

Article

Statistical Computation of Hjorth Competing Risks Using Binomial Removals in Adaptive Progressive Type II Censoring

Refah Alotaibi ¹, Mazen Nassar ^{2,3} and Ahmed Elshahhat ^{4,*}

¹ Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; rmalotaibi@pnu.edu.sa

² Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; mmohamad3@kau.edu.sa

³ Department of Statistics, Faculty of Commerce, Zagazig University, Zagazig 44519, Egypt

⁴ Faculty of Technology and Development, Zagazig University, Zagazig 44519, Egypt

* Correspondence: aelshahhat@ftd.zu.edu.eg

Abstract: In complex reliability applications, it is common for the failure of an individual or an item to be attributed to multiple causes known as competing risks. This paper explores the estimation of the Hjorth competing risks model based on an adaptive progressive Type II censoring scheme via a binomial removal mechanism. For parameter and reliability metric estimation, both frequentist and Bayesian methodologies are developed. Maximum likelihood estimates for the Hjorth parameters are computed numerically due to their intricate form, while the binomial removal parameter is derived explicitly. Confidence intervals are constructed using asymptotic approximations. Within the Bayesian paradigm, gamma priors are assigned to the Hjorth parameters and a beta prior for the binomial parameter, facilitating posterior analysis. Markov Chain Monte Carlo techniques yield Bayesian estimates and credible intervals for parameters and reliability measures. The performance of the proposed methods is compared using Monte Carlo simulations. Finally, to illustrate the practical applicability of the proposed methodology, two real-world competing risk data sets are analyzed: one representing the breaking strength of jute fibers and the other representing the failure modes of electrical appliances.



Academic Editor: Yuhlong Lio

Received: 10 May 2025

Revised: 11 June 2025

Accepted: 17 June 2025

Published: 18 June 2025

Citation: Alotaibi, R.; Nassar, M.; Elshahhat, A. Statistical Computation of Hjorth Competing Risks Using Binomial Removals in Adaptive Progressive Type II Censoring. *Mathematics* **2025**, *13*, 2010. <https://doi.org/10.3390/math13122010>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: reliability; Hjorth competing risks; censoring; binomial mechanism; Bayesian; Markov chains; data modeling

MSC: 62F10; 62F15; 62N01; 62N02; 62N05

1. Introduction

In various research fields, especially in medicine and engineering, unit failures often result from multiple competing causes. The data collected in such scenarios are known as competing risk data. Analyzing this type of data is essential for understanding the influence of one failure cause while accounting for the presence of others. Traditional methods typically focus only on failures attributed to a single cause, treating other failure types as censored data. However, this approach overlooks critical information about the underlying failure mechanisms, potentially leading to inaccurate reliability and survival estimates which are the key goals of these analyses. In studies involving competing risks, data typically comprise two key elements: the time until failure for the units under observation and a categorical indicator that identifies the specific cause of failure. Our methodology in this work adopts the latent failure time framework introduced by Cox [1], which assumes that

the different causes of failure are statistically independent. Within this field, researchers have predominantly applied various probability models to describe observed lifetime data. The primary focus of research has been parametric estimation methods, particularly employing maximum likelihood and Bayesian approaches. In contrast, reliability assessment and survival analysis have received relatively less attention in comparison. Several studies in this field include contributions by Kundu and Basu [2], Park [3], Wang [4], Park and Wang [5], and Alok et al. [6], among others.

A principal challenge in the analysis of competing risk data lies in selecting an appropriate statistical probability distribution for modeling such data. The statistical literature offers a wide array of models designed for various data types across multiple disciplines. This study focuses on the Hjorth (Hj) distribution, a relatively underexplored model in the context of competing risks, introduced by Hjorth [7]. The Hj distribution stands out for its remarkable flexibility, as it can accommodate four distinct hazard rate patterns: constant, increasing, decreasing, and bathtub-shaped. Despite being defined by only a single scale parameter and one shape parameter, it provides greater modeling versatility compared to other two-parameter distributions, such as the Weibull and gamma models. As noted by Hjorth [7], the reliability function (RF) of the Hj distribution can be expressed as the product of the RFs of the Rayleigh and Lomax distributions. This unique property allows the Hj distribution to serve as a natural candidate for modeling competing risks scenarios. Despite its demonstrated adaptability, there is limited research exploring the application of the Hj distribution in reliability and survival analysis, particularly in estimating its unknown parameters or related reliability metrics, like RF and hazard rate function (HRF). Notable exceptions include the works of Yadav et al. [8] and Elshahhat and Nassar [9]. The detailed description of the Hj distribution within the context of the competing risks model is discussed in the following section, which includes the definition of several structural functions, such as the probability density function (PDF), cumulative distribution function (CDF), RF, and HRF.

Another challenge researchers encounter when analyzing competing risk data is deciding whether to collect data on the failure of all units in the sample or to terminate the test before all units fail. In practice, gathering complete data on all tested units can be time-consuming and costly, particularly for highly reliable products which are now more common due to technological advancements. In such cases, researchers often opt for censoring sampling methods, which help reduce both costs and experimental duration. In the literature, various censoring schemes are available, each with its own unique features and advantages. One widely used censoring plan is the progressive Type II censoring (PTIIC) scheme, which allows researchers to remove some surviving units from the test according to a pre-specified censoring plan. More detailed information regarding the PTIIC can be found in Balakrishnan and Aggarwala [10], Balakrishnan and Hossain [11], Wu and Gui [12], and Baklizi [13]. A more flexible and efficient censoring plan was introduced by Ng et al. [14], designed to enhance the performance of statistical inference methods while ensuring that the test concludes with a fixed number of failures. This approach is known as the adaptive PTIIC (APTIIC) plan. Under the APTIIC framework, researchers can adjust the experimental design by changing the removal of test units once a specific time point is reached. This plan provides greater adaptability in handling real-world constraints. The APTIIC scheme has garnered significant attention in the literature, with numerous researchers exploring its applications; for instance, Haj Ahmad et al. [15], Panahi and Asadi [16], Lv et al. [17], and Sharma and Kumar [18]. Later, Anakha and Chacko [19] suggested a joint APTIIC plan.

The APTIIC plan typically assumes a fixed progressive censoring scheme throughout the testing process. Nonetheless, in certain real-world contexts, the removal of units may

occur randomly rather than adhering to a predetermined pattern. Yuen and Tse [20] emphasized that in some studies, researchers might determine it is inappropriate or unsafe to continue monitoring specific units, even if those units have not yet failed. In such instances, units are randomly removed following each failure. To model random removal patterns, three approaches have been employed in the literature: discrete uniform removals, binomial removals, and beta-binomial removals. In this study, we chose to model random removal using the binomial distribution for several important reasons. First, it naturally represents the process of a random number of removals occurring over a fixed number of independent trials, which aligns with the assumption that each item has an equal and independent chance of being removed. Second, the binomial distribution is simple and interpretable, requiring only one parameter, the probability of removal, which is easy to estimate and explain. In contrast, alternatives like the beta-binomial distribution introduce additional parameters, increasing model complexity. Similarly, the discrete uniform distribution assumes equal probability across all possible removal counts, which does not reflect realistic variability. For further details, see Singh et al. [21] and Elbatal et al. [22]. The random removal strategy is frequently considered more practical than fixed-pattern censoring, particularly within the framework of competing risks models. This approach has been employed in studies conducted by Chacko and Mohan [23] and Qin and Gui [24]. Recently, Elshahhat and Nassar [25] expanded the APTIIC framework by integrating binomial removals and investigated the estimation challenges associated with the Weibull lifetime model. Additionally, Alqasem et al. [26] analyzed the Burr-X competing risks model under the APTIIC scheme with binomial removals, further illustrating the applicability of this methodology in reliability analysis.

In this study, we utilize the APTIIC plan with a binomial removal pattern to analyze the H_j distribution within the context of a competing risks model. To the best of our knowledge, no prior research has explored the H_j distribution under the framework of competing risks models, despite its potential applicability in modeling real-world scenarios. Our work can be justified based on the following points:

1. The H_j distribution is highly significant in modeling diverse types of data due to its flexible HRF and its broad applicability across various fields.
2. The APTIIC strategy with binomial removals is more suitable for data collection compared to the traditional fixed removal approach, as it better accommodates random patterns in real-world data.
3. The use of the competing risks model is crucial for analyzing data involving multiple failure causes, which cannot be effectively addressed using conventional methods.

These considerations highlight the novelty and practical relevance of our proposed approach. The primary objective of this paper is to explore the estimation of the H_j competing risks model under the APTIIC scheme with binomial removals. To achieve this, we employ two estimation approaches: maximum likelihood estimation and Bayesian estimation. The analysis encompasses both the model parameters and two key reliability metrics, namely the RF and HRF. First, we derive the maximum likelihood estimates (MLEs) and construct approximate confidence intervals (ACIs). Additionally, we compute Bayes estimates (BEs) and Bayesian credible intervals (BCIs). The Bayesian estimation is performed using the squared error (SE) loss function, along with the Markov Chain Monte Carlo (MCMC) technique for computational implementation. To evaluate the performance of the proposed methods, we conduct a simulation study to compare the MLEs, BEs, ACIs, and BCIs. Finally, we validate the proposed estimation techniques by applying them to real-world data sets.

The remainder of this study is structured as follows: Section 2 presents the derivation of the H_j competing risks model and discusses the APTIIC methodology in conjunction

with binomial removals. Section 3 elaborates on the MLEs and ACIs for the parameters of the Hj competing risks model, as well as the associated reliability metrics. In Section 4, the BEs and their corresponding BCIs are examined. Section 5 outlines the simulation study conducted as part of this research. Section 6 analyzes two real data sets, while the paper concludes with Section 7.

2. Model Description and Notation

This section presents the model and the structure of the available data. Suppose n identical units are tested starting at time zero, with their lifetimes denoted by $X_1, \dots, X_n$. Under the assumption that failures can occur due to two distinct causes, it is assumed for $i = 1, \dots, n$ that $X_i = \min(X_{i1}, X_{i2})$, where X_i represents the observed failure time of the i th unit. Here, X_{ik} corresponds to the latent failure time of the i th unit due to the k th cause of failure ($k = 1, 2$). Additionally, it is assumed that X_{i1} and X_{i2} are independently distributed, with each following the Hj distribution. The PDF and CDF of the Hj distribution with scale parameter μ_k and shape parameter $\lambda_k, k = 1, 2$, can be expressed, respectively, as

$$g_k(x; \mu_k, \lambda_k) = (1+x)^{-(\lambda_k+1)} [\lambda_k + \mu_k x (1+x)] e^{-0.5\mu_k x^2}, x > 0, \mu_k, \lambda_k > 0, \quad (1)$$

and

$$G_k(x; \mu_k, \lambda_k) = 1 - (1+x)^{-\lambda_k} e^{-0.5\mu_k x^2}. \quad (2)$$

The RF and HRF of $X_i = \min(X_{i1}, X_{i2})$ can be simply derived, respectively, as

$$S(t; \boldsymbol{\psi}) = \prod_{k=1}^2 (1+t)^{-\lambda_k} e^{-0.5\mu_k t^2}, \quad (3)$$

and

$$h(t; \boldsymbol{\psi}) = \sum_{k=1}^2 [\mu_k t + \lambda_k (1+t)^{-1}], \quad (4)$$

where $\boldsymbol{\psi} = (\mu_1, \mu_2, \lambda_1, \lambda_2)^\top$ is the full parameter vector.

The APTIIC methodology under competing risks, incorporating binomial removals, operates as follows: We let $m (< n)$ denote the predetermined target number of failures and let $\mathbf{R} = (R_1, R_2, \dots, R_m)$ represent the pre-specified removal scheme established prior to experimentation. A time threshold $T \in (0, \infty)$ is defined as the intended maximum test duration, though the experiment may continue beyond T if necessary. At the occurrence of the first failure time $X_{1:m:n}$, R_1 surviving units are randomly removed from the remaining non-failed items. This procedure repeats iteratively: For the i th failure time $X_{i:m:n}$, R_i surviving units are eliminated randomly from the remaining units. In this case, we have two termination scenarios:

1. Early Termination: If the m th failure occurs before T (i.e., $X_{m:m:n} < T$), the experiment concludes at $X_{m:m:n}$, adhering to conventional PTIIC.
2. Time-Exceeded Adaptation: If $X_{m:m:n} > T$, the removal scheme is modified: All scheduled removals after the d th failure (where d denotes the number of failures observed before T) are suspended by setting $R_i = 0$ for $i = d+1, \dots, m-1$. Post-adjustment, no further removals occur until the m th failure time. At this stage, all remaining surviving units ($R^* = n - m - \sum_{i=1}^d R_i$) are removed.

The resulting APTIIC sample under competing risks consists of the following: (1) Failure times $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$, (2) Corresponding removal counts $\mathbf{R} = (R_1, R_2, \dots, R_d, 0, \dots, 0, R^*)$, and (3) Competing risk indicator $\delta_i, i = 1, \dots, m$ for each observed failure.

We let $x_i = x_{i:m:n}$ for simplicity, $\underline{x} = (x_1, \dots, x_d, x_{d+1}, \dots, x_m)$ is an observed APTIIC with competing risks sample obtained from the Hj population, and $I(\delta_i = k)$ refers to an indicator function such that

$$I(\delta_i = k) = \begin{cases} 1, & \delta_i = k \\ 0 & \text{else} \end{cases}.$$

Then, after ignoring the constant term, we can write the joint likelihood function of the observed data as follows:

$$\begin{aligned} L_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r}) &= \prod_{i=1}^m \{g_1(x_i)[1 - G_2(x_i)]\}^{I(\delta_i=1)} \{g_2(x_i)[1 - G_1(x_i)]\}^{I(\delta_i=2)} \\ &\times \prod_{i=1}^d \{[1 - G_1(x_i)][1 - G_2(x_i)]\}^{R_i} \{[1 - G_1(x_m)][1 - G_2(x_m)]\}^{R^*}, \end{aligned} \quad (5)$$

with $m_k = \sum_{i=1}^m I(\delta_i = k)$, for $k = 1, 2$, represents the count of observed failures attributed to cause k , with the total number of failures given by $m = m_1 + m_2$. We choose the APTIIC scheme over conventional censoring methods for several compelling reasons. First, the APTIIC dynamically adjusts the censoring process based on observed data, allowing the experiment to conclude near a predetermined threshold time. This is especially valuable for high-reliability products where failures are rare, as it prevents unnecessarily long test durations. Second, if failures occur more slowly than expected, the APTIIC halts the removal of remaining units after the threshold time and continues until the required number of failures is observed. Unlike the progressive Type I hybrid censoring scheme proposed by Kundu and Joarder [27], which may result in insufficient sample sizes when failures are infrequent, APTIIC guarantees the collection of the pre-specified number of failures. This ensures more reliable statistical inference by minimizing the risk of invalid or inefficient estimates due to inadequate data.

According to the methodology of Elshahhat and Nassar [25], the removal process R_i at the i th failure stage is modeled using a binomial distribution with parameter θ . In this framework, each surviving unit at the time of the i th failure has an identical probability θ of being removed from the experiment. The probability mass function (PMF) for the random removal R_i is defined by two parameters: (1) $n_i^* = n - m - \sum_{j=1}^{i-1} r_j$: The number of surviving units available just before the i th failure, adjusted for prior removals. (2) θ : The constant removal probability for each surviving unit. In this case, we can write the PMF of R_i as

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \dots, R_1 = r_1) = \binom{n_i^*}{r_i} \theta^{r_i} (1 - \theta)^{n - m - \sum_{k=1}^i r_k}, \quad (6)$$

where $0 \leq r_i \leq n_i^*$ for $i = 1, 2, \dots, d$. Accordingly, one can easily write the joint probability of $R_i = r_i$, $i = 1, 2, \dots, d$, as

$$L_2(\mathbf{R} = \mathbf{r}; \theta) = P(R_1 = r_1)P(R_2 = r_2 | R_1 = r_1) \times \dots \times P(R_d = r_d | R_{d-1} = r_{d-1}, \dots, R_1 = r_1). \quad (7)$$

It follows from (6) and (7) that

$$L_2(\mathbf{R} = \mathbf{r}; \theta) = \left[\prod_{i=1}^d \binom{n_i^*}{r_i} \right] \theta^{\sum_{i=1}^d r_i} (1 - \theta)^{d(n-m) - \sum_{i=1}^d (d-i+1)r_i}. \quad (8)$$

This binomial model ensures that the removal process accounts for the decreasing number of surviving units as the experiment progresses. From (5) and (8), it is evident that the binomial parameter is independent of the Hj competing risks model. Therefore,

as demonstrated in the following section, we separately derive the classical and Bayesian estimations for these two types of parameters.

3. Maximum Likelihood Estimation

We have an observed APTIIC sample $\underline{x}$ from a population subject to competing risks under the Hj model with removal counts $(R_1, \dots, R_d)$ generated via a binomial process. Then, the likelihood function for the unknown parameters $\boldsymbol{\psi}$ can be constructed as follows using the PDF defined in (1), the CDF in (2), and the likelihood formulation in (5), excluding constant terms:

$$L_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r}) = \exp \left\{ \sum_{k=1}^2 \sum_{i=1}^{m_k} \log[\lambda_k + \mu_k x_i (1 + x_i)] - (\lambda_1 + \lambda_2) W_1 - (\mu_1 + \mu_2) W_2 \right\}, \quad (9)$$

where

$$W_1 = \sum_{i=1}^m \log(1 + x_i) + \sum_{i=1}^d r_i \log(1 + x_i) + r^* \log(1 + x_m)$$

and

$$W_2 = 0.5 \left(\sum_{i=1}^m x_i^2 + \sum_{i=1}^d r_i x_i^2 + r^* x_m^2 \right).$$

The log likelihood function of (9) takes the following form:

$$l_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r}) = \sum_{k=1}^2 \sum_{i=1}^{m_k} \log[\lambda_k + \mu_k x_i (1 + x_i)] - (\lambda_1 + \lambda_2) W_1 - (\mu_1 + \mu_2) W_2. \quad (10)$$

Taking the first-order partial derivatives of the objective function defined in (10) with respect to the parameters results in a system of normal equations, these equations must be solved simultaneously. They are given by

$$\frac{\partial l_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_1} = \sum_{i=1}^{m_1} \frac{x_i(1 + x_i)}{\lambda_1 + \mu_1 x_i(1 + x_i)} - W_2 = 0, \quad (11)$$

$$\frac{\partial l_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_2} = \sum_{i=1}^{m_2} \frac{x_i(1 + x_i)}{\lambda_2 + \mu_2 x_i(1 + x_i)} - W_2 = 0, \quad (12)$$

$$\frac{\partial l_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r})}{\partial \lambda_1} = \sum_{i=1}^{m_1} \frac{1}{\lambda_1 + \mu_1 x_i(1 + x_i)} - W_1 = 0 \quad (13)$$

and

$$\frac{\partial l_1(\boldsymbol{\psi}; \underline{x} | \mathbf{R} = \mathbf{r})}{\partial \lambda_2} = \sum_{i=1}^{m_2} \frac{1}{\lambda_2 + \mu_2 x_i(1 + x_i)} - W_1 = 0. \quad (14)$$

From the preceding expressions, it is evident that the normal equations for the Hj competing risks model parameters do not yield a closed-form solution. Consequently, the MLEs of the parameters cannot be derived analytically. To address this issue, numerical approximation techniques, such as the Newton–Raphson method, may be employed. In this study, the Newton–Raphson method was utilized to compute the required MLEs. On the other hand, the MLE of the binomial parameter θ can be readily derived by maximizing the

likelihood function in (8). The log-likelihood function corresponding to (8), excluding the constant term, can be written as

$$l_2(\mathbf{R} = \mathbf{r}; \theta) = \log(\theta) \sum_{i=1}^d r_i + \log(1 - \theta) \left[d(n - m) - \sum_{i=1}^d (d - i + 1)r_i \right], \quad (15)$$

where $v_i = d - i + 1$. By maximizing (15) with respect to θ , one can easily obtain the MLE of θ as

$$\hat{\theta} = \frac{\sum_{i=1}^d r_i}{d(n - m) - \sum_{i=1}^d (v_i - 1)r_i}.$$

Once the MLEs of the parameters are acquired by simultaneously solving the aforementioned normal equations, the MLEs of the RF and the HRF can be computed utilizing the invariance property of the MLEs. We let $\hat{\mu}_k$ and $\hat{\lambda}_k, k = 1, 2$, represent the MLEs of the parameters μ_k and λ_k , respectively. It is evident that the RF and HRF of the Hj competing risks model are independent of the binomial parameter. Consequently, the MLEs of these functions are exclusively determined by the MLEs of the parameters μ_k and λ_k for $k = 1, 2$. Consequently, the MLEs of the RF and the HRF for a mission time $t > 0$, are given, respectively, by

$$\hat{S}(t) = \prod_{k=1}^2 (1 + t)^{-\hat{\lambda}_k} e^{-0.5\hat{\mu}_k t^2}$$

and

$$\hat{h}(t) = \sum_{k=1}^2 [\hat{\mu}_k t + \hat{\lambda}_k (1 + t)^{-1}].$$

An essential aspect of statistical inference is the construction of interval estimates for both model parameters, binomial parameter, and reliability indices. To this end, the Fisher information matrix is derived, enabling the construction of $100(1 - \alpha)\%$ confidence intervals. However, due to the analytical complexity often associated with obtaining the Fisher information matrix, the observed Fisher information matrix is commonly used in practice as an estimate of the variance–covariance structure. For the Hj competing risks model, this is achieved by inverting the observed Fisher information matrix as

$$\begin{aligned} \mathcal{I}^{-1}(\hat{\psi}) &= \begin{bmatrix} -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_1^2} & 0 & -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_1 \partial \lambda_1} & 0 \\ -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_2^2} & 0 & -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \mu_2 \partial \lambda_2} & 0 \\ -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \lambda_1^2} & 0 & -\frac{\partial^2 l_1(\psi; \mathbf{x} | \mathbf{R} = \mathbf{r})}{\partial \lambda_2^2} & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \hat{\sigma}_{11} & 0 & \hat{\sigma}_{13} & 0 \\ & \hat{\sigma}_{22} & 0 & \hat{\sigma}_{24} \\ & & \hat{\sigma}_{33} & 0 \\ & & & \hat{\sigma}_{44} \end{bmatrix}, \end{aligned} \quad (16)$$

where $\hat{\psi} = (\hat{\mu}_1, \hat{\mu}_2, \hat{\lambda}_1, \hat{\lambda}_2)^\top$ is the MLE of ψ and

$$\begin{aligned} \frac{\partial^2 l_1}{\partial \mu_1^2} &= -\sum_{i=1}^{m_1} \frac{[x_i(1 + x_i)]^2}{[\lambda_1 + \mu_1 x_i(1 + x_i)]^2}, & \frac{\partial^2 l_1}{\partial \mu_2^2} &= -\sum_{i=1}^{m_2} \frac{[x_i(1 + x_i)]^2}{[\lambda_2 + \mu_2 x_i(1 + x_i)]^2}, \\ \frac{\partial^2 l_1}{\partial \lambda_1^2} &= -\sum_{i=1}^{m_1} \frac{1}{[\lambda_1 + \mu_1 x_i(1 + x_i)]^2}, & \frac{\partial^2 l_1}{\partial \lambda_2^2} &= -\sum_{i=1}^{m_2} \frac{1}{[\lambda_2 + \mu_2 x_i(1 + x_i)]^2}, \end{aligned}$$

$$\frac{\partial^2 l_1}{\partial \mu_1 \partial \lambda_1} = - \sum_{i=1}^{m_1} \frac{x_i(1+x_i)}{[\lambda_1 + \mu_1 x_i(1+x_i)]^2}, \text{ and } \frac{\partial^2 l_1}{\partial \mu_2 \partial \lambda_2} = - \sum_{i=1}^{m_2} \frac{x_i(1+x_i)}{[\lambda_2 + \mu_2 x_i(1+x_i)]^2}.$$

Under the asymptotic theory of maximum likelihood estimation, the estimator $\hat{\psi}$ follows a multivariate normal distribution with mean ψ and estimated variance–covariance matrix $\mathcal{I}^{-1}(\hat{\psi})$ as presented in (16). Formally, $\hat{\psi} \xrightarrow{d} \mathcal{N}(\psi, \mathcal{I}^{-1}(\hat{\psi}))$, as $n \rightarrow \infty$. Thus, the $100(1 - \alpha)\%$ ACIs for the parameters μ_k and λ_k , ($k = 1, 2$) are derived as

$$\hat{\mu}_k \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{kk}}, \text{ and } \hat{\lambda}_k \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{pp}}, p = k + 2, k = 1, 2,$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution and $\hat{\sigma}_{jj}$, $j = 1, \dots, 4$ denotes the diagonal elements of $\mathcal{I}^{-1}(\hat{\psi})$ presented in (16). Regarding the binomial parameter, we first need to estimate the variance of the MLE of θ as follows:

$$\hat{\sigma}_{\theta} = \left[- \frac{d^2 l_2(\mathbf{R} = \mathbf{r}; \theta)}{d\theta^2} \right]_{\theta=\hat{\theta}}^{-1},$$

where

$$\frac{d^2 l_2(\mathbf{R} = \mathbf{r}; \theta)}{d\theta^2} = - \frac{\sum_{i=1}^d r_i}{\theta^2} - \frac{d(n-m) - \sum_{i=1}^d (d-i+1)r_i}{(1-\theta)^2}.$$

Then, the $100(1 - \alpha)\%$ ACI for θ follows

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{\theta}}.$$

To construct the $100(1 - \alpha)\%$ ACIs for the RF and HRF, it is necessary to estimate the variances of their respective MLEs. These variances are commonly approximated via the delta method. Prior to applying this method, the gradient vectors of the RF and HRF must first be derived using their expressions defined in (3) and (4) as

$$\hat{\Delta}_S = -\hat{S}(t) \left[0.5t^2, \log(1+t), 0.5t^2, \log(1+t) \right]^{\top},$$

and

$$\hat{\Delta}_h = \left[t, (1+t)^{-1}, t, (1+t)^{-1} \right]^{\top}.$$

Then, the approximated variances can be obtained as

$$\hat{\sigma}_S \approx \hat{\Delta}_S^{\top} \mathcal{I}^{-1}(\hat{\psi}) \hat{\Delta}_S \text{ and } \hat{\sigma}_h \approx \hat{\Delta}_h^{\top} \mathcal{I}^{-1}(\hat{\psi}) \hat{\Delta}_h.$$

Accordingly, the $100(1 - \alpha)\%$ ACIs for the RF and HRF are

$$\hat{S}(t) \pm z_{\alpha/2} \sqrt{\hat{\sigma}_S} \text{ and } \hat{h}(t) \pm z_{\alpha/2} \sqrt{\hat{\sigma}_h}.$$

4. Bayesian Estimation

This section derives Bayes estimators for the parameters and reliability characteristics of the H_j competing risks model under APTIIC data as well as the binomial parameter. In Bayesian point estimation, selecting an appropriate loss function is a key consideration. In this study, the symmetric squared error loss function is utilized, but other loss functions can also be seamlessly integrated. In Bayesian inference, selecting appropriate prior distributions is critical. For the H_j distribution, conjugate priors are not available. Therefore, independent gamma priors are assigned to the H_j parameters μ_k and λ_k ($k = 1, 2$). The gamma distribution is widely used as a prior in Bayesian analysis because of its mathematical convenience and flexibility in modeling positive continuous parameters.

Its shape and scale parameters allow it to represent a wide range of beliefs, from highly skewed to nearly symmetric distributions, while still ensuring analytical tractability in posterior computations. The gamma distribution also offers computational advantages which facilitate efficient sampling in MCMC algorithms. Its robustness and versatility make it particularly suitable for applications in reliability and survival analysis, reinforcing its status as a practical and reliable prior choice in Bayesian frameworks. We assume that $\mu_k \sim \text{Gamma}(a_k, b_k)$ and $\lambda_k \sim \text{Gamma}(c_k, d_k)$, ($k = 1, 2$). Then, the joint gamma prior distribution for the parameters μ_k and λ_k is defined as

$$\pi_1(\boldsymbol{\psi}) \propto \prod_{k=1}^2 \mu_k^{a_k-1} \lambda_k^{c_k-1} e^{-b_k \mu_k - d_k \lambda_k}, \mu_k, \lambda_k > 0, \quad (17)$$

where $a_k, b_k > 0$ and $c_k, d_k > 0$ are hyper-parameters. Gamma priors offer flexibility, encompassing diverse shapes that can approximate other distributions. Noninformative priors emerge as a special case when hyper-parameters a_k, b_k, c_k, d_k approach zero.

In relation to the binomial parameter θ , examining the likelihood in (8) reveals that the natural conjugate prior distribution for θ is the beta distribution. We assume that $\varepsilon_1, \varepsilon_2 > 0$ represent the hyper-parameters of the beta prior distribution. Then, the prior distribution of the parameter θ can be expressed in the following form:

$$\pi_2(\theta) \propto \theta^{\varepsilon_1-1} (1-\theta)^{\varepsilon_2-1}, \quad 0 < \theta < 1. \quad (18)$$

To derive the Bayes estimators for the Hj competing risks model and its associated reliability metrics, we first determine the posterior distribution of $\boldsymbol{\psi}$. This is achieved by combining the observed data, represented by the likelihood function in (9), with the joint prior distribution given in (17). The posterior distribution is then obtained by applying Bayes' rule as follows:

$$\begin{aligned} H_1(\boldsymbol{\psi} | \underline{x}, \mathbf{R} = \mathbf{r}) &= A_1 \left(\prod_{k=1}^2 \mu_k^{a_k-1} \lambda_k^{c_k-1} \right) \exp \left\{ \sum_{k=1}^2 \sum_{i=1}^{m_k} \log[\lambda_k + \mu_k x_i (1+x_i)] \right. \\ &\quad \left. - \sum_{k=1}^2 \lambda_k (d_k + W_1) - \sum_{k=1}^2 \mu_k (b_k + W_2) \right\}, \end{aligned} \quad (19)$$

where A_1 refers to the normalized constant. On the other hand, by combining (8) and (18) and applying Bayes' rule, the posterior distribution of the binomial parameter can be derived as

$$H_2(\theta | \mathbf{R} = \mathbf{r}) = \frac{1}{B(\varepsilon_1^*, \varepsilon_2^*)} \theta^{\varepsilon_1^*-1} (1-\theta)^{\varepsilon_2^*-1}, \quad (20)$$

where $B(\cdot, \cdot)$ is the beta function, $\varepsilon_1^* = \sum_{i=1}^d r_i + \varepsilon_1$, and $\varepsilon_2^* = d(n-m) - \sum_{i=1}^d (d-i+1)r_i + \varepsilon_2$.

Owing to the straightforward and well-known form of the posterior distribution of the parameter θ , which follows the beta distribution with shape parameters ε_1^* and ε_2^* , we begin by deriving the Bayes estimator and the BCI for θ . Under the squared error loss function, the Bayes estimator of θ is expressed as

$$\begin{aligned} \tilde{\theta} &= \int_0^1 \theta H_2(\theta | \mathbf{R} = \mathbf{r}) d\theta \\ &= \frac{\varepsilon_1^*}{\varepsilon_1^* + \varepsilon_2^*}. \end{aligned} \quad (21)$$

In addition, the $100(1 - \alpha)\%$ BCI of θ can be obtained using the posterior distribution in (20) as

$$[B(\varepsilon_1^*, \varepsilon_2^*, \alpha/2), B(\varepsilon_1^*, \varepsilon_2^*, 1 - \alpha/2)],$$

where $B(., ., .)$ is the quantile function of the beta distribution.

Regarding the Bayes estimators of the Hj competing risks model, we let $\varphi(\boldsymbol{\psi})$ be any function of the model parameters. Then, by employing the squared error loss function, the Bayes estimator of $\varphi(\boldsymbol{\psi})$ can be obtained using the posterior distribution in (19) as

$$\tilde{\varphi}(\boldsymbol{\psi}) = \int_0^\infty \varphi(\boldsymbol{\psi}) H_1(\boldsymbol{\psi} | \underline{\mathbf{x}}, \mathbf{R} = \mathbf{r}) d\boldsymbol{\psi}. \quad (22)$$

The Bayesian estimator defined in (22) requires solving integrals that are analytically intractable, primarily due to the high-dimensional and non-conjugate nature of the posterior distribution. To address this issue, we utilize the MCMC methods to directly sample from the posterior distribution outlined in (19). These samples are then used to approximate posterior expectations for Bayes estimates and to construct the BCIs. A key step in implementing MCMC involves deriving the full conditional posterior distributions for each parameter. From the joint posterior in (19), the conditional distributions for μ_k and λ_k ($k = 1, 2$) are as follows:

$$H_1(\mu_k | \lambda_k, \underline{\mathbf{x}}, \mathbf{R} = \mathbf{r}) \propto \mu_k^{a_k-1} \exp \left\{ \sum_{i=1}^{m_k} \log[\lambda_k + \mu_k x_i(1 + x_i)] - \mu_k(b_k + W_2) \right\} \quad (23)$$

and

$$H_1(\lambda_k | \mu_k, \underline{\mathbf{x}}, \mathbf{R} = \mathbf{r}) \propto \lambda_k^{c_k-1} \exp \left\{ \sum_{i=1}^{m_k} \log[\lambda_k + \mu_k x_i(1 + x_i)] - \lambda_k(d_k + W_1) \right\}. \quad (24)$$

Upon examining the conditional distributions in (23) and (24), it is clear that these distributions do not simplify to any standard forms. As a result, Gibbs sampling cannot be applied in this case. Consequently, we propose the use of the Metropolis–Hastings (MH) algorithm to sample from these distributions, utilizing a normal distribution as the proposal distribution. The MH algorithm is a well-established MCMC procedure for sampling from complex probability distributions. This approach generates candidate samples using a proposal distribution, which are then evaluated for acceptance based on their probability under the target posterior. In this analysis, the proposal distribution is a normal distribution, initialized with the MLEs of the parameters as its mean and variance. The algorithm iteratively proposes new parameter values from this distribution and computes their acceptance probabilities, ensuring convergence of the Markov chain to the target posterior distribution. Once sufficient samples are generated, it becomes feasible to compute the Bayes estimates and BCIs for μ_k and λ_k , $k = 1, 2$, as well as the reliability metrics. The MCMC sampling process follows the steps outlined below.

1. Put the initial values of $\mu_k^{(0)}$ and $\lambda_k^{(0)}$; here, we use the MLEs as starting values.
2. Put $j = 1$.
3. Perform the following MH steps to simulate $\mu_k^{(j)}$, with $k = 1, 2$:
 - (a) Generate $\mu_k^* \sim N(\hat{\mu}_k, \hat{\sigma}_{kk})$, where $\hat{\mu}_k$ is the MLE of μ_k and $\hat{\sigma}_{kk}$ is the estimated variance of $\hat{\sigma}_{kk}$ obtained from (16).

(b) Obtain the acceptance probability (AP):

$$AP_{\mu_k} = \min \left(1, \frac{H_1(\mu_k^* | \lambda^{(j-1)}, \underline{x})}{H_1(\mu_k^{(j-1)} | \lambda_k^{(j-1)}, \underline{x})} \right), k = 1, 2.$$

(c) Generate a random number u from the unit uniform distribution.

(d) Set the generated value μ_k^* as $\mu_k^{(j)}$ if $u \leq AP_{\mu_k}$, and $\mu_k^{(j)} = \mu_k^{(j-1)}$ if $u > AP_{\mu_k}$.

4. Repeat Step 3 to obtain $\lambda_k^{(j)}$ from (24).

5. Compute the RF and HRF at iteration j as

$$S^{(j)} = \prod_{k=1}^2 (1+t)^{-\lambda_k^{(j)}} e^{-0.5\mu_k^{(j)} t^2} \text{ and } h^{(j)} = \sum_{k=1}^2 (\mu_k^{(j)} t + \lambda_k^{(j)} (1+t)^{-1}).$$

6. Replace j by $j+1$.

7. Repeat Steps 3 to 6, $\mathcal{M}$ times.

8. Store the acquired sequence as

$$\{\mu_1^{(j)}, \mu_2^{(j)}, \lambda_1^{(j)}, \lambda_2^{(j)}, S^{(j)}, h^{(j)}\}, \quad j = 1, \dots, \mathcal{M}. \quad (25)$$

Under the squared error loss function, the Bayes estimates for the parameters of the H_j competing risks model, as well as the two reliability metrics, are calculated as the means of their respective MCMC sample values generated in (25). To ensure convergence and reduce the influence of initialization bias, the initial $\mathcal{B}$ iterations are discarded as a burn-in period. The Bayes estimates in this context are given as follows:

$$\tilde{\mu}_k = \frac{1}{\mathcal{M} - \mathcal{B}} \sum_{j=\mathcal{B}+1}^{\mathcal{M}} \mu_k^{(j)}, \quad \tilde{\lambda}_k = \frac{1}{\mathcal{M} - \mathcal{B}} \sum_{j=\mathcal{B}+1}^{\mathcal{M}} \lambda_k^{(j)}, \quad k = 1, 2,$$

$$\tilde{S}(t) = \frac{1}{\mathcal{M} - \mathcal{B}} \sum_{j=\mathcal{B}+1}^{\mathcal{M}} S^{(j)}, \text{ and } \tilde{h}(t) = \frac{1}{\mathcal{M} - \mathcal{B}} \sum_{j=\mathcal{B}+1}^{\mathcal{M}} h^{(j)}.$$

To compute the BCIs of, we first sort the acquired MCMC sequence in (25) as $\mu_k^{[\mathcal{B}+1]} < \dots < \mu_k^{[\mathcal{M}]}$, $\lambda_k^{[\mathcal{B}+1]} < \dots < \lambda_k^{[\mathcal{M}]}$, $S^{[\mathcal{B}+1]} < \dots < S^{[\mathcal{M}]}$ and $h^{[\mathcal{B}+1]} < \dots < h^{[\mathcal{M}]}$. Accordingly, the $100(1 - \alpha)\%$ BCIs are as follows:

$$\left\{ \mu_k^{[\alpha(\mathcal{M}-\mathcal{B})/2]}, \mu_k^{[(1-\alpha/2)(\mathcal{M}-\mathcal{B})]} \right\}, \left\{ \lambda_k^{[\alpha(\mathcal{M}-\mathcal{B})/2]}, \lambda_k^{[(1-\alpha/2)(\mathcal{M}-\mathcal{B})]} \right\}, \quad k = 1, 2,$$

$$\left\{ S^{[\alpha(\mathcal{M}-\mathcal{B})/2]}, S^{[(1-\alpha/2)(\mathcal{M}-\mathcal{B})]} \right\}, \text{ and } \left\{ h^{[\alpha(\mathcal{M}-\mathcal{B})/2]}, h^{[(1-\alpha/2)(\mathcal{M}-\mathcal{B})]} \right\}.$$

5. Monte Carlo Simulations

To assess the performance of the theoretical results—specifically the point and interval estimates of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ —we generate multiple adaptive Type II progressive hybrid censored datasets with binomial removals based on the H_j lifetime distribution under a competing risks setup.

5.1. Simulation Design

By assigning $(\lambda_1, \lambda_2) = (1.5, 0.5)$ and $(\mu_1, \mu_2) = (0.5, 1.5)$ in the H_j competing risks model, we simulate 1000 APTIIC datasets under a competing risks framework with bi-

nomial removals. These datasets are gathered across various configurations of binomial probability θ ($=0.4, 0.8$), threshold time T ($=0.3, 0.5$), total number of test units n ($=30, 50, 80$), and different effective sample sizes m for each n . At a distinct time $t = 0.1$, the corresponding actual values of $S(t)$ and $\bar{S}(t)$ are taken as 0.8182 and 0.8546, respectively. To perform the removal mechanism of the survived items r_i during the APTIIC strategy, perform the steps presented in Algorithm 1.

Algorithm 1 The APTIIC Generation Steps:

Step 1: Set the values of λ_i and μ_i (for $i = 1, 2$).

Step 2: Set the binomial probability value θ .

Step 3: Generate r_1 from $Bin(n - m, \theta)$.

Step 4: Generate r_i as

$$r_i \sim Bin\left(n - m - \sum_{v=1}^{i-1} r_v, \theta\right), \quad \text{for } i = \begin{cases} 2, 3, \dots, m-1 & \text{for } X_{m:m:n} < T; \\ 2, 3, \dots, d & \text{for } X_{m:m:n} > T. \end{cases}$$

Step 5: Find r^* according to the following relation:

$$r^* = \begin{cases} n - m - \sum_{v=1}^{m-1} r_v, & \text{if } \left(n - m - \sum_{v=1}^{m-1} r_v\right) > 0, \quad \text{for } X_{m:m:n} < T; \\ n - m - \sum_{v=1}^d r_v, & \text{if } \left(n - m - \sum_{v=1}^d r_v\right) > 0, \quad \text{for } X_{m:m:n} > T; \\ 0 & \text{for otherwise.} \end{cases}$$

Step 6: Given $\mathbf{R} = \mathbf{r}$, find a PTIIC sample with binomial removals as

(a) Generate Ω from $U(0, 1)$.

(b) Set $q_i = \Omega_i^{(i + \sum_{v=m-i+1}^m r_v)^{-1}}$, $i = 1, 2, \dots, m$.

(c) Set $U_i = 1 - q_m q_{m-1} \cdots q_{m-i+1}$ for $i = 1, 2, \dots, m$.

(d) Set $x_i = G^{-1}(u_i; \lambda_v)$, $i = 1, 2, \dots, m$, $v = 1, 2$.

Step 7: For x_i for $i = 1, 2, \dots, m$, determine d where $x_d < T < x_{d+1}$, and discard x_i for $i = d + 2, \dots, m$.

Step 8: From $\frac{g(x)}{S(x_{d+1})}$, obtain the first-order statistics $(x_{d+2}, \dots, x_m)$ with size $n - d - \sum_{v=1}^d r_v - 1$.

Step 9: For x_i for $i = 1, 2, \dots, m$, assign the cause of failure $c_i \in (1, 2)$, $i = 1, \dots, m$.

The idea of setting prior hyper-parameters (like the suggested gamma density priors) often revolves around encoding prior beliefs. In particular, the method suggested by Kundu [28] proposes choosing hyper-parameters such that the prior mean reflects the true value of the parameter based on domain knowledge. Thus, to evaluate all acquired Bayes theoretical results of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, or θ , the prior distributions for λ_i or μ_i for $i = 1, 2$, are chosen such that their means match the true values. Two informative prior sets for (a_i, b_i, c_i, d_i) , $i = 1, 2$ of λ_i and μ_i for $i = 1, 2$, are considered, namely the following:

- For (c_i, d_i) , $i = 1, 2$ of λ_i :
 - Prior-A:(7.5, 5) and Prior-B:(15, 10) of λ_1 ;
 - Prior-A:(2.5, 5) and Prior-B:(5, 10) of λ_2 ,
- For (a_i, b_i) , $i = 1, 2$ of μ_i :
 - Prior-A:(2.5, 5) and Prior-B:(5, 10) of μ_1 ;
 - Prior-A:(7.5, 5) and Prior-B:(15, 10) of μ_2 ,

On the other hand, to examine the hyper-parameter behavior of $(\varepsilon_1, \varepsilon_2)$ for θ , we consider the following sets:

- For $\theta = 0.4$: Prior-A:(0.2, 0.3) and Prior-B:(0.4, 0.6);

- For $\theta = 0.8$: Prior-A:(1.6, 0.4) and Prior-B:(3.2, 0.8).

The sensitivity analysis shows varying degrees of dependence on prior choice across parameters. Figure 1 shows how the Bayesian posterior summaries for four model parameters (λ_i and μ_i (for $i = 1, 2$)) change with different kinds of prior distributions: improper, informative, overdispersed, and weak. It states that λ_2 is highly robust, with posterior distributions nearly unaffected by prior type. In contrast, μ_2 demonstrates strong sensitivity: informative priors shift the posterior upward and reduce spread, while weak and overdispersed priors result in lower, more dispersed estimates. Both λ_1 and μ_1 exhibit moderate sensitivity, where informative priors slightly shift the posterior centers and reduce variability. These findings emphasize that while some parameters are data-driven and stable, others (particularly μ_2) require careful prior specification due to limited identifiability from the data alone.

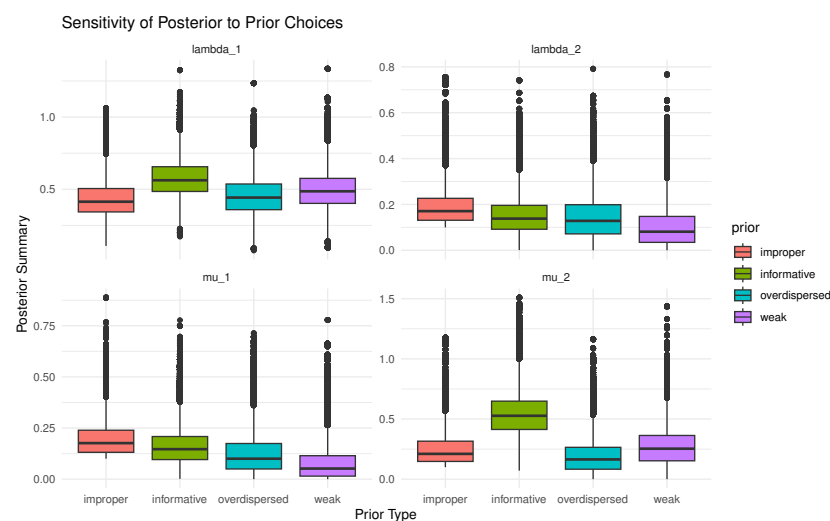


Figure 1. Posterior sensitivity to prior type.

For each simulated dataset, MLEs and their associated 95% ACI are computed for the H_j model parameters λ_i and μ_i (for $i = 1, 2$), the reliability measures $S(t)$ and $h(t)$, and the binomial parameters θ . The Newton–Raphson algorithm, implemented using the maxLik package (version 1.5-2.1) suggested by Henningsen and Toomet [29], is used to obtain all frequentist estimators. In contrast, Bayesian point estimates and 95% BCI estimates for the same quantities are obtained via MCMC sampling, using the coda package (version 0.19-4.1) developed by Plummer et al. [30]. To perform the MCMC estimation and associated 95% BCI for λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ (as outlined in Section 4), we generate 12,000 MCMC samples, discarding the first 2000 as burn-in.

To check the uniqueness of the calculated MLEs of λ_i and μ_i (for $i = 1, 2$) in the proposed Monte Carlo simulations, using just one replicate for illustration, the contour diagrams for log-likelihoods of pair parameters (λ_i, μ_i) for $i = 1, 2$ (when $(n, m, T, \theta) = (50, 40, 0.5, 0.8)$ as an example) are plotted and shown in Figure 2. As a result, the ML estimates of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ according to this simulation setting are (1.3539, 0.6198, 1.0384, 0.3717). The contour plots provide strong evidence that, in this simulation, the MLEs $\hat{\lambda}_i$ and $\hat{\mu}_i$ (for $i = 1, 2$) exist and are unique, as well as confirm stability and identifiability of the estimates in repeated sampling.

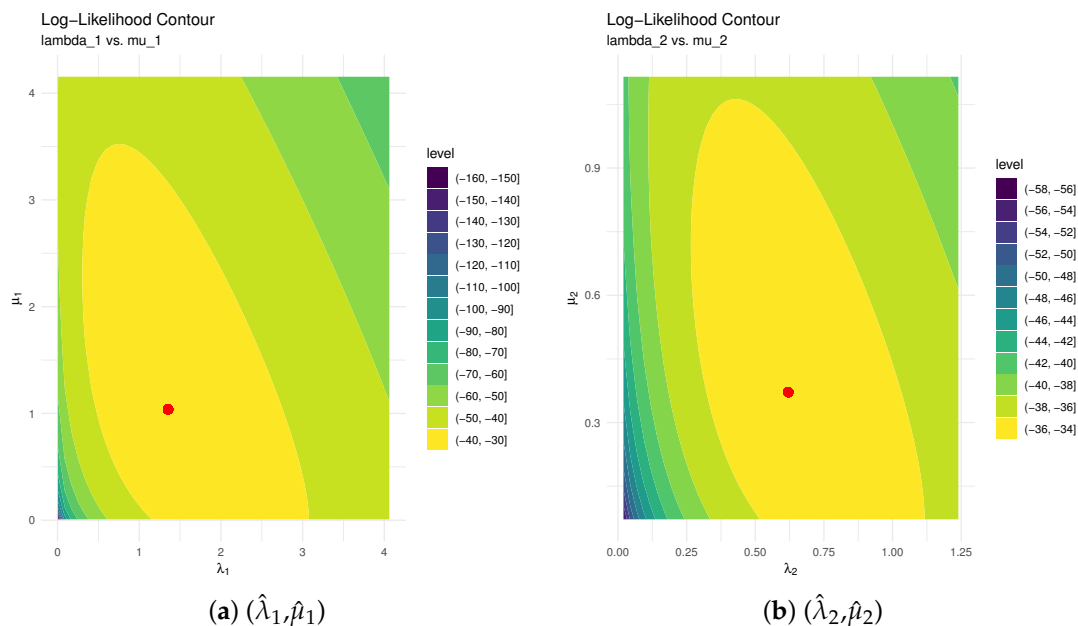


Figure 2. Contour diagrams of the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, parameters in Monte Carlo simulations.

To assess the convergence of the MCMC chains for $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, parameters, in this simulation, both Geweke and Heidelberger–Welch diagnostics are applied. The Geweke diagnostic evaluates the convergence of an MCMC chain by comparing the means of its early and late segments. If the chain is stationary (i.e., converged), these means should be statistically indistinguishable, producing a Z-score close to zero. The Heidelberger–Welch diagnostic performs a two-part test called (1) a stationarity test to verify whether the chain has reached equilibrium and (2) a half-width test to check whether the Monte Carlo error in the posterior mean is sufficiently small relative to the estimate, ensuring estimation precision. The Geweke Z-scores (shown in Figure 3) for all parameters lie within ± 2 ($\lambda_1 = -1.63, \lambda_2 = -0.29, \mu_1 = -0.44, \mu_2 = -0.77$), indicating no significant difference between early and late chain segments and thus supporting convergence. Consistently, the Heidelberger–Welch test confirms that all chains passed the stationarity and half-width criteria, with $\mathcal{P}$ -values ($\lambda_1 = 0.320, \lambda_2 = 0.498, \mu_1 = 0.871, \mu_2 = 0.404$) well above 0.05 and negligible Monte Carlo error. These results provide strong numerical evidence that the MCMC chains have adequately converged and that the resulting posterior estimates are both stable and reliable.

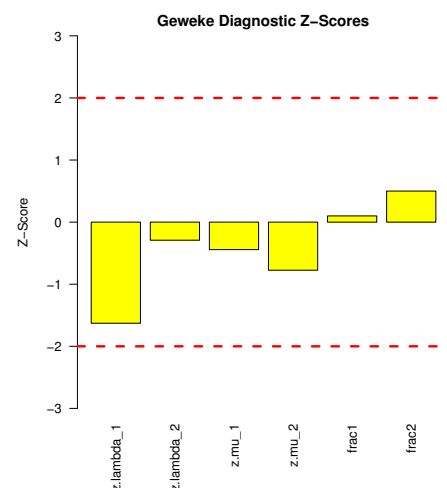


Figure 3. Geweke convergence diagnostics for the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, parameters in Monte Carlo simulations.

To visualize these facts, the trace (with its Gaussian kernel) and autocorrelation function (ACF) are plotted; see Figure 4. Figure 4 indicates that ACF values of λ_i or μ_i (for $i = 1, 2$) look like random noise and close to zero as the lag value increases, while Figure 5 implies that the simulated chains are substantially mixed. As a summary, Figures 4 and 5 support the numerical facts given by Geweke and Heidelberger–Welch diagnostics and confirm that the Markov chain draws of λ_i or μ_i (for $i = 1, 2$) are acceptable.

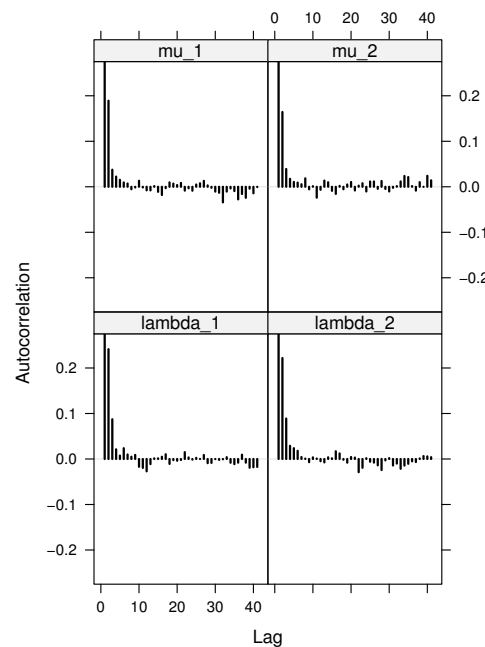


Figure 4. The ACF diagrams of the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, parameters in Monte Carlo simulations.

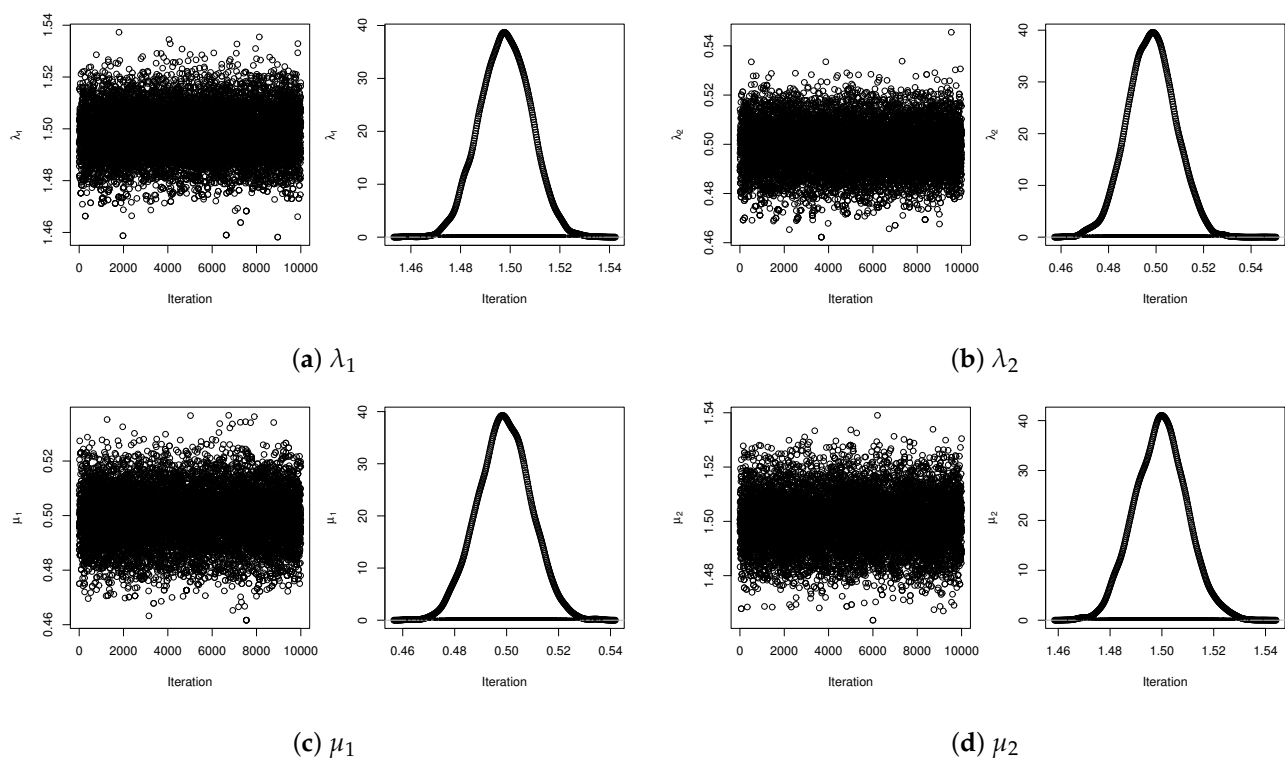


Figure 5. The trace diagrams of the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, parameters in Monte Carlo simulations.

The mean point estimate (MPE), root mean square error (RMSE), mean relative absolute bias (MRAB), average interval length (AIL), and coverage percentage (CP) of θ (for instance) are obtained as

$$\text{MPE}(\hat{\theta}) = \frac{1}{1000} \sum_{s=1}^{1000} \hat{\theta}^{[s]},$$

$$\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{1000} \sum_{s=1}^{1000} (\hat{\theta}^{[s]} - \theta)^2},$$

$$\text{MRAB}(\hat{\theta}) = \frac{1}{1000} \sum_{s=1}^{1000} \theta^{-1} |\hat{\theta}^{[s]} - \theta|,$$

$$\text{AIL}^{95\%}(\theta) = \frac{1}{1000} \sum_{s=1}^{1000} (\mathbb{U}_{\hat{\theta}^{[s]}} - \mathbb{L}_{\hat{\theta}^{[s]}}),$$

and

$$\text{CP}^{95\%}(\theta) = \frac{1}{1000} \sum_{s=1}^{1000} \mathbb{I}_{(\mathbb{L}_{\hat{\theta}^{[s]}}, \mathbb{U}_{\hat{\theta}^{[s]}})}(\theta),$$

respectively, where $\hat{\theta}^{[s]}$ represents the estimate of θ obtained at s th sample, $\mathbb{I}(\cdot)$ is the indicator, and $(\mathbb{L}(\cdot), \mathbb{U}(\cdot))$ denotes the estimated interval bounds. The simulated MPE, RMSE, MRAB, AIL, and CP values of (λ_i, μ_i) , $i = 1, 2$, $S(t)$ and $h(t)$ are readily obtained in a similar manner.

5.2. Simulation Results

All Monte Carlo results for λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ are presented in Tables 1–8. Now, based on lower RMSE, MRAB, AIL values, and higher CP values, the following observations are made:

- All estimators for λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ perform well overall.
- Increasing n (or m) improves the accuracy of both point and interval estimation results. A similar improvement is observed when $\sum_{i=1}^m r_i$ decreases.
- Bayesian MCMC estimates (or 95% BCI) using informative gamma and beta priors are more precise than others developed from the competitive likelihood method.
- For each unknown subject, Bayesian results under Prior-B outperform those under Prior-A due to the smaller prior variance in Prior-B. Both Bayesian priors outperform the frequentist estimators.
- As T (or θ) grows,
 - the RMSEs, MRABs, and AILs of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ decrease;
 - the CPs of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ increase. In most situations, the simulated CP values for both ACI and BCI intervals are close to the pre-specified 95% nominal level.
- Overall, based on two competing risks modeled via the APTIIC via binomial removals, the Bayesian approach is recommended for estimating the H_j distribution parameters and its reliability measures.

Table 1. The point estimation results of λ_i , $i = 1, 2$.

(n, m)	θ	$T = 0.3$									$T = 0.5$								
		MLE			Bayes						MLE			Bayes					
		Prior→			A			B						A			B		
		MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB
λ_1																			
(30, 10)	0.4	1.686	1.235	0.817	1.310	1.189	0.794	1.470	0.791	0.780	1.570	1.225	1.581	0.640	1.178	0.788	1.653	0.518	0.442
	0.8	1.425	1.146	0.754	1.634	1.012	0.661	1.574	0.796	0.442	1.524	1.120	1.469	0.854	0.989	0.648	1.446	0.711	0.400
(30, 20)	0.4	1.524	1.205	0.780	1.655	1.179	0.705	1.741	1.068	0.444	1.476	1.188	1.493	0.615	1.075	0.701	1.339	0.748	0.424
	0.8	1.500	1.078	0.712	1.387	0.986	0.646	1.851	0.777	0.413	1.310	1.033	1.580	0.548	0.943	0.619	1.606	0.704	0.398
(50, 20)	0.4	1.573	1.003	0.651	1.528	0.758	0.489	1.748	0.726	0.394	1.607	0.986	1.634	0.664	0.753	0.485	1.564	0.658	0.329
	0.8	1.493	0.916	0.592	1.390	0.716	0.451	1.696	0.596	0.321	1.584	0.908	1.590	0.901	0.694	0.440	1.483	0.542	0.292
(50, 40)	0.4	1.436	0.967	0.633	1.379	0.741	0.459	1.532	0.698	0.333	1.635	0.957	1.445	0.564	0.710	0.446	1.727	0.581	0.322
	0.8	1.485	0.888	0.574	1.611	0.693	0.437	1.708	0.584	0.293	1.488	0.870	1.586	0.394	0.677	0.435	1.531	0.529	0.281
(80, 30)	0.4	1.525	0.858	0.557	1.572	0.680	0.432	1.661	0.527	0.286	1.521	0.825	1.484	0.739	0.675	0.420	1.415	0.506	0.263
	0.8	1.629	0.715	0.432	1.399	0.667	0.415	1.544	0.488	0.256	1.572	0.693	1.419	0.433	0.580	0.365	1.527	0.468	0.235
(80, 60)	0.4	1.551	0.796	0.508	1.776	0.675	0.419	1.396	0.507	0.265	1.414	0.735	1.469	0.338	0.601	0.368	1.482	0.480	0.246
	0.8	1.499	0.639	0.400	1.685	0.601	0.383	1.740	0.463	0.249	1.628	0.621	1.390	0.285	0.517	0.315	1.413	0.443	0.227
λ_2																			
(30, 10)	0.4	0.554	0.492	0.793	0.429	0.400	0.790	0.523	0.380	0.752	0.584	0.464	0.791	0.429	0.389	0.755	0.516	0.368	0.749
	0.8	0.519	0.450	0.737	0.499	0.344	0.668	0.545	0.328	0.642	0.513	0.430	0.702	0.511	0.343	0.661	0.555	0.313	0.599
(30, 20)	0.4	0.526	0.457	0.773	0.512	0.398	0.706	0.417	0.352	0.688	0.508	0.443	0.740	0.520	0.368	0.702	0.512	0.328	0.640
	0.8	0.500	0.445	0.703	0.461	0.339	0.661	0.392	0.303	0.590	0.526	0.417	0.666	0.492	0.333	0.633	0.416	0.280	0.538
(50, 20)	0.4	0.491	0.421	0.664	0.474	0.323	0.632	0.451	0.288	0.507	0.630	0.409	0.644	0.528	0.320	0.590	0.480	0.253	0.408
	0.8	0.466	0.404	0.659	0.636	0.261	0.471	0.380	0.207	0.373	0.595	0.383	0.629	0.437	0.258	0.462	0.486	0.181	0.316
(50, 40)	0.4	0.541	0.409	0.662	0.467	0.286	0.526	0.472	0.210	0.377	0.662	0.399	0.634	0.517	0.263	0.479	0.513	0.205	0.359
	0.8	0.531	0.393	0.612	0.515	0.199	0.348	0.540	0.177	0.306	0.480	0.377	0.590	0.438	0.195	0.341	0.465	0.174	0.291
(80, 30)	0.4	0.539	0.297	0.519	0.529	0.193	0.324	0.455	0.174	0.288	0.629	0.197	0.334	0.304	0.187	0.289	0.457	0.168	0.243
	0.8	0.468	0.164	0.285	0.479	0.145	0.241	0.431	0.126	0.203	0.478	0.155	0.259	0.426	0.137	0.211	0.494	0.114	0.187
(80, 60)	0.4	0.481	0.248	0.303	0.496	0.182	0.274	0.480	0.166	0.237	0.698	0.187	0.296	0.470	0.167	0.272	0.518	0.142	0.217
	0.8	0.587	0.154	0.259	0.535	0.135	0.203	0.523	0.115	0.175	0.467	0.153	0.245	0.505	0.126	0.198	0.512	0.110	0.163

Table 2. The point estimation results of $\mu_i, i = 1, 2$.

(n, m)	θ	$T = 0.3$									$T = 0.5$									
		MLE			Bayes						MLE			Bayes						
					A			B						A			B			
		Prior→	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB
μ_1																				
(30, 10)	0.4	0.536	1.776	2.612	0.591	0.361	0.683	0.527	0.345	0.668	0.603	1.706	2.442	0.548	0.346	0.672	0.293	0.338	0.626	
	0.8	0.547	1.501	2.250	0.537	0.329	0.554	0.494	0.279	0.512	0.541	1.457	2.207	0.523	0.280	0.521	0.457	0.262	0.483	
(30, 20)	0.4	0.578	1.690	2.522	0.460	0.341	0.659	0.475	0.289	0.541	0.515	1.680	2.439	0.516	0.330	0.568	0.328	0.277	0.516	
	0.8	0.525	1.459	2.205	0.499	0.313	0.515	0.516	0.259	0.466	0.412	1.345	2.189	0.432	0.275	0.496	0.364	0.251	0.457	
(50, 20)	0.4	0.689	1.265	1.934	0.531	0.293	0.496	0.528	0.237	0.424	0.590	1.239	1.927	0.528	0.269	0.468	0.340	0.228	0.419	
	0.8	0.405	1.193	1.717	0.478	0.267	0.484	0.491	0.213	0.380	0.484	1.109	1.696	0.473	0.243	0.429	0.543	0.209	0.365	
(50, 40)	0.4	0.649	1.228	1.879	0.544	0.284	0.494	0.527	0.227	0.409	0.595	1.202	1.838	0.517	0.256	0.448	0.216	0.214	0.407	
	0.8	0.481	1.069	1.684	0.455	0.236	0.441	0.477	0.195	0.341	0.487	1.056	1.663	0.473	0.210	0.537	0.333	0.191	0.325	
(80, 30)	0.4	0.521	0.925	0.607	0.597	0.206	0.345	0.460	0.175	0.314	0.510	0.763	0.369	0.574	0.198	0.341	0.448	0.173	0.265	
	0.8	0.484	0.598	0.306	0.425	0.180	0.319	0.521	0.154	0.241	0.296	0.484	0.291	0.396	0.165	0.467	0.331	0.138	0.224	
(80, 60)	0.4	0.538	0.677	0.327	0.523	0.197	0.324	0.524	0.165	0.254	0.502	0.520	0.317	0.488	0.186	0.528	0.241	0.148	0.234	
	0.8	0.521	0.453	0.249	0.516	0.158	0.216	0.516	0.134	0.182	0.600	0.374	0.232	0.517	0.143	0.604	0.168	0.125	0.175	
μ_2																				
(30, 10)	0.4	1.681	1.249	0.815	1.384	1.177	0.766	1.727	1.162	0.757	1.653	1.215	0.804	1.594	1.164	0.760	1.629	1.085	0.593	
	0.8	1.627	1.115	0.730	1.478	1.008	0.646	1.322	0.977	0.575	1.508	1.073	0.654	1.436	1.008	0.584	1.692	0.887	0.509	
(30, 20)	0.4	1.710	1.156	0.758	1.689	1.099	0.674	1.653	1.031	0.586	1.531	1.131	0.741	1.722	1.046	0.664	1.515	1.023	0.577	
	0.8	1.678	1.076	0.697	1.590	0.949	0.528	1.400	0.793	0.484	1.474	1.048	0.619	1.638	0.903	0.504	1.531	0.725	0.452	
(50, 20)	0.4	1.621	0.930	0.534	1.579	0.856	0.492	1.659	0.679	0.423	1.497	0.898	0.507	1.586	0.768	0.480	1.479	0.668	0.403	
	0.8	1.526	0.888	0.492	1.576	0.715	0.443	1.559	0.572	0.351	1.667	0.840	0.449	1.499	0.591	0.352	1.644	0.560	0.325	
(50, 40)	0.4	1.673	0.902	0.501	1.556	0.794	0.472	1.322	0.590	0.365	1.520	0.870	0.498	1.471	0.764	0.463	1.589	0.577	0.354	
	0.8	1.513	0.869	0.460	1.550	0.576	0.338	1.579	0.523	0.313	1.475	0.785	0.446	1.462	0.547	0.315	1.472	0.507	0.289	
(80, 30)	0.4	1.623	0.803	0.454	1.744	0.557	0.307	1.473	0.500	0.247	1.628	0.780	0.312	1.568	0.524	0.302	1.526	0.466	0.225	
	0.8	1.465	0.482	0.262	1.526	0.446	0.225	1.469	0.367	0.204	1.517	0.449	0.229	1.464	0.439	0.222	1.660	0.259	0.187	
(80, 60)	0.4	1.568	0.798	0.297	1.681	0.492	0.247	1.376	0.444	0.223	1.552	0.493	0.289	1.648	0.455	0.235	1.486	0.324	0.219	
	0.8	1.474	0.416	0.213	1.438	0.397	0.180	1.648	0.285	0.164	1.663	0.398	0.198	1.548	0.393	0.176	1.549	0.233	0.152	

Table 3. The point estimation results of $S(t)$ and $h(t)$.

(n, m)	θ	$T = 0.3$									$T = 0.5$								
		MLE			Bayes						MLE			Bayes					
		Prior→			A			B						A			B		
		MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB
$S(t)$																			
(30, 10)	0.4	0.813	0.145	0.176	0.906	0.138	0.168	0.871	0.127	0.153	0.810	0.144	0.174	0.896	0.137	0.167	0.856	0.055	0.055
	0.8	0.818	0.127	0.151	0.869	0.110	0.136	0.864	0.082	0.115	0.817	0.120	0.147	0.871	0.096	0.124	0.866	0.046	0.045
(30, 20)	0.4	0.818	0.134	0.162	0.926	0.122	0.147	0.902	0.095	0.126	0.817	0.133	0.161	0.927	0.122	0.147	0.903	0.049	0.047
	0.8	0.818	0.118	0.143	0.889	0.094	0.122	0.877	0.064	0.101	0.943	0.118	0.133	0.925	0.091	0.115	0.892	0.045	0.045
(50, 20)	0.4	0.848	0.120	0.131	0.915	0.090	0.113	0.885	0.058	0.086	0.851	0.110	0.130	0.907	0.087	0.109	0.875	0.042	0.038
	0.8	0.832	0.100	0.120	0.888	0.081	0.092	0.883	0.050	0.064	0.830	0.092	0.109	0.879	0.073	0.089	0.874	0.032	0.029
(50, 40)	0.4	0.851	0.107	0.129	0.927	0.087	0.107	0.903	0.053	0.074	0.847	0.100	0.118	0.930	0.086	0.101	0.906	0.036	0.034
	0.8	0.831	0.092	0.109	0.928	0.072	0.083	0.915	0.049	0.059	0.828	0.091	0.108	0.924	0.071	0.082	0.912	0.029	0.028
(80, 30)	0.4	0.849	0.088	0.103	0.902	0.068	0.079	0.888	0.045	0.046	0.850	0.085	0.098	0.899	0.062	0.070	0.885	0.017	0.026
	0.8	0.828	0.073	0.086	0.907	0.058	0.067	0.903	0.037	0.031	0.827	0.064	0.074	0.917	0.054	0.059	0.911	0.018	0.018
(80, 60)	0.4	0.845	0.083	0.096	0.951	0.063	0.073	0.939	0.041	0.035	0.841	0.075	0.086	0.950	0.059	0.068	0.938	0.020	0.021
	0.8	0.828	0.058	0.064	0.962	0.052	0.058	0.956	0.034	0.027	0.826	0.057	0.063	0.961	0.044	0.049	0.955	0.016	0.015
$h(t)$																			
(30, 10)	0.4	0.947	0.827	0.954	0.857	0.803	0.937	0.479	0.765	0.891	0.726	0.815	0.935	0.903	0.780	0.824	0.571	0.450	0.437
	0.8	0.866	0.774	0.902	0.771	0.709	0.824	0.569	0.506	0.553	0.873	0.763	0.888	0.764	0.688	0.798	0.551	0.386	0.524
(30, 20)	0.4	0.864	0.786	0.914	0.684	0.758	0.882	0.150	0.529	0.625	0.870	0.799	0.913	0.696	0.757	0.880	0.173	0.424	0.591
	0.8	0.855	0.753	0.873	0.877	0.678	0.787	0.396	0.436	0.486	0.093	0.722	0.835	0.761	0.648	0.748	0.324	0.374	0.425
(50, 20)	0.4	0.676	0.708	0.822	0.812	0.639	0.735	0.348	0.434	0.446	0.647	0.702	0.812	0.853	0.613	0.732	0.416	0.366	0.396
	0.8	0.841	0.661	0.768	0.746	0.587	0.660	0.377	0.418	0.413	0.881	0.616	0.713	0.795	0.538	0.605	0.423	0.342	0.354
(50, 40)	0.4	0.651	0.694	0.806	0.741	0.613	0.731	0.227	0.422	0.426	0.697	0.654	0.751	0.708	0.590	0.681	0.182	0.358	0.375
	0.8	0.859	0.617	0.704	0.752	0.538	0.681	0.216	0.410	0.396	0.736	0.608	0.688	0.766	0.526	0.688	0.229	0.331	0.321
(80, 30)	0.4	0.663	0.606	0.684	0.867	0.499	0.760	0.337	0.361	0.375	0.657	0.595	0.671	0.881	0.478	0.733	0.352	0.329	0.310
	0.8	0.879	0.524	0.594	0.845	0.414	0.796	0.273	0.329	0.326	0.908	0.481	0.539	0.799	0.358	0.744	0.230	0.287	0.266
(80, 60)	0.4	0.706	0.598	0.675	0.741	0.468	0.814	0.102	0.347	0.347	0.751	0.557	0.623	0.738	0.458	0.684	0.101	0.294	0.286
	0.8	0.884	0.488	0.446	0.690	0.349	0.849	0.054	0.317	0.308	0.913	0.451	0.432	0.811	0.339	0.642	0.056	0.273	0.243

Table 4. The point estimation results of θ .

(n, m)	θ	MLE			Bayes					
					Prior-A			Prior-B		
		MPE	RMSE	MRAB	MPE	RMSE	MRAB	MPE	RMSE	MRAB
$T = 0.3$										
(30, 10)	0.4	0.391	0.400	0.472	0.368	0.364	0.454	0.303	0.354	0.398
	0.8	0.801	0.376	0.454	0.757	0.360	0.450	0.767	0.341	0.347
(30, 20)	0.4	0.402	0.389	0.460	0.441	0.363	0.453	0.437	0.346	0.378
	0.8	0.809	0.364	0.454	0.887	0.353	0.449	0.881	0.336	0.321
(50, 20)	0.4	0.426	0.356	0.451	0.329	0.346	0.448	0.299	0.329	0.289
	0.8	0.804	0.121	0.277	0.758	0.107	0.270	0.767	0.110	0.241
(50, 40)	0.4	0.402	0.326	0.448	0.442	0.336	0.446	0.439	0.312	0.243
	0.8	0.814	0.111	0.265	0.888	0.105	0.253	0.882	0.107	0.191
(80, 30)	0.4	0.410	0.109	0.259	0.389	0.103	0.244	0.292	0.097	0.182
	0.8	0.803	0.104	0.239	0.746	0.096	0.221	0.753	0.082	0.140
(80, 60)	0.4	0.415	0.106	0.253	0.443	0.102	0.242	0.441	0.085	0.177
	0.8	0.805	0.102	0.234	0.890	0.093	0.219	0.887	0.080	0.115
$T = 0.5$										
(30, 10)	0.4	0.407	0.389	0.456	0.397	0.362	0.451	0.353	0.349	0.376
	0.8	0.806	0.359	0.454	0.817	0.346	0.449	0.816	0.333	0.317
(30, 20)	0.4	0.423	0.376	0.455	0.406	0.356	0.450	0.380	0.340	0.368
	0.8	0.809	0.345	0.449	0.793	0.341	0.446	0.795	0.328	0.310
(50, 20)	0.4	0.408	0.334	0.445	0.279	0.337	0.446	0.330	0.321	0.265
	0.8	0.805	0.112	0.275	0.768	0.119	0.268	0.787	0.107	0.241
(50, 40)	0.4	0.426	0.313	0.440	0.644	0.326	0.444	0.392	0.309	0.243
	0.8	0.814	0.109	0.263	0.788	0.110	0.251	0.818	0.094	0.185
(80, 30)	0.4	0.403	0.104	0.259	0.290	0.103	0.244	0.293	0.082	0.177
	0.8	0.803	0.101	0.234	0.746	0.095	0.217	0.753	0.075	0.138
(80, 60)	0.4	0.410	0.103	0.251	0.443	0.101	0.240	0.441	0.079	0.170
	0.8	0.805	0.096	0.231	0.890	0.092	0.211	0.887	0.071	0.110

Table 5. The 95% interval estimation results of λ_i , $i = 1, 2$.

(n, m)	θ												
		$T = 0.3$								$T = 0.5$			
		ACI		BCI				ACI		BCI			
		Prior→		A		B				A		B	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
λ_1													
(30, 10)	0.4	3.188	0.847	1.142	0.912	0.987	0.914	2.924	0.850	0.999	0.913	0.928	0.915
	0.8	2.560	0.853	0.920	0.915	0.885	0.917	2.272	0.857	0.897	0.916	0.847	0.919
(30, 20)	0.4	2.818	0.851	0.993	0.914	0.951	0.916	2.571	0.854	0.982	0.915	0.900	0.917
	0.8	2.256	0.856	0.883	0.917	0.851	0.919	2.078	0.860	0.876	0.918	0.837	0.921
(50, 20)	0.4	1.915	0.860	0.877	0.918	0.809	0.920	1.893	0.862	0.864	0.919	0.785	0.922
	0.8	1.658	0.866	0.860	0.919	0.710	0.922	1.591	0.867	0.829	0.920	0.694	0.924
(50, 40)	0.4	1.897	0.864	0.874	0.918	0.754	0.921	1.662	0.865	0.856	0.920	0.709	0.923
	0.8	1.585	0.868	0.814	0.921	0.687	0.923	1.454	0.870	0.777	0.922	0.687	0.925
(80, 30)	0.4	1.439	0.872	0.779	0.923	0.678	0.925	1.415	0.872	0.768	0.924	0.671	0.927
	0.8	1.272	0.877	0.585	0.928	0.579	0.930	1.240	0.878	0.584	0.929	0.563	0.931
(80, 60)	0.4	1.329	0.875	0.678	0.925	0.619	0.927	1.253	0.876	0.637	0.926	0.590	0.928
	0.8	1.227	0.878	0.541	0.929	0.499	0.931	1.145	0.879	0.574	0.930	0.494	0.932

Table 5. Cont.

(n, m)	θ												
		$T = 0.3$						$T = 0.5$					
		ACI		BCI				ACI		BCI			
		Prior→		A		B				A		B	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
λ_2													
(30, 10)	0.4	2.442	0.864	0.814	0.924	0.726	0.925	2.349	0.866	0.737	0.925	0.701	0.927
	0.8	1.954	0.871	0.627	0.930	0.550	0.931	1.912	0.873	0.606	0.931	0.502	0.933
(30, 20)	0.4	2.395	0.866	0.716	0.927	0.639	0.928	2.184	0.868	0.653	0.928	0.625	0.930
	0.8	1.773	0.873	0.590	0.931	0.517	0.932	1.742	0.875	0.576	0.932	0.495	0.934
(50, 20)	0.4	1.629	0.876	0.578	0.932	0.488	0.933	1.614	0.878	0.529	0.933	0.451	0.935
	0.8	1.554	0.880	0.486	0.935	0.431	0.936	1.524	0.882	0.462	0.936	0.416	0.938
(50, 40)	0.4	1.600	0.878	0.533	0.933	0.451	0.934	1.593	0.879	0.508	0.934	0.427	0.936
	0.8	1.516	0.882	0.433	0.936	0.414	0.937	1.437	0.884	0.423	0.937	0.406	0.938
(80, 30)	0.4	1.464	0.883	0.391	0.937	0.377	0.938	1.398	0.885	0.387	0.938	0.368	0.939
	0.8	1.295	0.886	0.331	0.938	0.268	0.939	1.243	0.887	0.320	0.939	0.248	0.941
(80, 60)	0.4	1.392	0.885	0.370	0.937	0.356	0.938	1.328	0.886	0.360	0.938	0.337	0.940
	0.8	1.261	0.887	0.234	0.940	0.210	0.941	1.222	0.888	0.228	0.941	0.196	0.941

Table 6. The 95% interval estimation results of μ_i , $i = 1, 2$.

(n, m)	θ												
		$T = 0.3$						$T = 0.5$					
		ACI		BCI				ACI		BCI			
		Prior→		A		B				A		B	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
μ_1													
(30, 10)	0.4	3.852	0.842	0.857	0.907	0.693	0.908	3.533	0.844	0.814	0.909	0.640	0.910
	0.8	3.437	0.849	0.673	0.915	0.575	0.914	3.393	0.851	0.646	0.917	0.565	0.916
(30, 20)	0.4	3.585	0.847	0.778	0.911	0.612	0.912	3.428	0.849	0.690	0.913	0.566	0.914
	0.8	3.254	0.852	0.563	0.919	0.523	0.921	3.197	0.854	0.542	0.920	0.492	0.922
(50, 20)	0.4	2.995	0.855	0.546	0.921	0.501	0.922	2.845	0.857	0.538	0.923	0.475	0.924
	0.8	2.688	0.858	0.480	0.924	0.456	0.925	2.424	0.860	0.479	0.926	0.452	0.927
(50, 40)	0.4	2.871	0.857	0.515	0.922	0.490	0.923	2.742	0.859	0.506	0.924	0.467	0.925
	0.8	2.347	0.862	0.454	0.925	0.430	0.926	2.209	0.864	0.445	0.927	0.419	0.928
(80, 30)	0.4	2.191	0.864	0.439	0.926	0.406	0.927	1.908	0.866	0.409	0.928	0.363	0.929
	0.8	1.754	0.870	0.352	0.930	0.339	0.931	1.564	0.872	0.348	0.931	0.325	0.933
(80, 60)	0.4	1.985	0.867	0.404	0.928	0.361	0.929	1.745	0.869	0.382	0.930	0.356	0.931
	0.8	1.549	0.872	0.343	0.931	0.309	0.932	1.391	0.874	0.330	0.932	0.314	0.934
μ_2													
(30, 10)	0.4	2.634	0.885	1.737	0.889	1.531	0.893	2.593	0.886	1.633	0.891	1.489	0.894
	0.8	2.246	0.890	1.399	0.894	1.207	0.898	2.229	0.891	1.375	0.896	1.178	0.899
(30, 20)	0.4	2.571	0.887	1.665	0.891	1.315	0.895	2.455	0.888	1.563	0.893	1.282	0.896
	0.8	2.212	0.891	1.251	0.895	1.167	0.899	2.172	0.892	1.247	0.897	0.972	0.900
(50, 20)	0.4	2.100	0.892	1.162	0.896	1.101	0.900	2.072	0.893	1.141	0.898	0.916	0.901
	0.8	1.740	0.896	1.118	0.900	0.926	0.904	1.705	0.897	1.075	0.902	0.844	0.905
(50, 40)	0.4	2.073	0.893	1.139	0.897	0.954	0.901	1.974	0.894	1.115	0.899	0.862	0.902
	0.8	1.696	0.898	1.107	0.902	0.896	0.906	1.635	0.899	1.067	0.904	0.800	0.907
(80, 30)	0.4	1.511	0.901	1.084	0.906	0.869	0.909	1.479	0.902	1.001	0.907	0.792	0.910
	0.8	1.220	0.906	0.970	0.909	0.780	0.913	1.182	0.907	0.874	0.910	0.724	0.913
(80, 60)	0.4	1.447	0.903	1.004	0.908	0.853	0.911	1.342	0.904	0.959	0.909	0.784	0.912
	0.8	1.169	0.908	0.854	0.911	0.732	0.914	1.094	0.909	0.845	0.912	0.646	0.915

Table 7. The 95% interval estimation results of $S(t)$ and $h(t)$.

(n, m)	θ												
		$T = 0.3$						$T = 0.5$					
		ACI		BCI				ACI		BCI			
		Prior→		A		B				A		B	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
$S(t)$													
(30, 10)	0.4	0.462	0.932	0.117	0.944	0.104	0.946	0.289	0.934	0.107	0.946	0.095	0.948
	0.8	0.410	0.935	0.099	0.945	0.091	0.947	0.240	0.937	0.098	0.947	0.087	0.949
(30, 20)	0.4	0.434	0.933	0.102	0.944	0.097	0.946	0.275	0.935	0.102	0.946	0.090	0.948
	0.8	0.384	0.936	0.095	0.945	0.088	0.947	0.219	0.938	0.092	0.947	0.082	0.949
(50, 20)	0.4	0.275	0.939	0.093	0.945	0.083	0.948	0.190	0.941	0.091	0.947	0.079	0.950
	0.8	0.221	0.940	0.085	0.947	0.079	0.949	0.182	0.942	0.084	0.949	0.075	0.951
(50, 40)	0.4	0.230	0.940	0.090	0.946	0.080	0.948	0.184	0.941	0.089	0.948	0.078	0.950
	0.8	0.204	0.941	0.080	0.947	0.074	0.949	0.179	0.943	0.079	0.949	0.072	0.951
(80, 30)	0.4	0.193	0.941	0.077	0.948	0.071	0.950	0.173	0.943	0.077	0.950	0.071	0.952
	0.8	0.152	0.943	0.065	0.950	0.062	0.952	0.149	0.944	0.064	0.951	0.060	0.953
(80, 60)	0.4	0.186	0.942	0.071	0.949	0.068	0.951	0.158	0.944	0.070	0.951	0.067	0.953
	0.8	0.145	0.944	0.057	0.950	0.055	0.952	0.141	0.945	0.056	0.952	0.054	0.954
$h(t)$													
(30, 10)	0.4	1.109	0.942	0.785	0.946	0.689	0.949	1.093	0.943	0.733	0.947	0.632	0.950
	0.8	1.017	0.943	0.692	0.947	0.576	0.950	0.989	0.944	0.646	0.948	0.568	0.951
(30, 20)	0.4	1.084	0.943	0.726	0.947	0.624	0.950	1.002	0.944	0.714	0.948	0.603	0.951
	0.8	0.985	0.944	0.638	0.948	0.566	0.951	0.855	0.945	0.610	0.949	0.548	0.955
(50, 20)	0.4	0.781	0.947	0.559	0.950	0.518	0.953	0.760	0.948	0.521	0.951	0.491	0.954
	0.8	0.744	0.948	0.424	0.952	0.390	0.954	0.735	0.949	0.407	0.953	0.361	0.955
(50, 40)	0.4	0.767	0.947	0.546	0.951	0.487	0.954	0.754	0.948	0.515	0.952	0.371	0.955
	0.8	0.696	0.950	0.382	0.954	0.330	0.956	0.669	0.951	0.357	0.954	0.318	0.958
(80, 30)	0.4	0.659	0.951	0.335	0.955	0.316	0.958	0.623	0.952	0.322	0.956	0.305	0.959
	0.8	0.609	0.953	0.259	0.957	0.253	0.960	0.549	0.954	0.258	0.958	0.245	0.961
(80, 60)	0.4	0.631	0.953	0.310	0.957	0.275	0.959	0.598	0.954	0.295	0.958	0.275	0.960
	0.8	0.576	0.954	0.233	0.958	0.204	0.961	0.518	0.955	0.221	0.959	0.198	0.961

Table 8. The interval estimation results of θ .

(n, m)	θ	95% ACI		95% BCI			
				Prior-A		Prior-B	
		AIL	CP	AIL	CP	AIL	CP
$T = 0.3$							
(30, 10)	0.4	0.427	0.951	0.376	0.954	0.348	0.957
	0.8	0.388	0.953	0.322	0.956	0.279	0.959
(30, 20)	0.4	0.413	0.952	0.360	0.955	0.333	0.958
	0.8	0.359	0.953	0.298	0.956	0.268	0.959
(50, 20)	0.4	0.330	0.954	0.287	0.957	0.259	0.960
	0.8	0.287	0.956	0.256	0.959	0.240	0.962
(50, 40)	0.4	0.299	0.956	0.271	0.959	0.251	0.961
	0.8	0.273	0.957	0.246	0.960	0.226	0.962
(80, 30)	0.4	0.267	0.957	0.239	0.960	0.219	0.963
	0.8	0.245	0.958	0.217	0.961	0.167	0.964
(80, 60)	0.4	0.252	0.958	0.228	0.961	0.175	0.964
	0.8	0.213	0.960	0.197	0.962	0.153	0.965
$T = 0.5$							
(30, 10)	0.4	0.413	0.952	0.357	0.955	0.334	0.957
	0.8	0.378	0.954	0.318	0.957	0.269	0.960
(30, 20)	0.4	0.406	0.953	0.346	0.956	0.325	0.958
	0.8	0.368	0.954	0.286	0.957	0.257	0.960
(50, 20)	0.4	0.321	0.955	0.279	0.958	0.251	0.961
	0.8	0.269	0.957	0.250	0.960	0.236	0.963
(50, 40)	0.4	0.280	0.957	0.268	0.960	0.241	0.962
	0.8	0.257	0.958	0.239	0.961	0.221	0.963
(80, 30)	0.4	0.247	0.958	0.230	0.961	0.200	0.964
	0.8	0.222	0.960	0.211	0.962	0.157	0.965
(80, 60)	0.4	0.239	0.959	0.218	0.962	0.167	0.965
	0.8	0.197	0.961	0.186	0.964	0.148	0.966

6. Real Applications

To illustrate how the proposed inference techniques can be applied in practice, we analyze two sets of real competing risk data collected from the industrial field, one representing the breaking strength of jute fibers and the other representing the failure modes of electrical appliances.

6.1. Jute Fibers

Jute fiber is extracted from the jute plant and is important due to its sustainability, biodegradability, and low environmental impact, making it a preferred choice in eco-friendly materials and green manufacturing. Its advantages include high tensile strength, good insulating properties, low cost, and excellent breathability, making it suitable for textiles, geotextiles, and composites. In this application, we analyze a dataset comprising industrial observations of 58 breaking strength failures of the jute fiber was affected by two different gauge lengths, namely 15 (say, Cause 1) and 20 mm (say, Cause 2); see Xia et al. [31]. In Table 9, beyond dividing each breaking strength failure by one hundred (due to computational ease), the newly transformed jute fiber dataset is presented.

Table 9. Newly transformed jute fiber data.

Time [Cause]									
0.3675 [2]	0.4081 [2]	0.4558 [2]	0.4666 [1]	0.7009 [1]	0.7146 [2]	0.7638 [1]	0.8355 [2]	0.8364 [2]	0.8942 [1],
0.9072 [2]	1.1045 [2]	1.1385 [2]	1.1630 [1]	1.1896 [2]	1.1986 [2]	1.2781 [1]	1.3509 [1]	1.4596 [2]	1.4696 [1],
1.5029 [2]	1.6837 [1]	1.8266 [1]	1.9342 [1]	2.0275 [1]	2.2565 [1]	2.4443 [2]	2.4633 [2]	3.0000 [1]	3.1286 [2],
3.2831 [2]	3.6027 [1]	3.6084 [2]	3.7581 [2]	3.7822 [2]	4.0425 [1]	4.1902 [2]	4.2597 [1]	4.5771 [1]	4.6287 [1],
4.8966 [1]	4.9568 [1]	4.9794 [1]	5.4744 [2]	5.5042 [1]	5.6239 [1]	5.7486 [1]	5.7860 [2]	5.8557 [2]	5.9440 [1],
5.9670 [2]	6.4048 [1]	6.7262 [2]	6.8816 [2]	7.1636 [1]	7.6514 [2]	8.1387 [1]	16.6492 [2]		

First, at a 5% significance level, the Kolmogorov–Smirnov (KS) test is performed to check whether the Hj model fits the jute fiber data well. Given that the latent cause of failures has independent $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, populations, we apply the following null and alternative hypotheses, respectively, as follows:

- H_0 : The jute fiber data set follows a Hj distribution;
- H_1 : The jute fiber data set does not follow a Hj distribution.

To achieve this, the MLEs (with their standard errors (Std.Ers)) and 95% ACIs (with their interval widths (IW)) of Hj parameters λ_i and μ_i (for $i = 1, 2$), as well as the Kolmogorov–Smirnov (KS) distance (with its $\mathcal{P}$ -value), are computed; see Table 10. This indicates that there is no statistical reason to reject the null hypothesis, which states that jute fiber failure times follow an Hj distribution.

Table 10. Fitting the Hj model from jute fiber data.

Cause	Par.	MLE (Std.Er)	95% ACI [IW]	KS ($\mathcal{P}$ -Value)
Cause 1	λ	0.1139 (0.1117)	(0.0015, 0.3328) [0.3313]	0.1314 (0.6505)
	μ	0.1001 (0.0257)	(0.0496, 0.1505) [0.1009]	
Cause 2	λ	0.4439 (0.1468)	(0.1561, 0.7317) [0.5755]	0.1320 (0.6458)
	μ	0.0381 (0.0152)	(0.0084, 0.0678) [0.0594]	

To conduct a comprehensive visual assessment of the goodness-of-fit from the jute fiber datasets, we refer to Figure 6, which presents three types of diagnostic plots: the empirical and estimated survival functions $S(t)$, probability–probability (PP) plots, and contour diagrams of the fitted parameter surfaces. For both jute fiber datasets under consideration, the graphical comparisons between the empirical and theoretical curves indicate that the fitted RF model aligns closely with the empirical data. Specifically, the PP plots

exhibit a near-linear correspondence between the empirical and model-based probabilities, suggesting a good agreement between the observed and expected distributions under the fitted model. Additionally, the subplots in Figure 6 highlight the behavior of the estimated parameter surfaces, revealing the existence and uniqueness of the MLEs $\hat{\lambda}_i$ and $\hat{\mu}_i$ for $i = 1, 2$, corresponding to the two jute fiber datasets. These estimated values are reported in Table 10. Given these diagnostic findings, we recommend the use of the obtained parameter estimates $\hat{\lambda}_i$ and $\hat{\mu}_i$ for $i = 1, 2$ as initial values for future numerical optimization procedures and iterative estimation algorithms applied to jute fiber data.

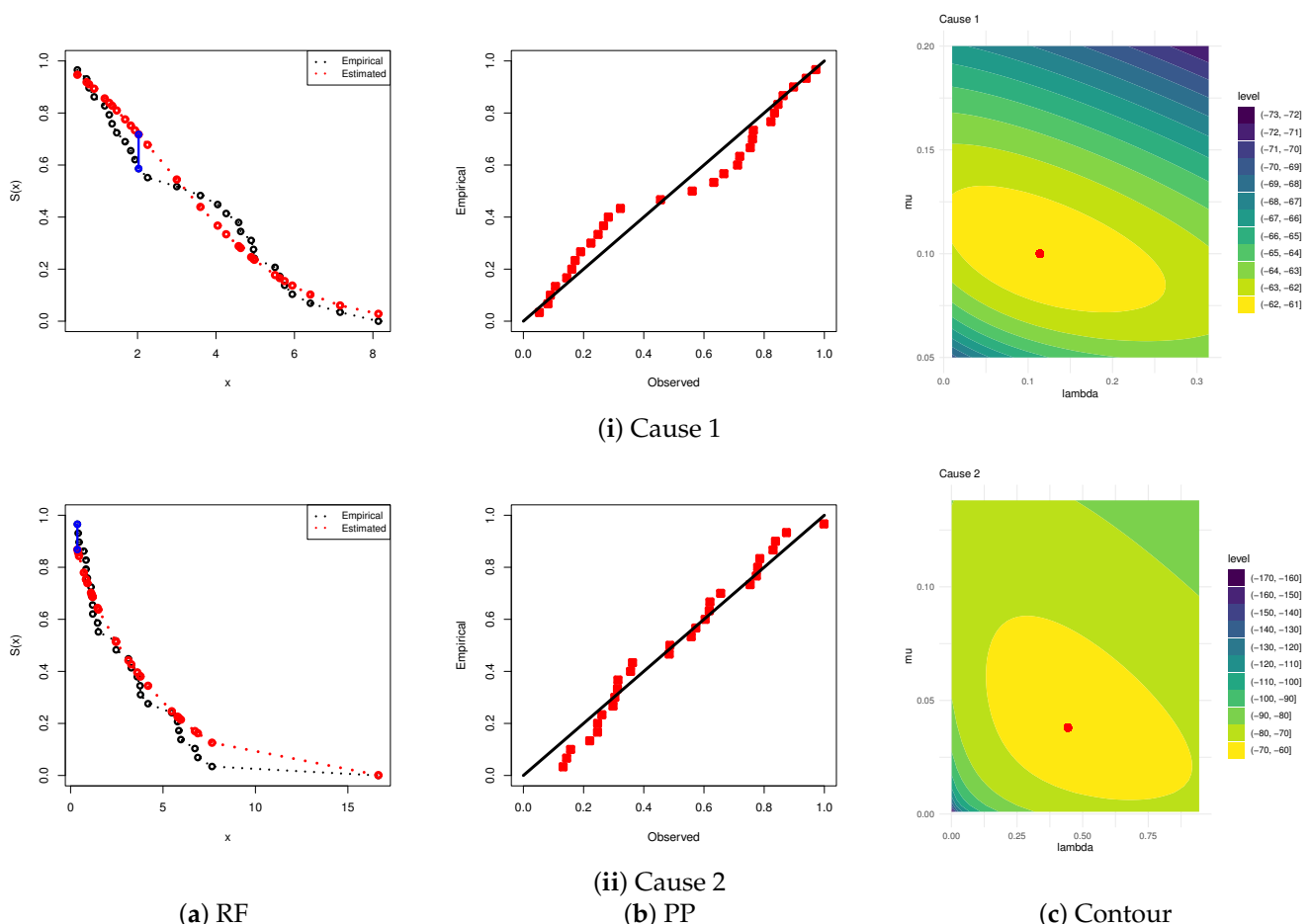


Figure 6. Fitting diagrams of the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, model from jute fiber data.

By setting $\theta = 0.5$ from the whole electrode data for different combinations of m and T , two APTIIC competing risk samples with binomial removals (denoted by $S[i]$, $i = 1, 2$) are formed; see Table 11. The Bayes calculations of λ_i , μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ use non-informative priors due to a lack of prior knowledge about the H_j parameters λ_i and μ_i (for $i = 1, 2$) and the binomial parameter θ . The MCMC sampler chain (detailed in Section 4) is run 50,000 times, and the first 10,000 results are discarded as a burn-in. The MCMC point and 95% BCI estimates of the H_j parameters λ_i and μ_i (for $i = 1, 2$) and reliability factors $S(t)$ and $h(t)$ are produced. However, using Table 11, the derived maximum likelihood and Bayes' MCMC estimates (with their Std.Ers), 95% ACI/BCI estimates (with their IWs) of λ_i , μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ are evaluated; see Table 12. As a sequence, it exhibits that the Bayes' estimates of λ_i , $i = 1, 2$, $R(t)$, or θ outperforms the others regarding the lowest standard error and width values.

Table 11. Two APTIIC competing risk samples with binomial removals from jute fiber data.

$\mathcal{S}[1]: T(d) = 0.8(3) \text{ and } R^* = 9$										
i	1	2	3	4	5	6	7	8	9	10
r_i	22	13	4	0	0	0	0	0	0	0
δ_i	2	1	1	2	2	2	2	1	2	2
x_i	0.3675	0.4666	0.7009	0.8355	0.9072	1.1385	1.1896	1.2781	1.4596	1.5029
$\mathcal{S}[2]: T(d) = 0.6(3) \text{ and } R^* = 7$										
i	1	2	3	4	5	6	7	8	9	10
r_i	17	11	3	0	0	0	0	0	0	0
δ_i	2	2	1	1	1	2	1	2	2	1
x_i	0.3675	0.4558	0.4666	0.7009	0.7638	0.8355	0.8942	0.9072	1.1385	1.1630
i	11	12	13	14	15	16	17	18	19	20
r_i	0	0	0	0	0	0	0	0	0	0
δ_i	1	1	2	1	1	1	1	2	1	2
x_i	1.2781	1.3509	1.5029	1.6837	1.9342	2.0275	2.2565	2.4633	3.0000	3.7822

Table 12. Estimates of λ_i , μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ from jute fiber data.

Sample	Par.	MLE Bayes		ACI BCI		
		Est.	S.Err	Lower	Upper	IW
$\mathcal{S}[1]$	λ_1	4.21×10^{-8}	5.00×10^{-7}	0.0000	1.02×10^{-6}	1.02×10^{-6}
		4.21×10^{-8}	1.00×10^{-10}	4.19×10^{-8}	4.23×10^{-8}	3.91×10^{-10}
	λ_2	2.64×10^{-9}	4.43×10^{-8}	0.0000	8.94×10^{-8}	8.94×10^{-8}
		3.99×10^{-5}	4.99×10^{-5}	1.54×10^{-6}	1.11×10^{-4}	1.09×10^{-4}
	μ_1	2.58×10^{-1}	1.49×10^{-1}	0.0000	5.49×10^{-1}	5.49×10^{-1}
		2.58×10^{-1}	4.98×10^{-5}	2.58×10^{-1}	2.58×10^{-1}	1.96×10^{-4}
	μ_2	6.08×10^{-1}	2.30×10^{-1}	1.57×10^{-1}	$0.11 \times 10^{+1}$	9.00×10^{-1}
		6.08×10^{-1}	4.97×10^{-5}	6.07×10^{-1}	6.08×10^{-1}	1.95×10^{-4}
	$S(t)$	8.97×10^{-1}	3.07×10^{-2}	8.37×10^{-1}	9.58×10^{-1}	1.20×10^{-1}
		8.97×10^{-1}	1.98×10^{-5}	8.97×10^{-1}	8.98×10^{-1}	5.27×10^{-5}
	$h(t)$	3.91×10^{-2}	2.70×10^{-2}	0.0000	9.21×10^{-2}	9.21×10^{-2}
		3.91×10^{-2}	9.28×10^{-6}	3.91×10^{-2}	3.92×10^{-2}	3.38×10^{-5}
	θ	2.23×10^{-1}	2.58×10^{-2}	1.72×10^{-1}	2.73×10^{-1}	1.01×10^{-1}
		1.82×10^{-1}	4.06×10^{-2}	1.34×10^{-1}	2.37×10^{-1}	1.03×10^{-1}
$\mathcal{S}[2]$	λ_1	4.25×10^{-8}	1.94×10^{-8}	4.45×10^{-9}	8.06×10^{-8}	7.62×10^{-8}
		4.25×10^{-8}	1.00×10^{-10}	4.23×10^{-8}	4.27×10^{-8}	3.93×10^{-10}
	λ_2	5.96×10^{-8}	1.23×10^{-6}	0.0000	2.46×10^{-6}	2.46×10^{-6}
		3.99×10^{-5}	5.01×10^{-5}	1.62×10^{-6}	1.13×10^{-4}	1.12×10^{-4}
	μ_1	3.96×10^{-1}	1.14×10^{-1}	1.72×10^{-1}	6.20×10^{-1}	4.48×10^{-1}
		3.96×10^{-1}	4.99×10^{-5}	3.96×10^{-1}	3.96×10^{-1}	1.94×10^{-4}
	μ_2	2.46×10^{-1}	8.71×10^{-2}	7.54×10^{-2}	4.17×10^{-1}	3.41×10^{-1}
		2.46×10^{-1}	4.92×10^{-5}	2.46×10^{-1}	2.46×10^{-1}	1.92×10^{-4}
	$S(t)$	9.23×10^{-1}	1.66×10^{-2}	8.90×10^{-1}	9.55×10^{-1}	6.50×10^{-2}
		9.23×10^{-1}	2.04×10^{-5}	9.23×10^{-1}	9.23×10^{-1}	5.47×10^{-5}
	$h(t)$	2.44×10^{-2}	1.11×10^{-2}	2.55×10^{-3}	4.62×10^{-2}	4.36×10^{-2}
		2.44×10^{-2}	8.76×10^{-6}	2.43×10^{-2}	2.44×10^{-2}	2.74×10^{-5}
	θ	6.59×10^{-2}	1.09×10^{-2}	4.45×10^{-2}	8.72×10^{-2}	4.27×10^{-2}
		6.18×10^{-2}	4.07×10^{-3}	4.30×10^{-2}	8.37×10^{-2}	4.07×10^{-2}

To show the existence and uniqueness features of the collected MLEs $\hat{\lambda}_i$ and $\hat{\mu}_i$ (for $i = 1, 2$) using $\mathcal{S}[i]$, $i = 1, 2$ gathered from jute fiber data, the log-likelihood contours of λ_i , μ_i , $i = 1, 2$ are shown in Figure 7. As a consequence, it confirms that the acquired MLEs $\hat{\lambda}_i$ and $\hat{\mu}_i$ (for $i = 1, 2$) exist and are unique. We thus recommend utilizing the

fitted values of $\hat{\lambda}_i$, $i = 1, 2$ and $\hat{\mu}_i$, $i = 1, 2$ (reported in Table 12) as initial guesses to run Bayes' computations.

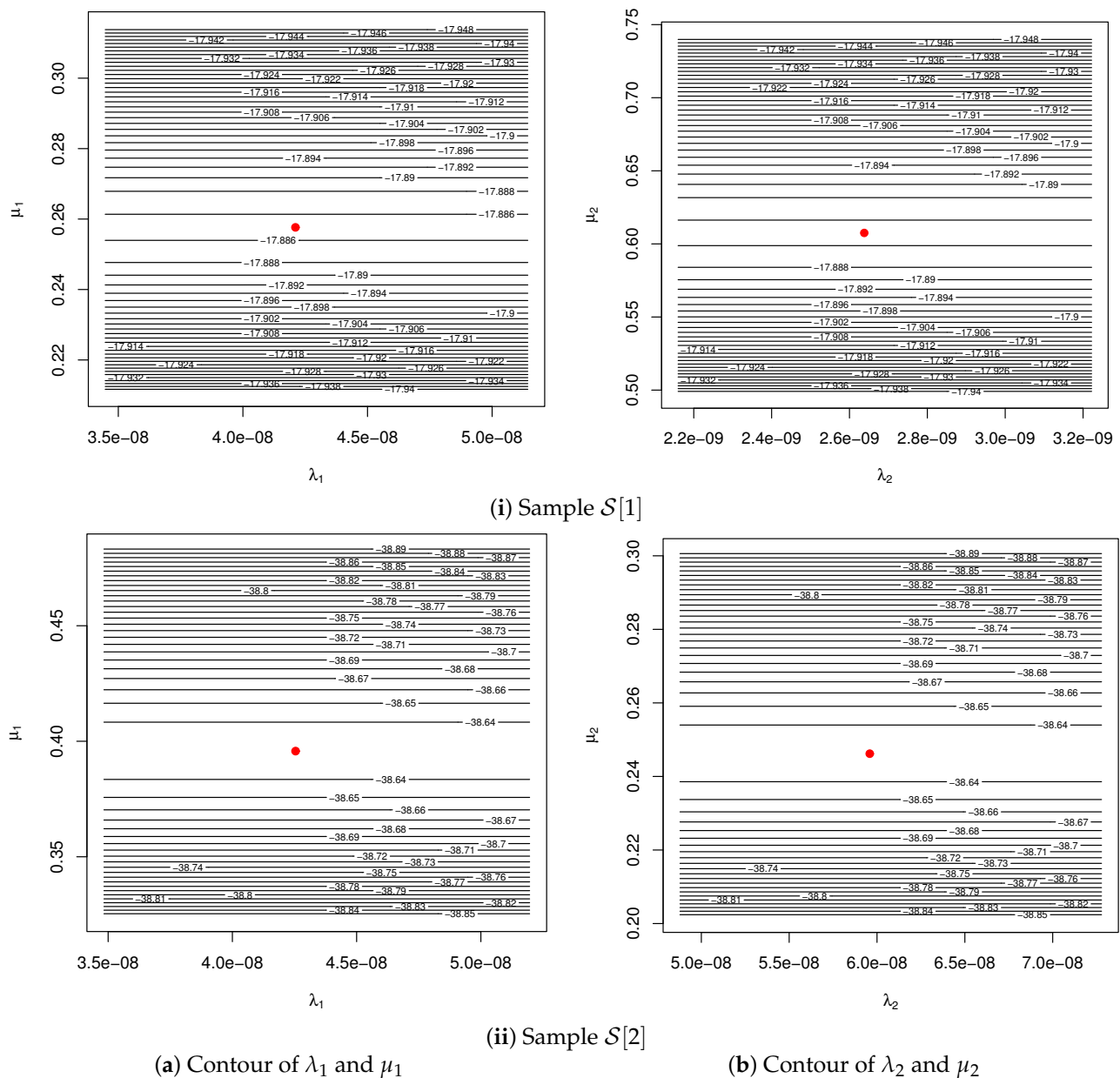


Figure 7. The log-likelihood contours for λ_i and μ_i (for $i = 1, 2$) from jute fiber data.

To further validate the Bayesian inference and to assess the convergence and mixing behavior of the associated MCMC iterations, we present in Figure 8 the trace and kernel density plots for λ_i , μ_i , $i = 1, 2$, $S(t)$, and $h(t)$ based on their remaining 40,000 MCMC simulated variables collected from $\mathcal{S}[1]$ as an example. These diagnostic plots exhibit good mixing and convergence, indicating that the MCMC chains are sampling effectively from the posterior distributions. The resulting posterior distributions for the parameters show that most are approximately symmetric. However, mild positive skewness is observed in the posterior of λ_2 , while the posterior of $S(t)$ exhibits slight negative skewness. From the posterior samples (remaining 40,000 MCMC iterates), we compute key summary statistics including the posterior mean, mode, the three quartiles (Q_i , $i = 1, 2, 3$), standard deviation (Std.D), and skewness (Sk.) of λ_1 , μ_i , $i = 1, 2$, $S(t)$, and $h(t)$ based on their remaining

40,000 MCMC variates. These results are compiled in Table 13, providing further support and consistency with the Bayesian estimates listed in Table 12.

Table 13. Summary for statistics of λ_i , μ_i , $i = 1, 2$, $S(t)$, and $h(t)$ from jute fiber data.

Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	Std.D	Sk.
$S[1]$	λ_1	4.21×10^{-8}	4.19×10^{-8}	4.20×10^{-8}	4.21×10^{-8}	4.22×10^{-8}	1.00×10^{-10}	8.83×10^{-3}
	λ_2	3.99×10^{-5}	1.17×10^{-5}	1.59×10^{-5}	3.40×10^{-5}	5.77×10^{-5}	3.00×10^{-5}	9.84×10^{-1}
	μ_1	2.58×10^{-1}	2.58×10^{-1}	2.58×10^{-1}	2.58×10^{-1}	2.58×10^{-1}	4.98×10^{-5}	2.56×10^{-2}
	μ_2	6.08×10^{-1}	6.08×10^{-1}	6.07×10^{-1}	6.08×10^{-1}	6.08×10^{-1}	4.97×10^{-5}	2.14×10^{-2}
	$S(t)$	8.97×10^{-1}	8.97×10^{-1}	8.97×10^{-1}	8.97×10^{-1}	8.97×10^{-1}	1.34×10^{-5}	-5.21×10^{-1}
	$h(t)$	3.91×10^{-2}	3.91×10^{-2}	3.91×10^{-2}	3.91×10^{-2}	3.91×10^{-2}	8.61×10^{-6}	4.46×10^{-2}
$S[2]$	λ_1	4.25×10^{-8}	4.27×10^{-8}	4.25×10^{-8}	4.25×10^{-8}	4.26×10^{-8}	1.00×10^{-10}	-6.80×10^{-3}
	λ_2	3.99×10^{-5}	4.81×10^{-5}	1.56×10^{-5}	3.38×10^{-5}	5.76×10^{-5}	3.04×10^{-5}	0.11×10^{-1}
	μ_1	3.96×10^{-1}	3.96×10^{-1}	3.96×10^{-1}	3.96×10^{-1}	3.96×10^{-1}	4.99×10^{-5}	3.69×10^{-2}
	μ_2	2.46×10^{-1}	2.46×10^{-1}	2.46×10^{-1}	2.46×10^{-1}	2.46×10^{-1}	4.92×10^{-5}	-2.39×10^{-2}
	$S(t)$	9.23×10^{-1}	9.23×10^{-1}	9.23×10^{-1}	9.23×10^{-1}	9.23×10^{-1}	1.40×10^{-5}	-5.39×10^{-1}
	$h(t)$	2.44×10^{-2}	2.44×10^{-2}	2.44×10^{-2}	2.44×10^{-2}	2.44×10^{-2}	7.02×10^{-6}	1.76×10^{-1}

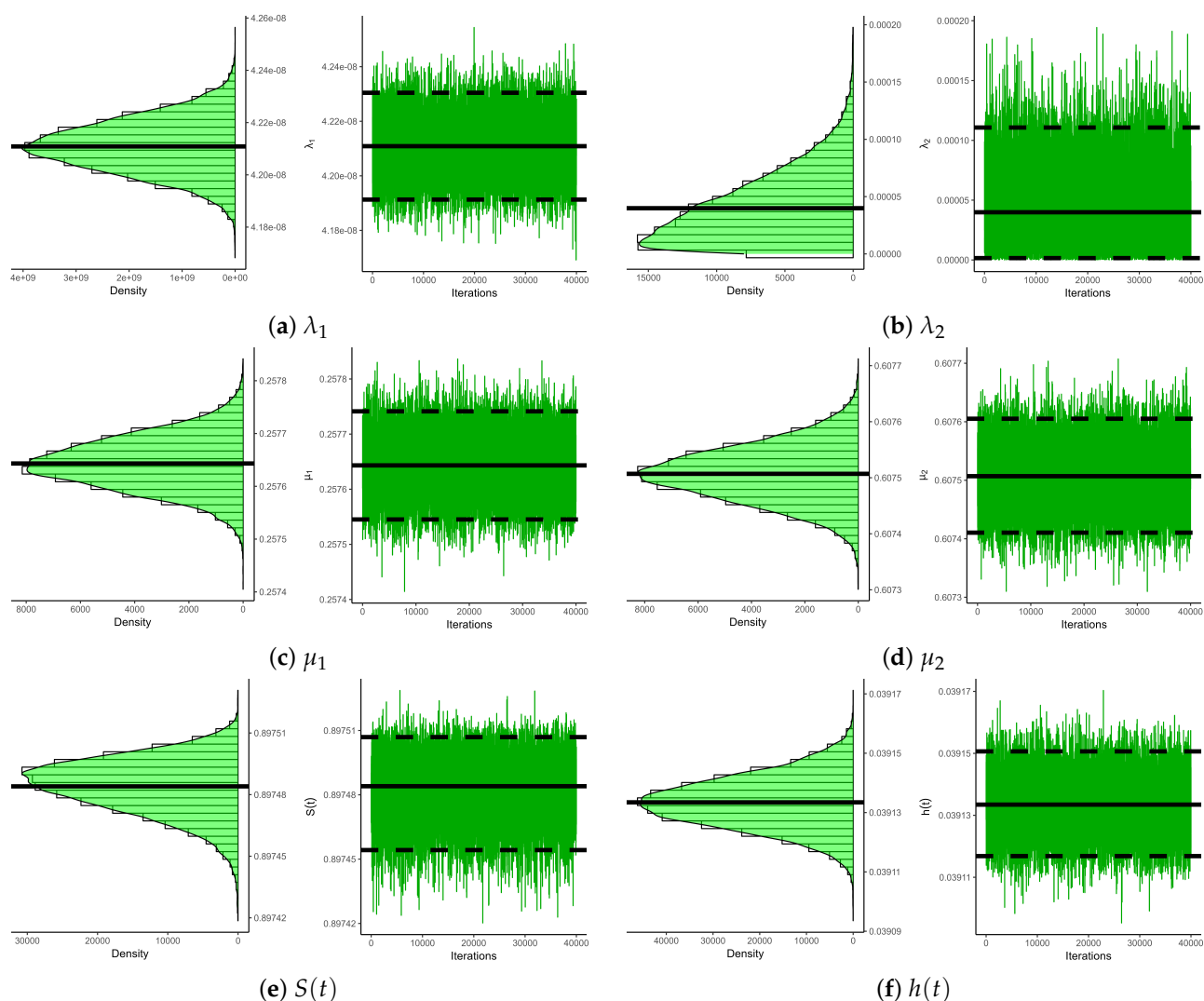


Figure 8. Two MCMC plots of λ_i , μ_i (for $i = 1, 2$), $S(t)$, and $h(t)$ from jute fiber data.

6.2. Electrical Appliances

Electrical appliances are essential in modern life as they enhance convenience, save time, improve efficiency in daily tasks, and support productivity both at home and in the workplace. This section examines real-world failure time statistics for electrical appliances

evaluated in eighteen operating modes, as published by Lawless [32]. Failures in Mode 11 were identified as Cause 1, whereas failures in all other modes were classified as Cause 2. In this part, to make calculation easier, each failure time in the electrical dataset is split by one hundred. Table 14 shows eight failures attributed to Cause 1 and thirteen to Cause 2. Alotaibi et al. [33] analyzed this dataset later. Before analyzing the proposed point and interval estimation findings of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ , it is crucial to evaluate whether the Hj distribution fits the electrical appliances data (provided in Table 14). Since the estimated $\mathcal{P}$ -values (in Table 15) exceed the pre-specified significance level of 5%, there is no statistical reason to reject the null hypothesis, which claims that electrical appliance failure times follow the Hj distribution.

Table 14. Times to failure of 21 electrical appliances.

Time [Cause]										
0.12 [2]	0.16 [2]	0.16 [2]	0.46 [2]	0.46 [2]	0.52 [2]	0.98 [1]	0.98 [2]	2.70 [2]	4.13 [1]	4.95 [1]
4.95 [2]	5.57 [2]	6.16 [2]	6.92 [1]	10.65 [1]	11.07 [2]	11.93 [1]	14.67 [1]	14.67 [2]	19.37 [1]	

Table 15. Fitting the Hj model from electrical appliance data.

Cause	Par.	MLE (Std.Er)	95% ACI [IW]	KS (p -Value)
Cause 1	λ	0.0750 (0.1030)	(0.0000, 0.2770) [0.2770]	0.1322 (0.9950)
	μ	0.0144 (0.0062)	(0.0022, 0.0266) [0.0244]	
Cause 2	λ	0.6843 (0.2473)	(0.1996, 1.1691) [0.9694]	0.2109 (0.6093)
	μ	0.0141 (0.0111)	(0.0014, 0.0358) [0.0344]	

Following the goodness-of-fit visualization tools discussed in Subsection 6.1, Figure 9 indicates that there is a strong relationship between the fitted and empirical RF (or PP) lines for both electrical appliances datasets. It also confirms that the estimated parameters $\hat{\lambda}_i$, $i = 1, 2$ and $\hat{\mu}_i$, $i = 1, 2$ exist and are unique.

Taking various configurations of m and T , using $\theta = 0.5$ across the complete electrode dataset, two APTIIC competing risks samples with binomial removals (denoted $S[i]$ for $i = 1, 2$) are generated (see Table 16). Bayesian estimation of the Hj parameters λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, and $h(t)$, and the binomial parameter θ is carried out using non-informative priors, reflecting the absence of prior information regarding the parameters λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, and θ . The proposed MCMC sampling, described in Section 4, is implemented with a total of 50,000 iterations, discarding the initial 10,000 samples as burn-in. Posterior point estimates and 95% BCIs are computed for the parameters λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, and $h(t)$. Additionally, Table 17 provides a comparative summary of the classical and Bayesian estimates, including their associated Std.Errs and IWs of the 95% ACI/BCI bounds. Results indicate that the Bayesian estimates of λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, $h(t)$, or θ demonstrated improved precision, evidenced by reduced standard errors and narrower interval widths relative to their likelihood counterparts.

To rigorously demonstrate the existence and uniqueness of the ML estimates $\hat{\lambda}_i$, $i = 1, 2$ and $\hat{\mu}_i$, $i = 1, 2$, derived from the electrical appliances datasets $S[i]$, $i = 1, 2$, we examine the joint log-likelihood contour plots presented in Figure 10. These contours depict the shape and topology of the log-likelihood function with respect to the parameter pairs (λ_i, μ_i) , $i = 1, 2$, and confirm the existence and uniqueness of the corresponding MLEs $\hat{\lambda}_i$, $i = 1, 2$ and $\hat{\mu}_i$, $i = 1, 2$. Subsequently, the obtained MLEs (as reported in Table 17) are recommended as suitable initial values for the Bayesian analysis from the electrical appliances datasets. Convergence and mixing properties of the MCMC chains are illustrated in Figure 11, which presents the posterior density and trace plots for λ_i , $i = 1, 2$, μ_i , $i = 1, 2$, $S(t)$, and $h(t)$ based on the remaining 40,000 post-burn-in samples from $S[1]$. The visual

diagnostics indicate that the chains are well-mixed, suggesting adequate convergence. Most posterior distributions of λ_i , $i = 1, 2$, μ_2 , $S(t)$, and $h(t)$ are approximately symmetric, except those of μ_1 , which exhibit slight positive skewness. Descriptive statistics (provided in Table 18) further support the Bayesian inference results summarized in Table 17.

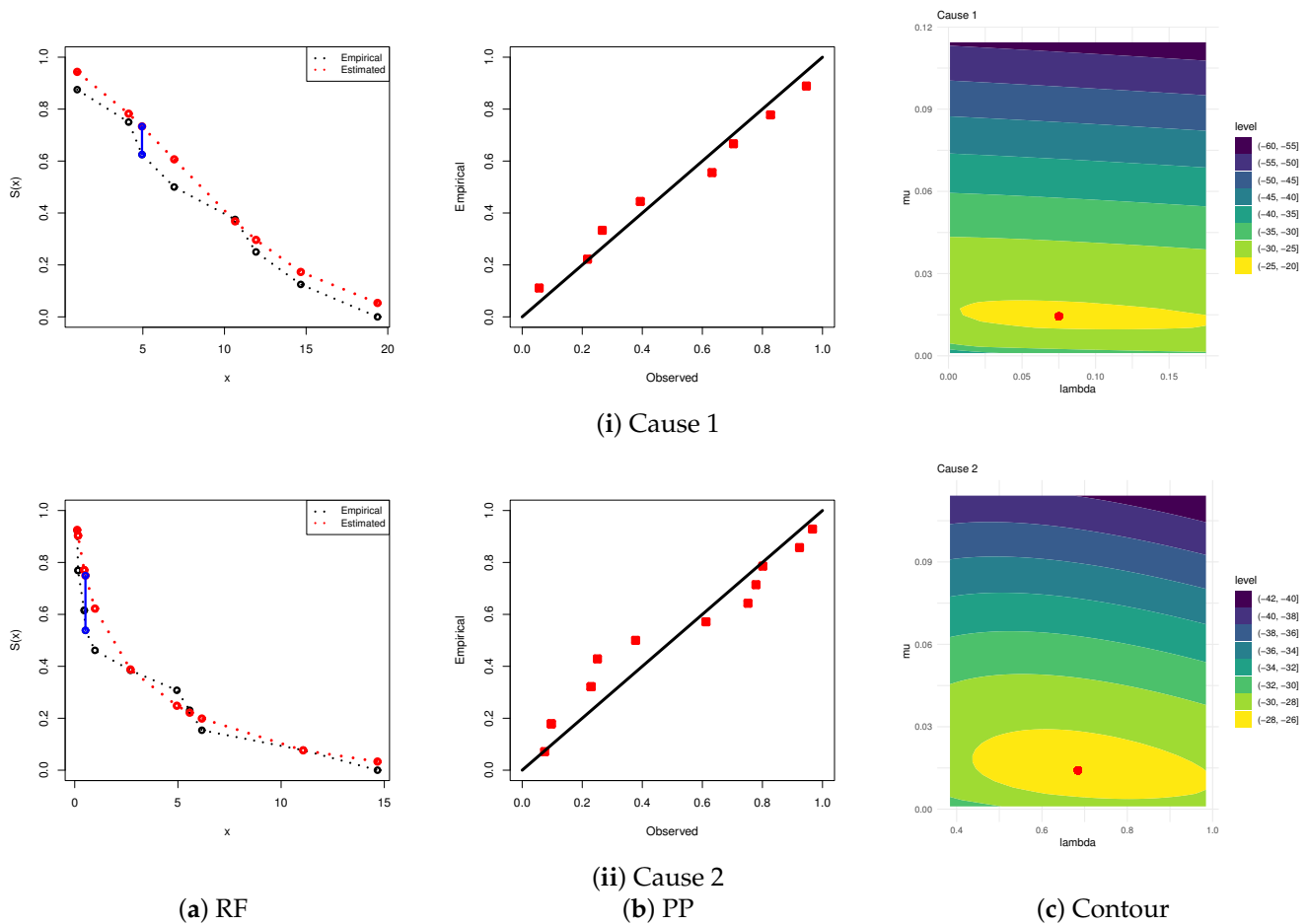


Figure 9. Fitting diagrams of the $H_j(\lambda_i, \mu_i)$, $i = 1, 2$, model from electrical appliance data.

Table 16. Two APTIC competing risk samples with binomial removals from electrical appliance data.

$\mathcal{S}[1]: T(d) = 0.2(2) \text{ and } R^* = 3$										
i	1	2	3	4	5	6	7	8	9	10
r_i	4	4	0	0	0	0	0	0	0	0
δ_i	2	2	2	2	1	2	1	2	2	2
x_i	0.12	0.16	0.46	0.52	0.98	2.70	4.13	4.95	5.57	6.16
$\mathcal{S}[2]: T(d) = 0.3(2) \text{ and } R^* = 2$										
i	1	2	3	4	5	6	7	8	9	10
r_i	2	2	0	0	0	0	0	0	0	0
δ_i	2	2	2	2	1	2	1	1	2	2
x_i	0.12	0.16	0.46	0.52	0.98	2.70	4.13	4.95	5.57	6.16
i	11	12	13	14	15					
r_i	0	0	0	0	0					
δ_i	1	1	1	1	1					
x_i	6.92	10.65	11.93	14.67	19.37					

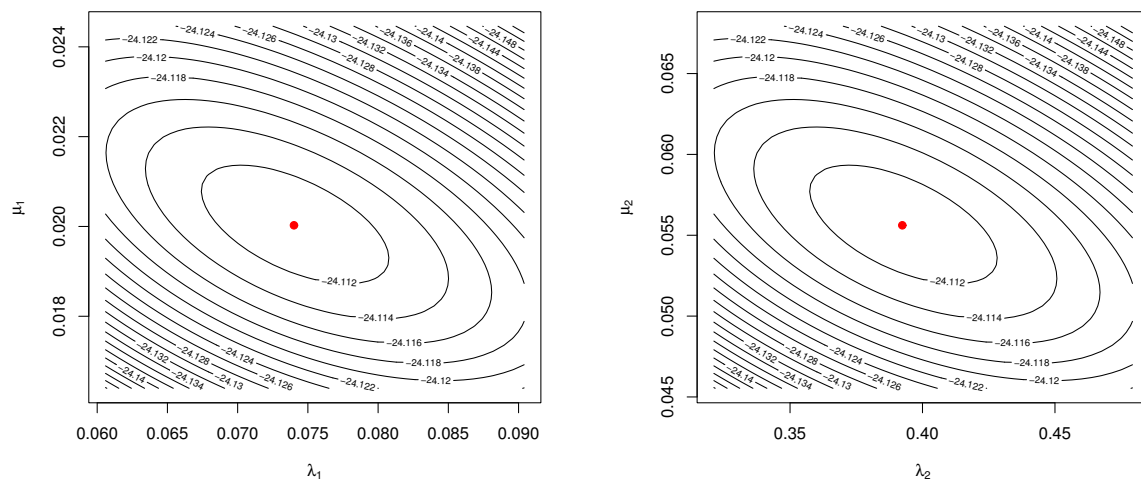
Table 17. Estimates of λ_i , μ_i , $i = 1, 2$, $S(t)$, $h(t)$, and θ from electrical appliance data.

Sample	Par.	MLE Bayes		ACI BCI		
		Est.	S.Err	Lower	Upper	IW
S[1]	λ_1	7.40×10^{-2}	1.32×10^{-1}	0.0000	3.33×10^{-1}	3.33×10^{-1}
		7.41×10^{-2}	9.98×10^{-3}	5.45×10^{-2}	9.38×10^{-2}	3.93×10^{-2}
	λ_2	3.92×10^{-1}	2.10×10^{-1}	0.0000	8.03×10^{-1}	8.03×10^{-1}
		3.92×10^{-1}	1.01×10^{-2}	3.73×10^{-1}	4.12×10^{-1}	3.91×10^{-2}
	μ_1	2.00×10^{-2}	2.65×10^{-2}	0.0000	7.19×10^{-2}	7.19×10^{-2}
		2.08×10^{-2}	8.74×10^{-3}	4.84×10^{-3}	3.87×10^{-2}	3.38×10^{-2}
	μ_2	5.56×10^{-2}	3.70×10^{-2}	0.0000	1.28×10^{-1}	1.28×10^{-1}
		5.56×10^{-2}	9.59×10^{-3}	3.69×10^{-2}	7.45×10^{-2}	3.76×10^{-2}
	$S(t)$	8.20×10^{-1}	8.07×10^{-2}	6.62×10^{-1}	9.78×10^{-1}	3.16×10^{-1}
		8.20×10^{-1}	4.87×10^{-3}	8.10×10^{-1}	8.29×10^{-1}	1.90×10^{-2}
	$h(t)$	1.72×10^{-2}	2.49×10^{-2}	0.0000	6.60×10^{-2}	6.60×10^{-2}
		1.73×10^{-2}	2.33×10^{-3}	1.28×10^{-2}	2.20×10^{-2}	9.13×10^{-3}
	θ	1.45×10^{-1}	4.41×10^{-2}	5.90×10^{-2}	2.32×10^{-1}	1.73×10^{-1}
		1.27×10^{-1}	1.85×10^{-2}	5.74×10^{-2}	2.19×10^{-1}	1.62×10^{-1}
S[2]	λ_1	3.68×10^{-2}	6.50×10^{-2}	0.0000	1.64×10^{-1}	1.64×10^{-1}
		3.68×10^{-2}	9.93×10^{-4}	3.48×10^{-2}	3.87×10^{-2}	3.90×10^{-3}
	λ_2	2.89×10^{-1}	1.09×10^{-1}	7.48×10^{-2}	5.03×10^{-1}	4.28×10^{-1}
		2.89×10^{-1}	1.00×10^{-3}	2.87×10^{-1}	2.91×10^{-1}	3.90×10^{-3}
	μ_1	1.41×10^{-2}	5.82×10^{-3}	2.67×10^{-3}	2.55×10^{-2}	2.28×10^{-2}
		1.41×10^{-2}	9.89×10^{-4}	1.21×10^{-2}	1.60×10^{-2}	3.89×10^{-3}
	μ_2	1.33×10^{-6}	1.60×10^{-5}	0.0000	3.27×10^{-5}	3.27×10^{-5}
		7.27×10^{-4}	9.12×10^{-4}	2.78×10^{-5}	2.05×10^{-3}	2.03×10^{-3}
	$S(t)$	8.75×10^{-1}	4.50×10^{-2}	7.87×10^{-1}	9.63×10^{-1}	1.76×10^{-1}
		8.75×10^{-1}	5.24×10^{-4}	8.74×10^{-1}	8.76×10^{-1}	2.03×10^{-3}
	$h(t)$	6.08×10^{-3}	8.49×10^{-3}	0.0000	2.27×10^{-2}	2.27×10^{-2}
		6.10×10^{-3}	1.62×10^{-4}	5.78×10^{-3}	6.41×10^{-3}	6.32×10^{-4}
	θ	5.41×10^{-2}	2.61×10^{-2}	2.81×10^{-3}	1.05×10^{-1}	1.02×10^{-1}
		5.13×10^{-2}	2.76×10^{-3}	1.43×10^{-2}	1.10×10^{-1}	9.53×10^{-2}

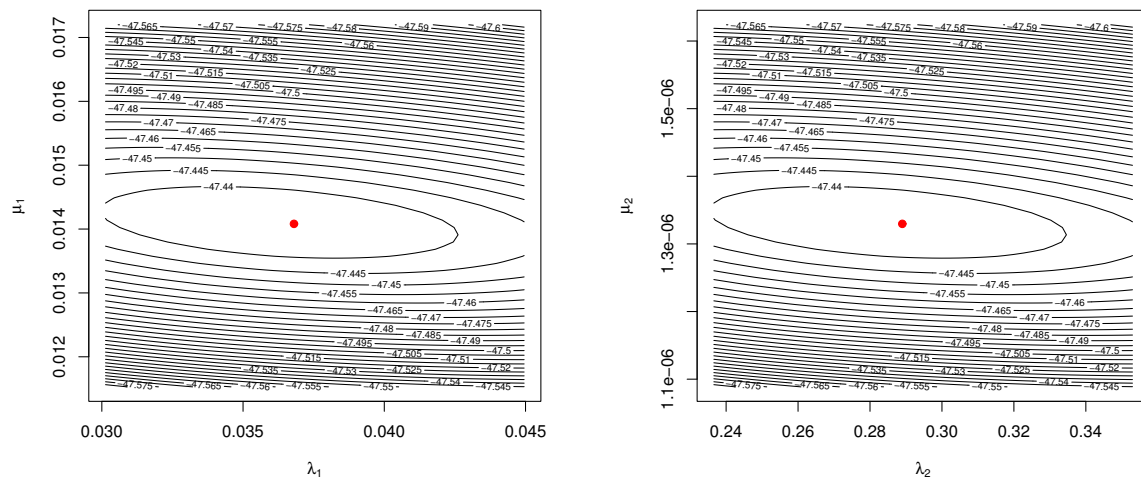
Table 18. Summary for statistics of λ_i , μ_i , $i = 1, 2$, $S(t)$, and $h(t)$ from electrical appliance data.

Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D	Sk.
S[1]	λ_1	7.41×10^{-2}	6.02×10^{-2}	6.73×10^{-2}	7.41×10^{-2}	8.09×10^{-2}	9.98×10^{-3}	1.34×10^{-2}
	λ_2	3.92×10^{-1}	3.79×10^{-1}	3.85×10^{-1}	3.92×10^{-1}	3.99×10^{-1}	1.01×10^{-2}	-6.51×10^{-3}
	μ_1	2.08×10^{-2}	1.89×10^{-2}	1.46×10^{-2}	2.04×10^{-2}	2.66×10^{-2}	8.71×10^{-3}	2.55×10^{-1}
	μ_2	5.56×10^{-2}	4.01×10^{-2}	4.90×10^{-2}	5.56×10^{-2}	6.20×10^{-2}	9.59×10^{-3}	2.66×10^{-2}
	$S(t)$	8.20×10^{-1}	8.19×10^{-1}	8.17×10^{-1}	8.20×10^{-1}	8.23×10^{-1}	4.87×10^{-3}	2.66×10^{-3}
	$h(t)$	1.73×10^{-2}	1.50×10^{-2}	1.57×10^{-2}	1.73×10^{-2}	1.89×10^{-2}	2.33×10^{-3}	9.83×10^{-2}
S[2]	λ_1	3.68×10^{-2}	3.79×10^{-2}	3.61×10^{-2}	3.68×10^{-2}	3.75×10^{-2}	9.93×10^{-4}	-1.63×10^{-2}
	λ_2	2.89×10^{-1}	2.88×10^{-1}	2.88×10^{-1}	2.89×10^{-1}	2.90×10^{-1}	1.00×10^{-3}	8.72×10^{-4}
	μ_1	1.41×10^{-2}	1.30×10^{-2}	1.34×10^{-2}	1.41×10^{-2}	1.48×10^{-2}	9.89×10^{-4}	-7.92×10^{-4}
	μ_2	7.27×10^{-4}	1.00×10^{-3}	2.89×10^{-4}	6.10×10^{-4}	1.06×10^{-3}	5.52×10^{-4}	0.11×10^{-1}
	$S(t)$	8.75×10^{-1}	8.75×10^{-1}	8.74×10^{-1}	8.75×10^{-1}	8.75×10^{-1}	5.17×10^{-4}	7.81×10^{-4}
	$h(t)$	6.10×10^{-3}	6.12×10^{-3}	5.99×10^{-3}	6.10×10^{-3}	6.21×10^{-3}	1.61×10^{-4}	-9.61×10^{-3}

The results from both the jute fiber and electrical appliance datasets support the applicability of the Hjorth competing risks model under the proposed estimation framework. These findings confirm that the model performs well when applied to real-world data collected through adaptive progressive Type II censoring with binomial removals. Overall, the proposed methods offer a practical and effective approach for analyzing complex lifetime data within this modeling context.



(i) Sample $\mathcal{S}[1]$



(ii) Sample $\mathcal{S}[2]$

(a) Contour of λ_1 and μ_1

(b) Contour of λ_2 and μ_2

Figure 10. The log-likelihood contours for λ_i and μ_i (for $i = 1, 2$) from electrical appliance data.

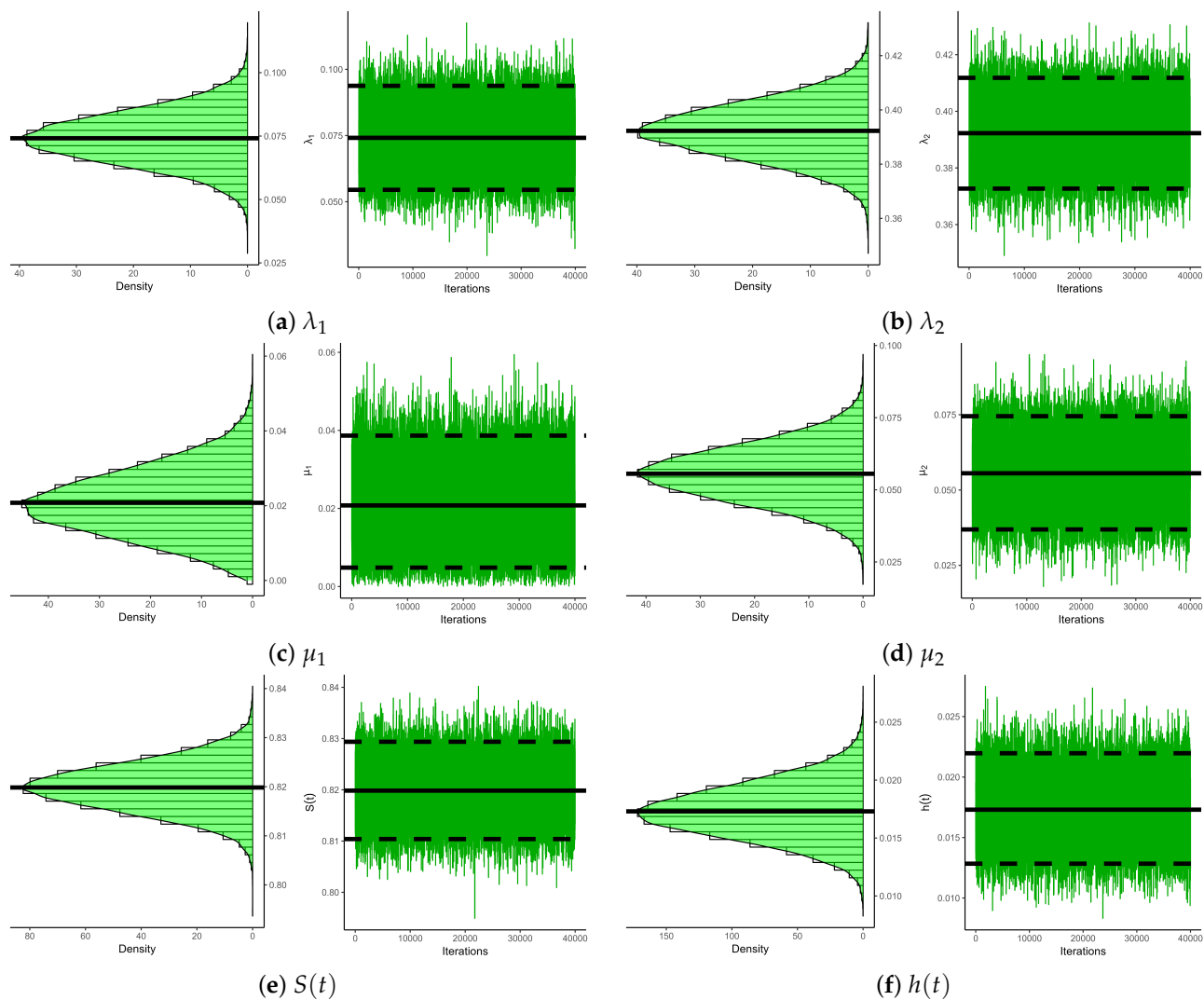


Figure 11. Two MCMC plots of λ_i , μ_i (for $i = 1, 2$), $S(t)$, and $h(t)$ from electrical appliance data.

7. Concluding Remarks

This study examined frequentist and Bayesian estimation frameworks for analyzing the Hjorth competing risks model under an adaptive progressive Type II censoring design. To address the limitations of conventional fixed removal schemes, we proposed a dynamic censoring approach that incorporated binomial removals, thereby enhancing adaptability in reliability assessments. The inferential focus included estimating the scale and shape parameters of the Hjorth distribution, the binomial removal probability, and key reliability metrics such as reliability and hazard rate functions. Within the frequentist paradigm, maximum likelihood estimation was utilized, with numerical optimization techniques employed to resolve the complex likelihood equations for the Hjorth parameters. The asymptotic normality of the MLEs facilitated the construction of confidence intervals for the parameters and reliability metrics. For Bayesian inference, independent gamma priors were specified for the Hjorth parameters, while a beta prior governed the binomial probability. Posterior estimates under squared error loss were derived through Markov Chain Monte Carlo algorithms, accompanied by symmetric Bayes credible intervals. A comprehensive simulation study was conducted to evaluate the precision of point and interval estimates, identifying scenarios where Bayesian or frequentist methods performed better. To highlight the practical relevance and adaptability of the proposed methodologies in diverse fields, two genuine real-world datasets, including jute fiber and electrical appliance datasets,

were used to evaluate two competing risks, illustrating the practical utility of the proposed estimation methods. These applications highlight the importance of accurately estimating the reliability function in the presence of multiple failure causes and complex censoring schemes. One limitation of the current study is the assumption that latent failure times are independent, which may not hold when the failure causes are correlated. In such scenarios, a dependent competing risks model would provide a more appropriate framework for data analysis. For future work, it would be valuable to explore alternative algorithms for generating samples from the full conditional distributions, such as the Hamiltonian Monte Carlo algorithm, instead of relying solely on the Metropolis–Hastings algorithm. The results obtained using Hamiltonian Monte Carlo can then be compared with those from the current study to assess potential improvements in efficiency and accuracy.

Author Contributions: Methodology, R.A., M.N. and A.E.; Funding acquisition, R.A.; Software, A.E.; Supervision, M.N.; Writing—original draft, R.A. and A.E.; Writing—review and editing, R.A. and M.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R50), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Cox, D.R. The analysis of exponentially distributed life-times with two types of failure. *J. R. Stat. Soc. Ser. Stat. Methodol.* **1959**, *21*, 411–421. [\[CrossRef\]](#)
2. Kundu, D.; Basu, S. Analysis of incomplete data in presence of competing risks. *J. Stat. Plan. Inference* **2000**, *87*, 221–239. [\[CrossRef\]](#)
3. Park, C. Parameter estimation of incomplete data in competing risks using the EM algorithm. *IEEE Trans. Reliab.* **2005**, *54*, 282–290. [\[CrossRef\]](#)
4. Wang, L. Inference for Weibull competing risks data under generalized progressive hybrid censoring. *IEEE Trans. Reliab.* **2018**, *67*, 998–1007. [\[CrossRef\]](#)
5. Park, C.; Wang, M. Parameter Estimation of Birnbaum-Saunders Distribution under Competing Risks Using the Quantile Variant of the Expectation-Maximization Algorithm. *Mathematics* **2024**, *12*, 1757. [\[CrossRef\]](#)
6. Alok, A.K.; Singh, G.N.; Chandra, P. Competing risks model with partially observed failure causes based on generalized lifetime family using minimum ranked set sampling. In *Quality Technology & Quantitative Management*; Taylor Francis: Abingdon, UK, 2025; pp. 1–31.
7. Hjorth, U. A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. *Technometrics* **1980**, *22*, 99–107. [\[CrossRef\]](#)
8. Yadav, A.S.; Bakouch, H.S.; Chesneau, C. Bayesian estimation of the survival characteristics for Hjorth distribution under progressive type-II censoring. *Commun. Stat.-Simul. Comput.* **2020**, *51*, 882–900. [\[CrossRef\]](#)
9. Elshahhat, A.; Nassar, M. Bayesian survival analysis for adaptive Type-II progressive hybrid censored Hjorth data. *Comput. Stat.* **2021**, *36*, 1965–1990. [\[CrossRef\]](#)
10. Balakrishnan, N.; Aggarwala, R. *Progressive Censoring: Theory, Methods, and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2000.
11. Balakrishnan, N.; Hossain, A. Inference for the Type II generalized logistic distribution under progressive Type II censoring. *J. Stat. Comput. Simul.* **2007**, *77*, 1013–1031. [\[CrossRef\]](#)
12. Wu, M.; Gui, W. Estimation and prediction for Nadarajah-Haghighi distribution under progressive type-II censoring. *Symmetry* **2021**, *13*, 999. [\[CrossRef\]](#)
13. Baklizi, A. Third Order Likelihood Inference in the Generalized Exponential Distribution under Progressive Type II Censoring. *J. Stat. Theory Appl.* **2025**, 1–28. [\[CrossRef\]](#)
14. Ng, H.K.T.; Kundu, D.; Chan, P.S. Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme. *Nav. Res. Logist.* **2009**, *56*, 687–698. [\[CrossRef\]](#)

15. Haj Ahmad, H.; Salah, M.M.; Eliwa, M.S.; Ali Alhussain, Z.; Almetwally, E.M.; Ahmed, E.A. Bayesian and non-Bayesian inference under adaptive type-II progressive censored sample with exponentiated power Lindley distribution. *J. Appl. Stat.* **2022**, *49*, 2981–3001. [\[CrossRef\]](#)
16. Panahi, H.; Asadi, P. Estimating the parameters of a generalized inverted exponential distribution based on adaptive type II hybrid progressive censoring with application. *J. Stat. Manag. Syst.* **2022**, *25*, 433–455. [\[CrossRef\]](#)
17. Lv, Q.; Tian, Y.; Gui, W. Statistical inference for Gompertz distribution under adaptive type-II progressive hybrid censoring. *J. Appl. Stat.* **2024**, *51*, 451–480. [\[CrossRef\]](#)
18. Sharma, H.; Kumar, P. On survival estimation of Lomax distribution under adaptive progressive type-II censoring. *Stat. Transit. New Ser.* **2025**, *26*, 51–67. [\[CrossRef\]](#)
19. Anakha, K.K.; Chacko, V.M. On comparative lifetime analysis with the generalized Lindley distribution: Insights from joint adaptive progressive Type-II censoring. *J. Stat. Comput. Simul.* **2025**, 1–28. [\[CrossRef\]](#)
20. Yuen, H.K.; Tse, S.K. Parameters estimation for Weibull distributed lifetimes under progressive censoring with random removals. *J. Stat. Comput. Simul.* **1996**, *55*, 57–71. [\[CrossRef\]](#)
21. Singh, S.K.; Singh, U.; Kumar, M. Bayesian estimation for Poisson-exponential model under progressive type-II censoring data with binomial removal and its application to ovarian cancer data. *Commun. Stat.-Simul. Comput.* **2016**, *45*, 3457–3475. [\[CrossRef\]](#)
22. Elbatal, I.; Hassan, A.S.; Diab, L.S.; Ben Ghorbal, A.; Elgarhy, M.; El-Saeed, A.R. Stress–strength reliability analysis for different distributions using Progressive Type-II censoring with Binomial removal. *Axioms* **2023**, *12*, 1054. [\[CrossRef\]](#)
23. Chacko, M.; Mohan, R. Bayesian analysis of Weibull distribution based on progressive type-II censored competing risks data with binomial removals. *Comput. Stat.* **2019**, *34*, 233–252. [\[CrossRef\]](#)
24. Qin, X.; Gui, W. Statistical inference of Burr-XII distribution under progressive Type-II censored competing risks data with binomial removals. *J. Comput. Appl. Math.* **2020**, *378*, 112922. [\[CrossRef\]](#)
25. Elshahhat, A.; Nassar, M. Analysis of adaptive Type-II progressively hybrid censoring with binomial removals. *J. Stat. Comput. Simul.* **2023**, *93*, 1077–1103. [\[CrossRef\]](#)
26. Alqasem, O.A.; Nassar, M.; Abd Elwahab, M.E.; Elshahhat, A. Analyzing Burr-X competing risk model using adaptive progressive Type-II censored binomial removal data with application to electrodes and electronics. *J. Radiat. Res. Appl. Sci.* **2024**, *17*, 101107. [\[CrossRef\]](#)
27. Kundu, D.; Joarder, A. Analysis of Type-II progressively hybrid censored data. *Comput. Stat. Data Anal.* **2006**, *50*, 2509–2528. [\[CrossRef\]](#)
28. Kundu, D. Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics* **2008**, *50*, 144–154. [\[CrossRef\]](#)
29. Henningsen, A.; Toomet, O. maxLik: A package for maximum likelihood estimation in R. *Comput. Stat.* **2011**, *26*, 443–458. [\[CrossRef\]](#)
30. Plummer, M.; Best, N.; Cowles, K.; Vines, K. CODA: Convergence diagnosis and output analysis for MCMC. *R news* **2006**, *6*, 7–11.
31. Xia, Z.P.; Yu, J.Y.; Cheng, L.D.; Liu, L.F.; Wang, W.M. Study on the breaking strength of jute fibres using modified Weibull distribution. *Compos. Part A Appl. Sci. Manuf.* **2009**, *40*, 54–59. [\[CrossRef\]](#)
32. Lawless, J.F. *Statistical Models and Methods For Lifetime Data*, 2nd ed.; John Wiley and Sons: Hoboken, NJ, USA, 2011.
33. Alotaibi, R.; Nassar, M.; Khan, Z.A.; Alajlan, W.A.; Elshahhat, A. Analysis and data modelling of electrical appliances and radiation dose from an adaptive progressive censored XGamma competing risk model. *J. Radiat. Res. Appl. Sci.* **2025**, *18*, 101188. [\[CrossRef\]](#)

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.