

Article

Risk Measure Examination for Large Losses

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Abstract: The risk measures such as value at risk, and conditional values at risk do not always account for the sensitivity of large losses with certainty, as large losses often break the homogeneity especially seen in an illiquidity risk. In this study, we examine the characteristics of large-loss sensitivity more holistically, including small probability, within the framework of risk measures. The analysis incorporates the certainty equivalent, generation of the optimal certainty equivalent formulation, divergence utility, and general utility functions in their original form, and their relationship with expectiles and elicibility. The discussion provides a summary in the understanding of risk measure status and sensitivity involving small probably cases. Additionally, we evaluate large-loss sensitivity in risk-sharing scenarios using the convex conjugation of the divergence utility. By clarifying the conditions affecting large-loss sensitivity, the findings highlight the limitations of existing risk measures and suggest directions for future improvement. Furthermore, these insights contribute to enhancing the stability of risk-sharing business models.

Keywords: large-loss; risk measure; risk management; risk-sharing; star-shaped

MSC: 91G70

1. Introduction

Risk measures have been instrumental in decision making for financial investments, pricing insurance, and identifying risks for management, control, and regulatory purposes. In insurance pricing, in the absence of a risk-neutral measure or a risk-hedging strategy, risk measures can be derived from the insured's utility. Various utility and risk measure models have been examined, yet the recognition of large losses remains an area of ongoing development [1]. While static and single-period risk measure models have been widely explored, research on dynamic risk measures and their optimization applications is still emerging (e.g., [2–5]), with a limited focus on large-loss sensitivity.

A holistic summarization of sensitivity has been missing from the literature. Thus, this study examines the certainty equivalent (CE) and the optimal certainty equivalent (OCE) methods, which include value at risk (VaR) and conditional VaR (CVaR) (or expected short fall, ES), as these represent fundamental risk measures and risk measure generation approaches for both static and dynamic settings, as well as for assessing losses and large losses (e.g., [5–9]). In particular, we explore the characteristics of large-loss risk measures, drawing on [1], which analyzed the homogeneity property of risk measures as an indicator of large-loss sensitivity. Within this framework, the star-shaped risk measure [10] serves as an intuitive example of the essential conditions for a meaningful sensitivity analysis [11].

Practically, an example of a large-loss risk measure characteristic issue is that, in terms of liquidity risk, a larger amount of a variable would make a larger risk (non-homogeneity). For instance, Refs. [12,13] investigate illiquidity risk as a typical non-homogeneity risk. We introduce a holistic perspective on sensitivity by examining probability changes and



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extend the discussion to optimal expected utility (OEU) risk measures [14] and utility-based shortfall risk [15]. The analysis of dynamic risk measures falls outside the scope of this study because the structure of sensitivity is decided almost in a one-period model. Here risk measures of expectiles and their elicibility are also discussed with the case of a single event/insured and with a pooling case [16–18]. The discussion also considers divergence utility (risk measure) (DV), defined as the convex conjugation of OCE [19], as a valuable tool for risk-sharing analyses, which is quite a novel relationship between DV and risk sharing. Another novel contribution of this study is the examination of risk-sharing mechanisms in the context of large-loss risk. In the real world economic scheme of pooling of insurance, Pareto optimal risk taking for each pool participant is reasonable but in the case of a large-loss occurrence, the pooling system needs to think about another solution. In addition, the worst-case risk treated by [20–22] and portfolio-type risk (pooling) analysis in [23] are discussed.

The remainder of this paper is structured as follows. Section 2 provides a review of relevant literature. Section 3 discusses small probability and large-loss sensitivity. Section 4 addresses risk sharing in the context of large losses, and Section 5 discusses other related topics of expectiles, elicibility, and the worst-case discussion. Section 6 concludes the paper.

2. Preparation and Literature Review

2.1. Notation and Risk Measure Characteristics

The mathematical notation used in this study is defined as follows, although in order to maintain the original depiction, some parts have been rendered redundant:

X, Y : Random variable, real number (X might mean the size of the losses. Some cases use $-X$ as the size of the losses and the utility used is $-u(-x)$).

$u(X)$: Utility function or loss function associated with X . The detailed functions are expressed in accordance with their original form with parameters or thresholds such as α, β, γ, c .

$\mathbb{E}[\cdot]$: Expectation under probability measure P . Q denotes another measure.

$\rho(X)$: Risk measure. The details are expressed with parameters or thresholds such as η (Optimizing variable. Estimator.)

$H(X_n)$: Risk-sharing rule for pooling of n members. $i = 1$ to n .

α, β, γ, c : Parameters shown in original forms. $\alpha, \beta, \gamma, c \in [0, 1]$ or $\alpha, \beta, \gamma, c > 0$.

A risk measure quantifies the amount of cash required to mitigate risk. Loss and utility functions serve as the basis for defining risk measures. Coherent and convex risk measures have been proposed as essential characteristics for effective risk assessment. Those characteristics are outlined below:

Monotonicity: $\rho(X) \geq \rho(Y)$ if $X \leq Y$.

Cash Invariance: $\rho(X + m) = \rho(X) - m$.

Normalization: $\rho(0) = 0$

means monetary by definition.

Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$

means convex by definition.

Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$

means coherent by definition, which can be further added to convexity.

In financial investment theory, coherency includes diversification characteristics and the portfolio theory fits this. Convex is a condition with no homogeneity, but risk aversion is still maintained (Refs. [24–27] provide a comprehensive review; see their references). However, the treatment of large losses requires a separate discussion, which is addressed in Section 3. (Other characteristics that are monetary and comonotonic-additive are Choquet

in nature, and law invariant Choquet risk measures are distortion risk measures, which are related but are not discussed in this paper).

Lastly, in the case of $\rho(\lambda X) \geq \lambda \rho(X)$ for $\lambda \geq 1$, ρ is called positively star-shaped (which is usually called, simply, star-shaped), and in the case of $\rho(\lambda X) \leq \lambda \rho(X)$ for $\lambda \geq 1$, ρ is called negatively star-shaped.

2.2. Risk Measures

CE ($\rho_{X,\mathbb{P}}^{CE}(X) \in \mathbb{R}$) is defined as follows:

$$\rho^{CE} = u^{-1}(\mathbb{E}[u(X)]). \quad (1)$$

Financially, this implies that, for a concave utility function (representing risk aversion), $|\mathbb{E}[X] - \rho^{CE}|$ denotes the amount which an individual is willing to pay for an insurance premium (or an option premium) more than the expected loss for a potential loss. As the utility function is concave, in the case where X expresses wealth size, ρ^{CE} is smaller than the expected wealth size.

Next, OCE ($\rho_{X,\mathbb{P}}^{OCE}(X), \eta_{X,\mathbb{P}}^{OCE} \in \mathbb{R}$) is defined as follows:

$$\rho^{OCE}(X) = \sup_{\eta^{OCE}} \{\eta^{OCE} + \mathbb{E}[u(X - \eta^{OCE})]\}. \quad (2)$$

This formulation determines the amount of money that should be reserved for uncertainty and the extent of risk exposure. Examples of [8] are presented in Table 1 and those are adequately showing financial investment practices.

Table 1. Examples of OCE.

	Shape of Utility	OCE
a.	$u(x) = x$	$\mathbb{E}[X]$
b.	$u(x) = \frac{1}{\beta} e^{\beta x} - \frac{1}{\beta}$	$\frac{1}{\beta} \mathbb{E}[e^{\beta X}]$
c.	$u(x) = -\frac{1}{\alpha}(-x)^+$	$\mathbb{E}[X X \leq q(\alpha)]$ (=CVaR(X))
d.	$u(x) = \frac{1}{4c} \left(x > \frac{1}{2c} \right), x - cx^2 \left(x \leq \frac{1}{2c} \right)$	$\mathbb{E}[X] - c \times VaR(X)$

The utility a. is risk neutral and this leads an OCE with an average of X . The utility b. is convex and this leads to an OCE that is generally larger than $\mathbb{E}[X]$. The utilities of c. and d. are concave and with a kink, leading the indicative index VaR to show up. In particular, d. is the target of a typical portfolio optimization.

DV (their $u_G(X)$, $c^{DV} \leftrightarrow$ here $\rho_{X, Z, \mathbb{P}}^{DV}(X)$, $\eta_{X, Z, \mathbb{P}}^{DV}$) is introduced by [19] as a risk measure, defined as follows, with modified notation:

$$\rho^{DV}(X) = \inf_{Z \in \mathcal{P}} \{\mathbb{E}[ZX + G(Z)]\}. \quad (3)$$

($\mathcal{P}$ is the set of positive random variables Z , $\mathbb{E}[Z] = 1$, and $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a convex function satisfying certain conditions.)

The interpretation is revealed below. It is a kind of OCE conjugate. The formulation below represents the convex conjugate of OCE and can also be expressed (with $c^{DV} \in \mathbb{R}$) as follows:

$$\rho^{DV} = \sup_{\eta^{DV}} \{\eta^{DV} - \mathbb{E}[F(\eta^{DV} - X)]\}, F(X) = \sup_{y \in \mathbb{R}} \{xy - G(y)\}. \quad (4)$$

In the financial investment world, DV is meaning a stress test. To find the worst case using scenarios (Q) other than usual scenario (P), but with some allowance (typically, radon

nikodym derivative dP/dQ is chosen), DV is defined in another way as shown below (See [19]).

$$\rho^{DV} = \sup_Q \{ \mathbb{E}^Q[-X] - \mathbb{E}[G(dQ/dP)] \}$$

An example of [19] is presented in Table 2.

Table 2. Example of DV.

Shape of Utility	DV	cf.
$G(x) = \frac{x \ln(x)}{\alpha}, F(x) = \frac{e^{\alpha x} - 1}{\alpha}$	$-\frac{\ln(\mathbb{E}[e^{-\alpha X}])}{\alpha}$	$\eta = \frac{1}{\alpha} - \frac{\ln(\mathbb{E}[e^{-\alpha X}])}{\alpha}$

This DV example shows a convex shape utility and a concave shape risk measure.

(Here X is assumed to be the size of losses and P, Q have smooth boundaries.)

The OEU risk measure, introduced by [14], is defined as follows, with a modified notation $\rho^{OEU}_{X, \mathbb{P}}(X), \eta^{OEU}_{X, \mathbb{P}} \in \mathbb{R}$ (ρ is treated as negative):

$$\rho^{OEU} = -\sup_{\eta^{OEU}} \{ -\beta \eta^{OEU} + \alpha u^{-1}(\mathbb{E}[u(X + \eta^{OEU})]) \}. \quad (5)$$

An alternative formulation, expressing marginal utility decisions directly, is given by the following (the parameters α and β in [14] are both set to 1):

$$u'(u^{-1}(\mathbb{E}[u(X + \eta^{OEU})])) = \mathbb{E}[u'(X + \eta^{OEU})]. \quad (6)$$

This risk measure means that the amount of capital required to make an investment in X appears favorable to him (See [14]). If the investor holds a financial position with a future net value X and must decide on the amount of risk capital that he reserves to achieve an optimal intertemporal allocation, he faces this optimization problem.

Examples of [14] are presented in Table 3. The interpretation is that the risk is compensated nicely by cash in terms of a utility point of view. Among those, some illustrative descriptions are in Figure 1.

Table 3. Examples of OEU.

	Shape of Utility	OEU
a.	$u(x) = \ln(x), x > 0$	$-\sup_{\eta^{OEU} \geq X_{min}} \{ -\beta \eta^{OEU} + \alpha \times \exp(\mathbb{E}[\ln(X + \eta^{OEU})]) \}$
b.	$u(x) = \frac{x^{1-\gamma}}{1-\gamma}$	$-\sup_{\eta^{OEU} \geq X_{min}} \{ -\beta \eta^{OEU} + \alpha \times \exp(\mathbb{E}[(X + \eta^{OEU})^{1-\gamma}]^{\frac{1}{1-\gamma}}) \}$
c.	$u(x) = \frac{1-e^{-\gamma x}}{\gamma}$	$(\alpha - \beta)X_{min} + \frac{\alpha}{\gamma} \ln(\mathbb{E}[e^{-\gamma X}])$
d.	$u(x) = x$	$(\alpha - \beta)X_{min} + \alpha \mathbb{E}[X]$
e.	$u(x) = 0$	$-\beta X_{min}$
f.	$u(x) = -(-x)^+$	$-\infty$ for $\alpha < 1, \mathbb{E}[-X]$ for $\alpha = 1, CVaR(X)$ for $\alpha > 1$ ($\beta = 1$)

The results of a., b., and c. are the mathematical expressions for typical utility functions. The OEU of d. and e. can be compared to OCE a. of Table 1. Also, the OEU of f. can be compared to OCE c. of Table 1. Regarding the former, as the range of the function u^{-1} starts at X_{min} , there is a difference of X_{min} among them. Regarding the latter, as the inverse function of $-(-x)^+$ is the same as $-(-x)^+$, the result is the same.

UBSR, discussed in [15], is defined as follows, with the modified notation ($\rho^{UBSR}_{X, \mathbb{P}}(X), \eta^{UBSR}_{X, \mathbb{P}}(z), z \in \mathbb{R}$):

$$\rho^{UBSR}(z) = \inf_{\eta^{UBSR}} \{ \mathbb{E}[u(X + \eta^{UBSR})] \leq z \}. \quad (7)$$

This means that how much should be prepared in order to escape the outcome X is above the target level (z). An example of [15] is presented in Table 4.

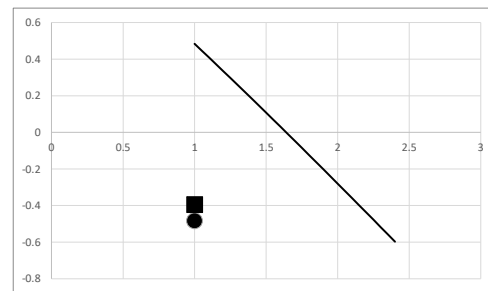


Figure 1. Risk measure OEU examples. OEU of Table 3 is treated as negative as in [14]. The line is the Sup function part of Table 3a, where the vertical axis is the value of the function, the horizontal axis is η , and the dot shows the OEU value of Table 3a, where $\alpha = \beta = 1$ and $X \in [0, 1]$ are uniformly distributed. The black square shows the OEU value of Table 3d. Its magnitude is smaller than that of Table 3a because Table 3a shows the risk-averse situation and Table 3d shows the risk-neutral situation.

Table 4. Example of USBR.

Shape of Utility	USBR
$u(x) = e^{\alpha x}$	$\frac{\ln(\mathbb{E}[e^{-\alpha X}]/z)}{\alpha}$

This example shows a convex utility case. In the case of risk-neutral utility $u(x) = x$, USBR is simple and it is $z - \mathbb{E}[X]$.

Notably, the entropy risk measure ([27,28] and its references) is a typical example of these measure formulations when an exponential utility function is used.

2.3. Optimization and Dynamic Risk Management

Continuous-time and multi-period optimization approaches for risk measures, including minimization strategies, have been examined (e.g., [2–5]). These studies fall within dynamic optimization, yet the fundamental treatment of large losses remains consistent with static or single-period models. Therefore, in this study, we focus on static analyses. In other words, the structure of sensitivity is decided almost in a one-period model, and the dynamic model treats no new sensitivity issues. Ref. [29] discussed a dynamic optimization issue with large-loss consideration in case of risk sharing related to Section 2.6. Also, Ref. [30] discussed BSDEs in a dynamic and star-shaped sense.

2.4. Sensitivity to Losses and Large Losses

Tail risk consideration is the key to losses and large losses. Refs. [31–33] discussed a good utility model for tail risk and its conditions. Ref. [1] examined the sensitivity of risk measures to large losses. Loss sensitivity and large-loss sensitivity are defined as follows:

“... defined on a suitable domain of random variables over a probability space $(\Omega, \mathcal{F}, P)$. Here, the elements of X represent positions, wealth, etc. R and U are assumed to be monotone and normalized, implying that a position with a worse loss profile has higher risk/lower utility and the risk/utility of the null position is set to zero by convention. Consequently, we cover a very wide spectrum of concrete functionals, including convex and nonconvex risk measures, concave and S-shaped expected utility functionals, and OCE functionals. Each risk or utility functional can be naturally associated with a corresponding utility or risk functional through... Let $S \subset X$. A risk functional R or a utility functional U is sensitive to losses on S if, for every $X \in S$ with $P(X < 0) > 0$, we

have $R(X) > 0$ and respective $U(X) < 0$. It is called sensitive to large losses on S if, for every $X \in S$ with $P(X < 0) > 0$, $\lambda X \in (0, \infty)$ exists such that $R(\lambda X) > 0$ and respective $U(\lambda X) < 0$, for each $\lambda \in (\lambda X, \infty)$”

The concept of star-shaped risk measures introduced by Ref. [10] is also discussed in Ref. [1]. However, in this study, we approach risk measures by including a different perspective, which is addressed in a later section.

2.5. Regulatory Framework

Solvency rules in insurance industries and Bank for International Settlements rules in banking industries incorporate risk measures to determine capital requirements. VaR or CVaR have been widely used in these regulations. However, Ref. [34] illustrated that risk management based on VaR or CVaR limits is ineffective for decision-makers with S-shaped utility functions. As discussed earlier, standard risk measures do not adequately capture cumulative large losses.

2.6. Risk—Sharing

The following is based on [35]’s summary. The universal nature of risk-sharing rules were examined by [36,37]. They proposed the ideal form of risk-sharing rules. Refs. [38–41] analyzed the general characteristics of risk sharing. Following [37], a standard risk-sharing theory involves n individuals forming a risk-sharing pool; X_i is set as the stochastically determined loss of an individual i ($i = 1$ to n). The losses of pool individuals in a certain period of time are considered to be shared among n individuals according to certain rules. X_i is a random variable in a general probability space (Ω, F, P) , and its mean value is finite. The loss vector $\mathbf{X}_n = (X_1, X_2, \dots, X_n)$ is called a pool. S_n is the total loss of the pool. It is decided in advance that i will bear the loss of $H(\mathbf{X}_n)$, which is realized after one period has passed (risk-sharing rule):

$$\mathbf{H}(\mathbf{X}_n) = (H_1(\mathbf{X}_n), H_2(\mathbf{X}_n), \dots, H_n(\mathbf{X}_n)).$$

$$\sum_{i=1}^n H_i(\mathbf{X}_n) = \sum_{i=1}^n X_i$$

For example, the case where everyone equally shares the risk (in this case, H_i^u is used) is as follows:

$$H_i^u(\mathbf{X}_n) = \frac{S_n}{n}, \quad (S_n = \sum_{i=1}^n X_i). \quad (8)$$

$$(\mathbb{E}[\sum_{i=1}^n H_i^u(\mathbf{X}_n)]) = \mathbb{E}[\sum_{i=1}^n \frac{S_n}{n}] = \mathbb{E}[\sum_{i=1}^n X_i]$$

In this case, however, each individual’s expected loss value differs before and after risk sharing:

$$\mathbb{E}[H_i^u(\mathbf{X}_n)] = \frac{S_n}{n} \neq \mathbb{E}[X_i].$$

How the pool members’ characteristics affect the pool quality and the pool participants’ difficulty to enter were discussed in [42].

Regarding large losses, for the risk-sharing pool, large losses are important. Ref. [43] discussed this excess loss distribution. This study addresses divergence utility, as introduced by [19], and it provides a risk-sharing optimization framework. The relevant optimization problem and its solution using a divergence utility formulation are discussed in later sections.

3. Sensitivity to Large Losses

3.1. Small Probability of Large Loss and Its Effect on CE

The impact of heavy-tailed distributions on utility maximizing (or risk- or loss-minimizing) optimization and research on a better utility function for the issue were examined in [31–33]. In this section, we analyze the behavior of CE as a risk measure, particularly when the large-loss probability is small. Empirically, it is often observed that a large loss happens with a very small probability (e.g., [44]). Calculations are performed using power utility (constant relative-risk aversion (CRRA) utility function) and exponential utility (constant absolute-risk aversion (CARA) utility function) under the following loss scenario:

$$X \sim [0, -L], P(X = -L) = p, P(X = 0) = 1 - p,$$

where X is the random variable representing risk, $-L$ is the large loss, and p is its probability.

Using the CRRA utility $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ (γ : risk averseness), CE (CE^{CRRA}) can be calculated as follows:

$$p \frac{(w-L)^{1-\gamma}}{1-\gamma} + (1-p) \frac{w^{1-\gamma}}{1-\gamma} = \frac{(w-CE^{CRRA})^{1-\gamma}}{1-\gamma} (w : \text{initial wealth}).$$

Applying the condition $p \ll 1$ and using Taylor expansion with $L \rightarrow w$, we obtain

$$CE^{CRRA} \approx pL(1 + \frac{\gamma p}{2(1-\gamma)^2}). \quad (9)$$

For CARA utility $u(x) = -\frac{e^{-\gamma x}}{\gamma}$ (γ : risk averseness), CE (CE^{CARA}) can be derived as follows:

$$p(-\frac{e^{-\gamma(w-L)}}{\gamma}) + (1-p)(-\frac{e^{-\gamma w}}{\gamma}) = -\frac{e^{-\gamma(w-CE^{CARA})}}{\gamma} (w : \text{initial wealth}).$$

Applying the condition $p \ll 1$ and using Taylor expansion, we obtain

$$CE^{CARA} \approx pL \frac{e^{\gamma L}}{\gamma L}. \quad (10)$$

Equation (9) implies that under CRRA utility, CE remains higher than the pure premium (pL), with the difference being proportional to p . In this case, as p decreases, the insurance risk buffer (difference between CE and pure premium) shrinks. By contrast, Equation (10) indicates that under CARA (exponential) utility, the difference between CE and the pure premium grows with L , irrespective of p . This highlights a key distinction: CRRA utility exhibits constant relative-risk aversion, whereas CARA utility maintains constant absolute-risk aversion, leading to different sensitivity behaviors. Practically, this indicates that insurance premium principle and CRRA utility clients can still require some insurance premium buffer, while CARA utility clients do not. Further studies offered complementary insights. Optimal policies for low-probability extreme events were examined in [45], while [46] analyzed cumulative probability trends.

Figure 2a,b shows the illustrative status of the above Equations (9) and (10). Regarding the size of CE relative to the expected loss size when the loss increases, the CRRA utility case Figure 2a shows a lower bound and the CARA utility case Figure 2b shows increases.

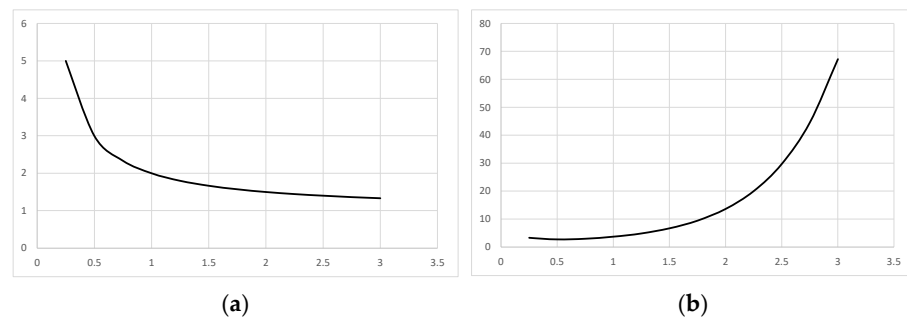


Figure 2. Relative size of CE. The left (a) shows the CRRA utility case and the right (b) is CARA utility case. In both figures, the vertical axis shows CE/pL and the horizontal axis shows L, supposing $pL = 1$ and $\gamma = 2$.

3.2. Large—Loss Sensitivity

Risk measures and utility sensitivity to large losses using the definitions outlined in Section 2.4 were examined in [1]. They investigated [10]’s star-shaped risk measures, cash additive risk measures, and some localized cases. The optimization of star-shaped risk measures, demonstrating their usefulness in evaluating large-loss sensitivity were summarized by [11].

In the context of this study’s risk measure categorization, Ref. [1] found that VaR generally fails to be sensitive to large losses. This is also true of ES (in this paper, CVaR and ES are treated the same). By contrast, expected utility functions, as well as (optimized) certainty equivalents, are sensitive to large losses for several standard choices of concave and nonconcave utility functions, including star-shaped utility functions. As the formulation of OEU is similar to OCE, its sensitivity characteristics are expected to be identical. Additionally, as UBSR is related to traditional risk measures (as described in Table 3), its sensitivity characteristics align accordingly. The above is summarized in Table 5.

Table 5. Conditions for large-loss sensitivity.

	Risk Measures	Sensitive to Large Losses?	Condition for Yes.
a.	VaR/CVaR	Yes for special cases.	If they are either properly adjusted or if the property is suitably localized.
b.	CE	Yes for a special case.	Negatively star shaped and $\lim_{x \rightarrow \infty} \text{Sup} \left\{ \frac{u(-x)}{u(x)} \right\} = -\infty$
c.	OCE DV OEU USBR	Yes for many cases. Valid for nonconcave utilities.	Risk of scalar multiplied is infinite or true below. $u(x) < x$ and; $\lim_{x \rightarrow -\infty} \text{Inf} \left\{ \frac{u(x)}{x} \right\} = \infty, \quad \lim_{x \rightarrow \infty} \text{Sup} \left\{ \frac{u(x)}{x} \right\} = 0.$
d.	Cash Additive	Yes for a special case.	When the risk is $\text{sup}(-X)$.
e.	Star-Shaped	Yes for many cases.	-

The general illustrative explanation is in the text. Risk/utility characteristics of large-loss sensitivity are summarized.

Further studies offer complementary insights. While uncertainty and ambiguity may influence large-loss sensitivity, Ref. [47] focused on ambiguity, and [48] decomposed risk, treating ambiguity separately from the loss size.

Figure 3a,b show illustrative explanation of Table 5, especially Table 5c. The risk measure OEU example is using Figure 3a function and it coincides with CVaR In this case, not large-loss sensitive. The risk measure OEU from using utility Figure 3b satisfies conditions of Table 5c and has a large-loss sensitive risk measure. Also as Figure 3a indicated, some special case, there is a boundary for variable X when utility or risk measure calculation.

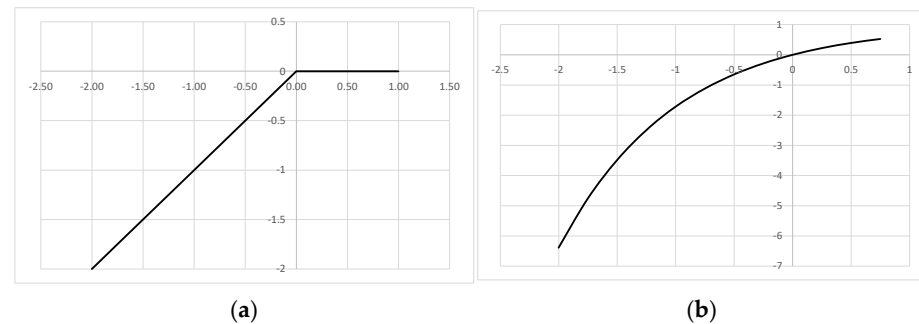


Figure 3. Utility examples used for OEU. The left (a) shows utility function $u(x) = -(-x)^+$ and the right (b) shows utility function $u(x) = 1 - e^{-x}$ with the vertical axis is utility and the horizontal axis is X .

3.3. Interpretation of Large—Loss Sensitivity

Regulatory frameworks, such as those based on VaR and CVaR, rely on coherent risk measures [10]. While CVaR is widely regarded as effective, it does not perfectly capture large-loss sensitivity. Potential alternatives include employing utility functions specifically designed for large-loss sensitivity (e.g., star-shaped functions) or using CVaR within constrained loss distribution models to improve sensitivity. For this issue, Ref. [49] could be the solution. This expanded the CVaR and introduced adjusted expected shortfall (adjusted CVaR). This is CVaR with simultaneously controlling expected losses associated with different portions of the tail distribution.

From financial investment point of view, the portfolio management wisdom values diversification and large-loss discussion above tells that there might be the status that, the larger size, the larger risk, like illiquidity risk, and it should be prepared. For example, use of CARA utility (exponential type) would be the candidate for the solution. In addition, considering large-loss sensitivity is a favorable characteristic. Just in case, among others, Ref. [14] observed that OEU exhibits smoother, more granular changes in response to large-loss distribution shifts compared to VaR and CVaR. In Section 3.1, the loss size is considered purely in terms of insurance coverage.

3.4. Policy Implications

Currently, CVaR (ES) is the standard risk measure in the fundamental review of the banking and other financial institutions business activities. Regulatory rules typically rely on VaR and CVaR [10], these measures do not perfectly capture large-loss sensitivity. Possible enhancements include using utility functions that account for large-loss sensitivity, such as star-shaped functions, applying CVaR within constrained loss distribution models or improved performance, or, under some conditions, CE, OCE, and others are good indicators.

3.5. Limitation

There would still be limitations for large-loss sensitivity use. One is that securing elicibility is not easy (See Section 6. CVaR also has a problem). The other issue is popularity. There are no standard or popular large-loss sensitive risk measures, as current popular risk measures like VaR, CVaR, or entropy risk measures (from exponential utility) have $\rho(\lambda X) = \lambda \rho(X)$ characteristics but they are easy to understand and popular.

4. Risk Sharing

4.1. Risk Sharing for Large Losses

The background of risk sharing for large losses is the following. In recent years, insurance and mutual assistance mechanisms based on pooling have been re-emerging through

an insurance business model known as P2P insurance (Lemonade in the US, Friendsurance in Germany, etc.). For such risk-sharing mechanisms, fair and reasonable methods of loss sharing within the pool are being explored. In other words, Pareto optimality is considered, but as will be described later, this is a solution that places a burden on individuals with low risk aversion. In casual terms, this is a situation in which those who are more likely to bear risk are the ones who suffer, and the problem becomes even more serious when huge losses occur.

Using [50,51] with some assumptions and arrangements, the following is denoting Pareto optimal risk sharing $H_i^{pareto}(\mathbf{X}_n)$:

$$H_i^{pareto}(\mathbf{X}_n) = \frac{S_n}{n} + \left(\frac{\frac{1}{\gamma_i}}{\sum_{i=1}^n \frac{1}{\gamma_i}} - \frac{1}{n} \right) S_n + E[(X_i - \frac{1}{\gamma_i} S_n) e^{-\frac{S_n}{\sum_{i=1}^n \frac{1}{\gamma_i}}}] / E[e^{-\frac{S_n}{\sum_{i=1}^n \frac{1}{\gamma_i}}}] \quad (11)$$

This shows that the distribution of risk is performed based on the degree of risk averseness of each individual. Some extended studies are [52], who examined the quantile base, Ref. [53], who made use of CVaR, and [54], who developed a computational method.

Equation (11) tells us that, for the sensitivity to a loss burden for each pool participant, the burden of each participant depends on their risk averseness. In terms of a large loss, for risk sharing, the large loss is the problem because there might be no option to distribute the risk burden among the pool members. Ref. [55] developed a practical method and [43] uses a conditional mean for the large loss. Regarding this paper discussion, using divergence utility [19], we consider the following optimization problem:

$$\text{Max } \sum_{i=1}^n \rho^{DV}(X_i), \text{ subject to } X = \sum_{i=1}^n X_i.$$

This intends to maximize the total utility for the individual participant (not always Pareto optimal). The solution of the risk-sharing rule $\mathbf{H}^{DV}(\mathbf{X}_n)$, incorporating individual-specific $G_i(X)$ functions of divergence utility and their derivatives $g_i(X)$, along with the derivative $f(X)$ of $F(X)$, is given by the following:

$$\mathbf{H}^{DV}(\mathbf{X}_n) = (c_1 - g_1 \circ f(c - X), c_2 - g_2 \circ f(c - X), \dots), \quad c = \sum_{i=1}^n c_i. \quad (12)$$

4.2. Interpretation and Implication

Here is an example of an insurance pool case. In the simplified case of Equation (12), where $g_i(x) = k_i x$, $f(y) = y$, Equation (12) implies the following (Also see Figure 4):

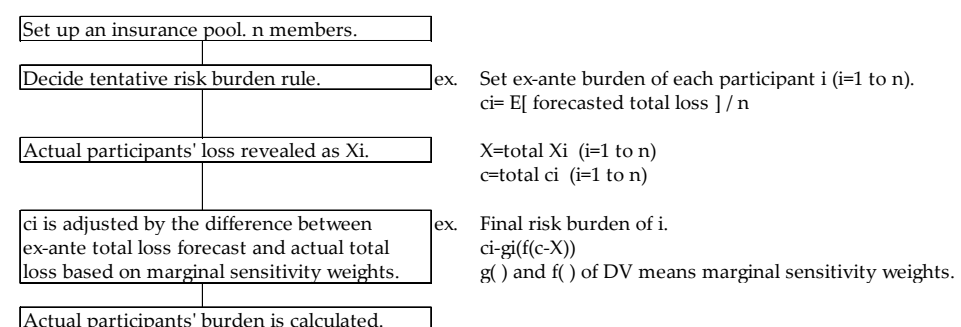


Figure 4. Decision Flow. The flow chart shows how to decide the risk burden of each pool participant.

- A predefined risk-sharing rule determines each individual's initial burden c_i .
- After the loss realization, the total burden (c_i) is adjusted to match the actual losses (X_i).
- The difference is redistributed based on marginal sensitivity weights (k_i).
- This approach combines predefined allocation with post-adjustment, making it a practical and flexible risk-sharing mechanism.

- More substantially, for example, using Table 3, in case $G(x) = \frac{x \ln(x)}{\alpha}$, $F(x) = \frac{e^{\alpha x} - 1}{\alpha}$ is used since $g(x) = \frac{\ln(x)+1}{\alpha}$, $f(x) = e^{\alpha x} - 1$, and $\mathbf{H}^{DV}(\mathbf{X}_n)$ is as below:

$$\mathbf{H}^{DV}(\mathbf{X}_n) = (c_1 - k_1(c - X), \dots).$$

This means, regarding the worst case, that no insurance effect is available if each individual bears a far larger loss burden than their own risk tolerance if actual losses are huge. For large losses, other backstop functions must be available. This implies that in the case of risk sharing, preparing for large losses requires a centralized system [56]. Other practical related issues include a post insurance premium adjustment, especially when large losses occur and the risk-sharing rule operates.

Table 6 shows illustrative risk-sharing examples. When larger than expected losses happen, even fair risk sharing seems does not seem favorable (Case A). In that sense, government support, like for Case B's #3 participant, would be helpful.

Table 6. Illustrative risk sharing.

	Ex-Ante Burden	Actual Each Participant Loss Size	Risk-Sharing with Same Sensitivity (Case A)	Case B Sensitivity Weights	Risk-Sharing (Case B)
#1 Participant	10	30	30	20%	22
#2 Participant	5	15	25	20%	17
#3 Participant	15	45	35	60%	51
Total	30	90	90	100%	90

Case A supposes sensitivity weights are the same among participants and Case B supposes the weights are 20%, 20%, and 60% for #1, #2, and #3, respectively. The total actual loss size was larger than the ex-ante burden total. The excess amount is distributed based on the sensitivity weights. Participant #3 in Case B would be a government.

5. Other Related Topics

5.1. Elicitability

As Refs. [18,57,58] described, the concept of elicibility has been deeply researched. For instance, from a back-testing point of view, the risk measure CVaR is less reliable than the risk measure VaR, as it is intuitively natural that the quantile could be observable and measurable by data but the loss magnitude beyond the quantile threshold might be unforecastable. This elicibility was generally discussed by [59–61].

5.2. Expectiles

The concept of an expectile as a risk measure is well researched [62–64]. The definition is as below (not used original parameters):

$$\min_{\eta} \mathbb{E}[\alpha \max(X - \eta, 0)^2 + (1 - \alpha) \max(\eta - X, 0)^2]. \quad (13)$$

The characteristics of expectiles and other risk measures are as below in Table 7 [64].

Table 7. Characteristics of expectiles and other risk measures.

	Coherence	Elicitability
Variance	X	X
VaR	X	O
CVaR (ES)	O	X
Expectiles ($\alpha \geq 1/2$)	O	O

"O" means true and "X" means false.

5.3. Large Losses and Pooling Discussions Related to Elicitability and Expectiles

Regarding large losses, Ref. [17] investigated elicibility for tail risk measures. For pooling, Ref. [65] discussed the elicibility of the risk measure for risk pooling. Diversification measurement of risk pooling regarding expectiles was shown in [23]. In those ways, as a quality point of view, elicibility and expectile analyses gives us useful additional large-loss sensitivity information.

5.4. Worst-Case Risk

By the notation of [66] (Y : random variable),

$$VaR_{\alpha}(Y) = -\inf_{x \in \mathbb{R}} (F_Y(x) \geq \alpha),$$

and, using the α th expectile, the

$$\text{worst-case value of the expectile} = \sup_{F_Y} (\arg \min_{\theta} \mathbb{E}[\eta_{\alpha}(Y - \theta) - \eta_{\alpha}(Y)])$$

This is coherent and is a law invariant based on the Kusuoka representation [67], and Ref. [21] discussed the asymptotic behavior of it. However, for the discussion of large losses, the worst case is not always the matter, as it is not a sensitivity discussion. The worst-case risk measure by expectiles is also discussed in [21]. The worst-case is to some extent the issue of large losses but the sensitivity of large losses might be more useful for actual uses.

6. Conclusions

Standard risk measures such as VaR and CVaR do not always ensure sensitivity to large losses. This study explored various loss and probability sensitivity characteristics, focusing on OCE formulations, UBSR, and star-shaped risk measures. Additionally, we examined risk-sharing mechanisms for large losses using convex conjugation of OCE. By clarifying the conditions for large-loss sensitivity and their variations, the analysis highlighted the limitations of existing regulatory risk measures and suggested potential improvements. The findings can also contribute to more stable risk-sharing frameworks in business applications. While regulatory rules typically rely on VaR and CVaR [10], these measures do not perfectly capture large-loss sensitivity. Possible enhancements include using utility functions that account for large-loss sensitivity, such as star-shaped functions, applying CVaR within constrained loss distribution models or improved performance, or, under some conditions, CE, OCE, and others are good indicators. In addition, the risk measures of expectiles and their elicibility are also discussed. Expectiles are useful from the point of large-loss sensitivity. Regarding elicibility, VaR is elicitable although CVaR is not. Finally, divergence-based risk measures naturally lead to a predefined and post-adjusted risk-sharing approach, which aligns well with practical applications like peer-to-peer (risk-sharing) insurance and ex-post insurance premium adjustment. Future research areas would be dynamic risk measure setting [30], relationships with uncertainty [68], and worst-case risk cases [21].

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