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# Investigating Monogenity in a Family of Cyclic Sextic Fields

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**Abstract:** Jones characterized, among others, monogenity of a family of cyclic sextic polynomials. Our purpose is to study monogenity of the family of corresponding sextic number fields. We show that several of these number fields are monogenic, despite the defining polynomial of their generating element being non-monogenic. In the monogenic fields, there are several inequivalent generators of power integral bases. Our calculation also provides the first non-trivial application of the method described earlier to study monogenity in totally real extensions of imaginary quadratic fields, emphasizing the efficiency of that algorithm.

**Keywords:** monogenity; power integral basis; sextic fields; relative cubic extension; quadratic subfield; Thue equations

**MSC:** 11Y50; 11R04; 11D25

## 1. Introduction

Monogenity and power integral bases are a classical topic in algebraic number theory, which is intensively studied even today. For the classical results, we refer to [1], and for several recent results to [2], as well as the references cited there, where more details can be found. Even in recent years, a large number of results have contributed to this area. Here, we briefly summarize the most important definitions.

A number field $K$ of degree $n$ with ring of integers $\mathbb{Z}_K$ is called *monogenic* (cf. [2]) if there exists $\xi \in \mathbb{Z}_K$ such that $(1, \xi, \dots, \xi^{n-1})$ is an integral basis, called a *power integral basis*. We call $\xi$ the *generator* of this power integral basis. Additionally, $\alpha, \beta \in \mathbb{Z}_K$ are called *equivalent* if $\alpha + \beta \in \mathbb{Z}$ or $\alpha - \beta \in \mathbb{Z}$. Obviously, $\alpha$ generates a power integral basis in $K$ if and only if any $\beta$, equivalent to $\alpha$, does. As is known, any algebraic number field admits, up to equivalence, only finitely many generators of power integral bases.

An irreducible polynomial $f(x) \in \mathbb{Z}[x]$ is called *monogenic* if a zero $\xi$ of $f(x)$ generates a power integral basis in $K = \mathbb{Q}(\xi)$. If $f(x)$ is monogenic, then $K$ is also monogenic, but the converse is not true.

For $\alpha \in \mathbb{Z}_K$ (generating $K$ over $\mathbb{Q}$), the module index

$$I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$$

is called the *index* of $\alpha$. The element $\alpha$ generates a power integral basis in $K$ if and only if $I(\alpha) = 1$. If $\alpha^{(i)}$ $(1 \le i \le n)$ are the conjugates of $\alpha$ in $K$ of degree $n$, then

$$I(\alpha) = \frac{1}{\sqrt{|D_K|}} \prod_{1 \le i < j \le n} |\alpha^{(i)} - \alpha^{(j)}|,$$

where $D_K$ is the discriminant of $K$. Searching for elements of $\mathbb{Z}_K$, generating power integral bases, leads to a Diophantine equation. Let $(1, \omega_2, \dots, \omega_n)$ be an integral basis of $K$, and let

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$L^{(i)}(x_2,\dots,x_n) = x_2\omega_2^{(i)} + \dots + x_n\omega_n^{(i)}$ be the conjugates of the linear form $L(x_2,\dots,x_n) = x_2\omega_2 + \dots + x_n\omega_n$ $(1 \le i \le n)$. The polynomial

$$I(x_2,\dots,x_n) = \frac{1}{\sqrt{|D_K|}} \prod_{1\le i<j\le n} (L^{(i)}(x_2,\dots,x_n) - L^{(j)}(x_2,\dots,x_n))$$

has rational integer coefficients and degree $n(n-1)/2$. It is called the *index form* corresponding to the integral basis $(1,\omega_2,\dots,\omega_n)$. Obviously, $\gamma = x_1 + \omega_2 x_2 + \dots + \omega_n x_n \in \mathbb{Z}_K$ generates a power integral basis in $K$ if and only if $(x_2,\dots,x_n) \in \mathbb{Z}^{n-1}$ is a solution of the *index form equation*

$$I(x_2,\dots,x_n) = \pm 1$$

(independently of $x_1$).

Recently, several authors have applied the method of Newton polygons and the Dedekind criterion, which can be used to prove the monogenity or non-monogenity of polynomials and number fields. It is also an important problem to explicitly determine all inequivalent generators of power integral bases of a number field. The resolution of the index form equations requires numerical algorithms. Up to now, we only have general efficient algorithms for determining all inequivalent generators of power integral bases only in cubic, quartic, and special types of sextic and octic number fields (cf. [2]). The reason is that in these cases, the relevant index form equations lead to Thue equations or relative Thue equations, which can be easily solved, at least when their coefficients are moderate.

It may take an extremely long time to run the algorithms for the "complete resolution" of index form equations. There are also some very fast methods to determine generators of power integral bases with "small" coefficients, say, $<10^{100}$ in absolute value, with respect to an integral basis. These solutions cover all solutions with high probability, certainly all generators that can be used in practice for further calculations. It is usual to apply such algorithms also if we need to solve a large number of equations.

The general method for the complete resolutions of index form equations in sextic fields (cf. [2]) requires a very long running time. However, for some special types of sextic fields, we have developed very efficient methods. One of the most interesting cases is represented by composites of real cubic and imaginary quadratic fields [3]. In this paper, we give the first non-trivial application of that method to a parametric family of cyclic sextic fields, studied by L. Jones [4].

For further results on the monogenity of cyclic sextic fields, we refer to [5–7].

Let $n \in \mathbb{Z}$ and consider

$$f(x) = x^6 + (n^2+5)x^4 + (n^2+2n+6)x^2 + 1. \tag{1}$$

Let $\xi$ be a zero of $f(x)$ and set $K = \mathbb{Q}(\xi)$ with ring of integers $\mathbb{Z}_K$ and discriminant $D_K$.

The purpose of the present paper is to describe the results of our extensive calculations, which we performed to find generators of power integral bases in the number fields $K$, for a wide range of parameters $n$, using the algorithm of [3]. We prove that in the case that $f(x)$ is monogenic, there are often further inequivalent generators of power integral bases of $K$ in addition to $\xi$. Moreover, there are parameters $n$, for which $f(x)$ is not monogenic, but $K$ is monogenic.

## 2. The Family of Cyclic Sextic Fields

Jones [4] studied the cyclic sextic polynomials $f(x)$. Among other points, he proved the following:

**Lemma 1.** *$f(x)$ is irreducible for all $n \in \mathbb{Z}$ and is monogenic exactly for $n = -2, -1, 0, 1$.*

Let

$$g(x) = x^3 + (n^2+5)x^2 + (n^2+2n+6)x + 1. \tag{2}$$

Denote by $\alpha$ a zero of $g(x)$, and let $L = \mathbb{Q}(\alpha)$ with ring of integers $\mathbb{Z}_L$ and discriminant $D_L$. We have $\xi^2 = \alpha$; therefore, $L$ is a subfield of $K$.

**Lemma 2.** *$K$ is a composite of the totally real cubic number field $L$ and the imaginary quadratic field $M = \mathbb{Q}(i)$, where the discriminants of $L$ and $M$ are coprime.*

**Proof.** Substituting $x = y - (n^2+5)/3$ into $g(x)$ and writing the resulting polynomial in the form $h(y) = y^3 + py + q$, we obtain

$$D(h) = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = -\frac{1}{108}(n^2+n+7)^2(n^2+n-1)^2 < 0,$$

where the zeros $\alpha_1, \alpha_2, \alpha_3$ of $g(x)$ are real numbers: $L$ is a totally real cubic number field.

Set

$$\zeta = (-n^2 - 2n^2 - 3n + 3)\xi + (-n^3 + n^2 - 5n + 4)\xi^3 + (-n+1)\xi^5.$$

A Maple calculation implies

$$\zeta^2 = (-1)(n^2+n-1)^2,$$

where $\zeta = i \cdot (n^2+n-1)$. Hence, $M = \mathbb{Q}(i)$ is also a subfield of $K$.

An integral basis of $M$ is $(1, i)$, and the discriminant of $M$ is $D_M = -4$. The discriminant of $g(x)$ is

$$D(g) = (n^2+n+7)^2(n^2+n-1)^2,$$

which is always an odd number. $D_L$ is a divisor of $D(g)$; hence, $D_L$ is also odd. These imply that $(D_M, D_L) = 1$. □

As a consequence of $(D_M, D_L) = 1$, we obtain:

**Lemma 3.** *If*

$$(1, \beta_2, \beta_3)$$

*is an integral basis of $L$, then*

$$(1, \beta_2, \beta_3, i, i\beta_2, i\beta_3)$$

*is an integral basis of $K$.*

The properties of $K$ allow us to apply the method described in [3].

Denote by $\alpha^{(j)}$ $(j = 1, 2, 3)$ the conjugates of $\alpha$, and let $\beta_2^{(j)}, \beta_3^{(j)}$ be the conjugates of $\beta_2, \beta_3$, respectively, corresponding to $\alpha^{(j)}$.

Let

$$\gamma = x_1 + x_2\beta_2 + x_3\beta_3 + iy_1 + iy_2\beta_2 + iy_3\beta_3 \in \mathbb{Z}_K \tag{3}$$

be arbitrary with $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$. Then,

$$\gamma^{(1,j)} = x_1 + x_2\beta_2^{(j)} + x_3\beta_3^{(j)} + iy_1 + iy_2\beta_2^{(j)} + iy_3\beta_3^{(j)}$$

and

$$\gamma^{(2,j)} = x_1 + x_2\beta_2^{(j)} + x_3\beta_3^{(j)} - iy_1 - iy_2\beta_2^{(j)} - iy_3\beta_3^{(j)}$$

($j = 1, 2, 3$) are the conjugates of $\gamma$.

## 3. Auxiliary Results

We formulate the following general results of Theorem 1.6 of [2] for our special case of a sextic field $K$ being a composite of a totally real cubic field $L$ and an imaginary quadratic subfield $M = \mathbb{Q}(i)$.

**Lemma 4.** *For $\gamma \in \mathbb{Z}_K$ generating $K$ over $\mathbb{Q}$, we have*

$$I(\gamma) = I_{K/M}(\gamma) \cdot J(\gamma)$$

*where*

$$I_{K/M}(\gamma) = (\mathbb{Z}_K : \mathbb{Z}_M[\gamma]) = \frac{1}{\sqrt{|N_{M/\mathbb{Q}}(D_{K/M})|}} \prod_{i=1}^{2} \prod_{1 \le j_1 < j_2 \le 3} |\gamma^{(i,j_1)} - \gamma^{(i,j_2)}|$$

*is the relative index of $\gamma$ and*

$$J(\gamma) = \frac{1}{|D_M|^{3/2}} \prod_{j_1=1}^{3} \prod_{j_2=1}^{3} |\gamma^{(1,j_1)} - \gamma^{(2,j_2)}|.$$

Note that both the relative index and $J(\gamma)$ are integers. If $\gamma$ generates a power integral basis in $K$, that is $I(\gamma) = 1$, then $I_{K/M}(\gamma) = 1$ and $J(\gamma) = 1$. Let $I_L(x_2, x_3) \in \mathbb{Z}[x, y]$ be the index form corresponding to the integral basis $(1, \beta_2, \beta_3)$ of $L$.

In [3], we showed that in our special case:

**Lemma 5.** *If $I_{K/M}(\gamma) = 1$, then*

$$|I_L(x_2, x_2)| \le 1, \quad |I_L(y_2, y_3)| \le 1. \tag{4}$$

This is the main power of the method described in [3]. The relative index form equation $I_{K/M}(\gamma) = 1$ (in our case, a cubic relative Thue equation over the quadratic subfield $M$, cf. [2]) implies absolute index form equations in $L$, which are cubic Thue equations. Moreover, in the special case of our number field $K$, $J(\gamma)$ also factorizes:

**Lemma 6.** *If $J(\gamma) = 1$, then*

$$N_{L/Q}(y_1 + \beta_2 y_2 + \beta_3 y_3) = \pm 1 \tag{5}$$

*and*

$$P(\gamma) = \prod_{\substack{1 \le j_1, j_2 \le 3 \\ j_1 \ne j_2}} |\gamma^{(1,j_1)} - \gamma^{(2,j_2)}| = \pm 1. \tag{6}$$

**Proof.** In our case, $|D_M| = 4$, and

$$\frac{1}{2^3} \prod_{j=1}^{3} (\gamma^{(1,j)} - \gamma^{(2,j)})$$

is a symmetric polynomial with rational integer coefficients, equal to $N_{L/Q}(y_1 + \beta_2 y_2 + \beta_3 y_3)$. The corresponding factor of $J(\gamma)$ is $P(\gamma)$, also having rational integer coefficients. □

## 4. The Algorithm

In view of the above statements, in order to determine all non-equivalent generators of power integral bases of $K$, we perform the following steps for each parameter value $n$:

1. Calculate an integer basis $(1, \beta_2, \beta_3)$ of $L$.
2. Solve $I_L(x_2, x_3) = \pm 1$. Let $H$ be the set of solutions $(x_2, x_3)$.
3. Let $H_0 = H \cup \{(0,0)\}$.
4. For all $(y_1, y_2) \in H_0$, calculate the corresponding $y_1$. Let $H_1$ be the set of possible triples $(y_1, y_2, y_3)$.
5. For all $(x_1, x_2) \in H_0$ and for all $(y_1, y_2, y_3) \in H_1$, construct $\gamma$ (cf. (3)) and test if $I_{K/M}(\gamma) = 1$ and $P(\gamma) = 1$ hold.

We performed all calculations in Maple (version 2020.1). The integral basis in Step 1 was also calculated by Maple.

In Step 2, we calculated the solutions $(x_2, x_3)$ of

$$I_L(x_2, x_3) = \pm 1, \quad x_2, x_3 \in \mathbb{Z}, \text{ with } |x_2|, |x_3| \leq 10^{100}. \tag{7}$$

For larger parameters, the coefficients of $I_L$ become extremely large; therefore, the complete resolution of (7) would have been very time consuming.

In Step 4, for given $(y_2, y_3)$, we calculated $y_1$ using Equation (5), which is then a polynomial equation in $y_1$ with integer coefficients.

Step 5 is necessary to select the solutions from the set of possible solutions.

As an example for Equation (7), for $n = 140$, we provide here the integral basis $(1, x, (x^2 + 58941x + 118925)/138173)$ of $L$ and the corresponding index form equation:

$$I(x_2, x_3) = -138173x_2^3 - 137613x_2^2x_3 - 44758x_2x_3^2 - 4777x_3^3 = \pm 1.$$

The usual way to solve such equations is to multiply by the square of the leading coefficient $c = 138173$ and to substitute $x_2 = z/c$ in order to have a leading coefficient 1 in the new variable $z$. We obtain

$$-z^3 - 137613\, z^2\, x_3 - 6184347134\, z\, x_3^2 - 91201423166833\, x_3^3 = \pm 19091777929,$$

which clearly shows the difficulties in solving these equations for larger parameters.

We performed two series of explicit numerical calculations.

**A.** $-100 \leq n \leq 100$.
Calculating the solutions of (7) with $-100 \leq n \leq 100$ took about 30 min, out of which the calculation for the interval $-50 \leq n \leq 50$ took only 1.5 min. This shows how the large coefficients slow down the calculations.

**B.** $n \in S$, where

$$S = \{n \,:\, n \in [-1000, -100) \cup (100, 1000],\ \ n^2 + n + 7 \text{ square free}\}.$$

The set $S$ contains 1110 parameters $n$. The reason to consider this set is that for all $n \in S$, we have the same type of integer basis. Hence, we can write Equation (7) in a parametric form, and we can also perform Step 4 and Step 5 in a parametric form. It took 39 min to find the solution of $I(x_2, x_3) = \pm 1$ with $|x_2|, |x_3| \leq 10^{100}$ for all the 1110 parameters $n \in S$.

## 5. Results

For the set **A** of parameters $n$, our explicit calculations imply:

**Theorem 1.** *$n = -56, -14, -7, -5, -2, -1, 0, 1, 4, 6, 13$ are the only values of $n$ with $-100 \leq n \leq 100$, such that $K$ admits generators of power integral bases with coefficients $\leq 10^{100}$ in the integral basis.*

This statement was proved by explicit numerical calculations, following the above algorithm. All data of generators of power integral bases in the monogenic fields are listed in Section 6.

Theorem 1 should be compared with Lemma 1 of Jones, stating that the polynomials $f(x)$ are only monogenic for $n = -2, -1, 0, 1$. The table in Section 6 also shows that all these monogenic fields admit several inequivalent generators of power integral bases.

For the set **B** of parameters $n$, we only calculated the solutions of Equation (7) using Maple, and the remaining calculations were made in a parametric form.

**Theorem 2.** *For $n \in S$, there are no generators $\gamma$ (cf. (3)) of power integral bases of $K$ with coefficients $x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$ having absolute values $< 10^{100}$.*

**Remark 1.** *We cannot exclude the existence of monogenic fields $K$ for $|n| > 100$, but they are certainly not of type S.*

**Proof.** Our explicit calculations showed that for all $n \in S$, an integral basis of $L$ is given by

$$(1, \alpha, \beta),$$

where

$$\beta = \frac{\alpha^2 + (n^2 - n + 3)\alpha + n^2}{n^2 + n - 1}.$$

The index form of $L$ corresponding to this integral basis is

$$I_L(x_2, x_3) = (n^2 + n - 1)x_2^2 + (n^2 - 3n - 1)x_2^2 x_3 + (2 - 2n)x_2 x_3^2 + x_3^3.$$

Let

$$T := \{(0, 0), (\pm 1, \pm n), (0, \pm 1), (\pm 1, \pm(n - 1))\}. \tag{8}$$

It is easy to check whether $(x_2, x_3) \in T$ are solutions of $|I(x_2, x_3)| \leq 1$. For $n \in S$, we did not find any further solutions of (7).

For $n \in S$, $(0, 0) \neq (y_2, y_3) \in T$, there exists no corresponding $y_1$. This can be shown by explicit calculations using symmetric polynomials. For example, for $y_2 = 1, y_3 = n$, the left-hand side of (5) is

$$N = y_1^3 + (2n - 5)y_1^2 + (n^2 - 7n + 6)y_1 - 2n^2 + 4n - 1.$$

We have

$$N + 1 = (ny_1 + y_1^2 - 2n - 3y_1)(y_1 + n - 2)$$

$$N - 1 = (y_1 + n - 1)(ny_1 + y_1^2 - 2n - 4y_1 + 2)$$

The above second-degree factors are non-zero for $n \in S$. There remains $y_1 = 2 - n, 1 - n$ to test. For these triplets $(y_1, y_2, y_3)$ and for all $(x_2, x_3) \in T$, we calculated $P(\gamma)$, again using symmetric polynomials. These $P(\gamma)$ are polynomials of $n$ with integer coefficients, for which neither $P(\gamma) + 1$ nor $P(\gamma) - 1$ has integer zeros in $n$. The other possible non-zero pairs $(y_2, y_3)$ were considered similarly.

For $(y_2, y_2) = (0, 0)$, we obviously have $y_1 = \pm 1$. For $(y_1, y_2, y_3) = (\pm 1, 0, 0)$, we again tested all $(x_2, x_3) \in T$ and found that neither $P(\gamma) + 1$ nor $P(\gamma) - 1$ has integer zeros in $n$. □

## 6. Table

Here, we list the values of $n$ with $-100 \leq n \leq 100$ for which we found generators $\gamma$ of power integral bases of $K$ with coefficients $< 10^{100}$ in absolute value in the integral bases. We display $n$ and the integral basis $(1, \beta_1, \beta_2)$ of $L$. We display the coefficients $(x_2, x_3, y_1, y_2, y_3)$ of non-equivalent generators $\gamma$ (cf. (3)) of power integral bases with respect to the integral basis $(1, \beta_1, \beta_2, i, i\beta_1, i\beta_2)$.

$n = -56$, integral basis of $L$: $\left(1, \frac{x+4}{7}, \frac{x^2+9353x+18531}{150871}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,9,-49,170), (0,0,10,-49,170), (0,0,-4,17,-59)$,
$(0,0,-3,17,-59), (0,0,-5,32,-111), (0,0,-4,32,-111)$

$n = -14$, integral basis of $L$: $\left(1, \frac{x+1}{3}, \frac{x^2+32x+1282}{1629}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,12,-5,-16), (0,0,13,-5,-16), (0,0,11,-4,-13)$,
$(0,0,12,-4,-13), (0,0,-25,9,29), (0,0,-24,9,29)$

$n = -7$, integral basis of $L$: $\left(1, x, \frac{x^2+141x+254}{287}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-19,-7,23), (0,0,-18,-7,23), (0,0,7,3,-10)$,
$(0,0,8,3,-10), (0,0,11,4,-13), (0,0,12,4,-13)$

$n = -5$, integral basis of $L$: $\left(1, x, \frac{x^2+14x+25}{57}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,0,-2,-7), (0,0,1,-2,-7), (0,0,1,-1,-4)$,
$(0,0,2,-1,-4), (0,0,-4,3,11), (0,0,-3,3,11)$

$n = -2$, integral basis of $L$: $(1, x, x^2)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-5,-17,-2), (0,0,-4,-17,-2), (0,0,-4,-9,-1)$,
$(0,0,-3,-9,-1), (0,0,9,26,3), (0,0,10,26,3)$

$n = -1$, integral basis of $L$: $(1, x, x^2)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-4,-6,-1), (0,0,-3,-6,-1), (0,0,-2,-5,-1)$,
$(0,0,-1,-5,-1), (0,0,4,11,2), (0,0,5,11,2)$

$n = 0$, integral basis of $L$: $(1, x, x^2)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-1,-3,-1), (0,0,0,-3,-1), (0,0,-2,-1,0)$,
$(0,0,-1,-1,0), (0,0,2,4,1), (0,0,3,4,1)$

$n = 1$, integral basis of $L$: $(1, x, x^2)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-1,-3,-1), (0,0,0,-3,-1), (0,0,-3,-1,0)$,
$(0,0,-2,-1,0), (0,0,1,4,1), (0,0,2,4,1)$

$n = 4$, integral basis of $L$: $\left(1, x, \frac{x^2+53x+16}{57}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,0,-4,7), (0,0,1,-4,7), (0,0,-1,1,-2)$,
$(0,0,0,1,-2), (0,0,1,3,-5), (0,0,2,3,-5)$

$n = 6$, integral basis of $L$: $\left(1, x, \frac{x^2+74x+200}{287}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,-11,-2,17), (0,0,-10,-2,17), (0,0,4,1,-9)$,
$(0,0,5,1,-9), (0,0,5,1,-8), (0,0,6,1,-8)$

$n = 13$, integral basis of $L$: $\left(1, x, \frac{x^2+521x+169}{1629}\right)$
$(x_2, x_3, y_1, y_2, y_3) = (0,0,4,-16,25), (0,0,5,-16,25), (0,0,-3,7,-11)$,
$(0,0,-2,7,-11), (0,0,-2,9,-14), (0,0,-1,9,-14)$

## 7. Conclusions

Our numerical results show that in this family of cyclic sextic fields, there are several monogenic fields despite the defining polynomial of their generating element being nonmonogenic. In these monogenic fields, we found several inequivalent generators of power integral bases. The calculations demonstrated the efficiency of the method of [3], which allowed one to reduce the resolution of index form equations in sextic fields to the resolution of index form equations in cubic fields. This yields a resolution of cubic Thue equations, which is very fast, as long as the coefficients are moderate. The question whether there exist infinitely many monogenic cyclic sextic fields remains open.

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