m\lambda \cos^2 \lambda_c d\lambda_c = \left[\frac{\sin((3m+2)\lambda_c)}{16(3m+2)} + \frac{\sin((3m-2)\lambda_c)}{16(3m-2)} + \right. \\ &\left. + \frac{3\sin((m+2)\lambda_c)}{16(m+2)} + \frac{\sin 3m\lambda_c}{24m} + \frac{3\sin m\lambda_c}{8m} + \frac{3\sin((m-2)\lambda_c)}{16(m-2)}\right]_0^{2\pi} = 0 \ , \ m \neq 0 \end{aligned} \tag{A63}$$

$$\begin{aligned} &\int_0^{2\pi} \sin m\lambda_c \cos^2 \lambda_c d\lambda_c = \left[-\frac{2m\cos \lambda_c \sin \lambda_c \sin m\lambda_c}{m^3 - 4m} + \right. \\ &\left. + \frac{(m^2\cos^2 \lambda_c - 2)\cos m\lambda_c}{m^3 - 4m}\right]_0^{2\pi} = 0 \ , \ m \neq 0 \end{aligned} \tag{A64}$$

$$\int_0^{2\pi} \sin^3 m\lambda \cos^2 \lambda_c d\lambda_c = \left[\frac{\cos((3m+2)\lambda_c)}{16(3m+2)} + \frac{\cos((3m-2)\lambda_c)}{16(3m-2)} - \right.$$
$$\left. \frac{3\cos((m+2)\lambda_c)}{16(m+2)} + \frac{\cos 3m\lambda_c}{24m} - \frac{3\cos m\lambda_c}{8m} - \frac{3\cos((m-2)\lambda_c)}{16(m-2)} \right]_0^{2\pi} = 0 \ , \ m \neq 0 \tag{A65}$$

$$\int_0^{2\pi} \cos^2 \lambda_c d\lambda_c = \left[\frac{\cos \lambda_c \sin \lambda_c + \lambda_c}{2} \right]_0^{2\pi} = \frac{\pi}{2} \tag{A66}$$

The remaining integrals which appear in the right hand side of relation (64) are given in Appendix A. Since the eccentricities of the ellipsoid are very small, we make the following substitution,

$$\left(1 + \frac{e_x^2}{2-e_e^2} \cos^2 \theta' + \frac{e_e^2}{2-e_e^2} \cos^2 \theta' \cos^2 \lambda_c \right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} =$$
$$= 1 - \frac{1+\sqrt{9+4n(n+1)}}{2} \frac{e_x^2}{2-e_e^2} \cos^2 \theta' - \frac{1+\sqrt{9+4n(n+1)}}{2} \frac{e_e^2}{2-e_e^2} \cos^2 \theta' \cos^2 \lambda_c \tag{A67}$$

The right hand side of Equation (64) becomes

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{n_0} b^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \left(1 - \frac{e_e^2}{2}\right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \\
&\left(1 - \frac{1+\sqrt{9+4n(n+1)}}{2} \frac{e_x^2}{2-e_e^2} \cos^2 \theta' - \right. \\
&\left. - \frac{1+\sqrt{9+4n(n+1)}}{2} \frac{e_e^2}{2-e_e^2} \cos^2 \theta' \cos^2 \lambda_c \right) \\
&\sum_{m=0}^{n} \{ a_{nm} [P_{nm}(\sin\theta') \cos m\lambda_c - e_e^2 P_{nm}(\sin\theta') \cos m\lambda_c + \\
&\quad + e_e^2 P_{nm}(\sin\theta') \cos^3 m\lambda_c + P'_{nm}(\sin\theta') \cdot \\
&(e_x^2 \sin\theta' \cos^2\theta' + e_e^2 \sin^3\theta' - e_e^2 \sin\theta' \cos^2\theta' \sin^2\lambda_c) \cos m\lambda_c] \} + \\
&\quad + b_{nm} \{ [P_{nm}(\sin\theta') \sin m\lambda_c + e_e^2 P_{nm}(\sin\theta') \sin m\lambda_c - \\
&\quad - e_e^2 P_{nm}(\sin\theta') \sin^3 m\lambda_c + P'_{nm}(\sin\theta') \cdot \\
&(e_x^2 \sin\theta' \cos^2\theta' + e_e^2 \sin^3\theta' - e_e^2 \sin\theta' \cos^2\theta' \sin^2\lambda_c) \sin m\lambda_c] \} \} \cdot \\
&P_{jk}(\sin\theta') \cos m\lambda_c \cos\theta' d\theta' d\lambda_c
\end{aligned} \tag{A68}$$

The integrals of the form [20]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{nm}(\sin\theta') P_{jk}(\sin\theta') \cos\theta' d\theta' = \int_{-1}^{1} P_{nm}(x) P_{jk}(x) dx =$$
$$= \frac{2(n+m)!}{(2n+1)(n-m)!} \delta_{nj}, \ \ n = j \ , \ m = k \tag{A69}$$

are determined according to the rule of the orthogonality condition. The remaining integrals that appear on the right hand side of (64) and we have not yet addressed are of the following forms,

$$\int_{-1}^{1} xP_{jk}(x)P'_{nm}(x)dx \tag{A70}$$

and

$$\int_{-1}^{1} x^3P_{jk}(x)P'_{nm}(x)dx. \tag{A71}$$

The first integral [21] is equal to

$$\int_{-1}^{1} xP_{jk}(x)P'_{nm}(x)dx = \delta_{0m} - \frac{(n+m)!}{(2n+1)(n-m)!}\,,\ n=j\,,\ m=k. \tag{A72}$$

Using the recurrence properties of associated Legendre functions repeatedly,

$$(2j+1)xP_{jk}(x) = (j+k)P_{j-1k} + (j-k+1)P_{j+1k}(x) \tag{A73}$$

we have that

$$\begin{aligned} x[x^2P_{jk}(x)dx] &= \tfrac{(j+k)(j-1+k)}{(2j-1)(2j+1)}xP_{j-2k}(x) + \\ &+ \tfrac{(j+k)(j-k)}{(2j-1)(2j+1)}xP_{jk}(x) + \tfrac{(j-k+1)(j+1+k)}{(2j+3)(2j+1)}xP_{jk}(x) + \\ &+ \tfrac{(j-k+1)(j+2-k)}{(2j+3)(2j+1)}xP_{j+2k}(x) \end{aligned}\ . \tag{A74}$$

Hence Equation (A71) also involves the following integrals

$$\begin{aligned} &\int_{-1}^{1} xP_{j-2k}(x)P'_{nm}(x)dx \overset{l=1}{=} \int_{-1}^{1} xP_{j-2lk}(x)P'_{nm}(x)dx = \\ &= \delta_{0m} + \frac{(n-2l+m)!}{(n-2l-m)!} \overset{l=1}{=} \delta_{0m} + \frac{(n-3+m)!}{(n-3-m)!}\,,\ j=n,\ m=k \end{aligned}\ . \tag{A75}$$

$$\begin{aligned} &\int_{-1}^{1} xP_{j+2k}(x)P'_{nm}(x)dx \overset{l=1}{=} \int_{-1}^{1} xP_{j+2lk}(x)P'_{nm}(x)dx = \\ &= \delta_{0m} - \frac{(n+m)!}{(n-m)!},\ j=n,\ m=k \end{aligned} \tag{A76}$$

Finally, the derivatives of the associated Legendre functions [22] up to degree 4 for even values of n and m are needed, those being

$$\begin{aligned} P'_{00}(\sin\theta') &= 0 \\ P'_{20}(\sin\theta') &= 3\sin\theta' \\ P'_{22}(\sin\theta') &= -6\sin\theta' \\ P'_{40}(\sin\theta') &= \tfrac{140}{8}\sin^3\theta' - \tfrac{60}{8}\sin\theta' \\ P'_{42}(\sin\theta') &= -210\sin^3\theta' + 120\sin\theta' \\ P'_{44}(\sin\theta') &= -420\sin\theta'\cos^2\theta' \end{aligned} \tag{A77}$$

Appendix C

We are going to determine the leading term a_{00}. In addition, we will show the special cases of an oblate spheroid and of that of a sphere. The left hand side of the boundary condition is as follows.

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\gamma_{a_x}+$$
$$+\delta\gamma_x\cos^2\theta'\sin^2\lambda_c+\delta\gamma_x(-2e_x^2\cos^2\theta'\sin^2\lambda_c+2e_x^2\cos^4\theta'\sin^2\lambda_c+$$
$$+2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+2e_e^2\cos^4\theta'\sin^2\lambda_c\cos^2\lambda_c)-\frac{\delta\gamma_x}{2}e_e^2\cos^2\theta'\sin^2\lambda_c+ \quad (A78)$$
$$+\delta\gamma_z\sin^2\theta'+\delta\gamma_z(2e_x^2\sin^2\theta'\cos^2\theta'-2e_e^2\sin^2\theta'\cos^2\theta'\sin^2\lambda_c)-\frac{\delta\gamma_z e_x^2}{2}\sin^2\theta'+$$
$$+\frac{\delta\gamma_x e_e^2}{2}\cos^4\theta'\sin^4\lambda_c+\frac{\delta\gamma_x e_x^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+$$
$$+\frac{\delta\gamma_z e_e^2}{2}\sin^2\theta'\cos^2\theta'\sin^2\lambda_c+\frac{\delta\gamma_z e_x^2}{2}\sin^4\theta'\Big]\cos\theta' d\theta' d\lambda_c$$

We will determine the above integrals, and we will sum them.

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\gamma_{a_x}\cos\theta' d\theta' d\lambda_c = 4\pi\gamma_{a_x} \quad (A79)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\delta\gamma_x\cos^3\theta'\sin^2\lambda_c d\theta' d\lambda_c = \frac{4}{3}\pi\delta\gamma_x \quad (A80)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[(-2e_x^2)\delta\gamma_x\cos^3\theta'\sin^2\lambda_c d\theta' d\lambda_c = -\frac{8}{3}\pi\delta\gamma_x e_x^2 \quad (A81)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[2e_x^2\delta\gamma_x\cos^5\theta'\sin^2\lambda_c d\theta' d\lambda_c = \frac{32}{15}\pi\delta\gamma_x e_x^2 \quad (A82)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[2e_e^2\delta\gamma_x\sin^2\theta'\cos^3\theta'\sin^2\lambda_c d\theta' d\lambda_c = \frac{8\pi}{15}e_e^2\delta\gamma_x \quad (A83)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[2e_e^2\delta\gamma_x\cos^5\theta'\sin^2\lambda_c\cos^2\lambda_c d\theta' d\lambda_c = \frac{30\pi}{60}e_e^2\delta\gamma_x = \quad (A84)$$
$$= \frac{\pi}{2}e_e^2\delta\gamma_x$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[2e_e^2\delta\gamma_x\sin^2\theta'\cos^3\theta'\sin^2\lambda_c d\theta' d\lambda_c = \frac{8\pi}{15}e_e^2\delta\gamma_x \quad (A85)$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \delta\gamma_z \sin^2\theta' \cos\theta' d\theta' d\lambda_c = \frac{4\pi}{3}\delta\gamma_z \tag{A86}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e_x^2\delta\gamma_z \sin^2\theta' \cos^3\theta' d\theta' d\lambda_c = \frac{16\pi}{15}e_x^2\delta\gamma_z \tag{A87}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2e_e^2\delta\gamma_z \sin^2\theta' \cos^3\theta' \sin^2\lambda_c d\theta' d\lambda_c = -\frac{8\pi}{15}e_e^2\delta\gamma_z \tag{A88}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{\delta\gamma_z e_x^2}{2} \sin^2\theta' \cos\theta' d\theta' d\lambda_c = -\frac{2\pi}{3}\delta\gamma_z e_x^2 \tag{A89}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta\gamma_x e_e^2}{2} \cos^5\theta' \sin^4\lambda_c d\theta' d\lambda_c = \frac{6\pi}{15}\delta\gamma_x e_e^2 \tag{A90}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta\gamma_x e_x^2}{2} \sin^2\theta' \cos^3\theta' \sin^2\lambda_c d\theta' d\lambda_c = \frac{2\pi}{15}\delta\gamma_x e_x^2 \tag{A91}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta\gamma_z e_e^2}{2} \sin^2\theta' \cos^3\theta' \sin^2\lambda_c d\theta' d\lambda_c = \frac{2\pi}{15}\delta\gamma_z e_e^2 \tag{A92}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta\gamma_z e_x^2}{2} \sin^4\theta' \cos\theta' d\theta' d\lambda_c = \frac{2\pi}{5}\delta\gamma_z e_x^2 \tag{A93}$$

Therefore, the summation of the above results is

$$\begin{aligned}
&4\pi\gamma_{a_x} + \tfrac{4}{3}\pi\delta\gamma_x - \tfrac{8}{3}\pi\delta\gamma_x e_x^2 + \tfrac{32}{15}\pi\delta\gamma_x e_x^2 + \tfrac{8\pi}{15}e_e^2\delta\gamma_x + \tfrac{\pi}{2}e_e^2\delta\gamma_x + \tfrac{8\pi}{15}e_e^2\delta\gamma_x + \tfrac{4\pi}{3}\delta\gamma_z + \\
&+\tfrac{16\pi}{15}e_x^2\delta\gamma_z - \tfrac{8\pi}{15}e_e^2\delta\gamma_z - \tfrac{2\pi}{3}\delta\gamma_z e_x^2 + \tfrac{6\pi}{15}\delta\gamma_x e_e^2 + \tfrac{2\pi}{15}\delta\gamma_x e_x^2 + \tfrac{2\pi}{15}\delta\gamma_z e_e^2 + \tfrac{2\pi}{5}\delta\gamma_z e_x^2 = \\
&= 4\pi\gamma_{a_x} + \tfrac{4}{3}\pi\delta\gamma_x + \tfrac{4\pi}{3}\delta\gamma_z + \left(-\tfrac{8}{3}\pi e_x^2 + \tfrac{32}{15}\pi e_x^2 + \tfrac{16\pi}{15}e_e^2 + \tfrac{\pi}{2}e_e^2 + \tfrac{6\pi}{15}e_e^2 + \tfrac{2\pi}{15}e_x^2\right)\delta\gamma_x + \\
&+\left(\tfrac{16\pi}{15}e_x^2 - \tfrac{8\pi}{15}e_e^2 - \tfrac{2\pi}{3}e_x^2 + \tfrac{2\pi}{15}e_e^2 + \tfrac{2\pi}{5}e_x^2\right)\delta\gamma_z = \\
&= 4\pi\gamma_{a_x} + \tfrac{4}{3}\pi\delta\gamma_x + \tfrac{4\pi}{3}\delta\gamma_z + \left(-\tfrac{6\pi}{15}e_x^2 + \tfrac{59\pi}{30}e_e^2\right)\delta\gamma_x + \left(\tfrac{12\pi}{15}e_x^2 - \tfrac{6\pi}{15}e_e^2\right)\delta\gamma_z
\end{aligned} \tag{A94}$$

The right hand side of Equation (64) is

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{n=0}^{+\infty} b^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\left(1-\frac{e_e^2}{2}\right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\left(1+\frac{e_x^2}{2-e_e^2}\cos^2\theta'+\frac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c\right)^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}$$
$$\{\sum_{m=0}^{n}\{a_{nm}[P_{nm}(\sin\theta')\cos m\lambda_c - e_e^2 P_{nm}(\sin\theta')\cos m\lambda_c + e_e^2 P_{nm}(\sin\theta')\cos^3 m\lambda_c +$$
$$+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta' + e_e^2\sin^3\theta' - e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\cos m\lambda_c]\}+$$
$$+b_{nm}\{[P_{nm}(\sin\theta')\sin m\lambda_c + e_e^2 P_{nm}(\sin\theta')\sin m\lambda_c - e_e^2 P_{nm}(\sin\theta')\sin^3 m\lambda_c +$$
$$+P'_{nm}(\sin\theta')(e_x^2\sin\theta'\cos^2\theta' + e_e^2\sin^3\theta' - e_e^2\sin\theta'\cos^2\theta'\sin^2\lambda_c)\sin m\lambda_c]\}\}\cdot$$
$$P_{jk}(\sin\theta')\cos m\lambda_c\cos\theta' d\theta' d\lambda_c \tag{A95}$$

For $n = 0$ the above relation becomes

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a_{00}b^{-2}\left(1-\frac{e_e^2}{2}\right)^{-2}\left(1+\frac{e_x^2}{2-e_e^2}\cos^2\theta'+\frac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c\right)^{-2}\cos\theta' d\theta' d\lambda_c \tag{A96}$$

but

$$\left(1+\frac{e_x^2}{2-e_e^2}\cos^2\theta'+\frac{e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c\right)^{-2} = \\ = 1-\frac{2e_x^2}{2-e_e^2}\cos^2\theta'+\frac{2e_e^2}{2-e_e^2}\cos^2\theta'\cos^2\lambda_c \tag{A97}$$

putting

$$A_{00} = a_{00}b^{-2}\left(1-\frac{e_e^2}{2}\right)^{-2}, \tag{A98}$$

we end up to the following integrals

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\cos\theta' d\theta' d\lambda_c - \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\frac{2e_x^2}{2-e_e^2}\cos^3\theta' d\theta' d\lambda_c - \\ -\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\frac{2e_e^2}{2-e_e^2}\cos^3\theta'\cos^2\lambda_c d\theta' d\lambda_c \tag{A99}$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\cos\theta' d\theta' d\lambda_c = 4\pi A_{00} \tag{A100}$$

$$-\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\frac{2e_x^2}{2-e_e^2}\cos^3\theta' d\theta' d\lambda_c = -\frac{16\pi e_x^2}{3(2-e_e^2)}A_{00} \tag{A101}$$

$$-\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_{00}\frac{2e_e^2}{2-e_e^2}\cos^3\theta'\cos^2\lambda_c d\theta' d\lambda_c = -\frac{4\pi e_e^2}{3(2-e_e^2)}A_{00}. \tag{A102}$$

Hence,

$$4\pi\gamma_{a_x} + \frac{4}{3}\pi\delta\gamma_x + \frac{4\pi}{3}\delta\gamma_z + \left(-\frac{6\pi}{15}e_x^2 + \frac{59\pi}{30}e_e^2\right)\delta\gamma_x + \\ + \left(\frac{12\pi}{15}e_x^2 - \frac{6\pi}{15}e_e^2\right)\delta\gamma_z = \left[4\pi - \frac{16\pi e_x^2}{3(2-e_e^2)} - \frac{4\pi e_e^2}{3(2-e_e^2)}\right]A_{00} \quad . \tag{A103}$$

Using relation (A98) the leading term a_{00} is expressed as

$$a_{00} = \frac{b^2\left(1-\frac{e_e^2}{2}\right)^2\left[4\pi\gamma_{a_x} + \frac{4}{3}\pi\delta\gamma_x + \frac{4\pi}{3}\delta\gamma_z + \left(-\frac{6\pi}{15}e_x^2 + \frac{59\pi}{30}e_e^2\right)\delta\gamma_x + \left(\frac{12\pi}{15}e_x^2 - \frac{6\pi}{15}e_e^2\right)\delta\gamma_z\right]}{\left[4\pi - \frac{16\pi e_x^2}{3(2-e_e^2)} - \frac{4\pi e_e^2}{3(2-e_e^2)}\right]} \tag{A104}$$

or

$$a_{00} = \frac{b^2\left(1-\frac{e_e^2}{2}\right)^2\left[4\gamma_{a_x} + \frac{4}{3}\delta\gamma_x + \frac{4}{3}\delta\gamma_z + \left(-\frac{6}{15}e_x^2 + \frac{59}{30}e_e^2\right)\delta\gamma_x + \left(\frac{12}{15}e_x^2 - \frac{6}{15}e_e^2\right)\delta\gamma_z\right]}{\left[4 - \frac{16e_x^2}{3(2-e_e^2)} - \frac{4e_e^2}{3(2-e_e^2)}\right]} \tag{A105}$$

For the case of an oblate spheroid, we have $e_e = 0$, $\delta\gamma_x = 0$ ($e_x = e$) and the above relation becomes

$$a_{00} = \frac{b^2\left(4\gamma_{a_x} + \frac{4}{3}\delta\gamma_z + \frac{12}{15}e^2\delta\gamma_z\right)}{\left(4 - \frac{16e^2}{6}\right)} = \frac{b^2\left(\gamma_{a_x} + \frac{1}{3}\delta\gamma_z + \frac{1}{5}e^2\delta\gamma_z\right)}{\left(1 - \frac{2e^2}{3}\right)} \tag{A106}$$

Finally, for case of a sphere of radius $R = a_x$ ($a_x = a_y = b$) containing the Earth's mass M we have that

$$a_{00} = R^2\gamma_{a_x} = R^2\frac{GM}{R^2} \Rightarrow a_{00} = GM \tag{A107}$$

where G is the gravitational constant.

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