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Geometric and Structural Properties of Indefinite Kenmotsu Manifolds Admitting Eta-Ricci–Bourguignon Solitons

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Abstract: This paper undertakes a detailed study of η-Ricci–Bourguignon solitons on ϵ-Kenmotsu manifolds, with particular focus on three special types of Ricci tensors: Codazzi-type, cyclic parallel and cyclic η-recurrent tensors that support such solitonic structures. We derive key curvature conditions satisfying Ricci semi-symmetric $(R \cdot E = 0)$, conharmonically Ricci semi-symmetric $(C(\xi, \beta_X) \cdot E = 0)$, ξ-projectively flat $(P(\beta_X, \beta_Y)\xi = 0)$, projectively Ricci semi-symmetric $(L \cdot P = 0)$ and W_5-Ricci semi-symmetric $(W(\xi, \beta_Y) \cdot E = 0)$, respectively, with the admittance of η-Ricci–Bourguignon solitons. This work further explores the role of torse-forming vector fields and provides a thorough characterization of ϕ-Ricci symmetric indefinite Kenmotsu manifolds admitting η-Ricci–Bourguignon solitons. Through in-depth analysis, we establish significant geometric constraints that govern the behavior of these manifolds. Finally, we construct explicit examples of indefinite Kenmotsu manifolds that satisfy the η-Ricci–Bourguignon solitons equation, thereby confirming their existence and highlighting their unique geometric properties. Moreover, these solitonic structures extend soliton theory to indefinite and physically meaningful settings, enhance the classification and structure of complex geometric manifolds by revealing how contact structures behave under advanced geometric flows and link the pure mathematical geometry to applied fields like general relativity. Furthermore, η-Ricci–Bourguignon solitons provide a unified framework that deepens our understanding of geometric evolution and structure-preserving transformations.

Keywords: ϵ-Kenmotsu manifold; Ricci soliton; η-Ricci–Bourguignon solitons; ϕ-Ricci symmetric; conharmonic curvature tensor

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1. Introduction

In contemporary Riemannian geometry, geometric flows have become fundamental tools for analyzing the evolution of geometric structures. Among these, a particularly important class consists of metric evolutions governed by combined scaling and diffeomorphism transformations. The solutions to these flows, called solitons, have gained prominence for their ability to model and characterize singularity formation in geometric

evolution processes. Consequently, solitons now represent crucial paradigms in the study of geometric flows, providing key insights into the nature of singularities and serving as essential models for understanding the long-term behavior of such evolutionary systems.

The R–B flow serves as an important extension of the classical Ricci flow [1], first originated by Bourguignon [2] based on foundational work by Lichnerowicz (unpublished) and key results from Aubin [3]. Being an intrinsic geometric flow on Riemannian manifolds, its fixed points inherently correspond to solutions of the R–B soliton equation. R–B solitons are of particular significance, as they offer self-similar solutions to the flow, as demonstrated in [4]. These solitons play a crucial role in modeling the flow's singularities and analyzing its long-term behavior; they are defined by the relation

$$\frac{\partial g}{\partial t} + 2E = 2\theta s g, \qquad g(0) = g_0. \tag{1}$$

In this context, E denotes the Ricci curvature of the Riemannian manifold, s represents the scalar curvature associated with the metric g and $\theta \in \mathbb{R}$ is a constant. From Equation (1), the partial differential equation (PDE) defines the evolution equation, as illustrated in the following table, which is derived in [5] (Table 1).

Table 1. Evaluation equations by PDE.

Constant (θ)	Conditions for $\frac{\partial g}{\partial t}$	Tensors	Solitons
0	$-2E$	Ricci tensor	Ricci soliton [1]
$\frac{1}{2}$	$E - \frac{s}{2}g$	Einstein tensor	Einstein soliton [6]
$\frac{1}{n}$	$E - \frac{s}{n}g$	Traceless Ricci tensor	Trace-free Ricci soliton
$\frac{1}{2(n-1)}$	$E - \frac{s}{2(n-1)}$	Schouten tensor	Schouten soliton [5]

In two dimensions, the last three tensors in the table vanish, leading to a static flow. In higher dimensions, the values of θ are presented above.

The work of [5] establishes that the R–B flow, given in (1), admits a unique solution for sufficiently small $t > 0$ when the parameter θ satisfies $\theta < \frac{1}{2(n-1)}$, where $n \geq 2$. Moreover, as demonstrated in [1,7], quasi-Einstein metrics (including Ricci solitons) emerge as special solutions to the classical Ricci flow ($\theta = 0$), satisfying $E + Hess\, f = \lambda g$ for some smooth potential function f and constant λ. Aubin [3] was the pioneer in introducing the R–B flow on complete Riemannian manifolds. Subsequent contributions from researchers such as De et al. [8] and Siddiqi [9] have further advanced the theory of R–B solitons.

A (semi-)Riemannian manifold of dimension $n \geq 3$ is named as an R–B soliton [3] if it satisfies the equation

$$\pounds_{\beta_V} g + 2E = 2(\theta s - \lambda)g. \tag{2}$$

In this equation, $\pounds_{\beta_V}$ represents the Lie derivative with respect to the vector field β_V, referred to as the soliton or potential vector field. The parameter θ is a non-zero real constant that governs the contribution of the scalar curvature, while λ is an arbitrary real constant characterizing the nature of the soliton. Analogous to the classification of Ricci solitons, R–B soliton is categorized as expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$, respectively.

By modifying the defining equation of the R–B soliton in (2) through the addition of a multiple of the specific (0,2)-tensor field $\eta \otimes \eta$, we arrive at a broader framework referred to as η-R–B solitons [9]. These solitons are characterized by the equation

$$\pounds_{\beta_V} g + 2E = 2[(\theta s - \lambda)g - \mu \eta \otimes \eta], \tag{3}$$

where η is a 1-form and μ is a real constant. If $\theta = 0$, then Equation (3) reduces to the η-Ricci soliton (see [10–20]). Numerous contributions on η-R–B solitons were made by [21–24] and many other authors.

This paper aims to explore η-R–B solitons within the ϵ-Kenmotsu geometry in the following manner: Section 2 establishes the mathematical foundation, detailing the key properties of ϵ-Kenmotsu manifolds and requisite geometric tools. Section 3 develops the theoretical framework for η-R–B solitons on ϵ-Kenmotsu manifolds, with particular attention given to the functional relationship between the soliton constants λ and μ. Section 4 conducts a specialized analysis of η-R–B solitons-compatible Ricci tensors, focusing on three fundamental types: Codazzi-type, cyclic parallel and cyclic η-recurrent. Section 5 extends the investigation to indefinite Kenmotsu manifolds, analyzing η-R–B solitons in the context of projective, conharmonic and W_5-curvature tensors. Section 6 elucidates the geometric consequences of torse-forming vector fields in η-R–B soliton structures. Section 7 provides a complete characterization of ϕ-Ricci symmetric indefinite Kenmotsu manifolds that admit η-R–B soliton solutions. The concluding section demonstrates the physical realizability of these structures through explicit examples of η-R–B solitons on ϵ-Kenmotsu manifolds.

2. Fundamental Concepts

An n-dimensional smooth manifold $(\mathcal{F}, g)$ is characterized as an ϵ-almost contact metric manifold [25] if it is equipped with a (1,1)-tensor field ϕ, a structure vector field ξ, a 1-form η and an indefinite metric g, which collectively satisfy the following set of conditions:

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{4}$$

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad rank(\phi) = n - 1,$$

$$\begin{cases} \epsilon g(\beta_X, \xi) = \eta(\beta_X), \\ g(\xi, \xi) = \epsilon, \\ g(\phi\beta_X, \phi\beta_Y) = g(\beta_X, \beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y), \end{cases} \tag{5}$$

for all vector fields $\beta_X, \beta_Y \in \chi(\mathcal{F})$, where $\epsilon = 1$ or -1, depending on whether ξ is a spacelike or timelike vector field, respectively, and the symbol $\chi(\mathcal{F})$ designates the space of all differentiable vector fields on the manifold $(\mathcal{F}, g)$. If

$$d\eta(\beta_X, \beta_Y) = g(\beta_X, \phi\beta_Y), \qquad g(\beta_X, \phi\beta_Y) = -g(\phi\beta_X, \beta_Y),$$

where d is an exterior derivative, then we say that $(\mathcal{F}, g)$ is an ϵ-contact metric manifold. We also have

$$(D_{\beta_X}\phi)(\beta_Y) = -g(\beta_X, \phi\beta_Y) - \epsilon\eta(\beta_Y)\phi\beta_X,$$

where D denotes the Levi-Civita connection associated with the metric g and the manifold $(\mathcal{F}, g)$ is known as an ϵ-Kenmotsu manifold [26]. An ϵ-almost contact metric manifold is an ϵ-Kenmotsu manifold if and only if it meets the following conditions:

$$D_{\beta_X}\xi = \epsilon[\beta_X - \eta(\beta_X)\xi], \qquad D_\xi\xi = 0. \tag{6}$$

Moreover, an ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ holds the following relations:

$$(D_{\beta_X}\eta)(\beta_Y) = g(\beta_X, \beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y), \qquad (D_\xi\eta)(\beta_Y) = 0, \tag{7}$$

$$g(R(\beta_X,\beta_Y)\beta_Z,\xi) = \eta(R(\beta_X,\beta_Y)\beta_Z)$$
$$= \epsilon[g(\beta_X,\beta_Z)\eta(\beta_Y) - g(\beta_Y,\beta_Z)\eta(\beta_X)], \tag{8}$$
$$R(\xi,\beta_X)\beta_Y = \eta(\beta_Y)\beta_X - \epsilon g(\beta_X,\beta_Y)\xi, \tag{9}$$
$$R(\beta_X,\beta_Y)\xi = \eta(\beta_X)\beta_Y - \eta(\beta_Y)\beta_X, \tag{10}$$
$$R(\xi,\beta_X)\xi = -R(\beta_X,\xi)\xi = \beta_X - \eta(\beta_X)\xi, \tag{11}$$
$$E(\beta_X,\xi) = -(n-1)\eta(\beta_X), \quad E(\xi,\xi) = -(n-1), \tag{12}$$
$$L\beta_X = -\epsilon(n-1)\beta_X, \quad L\xi = -\epsilon(n-1)\xi, \tag{13}$$

where R represents the curvature tensor, E is the Ricci tensor and L is the Ricci operator defined by $g(L\beta_X,\beta_Y) = E(\beta_X,\beta_Y)$ [26], respectively. If $\epsilon = 1$, then an ϵ-Kenmotsu manifold reduces to a Kenmotsu manifold [27].

Definition 1. *The projective curvature tensor P [28], the conharmonic curvature tensor C [29] and the W_5-curvature tensor W [30], respectively, are defined on an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F},g)$ in the following manner:*

$$P(\beta_X,\beta_Y)\beta_Z = R(\beta_X,\beta_Y)\beta_Z - \frac{1}{n-1}[g(L\beta_Y,\beta_Z)\beta_X - g(L\beta_X,\beta_Z)\beta_Y], \tag{14}$$

$$C(\beta_X,\beta_Y)\beta_Z = R(\beta_X,\beta_Y)\beta_Z + \frac{1}{n-2}[E(\beta_X,\beta_Z)\beta_Y$$
$$- E(\beta_Y,\beta_Z)\beta_X + g(\beta_X,\beta_Z)L\beta_Y - g(\beta_Y,\beta_Z)L\beta_X], \tag{15}$$

$$W(\beta_X,\beta_Y)\beta_Z = R(\beta_X,\beta_Y)\beta_Z + \frac{1}{n-1}[g(\beta_X,\beta_Z)L\beta_Y - g(\beta_Y,\beta_Z)L\beta_X] \tag{16}$$

for all the vector fields β_X,β_Y,β_Z on the manifold $(\mathcal{F},g)$.

Definition 2. *The manifold $(\mathcal{F},g)$, equipped with an ϵ-Kenmotsu structure is called an η-Einstein manifold when its Ricci tensor E can be written in the form*

$$E = f_1 g(\beta_X,\beta_Y) + f_2\eta(\beta_X)\eta(\beta_Y),$$

where f_1 and f_2 are scalar functions defined on the manifold $(\mathcal{F},g)$.

3. Indefinite Kenmotsu Manifolds with η-R–B Solitonic Structures

Let us consider that an ϵ-Kenmotsu manifold $(\mathcal{F},g)$ admits η-R–B solitons $(g,\xi,\lambda,\theta,\mu)$. Then from (3), we can write

$$(\pounds_\xi g)(\beta_X,\beta_Y) + 2E(\beta_X,\beta_Y) + 2(\lambda-\theta s)g(\beta_X,\beta_Y) + 2\mu\eta(\beta_X)\eta(\beta_Y) = 0 \tag{17}$$

for all β_X,β_Y on $(\mathcal{F},g)$.

We know that $(\pounds_\xi g)(\beta_X,\beta_Y) = g(D_{\beta_X}\xi,\beta_Y) + g(\beta_X,D_{\beta_Y}\xi)$, then using (6) in (17), we have

$$(\pounds_\xi g)(\beta_X,\beta_Y) = 2\epsilon[g(\beta_X,\beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y)]. \tag{18}$$

Now, from (17) and (18), we get

$$E(\beta_X,\beta_Y) = (\theta s - \lambda - \epsilon)g(\beta_X,\beta_Y) - (\mu-1)\eta(\beta_X)\eta(\beta_Y). \tag{19}$$

Again, setting $\beta_Y = \xi$ in (19), we obtain

$$E(\beta_X, \xi) = [\epsilon(\theta s - \lambda) - \mu]\eta(\beta_X), \quad L\xi = [\epsilon(\theta s - \lambda) - \mu]\xi. \tag{20}$$

Comparing (12) and (20), we find

$$\lambda = \theta s + \epsilon(n - 1 - \mu). \tag{21}$$

As a result, we derive the following.

Theorem 1. *Suppose that an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ admits an η-R–B solitons structure defined by $(g, \xi, \lambda, \theta, \mu)$. In this case, $(\mathcal{F}, g)$ is considered to be an η-Einstein manifold given in (19) and the parameters λ and μ are connected through the relation given in (21).*

Moreover, if we take $\mu = 0$, then from (19) and (21), we have

$$E(\beta_X, \beta_Y) = (\theta s - \lambda - \epsilon)g(\beta_X, \beta_Y) + \eta(\beta_X)\eta(\beta_Y)$$

and

$$\lambda = \theta s + \epsilon(n - 1).$$

From this, we can deduce the following.

Corollary 1. *Suppose that an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ admits an η-R–B solitons structure defined by $(g, \xi, \lambda, \theta, \mu)$. Then $(\mathcal{F}, g)$ is necessarily an η-Einstein manifold, with the soliton constant λ satisfying $\lambda = \theta s + \epsilon(n - 1)$, where s denotes the scalar curvature. The soliton's nature is determined by the causal character of the structure vector field ξ:*

(i) For spacelike ξ: expanding if $\theta s > 1 - n$, steady if $\theta s = 1 - n$ and shrinking if $\theta s < 1 - n$;
(ii) For timelike ξ: expanding if $\theta s > n - 1$, steady if $\theta s = n - 1$ and shrinking if $\theta s < n - 1$.

Next, we seek to establish a condition involving a second-order symmetric parallel tensor that determines when an ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ admits an η-R–B solitons. To achieve this, we define the second-order tensor h on the manifold $(\mathcal{F}, g)$ as follows:

$$h = \pounds_\xi g + 2E + 2\mu\eta(\beta_X)\eta(\beta_Y). \tag{22}$$

In view of (18) and (19), Equation (22) becomes

$$h(\beta_X, \beta_Y) = 2(\theta s - \lambda)g(\beta_X, \beta_Y). \tag{23}$$

Setting $\beta_X = \beta_Y = \xi$ in (23), we obtain

$$h(\xi, \xi) = 2\epsilon(\theta s - \lambda). \tag{24}$$

Given that h is a second-order symmetric parallel tensor, that is, $Dh = 0$, we obtain the relation

$$h(R(\beta_X, \beta_Y)\beta_Z, \beta_U) + h(\beta_Z, R(\beta_X, \beta_Y)\beta_U) = 0 \tag{25}$$

for all $\beta_X, \beta_Y, \beta_Z, \beta_U \in \chi(\mathcal{F})$.

Again, setting $\beta_X = \beta_Z = \beta_U = \xi$ in (25), we obtain

$$h(R(\xi, \beta_Y)\xi, \xi) + h(\xi, R(\xi, \beta_Y)\xi) = 0. \tag{26}$$

Inserting (11) in (26), we get

$$h(\beta_Y, \xi) = h(\xi, \xi)\eta(\beta_Y). \tag{27}$$

By taking the covariant derivative of (27) with respect to β_X, we arrive at

$$h(\beta_Y, D_{\beta_X}\xi) = h(\xi, \xi)(D_{\beta_X}\eta)(Y) + 2h(D_{\beta_X}\xi, \xi)\eta(\beta_Y). \tag{28}$$

From (6), (7) and (28), we have

$$h(\beta_X, \beta_Y) = \epsilon h(\xi, \xi)g(\beta_X, \beta_Y). \tag{29}$$

Thus, in view of (22), (24) and (29), we finally obtain

$$(\pounds_\xi g)(\beta_X, \beta_Y) + 2E(\beta_X, \beta_Y) + 2(\lambda - \theta s)g(\beta_X, \beta_Y) + 2\mu\eta(\beta_X)\eta(\beta_Y) = 0.$$

Therefore, we can conclude the following.

Theorem 2. *Suppose that an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$. If the second-order symmetric (0,2)-tensor field $h = \pounds_\xi g + 2E + 2\mu\eta \otimes \eta$ is parallel to the Levi-Civita connection D associated with g, then the manifold $(\mathcal{F}, g)$ admits an η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$.*

4. Indefinite Kenmotsu Manifolds Admitting η-R–B Solitons with Certain Types of Ricci Tensor

Definition 3 ([31]). *An ϵ-Kenmotsu manifold is said to possess a Codazzi-type Ricci tensor if its non-zero Ricci tensor satisfies the differential symmetry condition*

$$(D_{\beta_X}E)(\beta_Y, \beta_Z) = (D_{\beta_Y}E)(\beta_X, \beta_Z) \tag{30}$$

for all $\beta_X, \beta_Y, \beta_Z \in \chi(\mathcal{F})$.

Let us examine an ϵ-Kenmotsu manifold with a Codazzi-type Ricci tensor that supports η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$. In this case, Equation (19) holds. By taking the covariant derivative of (19) and utilizing (7), we obtain the following expression:

$$\begin{aligned}(D_{\beta_X}E)(\beta_Y, \beta_Z) = \theta ds(\beta_X)g(\beta_Y, \beta_Z) - (\mu - 1)[g(\beta_X, \beta_Y)\eta(Z) \\ + g(\beta_X, \beta_Z)\eta(\beta_Y) - 2\epsilon\eta(\beta_X)\eta(\beta_Y)\eta(\beta_Z)].\end{aligned} \tag{31}$$

Due to the fact that the Ricci tensor exhibits Codazzi-type symmetry, Equation (31) follows from (30) and becomes

$$[\theta ds(\beta_X) + (\mu - 1)\eta(\beta_X)]g(\beta_Y, \beta_Z) - [\theta ds(\beta_Y) + (\mu - 1)\eta(\beta_Y)]g(\beta_X, \beta_Z) = 0.$$

Since $ds(\beta_X)$ is constant, that is, $ds(\beta_X) = 0$. Therefore, the above equation becomes

$$(\mu - 1)[g(\beta_Y, \beta_Z)\eta(\beta_X) - g(\beta_X, \beta_Z)\eta(\beta_Y)] = 0. \tag{32}$$

Since $g(\beta_Y, \beta_Z)\eta(\beta_X) - g(\beta_X, \beta_Z)\eta(\beta_Y) \neq 0$ in (32) and therefore, $\mu = 1$. Thus, from (21), it follows that

$$\lambda = \theta s + \epsilon(n - 2).$$

This leads to the formulation of the following.

Theorem 3. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons associated with the tuple $(g, \xi, \lambda, \theta, \mu)$. If the Ricci tensor on this manifold satisfies the Codazzi condition, then the soliton parameters are given by $\lambda = \theta s + \epsilon(n-2)$ and $\mu = 1$.*

Corollary 2. *Consider an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ that admits an η-R–B solitons characterized by the tuple $(g, \xi, \lambda, \theta, \mu)$ and suppose that its Ricci tensor is of Codazzi type. Then the nature of the soliton can be classified based on the causal character of the vector field ξ as follows:*

(i) If ξ is a spacelike vector field, the soliton is said to be expanding, steady, or shrinking depending on whether $\theta s + n > 2$, $\theta s + n = 2$, or $\theta s + n < 2$, respectively.
(ii) If ξ is timelike, the soliton will be expanding, steady, or shrinking according to whether $\theta s + 2 > n$, $\theta s + 2 = n$, or $\theta s + 2 < n$, respectively.

Definition 4 ([31])**.** *An ϵ-Kenmotsu manifold is said to possess a cyclic parallel Ricci tensor if its non-vanishing Ricci tensor E satisfies the cyclic covariant derivative condition*

$$(D_{\beta_X} E)(\beta_Y, \beta_Z) + (D_{\beta_Y} E)(\beta_X, \beta_Z) + (D_{\beta_Z} E)(\beta_X, \beta_Y) = 0 \tag{33}$$

for all $\beta_X, \beta_Y, \beta_Z \in \chi(\mathcal{F})$.

Assume that an ϵ-Kenmotsu manifold with a cyclic parallel Ricci tensor admits an η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$. In this case, Equation (19) holds.

From (31) and (33), we have

$$\begin{aligned} &\theta\Big[ds(\beta_X)g(\beta_Y, \beta_Z) + ds(\beta_Y)g(\beta_Z, \beta_X) + ds(\beta_Z)g(\beta_X, \beta_Y)\Big] \\ &- 2(\mu - 1)\Big[g(\beta_X, \beta_Y)\eta(\beta_Z) + g(\beta_Y, \beta_Z)\eta(\beta_X) \\ &+ g(\beta_X, \beta_Z)\eta(\beta_Y) - 3\epsilon\eta(\beta_X)\eta(\beta_Y)\eta(\beta_Z)\Big] = 0. \end{aligned} \tag{34}$$

Since $ds(\beta_X)$ is constant and upon replacing β_Z with ξ in (34), we get

$$2(1 - \mu)[g(\beta_X, \beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y)] = 0.$$

The above equation indicates that $\mu = 1$ and from (21), it follows that $\lambda = \theta s + \epsilon(n-2)$. Therefore, the statement can be written as.

Theorem 4. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons described by the tuple $(g, \xi, \lambda, \theta, \mu)$. If the Ricci tensor of the manifold is cyclic parallel, then the parameters satisfy $\lambda = \theta s + \epsilon(n-2)$ and $\mu = 1$.*

Definition 5. *A cyclic η-recurrent Ricci tensor on an ϵ-Kenmotsu manifold is defined by the requirement that the non-zero Ricci tensor E satisfies*

$$\begin{aligned} &(D_{\beta_X} E)(\beta_Y, \beta_Z) + (D_{\beta_Y} E)(\beta_X, \beta_Z) + (D_{\beta_Z} E)(\beta_X, \beta_Y) \\ &\quad = \eta(\beta_X)E(\beta_Y, \beta_Z) + \eta(\beta_Y)E(\beta_Z, \beta_X) + \eta(\beta_Z)E(\beta_X, \beta_Y) \end{aligned} \tag{35}$$

for all $\beta_X, \beta_Y, \beta_Z \in \chi(\mathcal{F})$.

Suppose an ϵ-Kenmotsu manifold with a cyclic η-recurrent Ricci tensor admits η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$. In this case, Equation (19) is satisfied. Thus, in view of (19), (31) and (35), we obtain

$$\begin{aligned}\theta[ds(\beta_X)g(\beta_Y,\beta_Z) + ds(\beta_Y)g(\beta_Z,\beta_X) + ds(\beta_Z)g(\beta_X,\beta_Y)] \\ + [\lambda - \theta s + \epsilon - 2(\mu-1)][g(\beta_Y,\beta_Z)\eta(\beta_X) + g(\beta_X,\beta_Z)\eta(\beta_Y) \\ + g(\beta_X,\beta_Y)\eta(\beta_Z)] + 3(\mu-1)(1+2\epsilon)\eta(\beta_X)\eta(\beta_Y)\eta(\beta_Z) = 0.\end{aligned} \tag{36}$$

Then Equation (36) becomes

$$\begin{aligned}[\lambda - \theta s + \epsilon - 2(\mu-1)][g(\beta_Y,\beta_Z)\eta(\beta_X) + g(\beta_X,\beta_Z)\eta(\beta_Y) \\ + g(\beta_X,\beta_Y)\eta(\beta_Z)] + 3(\mu-1)(1+2\epsilon)\eta(\beta_X)\eta(\beta_Y)\eta(\beta_Z) = 0.\end{aligned} \tag{37}$$

Now, setting $\beta_Y = \beta_Z = \xi$ in (37), we get

$$3[\epsilon\{\lambda - \theta s + \epsilon - 2(\mu-1)\} + (\mu-1)(1+2\epsilon)]\eta(\beta_X) = 0. \tag{38}$$

Since $\eta(\beta_X) \neq 0$ (in general), then from (38), we get

$$\lambda = \theta s - \epsilon\mu.$$

Thus, we can conclude with the following statement.

Theorem 5. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold admitting an η-R–B solitons given by the tuple $(g, \xi, \lambda, \theta, \mu)$. If the Ricci tensor of the manifold is cyclic η-recurrent, then the soliton parameter λ satisfies the relation $\lambda = \theta s - \epsilon\mu$. Furthermore, the behavior of the soliton depends on the causal character of the vector field ξ as follows:*

- (i) *If ξ is spacelike, the soliton is expanding when $\theta s > \mu$, steady when $\theta s = \mu$, and shrinking when $\theta s < \mu$.*
- (ii) *If ξ is timelike, the soliton is expanding if $\theta s + \mu > 0$, steady if $\theta s + \mu = 0$, and shrinking if $\theta s + \mu < 0$.*

Corollary 3. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an R–B soliton characterized by the quadruple $(g, \xi, \lambda, \theta)$. If the Ricci tensor of the manifold is cyclic η-recurrent, then the soliton constant λ is uniquely determined by the relation $\lambda = \theta s$.*

5. Indefinite Kenmotsu Manifolds Admitting η-R–B Solitons with Certain Curvature Conditions

Definition 6. *An n-dimensional ϵ-Kenmotsu manifold is said to be Ricci semi-symmetric if the following condition satisfies*

$$(R(\beta_X, \beta_Y) \cdot E)(\beta_Z, \beta_U) = 0$$

for all $\beta_X, \beta_Y, \beta_Z, \beta_U \in \chi(\mathcal{F})$.

Consider an n-dimensional ϵ-Kenmotsu manifold admitting η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$ and suppose the manifold is Ricci semi-symmetric, meaning $R \cdot E = 0$. Then we have

$$E(R(\beta_X, \beta_Y)\beta_Z, \beta_V) + E(\beta_Z, R(\beta_X, \beta_Y)\beta_V) = 0. \tag{39}$$

Setting $\beta_V = \xi$ and using (12) in (39), we have

$$-(n-1)\eta((R(\beta_X, \beta_Y)\beta_Z)) + E(\beta_Z, R(\beta_X, \beta_Y)\xi) = 0. \tag{40}$$

From (8), (10) and (40), we have

$$\eta(\beta_X)E(\beta_Y,\beta_Z)-\eta(\beta_Y)E(\beta_X,\beta_Z)+\epsilon(n-1)[g(\beta_Y,\beta_Z)-g(\beta_X,\beta_Z)]=0. \tag{41}$$

Using the expression in (19), we can rewrite Equation (41) as follows

$$[\theta s-\lambda+\epsilon(n-2)][\eta(\beta_X)g(\beta_Y,\beta_Z)-\eta(\beta_Y)g(\beta_X,\beta_Z)]=0. \tag{42}$$

Now, putting $\beta_X=\xi$ and with the help of (4) and (5) in (42), we have

$$[\theta s-\lambda+\epsilon(n-2)]g(\phi\beta_Y,\phi\beta_Z)=0. \tag{43}$$

Since $g(\phi\beta_Y,\phi\beta_Z)\neq 0$, we can conclude from (43) that

$$\lambda=\theta s+\epsilon(n-2).$$

Then from (21), we have $\mu=1$. Therefore, the following statement arises.

Theorem 6. *Let $(\mathcal{F},g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons $(g,\xi,\lambda,\theta,\mu)$. If the manifold is Ricci semi-symmetric, meaning that $R\cdot E=0$, then the soliton parameters satisfy $\lambda=\theta s+\epsilon(n-2)$ and $\mu=1$. Furthermore, the nature of the soliton depends on the causal character of the vector field ξ, if*

(*i*) *ξ is spacelike, the soliton is expanding if $\theta s>2-n$, steady if $\theta s=2-n$, shrinking if $\theta s<2-n$;*

(*ii*) *ξ is timelike, the soliton is expanding if $\theta s+2-n>0$, steady if $\theta s+2-n=0$, shrinking if $\theta s+2-n<0$.*

We thus set the following definition.

Definition 7. *An n-dimensional ϵ-Kenmotsu manifold is said to be conharmonically Ricci semi-symmetric if the following relation holds*

$$(C(\beta_X,\beta_Y)\cdot E)(\beta_Z,\beta_U)=0$$

for all $\beta_X,\beta_Y,\beta_Z,\beta_U\in\chi(\mathcal{F})$.

Let us now consider an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F},g)$ admitting an η-R–B solitons $(g,\xi,\lambda,\theta,\mu)$ that fulfills conharmonically Ricci semi-symmetric, i.e., $C(\xi,\beta_X)\cdot E=0$. Under these conditions, we obtain the following equation

$$E(C(\xi,\beta_X)\beta_Y,\beta_Z)+E(\beta_Y,C(\xi,\beta_X)\beta_Z)=0. \tag{44}$$

From (15), we can write

$$C(\xi,\beta_X)\beta_Y=-\frac{n}{n-2}\eta(\beta_Y)\beta_X-\epsilon g(\beta_X,\beta_Y)\xi. \tag{45}$$

From (44) and (45), we have

$$\begin{aligned}-\frac{n}{n-2}&[\eta(\beta_Y)E(\beta_X,\beta_Z)+\eta(\beta_Z)E(\beta_X,\beta_Y)]\\&=\epsilon[g(\beta_X,\beta_Y)E(\xi,\beta_Z)+g(\beta_X,\beta_Z)E(\beta_Y,\xi)].\end{aligned} \tag{46}$$

Again, from (12) and (46), we get

$$\frac{n}{n-2}[\eta(\beta_Y)E(\beta_X,\beta_Z)+\eta(\beta_Z)E(\beta_X,\beta_Y)] = \epsilon(n-1)[\eta(\beta_Z)g(\beta_X,\beta_Y)+\eta(\beta_Y)g(\beta_X,\beta_Z)]. \tag{47}$$

Setting $\beta_Z = \xi$ in (47) and using (4), (5) and (12) in (47), we get

$$E(\beta_X,\beta_Y) = \frac{n-2}{n}[-\epsilon g(\beta_X,\beta_Y)+(n-1)(\frac{n}{n-2}+1)\eta(\beta_X)\eta(\beta_Y)]. \tag{48}$$

Now, in view of (19) and (48), we derive

$$\frac{1}{n}[\{n(\theta s-\lambda)-2\epsilon\}g(\beta_X,\beta_Y)+\{n(1-\mu)-2(n-1)^2\}\eta(\beta_X)\eta(\beta_Y)] = 0. \tag{49}$$

Again, taking $\beta_Y = \xi$ in (49), we have

$$[\epsilon n(\theta s-\lambda)-2+n(1-\mu)-2(n-1)^2]\eta(\beta_X) = 0.$$

Based on the above equation, we are able to get

$$\lambda = \frac{1}{n}[\theta sn-\epsilon\{2+n(1-\mu)-2(n-1)^2\}].$$

This leads us to the following conclusion.

Theorem 7. *Let $(\mathcal{F},g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons specified by the tuple $(g,\xi,\lambda,\theta,\mu)$. If the manifold satisfies conharmonically Ricci semi-symmetric, that is, $C(\xi,\beta_X)\cdot E = 0$, then the manifold is an η-Einstein and the soliton constants λ and μ are related by the expression*

$$\lambda = \frac{1}{n}\Big[\theta sn-\epsilon\Big\{2+n(1-\mu)-2(n-1)^2\Big\}\Big].$$

Definition 8. *An n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F},g)$ is called ξ-projectively flat if*

$$P(\beta_X,\beta_Y)\xi = 0$$

for all $\beta_X,\beta_Y \in \chi(\mathcal{F})$, where P denotes the projective curvature tensor.

Again, we consider an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F},g)$ satisfying $P(\beta_X,\beta_Y)\xi = 0$ and admitting an η-R–B solitons $(g,\xi,\lambda,\theta,\mu)$. Then setting $\beta_Z = \xi$ in (14), we obtain

$$P(\beta_X,\beta_Y)\xi = R(\beta_X,\beta_Y)\xi-\frac{1}{n-1}[E(\beta_Y,\xi)\beta_X-E(\beta_X,\xi)\beta_Y]. \tag{50}$$

Using (10) and (20) in (50), we have

$$P(\beta_X,\beta_Y)\xi = \frac{1}{n-1}[\epsilon(\lambda-\theta s)+\mu-n+1][\eta(\beta_Y)\beta_X-\eta(\beta_X)\beta_Y]. \tag{51}$$

In view of (21) and (51), we obtain $P(\beta_X,\beta_Y)\xi = 0$. Thus, we have the following theorem.

Theorem 8. *An n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F},g)$ admitting an η-R–B solitons $(g,\xi,\lambda,\theta,\mu)$ is ξ-projectively flat.*

Now, we define another definition as follows.

Definition 9. *An n-dimensional ϵ-Kenmotsu manifold is said to be projectively Ricci semi-symmetric if the following condition holds*

$$L \cdot P = 0,$$

where P is the projective curvature tensor and L is the Ricci operator defined by $g(L\beta_X, \beta_Y) = E(\beta_X, \beta_Y)$, respectively.

Theorem 9. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons defined by the tuple $(g, \xi, \lambda, \theta, \mu)$. If the manifold satisfies projectively Ricci semi-symmetric, that is, $L \cdot P = 0$, then $(\mathcal{F}, g)$ is an η-Einstein and the soliton parameters λ and μ satisfy the relation*

$$\lambda = \theta s + \epsilon(n - \mu - 1).$$

Proof. Let an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ supporting η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$ be projectively Ricci semi-symmetric, i.e., consider $L \cdot P = 0$. This type of curvature condition is also studied by [32]. Then we can write

$$L(P(\beta_X, \beta_Y)\beta_Z) - P(L\beta_X, \beta_Y)\beta_Z - P(\beta_X, L\beta_Y)\beta_Z - P(\beta_X, \beta_Y)L\beta_Z = 0. \tag{52}$$

From (14) and (52), we get

$$\begin{aligned} L(R(\beta_X,\beta_Y)\beta_Z) - R(L\beta_X, \beta_Y)\beta_Z - R(\beta_X, L\beta_Y)\beta_Z \\ - R(\beta_X, \beta_Y)L\beta_Z + 2\epsilon[E(\beta_X, \beta_Z)\beta_Y - E(\beta_Y, \beta_Z)\beta_X] = 0. \end{aligned} \tag{53}$$

Applying the inner product of (53) with ξ, we arrive at

$$\begin{aligned} \epsilon[\eta(L(R(\beta_X, \beta_Y)\beta_Z)) - \eta(R(L\beta_X, \beta_Y)\beta_Z) - \eta(R(\beta_X, L\beta_Y)\beta_Z) \\ - \eta(R(\beta_X, \beta_Y)L\beta_Z) + 2[E(\beta_X, \beta_Z)\eta(\beta_Y) - E(\beta_Y, \beta_Z)\eta(\beta_X)] = 0. \end{aligned} \tag{54}$$

Putting $\beta_Y = \xi$ in (54), we obtain

$$\begin{aligned} \epsilon[\eta(L(R(\beta_X, \xi)\beta_Z)) - \eta(R(L\beta_X, \xi)\beta_Z) - \eta(R(\beta_X, L\xi)\beta_Z) \\ - \eta(R(\beta_X, \xi)L\beta_Z) + 2[E(\beta_X, \beta_Z)\eta(\xi) - E(\xi, \beta_Z)\eta(\beta_X)] = 0. \end{aligned}$$

From (4), (9) and (13), we get

$$E(\beta_X, \beta_Z) = (n - 1)[g(\beta_X, \beta_Z) - \epsilon\eta(\beta_X)\eta(\beta_Z)]. \tag{55}$$

Comparing (55) and (19), we get

$$(\theta s - \lambda - \epsilon - n + 1)g(\beta_X, \beta_Z) + [(\epsilon + 1)(n - 1) - \mu + 1]\eta(\beta_X)\eta(\beta_Z) = 0. \tag{56}$$

Again, setting $\beta_Z = \xi$ in (56), we have

$$[\epsilon(\theta s - \lambda) + n - \mu - 1]\eta(\beta_X) = 0 \tag{57}$$

for all $\beta_X \in \chi(\mathcal{F})$. Since $\eta(\beta_X) \neq 0$ always, then from (57), we can conclude that

$$\lambda = \theta s + \epsilon(n - \mu - 1).$$

Thus, the proof is completed. □

Once again, we present the following definition.

Definition 10. *An n-dimensional ϵ-Kenmotsu manifold is said to be W_5-Ricci semi-symmetric if the following expression satisfies*

$$(W(\beta_X, \beta_Y) \cdot E)(\beta_Z, \beta_U) = 0$$

for all $\beta_X, \beta_Y, \beta_Z, \beta_U \in \chi(\mathcal{F})$, where W is the W_5-curvature tensor.

Theorem 10. *Let $(\mathcal{F}, g)$ be an n-dimensional ϵ-Kenmotsu manifold that admits an η-R–B solitons represented by the tuple $(g, \xi, \lambda, \theta, \mu)$. If the manifold satisfies W_5-Ricci semi-symmetric, meaning $W(\xi, \beta_Y) \cdot E = 0$, then it is an Einstein and the constants λ and μ are related by*

$$\lambda = \theta s + \epsilon(n - \mu - 1).$$

Proof. Suppose an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ admits an η-R–B solitons $(g, \xi, \lambda, \theta, \mu)$ that satisfy the W_5-Ricci semi-symmetric condition, that is, $W(\xi, \beta_Y) \cdot E = 0$, where W is the W_5-curvature tensor. Then we have the following relation

$$E(W(\xi, \beta_Y)\beta_Z, \beta_U) + E(\beta_Z, W(\xi, \beta_Y)\beta_Z) = 0.$$

Setting $\beta_U = \xi$ in the latest equation, we get

$$E(W(\xi, \beta_Y)\beta_Z, \xi) + E(\beta_Z, W(\xi, \beta_Y)\xi) = 0. \tag{58}$$

Also, taking $\beta_X = \xi$ in (16), we get

$$W(\xi, \beta_Y)\beta_Z = R(\xi, \beta_Y)\beta_Z + \frac{1}{n-1}[\epsilon\eta(\beta_Z)L\beta_Y - E(\xi, \beta_Z)\beta_Y]. \tag{59}$$

In view of (9), (12), (13) and (59), we obtain

$$W(\xi, \beta_Y)\beta_Z = \epsilon g(\beta_Y, \beta_Z)\xi - \eta(\beta_Z)\beta_Y. \tag{60}$$

Again, performing $\beta_Z = \xi$ in (60), we have

$$W(\xi, \beta_Y)\xi = \eta(\beta_Y)\xi - \beta_Y. \tag{61}$$

Using (60) and (61) in (58), we have

$$E(\beta_Y, \beta_Z) = -\epsilon(n-1)g(\beta_Y, \beta_Z), \tag{62}$$

which shows that $(\mathcal{F}, g)$ is an Einstein manifold. Comparing (62) and (19), we get

$$[\theta s - \lambda + \epsilon(n-2)]g(\beta_Y, \beta_Z) - (\mu - 1)\eta(\beta_Y)\eta(\beta_Z) = 0. \tag{63}$$

Again, setting $\beta_Y = \xi$ in (63) and with the help of (4) and (5), we get

$$[\epsilon(\theta s - \lambda) + n - \mu - 1]\eta(\beta_Z) = 0. \tag{64}$$

Since $\eta(\beta_Z) \neq 0$, then from (64), we can say that

$$\lambda = \theta s + \epsilon(n - \mu - 1).$$

Thus, this completes the proof. □

6. Indefinite Kenmotsu Manifolds Possessing η-R–B Solitons Associated with a Torse-Forming Vector Field

Definition 11 ([33]). *A vector field β_U on an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$ is called a torse-forming vector field if*

$$D_{\beta_X}\beta_U = m\beta_X + \tau(\beta_X)\beta_U \tag{65}$$

for all $\beta_U \in \chi(\mathcal{F})$, where m is a smooth function and τ is a 1-form.

Theorem 11. *Let $(g, \xi, \lambda, \theta, \mu)$ define an η-R–B solitons on an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$, and suppose that the Reeb vector field ξ is torse-forming. Then the manifold $(\mathcal{F}, g)$ satisfies the condition of being an η-Einstein manifold.*

Proof. Let $(g, \xi, \lambda, \theta, \mu)$ represent an η-R–B solitons on an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$, and suppose that the Reeb vector field ξ is torse-forming. Then using Equation (65), we arrive at

$$D_{\beta_X}\xi = m\beta_X + \tau(\beta_X)\xi. \tag{66}$$

By taking the inner product of (6) with respect to ξ, we obtain

$$g(D_{\beta_X}\xi, \xi) = \epsilon g(\beta_X, \xi) - \epsilon\eta(\beta_X)g(\xi, \xi).$$

In view of (5), the above equation takes the form

$$g(D_{\beta_X}\xi, \xi) = 0. \tag{67}$$

Similarly, when we apply the inner product of Equation (66) with ξ, it leads to

$$g(D_{\beta_X}\xi, \xi) = \epsilon[m\eta(\beta_X) + \tau(\beta_X)]. \tag{68}$$

Combining equations (67) and (68), we obtain $\tau = -m\eta$. Therefore, in the context of ϵ-Kenmotsu manifolds, when the vector field ξ is torse-forming, the following relation holds

$$D_{\beta_X}\xi = m[\beta_X - \eta(\beta_X)\xi]. \tag{69}$$

Therefore, from (3), we can write the equation as

$$\begin{aligned} g(D_{\beta_X}\xi, \beta_Y) + g(\beta_X, D_{\beta_Y}\xi) + 2E(\beta_X, \beta_Y) \\ + 2(\lambda - \theta s)g(\beta_X, \beta_Y) + 2\mu\eta(\beta_X)\eta(\beta_Y) = 0. \end{aligned} \tag{70}$$

Thus, in view of (69), Equation (70) becomes

$$E(\beta_X, \beta_Y) = (\theta s - \lambda - m)g(\beta_X, \beta_Y) + (\epsilon m - \mu)\eta(\beta_X)\eta(\beta_Y). \tag{71}$$

This shows that (71) is an η-Einstein manifold. Thus, the proof completes. □

We can now deduce the following corollary.

Corollary 4. *Let $(g, \xi, \lambda, \theta, \mu)$ be an η-R–B solitons on an n-dimensional ϵ-Kenmotsu manifold $(\mathcal{F}, g)$, where the vector field ξ is torse-forming. Then according to Equation (71), the manifold $(\mathcal{F}, g)$ is Einstein if and only if*

(i) $\xi = 1$, that is, spacelike, and $\mu = m$, or

(ii) $\xi = -1$, *that is, timelike, and* $\mu = -m$, *respectively.*

7. ϕ-Ricci Symmetric Indefinite Kenmotsu Manifolds Admitting η-R–B Solitons

Definition 12 ([34]). *A Riemannian manifold equipped with an* ϵ*-Kenmotsu structure is said to be* ϕ*-Ricci symmetric if its Ricci operator* L *satisfies the condition*

$$\phi^2(D_{\beta_X}L)(\beta_Y) = 0 \tag{72}$$

for all vector fields $\beta_X, \beta_Y \in \chi(\mathcal{F})$, *where the Ricci tensor* E *is given by* $E(\beta_X, \beta_Y) = g(L\beta_X, \beta_Y)$.

Theorem 12. *If an* n*-dimensional* ϵ*-Kenmotsu manifold endowed with a* ϕ*-Ricci symmetric structure admits an* η*-R–B solitons* $(g, \xi, \lambda, \theta, \mu)$, *then the manifold is necessarily an* η*-Einstein.*

Proof. Suppose that $(\mathcal{F}, g)$ is an n-dimensional ϕ-Ricci symmetric ϵ-Kenmotsu manifold that supports an η-R–B solitons defined by the tuple $(g, \xi, \lambda, \theta, \mu)$. Then by combining Equations (72) and (4), we obtain

$$-(D_{\beta_X}L)(\beta_Y) + \eta((D_{\beta_X}L)(\beta_Y))\xi = 0. \tag{73}$$

Computing the inner product of (73) with β_U and invoking Equation (5), we arrive at

$$-g((D_{\beta_X}L)(\beta_Y), \beta_U) + \epsilon\eta((D_{\beta_X}L)(\beta_Y))\eta(\beta_U) = 0,$$

which implies that

$$-g(D_{\beta_X}L\beta_Y, \beta_U) + E(D_{\beta_X}\beta_Y, \beta_U) + \epsilon\eta((D_{\beta_X}L)(\beta_Y))\eta(\beta_U) = 0. \tag{74}$$

Setting $\beta_Y = \xi$ in (74) and using (5), (6) and (20) in (74), we obtain

$$E(\beta_X, \beta_U) = -a[g(\beta_X, \beta_U) - (\epsilon - 1)\eta(\beta_X)\eta(\beta_U)], \tag{75}$$

where $a = \epsilon(\lambda - \theta s + \epsilon) + \mu - 1$. For $\epsilon \neq 1$, Equation (75) implies that the manifold $(\mathcal{F}, g)$ possesses an η-Einstein structure. □

Now, from (17) and (75), we have

$$\begin{aligned} g(D_{\beta_X}\xi,\beta_U) + g(\beta_X, D_{\beta_U}\xi) + 2(\lambda - \theta s)g(\beta_X, \beta_U) \\ + 2\mu\eta(\beta_X)\eta(\beta_U) = 2a[g(\beta_X, \beta_U) - (\epsilon - 1)\eta(\beta_X)\eta(\beta_U)]. \end{aligned} \tag{76}$$

Setting $\beta_U = \xi$ and using (4), (5) and (6) in (76), we obtain

$$[\epsilon(\lambda - \theta s) + \mu - a]\eta(\beta_X) = 0, \tag{77}$$

where $\eta(\beta_X) \neq 0$. Therefore, Equation (77) becomes

$$\lambda = \theta s - \epsilon\mu.$$

Hence, we establish the following.

Theorem 13. *Let* $(g, \xi, \lambda, \theta, \mu)$ *define an* η*-R–B solitons on an* n*-dimensional* ϵ*-Kenmotsu manifold* $(\mathcal{F}, g)$ *endowed with a* ϕ*-Ricci symmetric structure. Then the constants* λ *and* μ *satisfy the relation* $\lambda = \theta s - \epsilon\mu$. *Furthermore, the nature of the soliton depends on the causal character of* ξ*:*

1. *If ξ is spacelike, the soliton is classified as expanding, steady, or shrinking according to whether $\theta s > \mu$, $\theta s = \mu$, or $\theta s < \mu$, respectively.*
2. *If ξ is timelike, then the soliton is expanding, steady, or shrinking depending on whether $\theta s + \mu > 0$, $\theta s + \mu = 0$, or $\theta s + \mu < 0$, respectively.*

8. Examples of an Indefinite Kenmotsu Manifold Admitting η-R–B Solitons

Example 1. *We consider a 3-dimensional manifold $\mathcal{F} = [(l_1, l_2, l_3) \in \mathbb{R}^3 : \ l_3 \neq 0]$, where (l_1, l_2, l_3) are the Cartesian coordinates in $\mathbb{R}^3$. We choose the vector fields as*

$$k_1 = e^{-l_3}(\frac{\partial}{\partial l_1} + \frac{\partial}{\partial l_2}), \quad k_2 = e^{-l_3}(\frac{\partial}{\partial l_2} - \frac{\partial}{\partial l_1}), \quad k_3 = \frac{\partial}{\partial l_3},$$

which are linearly independent at every point of $\mathcal{F}$. Let g be the Riemannian metric defined by

$$g(k_i, k_j) = 0 \quad for \quad i \neq j, \quad g(k_i, k_j) = \epsilon \quad for \quad i = j,$$

where $i, j = 1, 2, 3$ and $\epsilon = \pm 1$. One can express the metric in the following manner:

$$g = \epsilon[\frac{1}{2e^{-l_3}}(dl_1 \otimes dl_1 + dl_2 \otimes dl_2) + dl_3 \otimes dl_3].$$

Let η be the 1-form defined by $\eta(\beta_X) = \epsilon g(\beta_X, k_3)$ for any vector field β_X on $(\mathcal{F}, g)$. Let ϕ be the (1,1)-type tensor field defined by

$$\phi(k_1) = k_2, \quad \phi(k_2) = -k_1, \quad \phi(k_3) = 0.$$

By the linearity property of ϕ and g, we obtain

$$\phi^2\beta_X = -\beta_X + \eta(\beta_X)k_3, \quad \eta(k_3) = \epsilon, \quad g(\phi\beta_X, \phi\beta_Y) = g(\beta_X, \beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y).$$

Therefore, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost-contact structure on the manifold $\mathcal{F}$. Let D be the Levi-Civita connection corresponding to the metric g. Then we have

$$[k_1, k_2] = 0, \quad [k_1, k_3] = k_1, \quad [k_2, k_3] = k_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(D_{\beta_X}\beta_Y, \beta_Z) = {} & \beta_X g(\beta_Y, \beta_Z) + \beta_Y g(\beta_Z, \beta_X) - \beta_Z g(\beta_X, \beta_Y) \\ & - g([\beta_Y, \beta_Z], \beta_X) + g([\beta_Z, \beta_X], \beta_Y) + g([\beta_X, \beta_Y], \beta_Z) \end{aligned} \tag{78}$$

for arbitrary vector fields $\beta_X, \beta_Y, \beta_Z \in \chi(\mathcal{F})$. With the help of Equation (78), we have

$$\begin{aligned} &D_{k_1}k_3 = \epsilon k_1, && D_{k_2}k_3 = \epsilon k_2, && D_{k_3}k_3 = 0, \\ &D_{k_1}k_2 = 0, && D_{k_2}k_2 = -\epsilon k_3, && D_{k_3}k_2 = 0, \\ &D_{k_1}k_1 = -\epsilon k_3, && D_{k_2}k_1 = 0, && D_{k_3}k_1 = 0. \end{aligned}$$

Utilizing the preceding expressions, it follows that for any vector field β_X on $(\mathcal{F}, g)$, we obtain

$$D_{\beta_X}\xi = \epsilon[\beta_X - \eta(\beta_X)\xi]$$

for $\xi = k_3$. Hence, the manifold $(\mathcal{F}, g)$ under consideration is an ϵ-Kenmotsu manifold of dimension 3. The only non-zero components of the curvature tensor and the Ricci tensor are specified as follows:

$$R(k_i, k_j)k_j = -k_1 \qquad for \quad i = 1,\ j = 2,3,$$

$$R(k_i, k_j)k_j = -k_2 \qquad for \quad i = 2,\ j = 1,3,$$

$$R(k_i, k_j)k_j = -k_3 \qquad for \quad i = 3,\ j = 1,2$$

and

$$E(k_i, k_i) = -2\epsilon \quad for \quad i = 1,2,3.$$

Thus, the scalar curvature s is given by

$$s = 3E(k_i, k_i) = -6\epsilon.$$

Based on the preceding analysis, we observe that Equation (19) holds when $\lambda = 2 - \epsilon(1 + 6\theta)$ and $\mu = 3 - 2\epsilon$. Consequently, the quintuple $(g, \xi, \lambda, \theta, \mu)$ defines an η-R–B solitons structure on the 3-dimensional manifold $(\mathcal{F}^3, \phi, \xi, \eta, g, \epsilon)$.

Example 2. *Let $\mathcal{F} = [(q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : \ q_5 \neq 0]$ be a 5-dimensional manifold, where $(q_1, q_2, q_3, q_4, q_5)$ are the standard coordinates in $\mathbb{R}^5$. Define a set of vector fields $\{k_i : 1 \leq i \leq 5\}$ on the manifold $\mathcal{F}$ given by*

$$k_1 = e^{-\epsilon q_5}\frac{\partial}{\partial q_1}, \quad k_2 = e^{-\epsilon q_5}\frac{\partial}{\partial q_2}, \quad k_3 = e^{-\epsilon q_5}\frac{\partial}{\partial q_3},$$
$$k_4 = e^{-\epsilon q_5}\frac{\partial}{\partial q_4}, \quad k_5 = e^{-\epsilon q_5}\frac{\partial}{\partial q_5} = \xi,$$

which are linearly independent at each point of $\mathcal{F}$. Let us define the indefinite metric g on $\mathcal{F}$ by

$$g(k_i, k_j) = \epsilon \quad for \quad i = j, \quad g(k_i, k_j) = 0 \quad for \quad i \neq j,$$

where $i, j = 1, 2, 3, 4, 5$ and $\epsilon = \pm 1$.

Let η be the 1-form defined by $\eta(\beta_X) = \epsilon g(\beta_X, k_5) = \epsilon g(\beta_X, \xi)$ for any vector field β_X on $(\mathcal{F}, g)$. Let ϕ be the (1,1)-type tensor field defined by

$$\phi(k_1) = \epsilon k_2, \quad \phi(k_2) = \epsilon k_1, \quad \phi(k_3) = \epsilon k_4, \quad \phi(k_4) = \epsilon k_3, \quad \phi(k_5) = 0.$$

From the linearity property of ϕ and g, we have

$$\phi^2\beta_X = -\beta_X + \eta(\beta_X)\xi, \quad \eta(k_5) = 1,$$
$$g(\phi\beta_X, \phi\beta_Y) = g(\beta_X, \beta_Y) - \epsilon\eta(\beta_X)\eta(\beta_Y)$$

for any vector fields β_X, β_Y on $\chi(\mathcal{F})$.

Therefore, the tuple $(\phi, \xi, \eta, g, \epsilon)$ induces an indefinite almost-contact structure on the manifold $\mathcal{F}$, provided that $k_5 = \xi$. Let D be the Levi-Civita connection with respect to the metric g, then the components of Lie brackets are given by

$$[k_i, k_5] = \epsilon k_i \quad for \quad i = 1,2,3,4 \ \ and\ all\ other \ \ [k_i, k_j] \ \ vanishes.$$

Now, with the help of Equation (78), we obtain

$$D_{k_i}k_i = -\epsilon k_5, \quad D_{k_i}k_5 = \epsilon k_i \quad for \quad i = 1,2,3,4$$

and all other $D_{k_i}k_j$ vanishes.

Using the above relations, for any vector field β_X on $(\mathcal{F}, g)$, we have

$$D_{k_i}\xi = \epsilon[k_i - \eta(k_i)\xi]$$

for all k_i, $i = 1, 2, 3, 4, 5$.

Accordingly, the structure $(\mathcal{F}, g)$ represents a 5-dimensional ϵ-Kenmotsu manifold. The significant components of the curvature tensor R that do not vanish can be determined as

$$\begin{aligned}
&R(k_i, k_j)k_j = -k_1 \quad \textit{for} \quad i = 1,\ j = 2, 3, 4, 5,\\
&R(k_1, k_2)k_1 = k_2,\\
&R(k_i, k_j)k_i = k_3 \quad \textit{for} \quad i = 1, 2, 5,\ j = 3,\\
&R(k_i, k_j)k_j = -k_2 \quad \textit{for} \quad i = 2,\ j = 3, 4, 5,\\
&R(k_3, k_4)k_4 = -k_3,\\
&R(k_i, k_j)k_i = k_5 \quad \textit{for} \quad i = 1, 2, 3, 4,\ j = 5,\\
&R(k_i, k_j)k_i = k_4 \quad \textit{for} \quad i = 1, 2, 3, 5,\ j = 4.
\end{aligned}$$

The Ricci tensor E yields the following non-zero terms upon computation

$$E(k_i, k_i) = -4 \quad \textit{for} \quad i = 1, 2, 3, 4, 5.$$

Also, the scalar curvature s is obtained as follows

$$s = 5E(k_i, k_i) = -20.$$

In light of the previous finding, it is evident that Equation (19) holds for $\lambda = 3\epsilon - 20\theta$ and $\mu = 1$. Therefore, the data $(g, \xi, \lambda, \theta, \mu)$ correspond to a 5-dimensional η-R–B soliton on the manifold $(\mathcal{F}^5, \phi, \xi, \eta, g, \epsilon)$.

9. Conclusions and Future Directions

This study has provided a comprehensive examination of an η-R–B solitons on ϵ-Kenmotsu manifolds, particularly under the influence of special Ricci tensor conditions such as Codazzi-type, cyclic parallel and cyclic η-recurrent structures. By analyzing various curvature tensors, including the projective, conharmonic and W_5 tensors, we have identified several key geometric constraints that govern the existence and nature of these solitons. The role of torse-forming vector fields was also explored in detail, offering deeper insight into their interaction with solitonic structures. Moreover, we presented a characterization of ϕ-Ricci symmetric indefinite Kenmotsu manifolds admitting such solitons, further enriching the geometric context.

To validate our theoretical findings, explicit examples of indefinite Kenmotsu manifolds admitting η-R–B solitons were constructed. These examples not only demonstrate the existence of such solitons but also highlight the distinctive geometric features they introduce.

Looking ahead, several promising directions emerge for future research. One natural extension is to study η-R–B solitons on broader classes of contact metric manifolds, such as (κ, μ)- or (ϵ, μ)-spaces. Another avenue is to investigate the evolution and stability of these solitons under various geometric flows, including the R–B flow or its conformal variants. Additionally, exploring solitonic behavior on warped product manifolds or in pseudo-Riemannian contexts may yield further insights with potential applications in

mathematical physics and relativity. These future investigations could significantly deepen our understanding of the interplay between soliton theory and contact metric geometry.

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Abbreviations

The following abbreviations are used in this manuscript:

R–B	Ricci–Bourguignon
η-R–B	η-Ricci–Bourguignon
PDE	Partial Differential Equation

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