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On the Transition Density of the Time-Inhomogeneous 3/2 Model: A Unified Approach for Models Related to Squared Bessel Process

Rattiya Meesa ¹, Ratinan Boonklurb ^{1,*} and Phiraphat Sutthimat ^{2,3,*}¹ Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand; rattiya.me@chula.ac.th² Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand³ Financial Mathematics, Data Science and Computational Innovations Research Unit (FDC), Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand

* Correspondence: ratinan.b@chula.ac.th (R.B.); phiraphat.sut@ku.th (P.S.)

Abstract: We derive an infinite-series representation for the transition probability density function (PDF) of the time-inhomogeneous 3/2 model, expressing all coefficients in terms of Bell-polynomial and generalized Laguerre-polynomial formulas. From this series, we obtain explicit expressions for all conditional moments of the variance process, recovering the familiar time-homogeneous formulas when parameters are constant. Numerical experiments illustrate that both the density and moment series converge rapidly, and the resulting distributions agree with high-precision Monte Carlo simulations. Finally, we demonstrate that the same approach extends to a broad family of non-affine, time-varying diffusions, providing a general framework for obtaining transition PDFs and moments in advanced models.

Keywords: characteristic function; 3/2 model; noncentral Chi-square; squared Bessel process; transition density function

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1. Introduction

In this paper, we analyze a time-inhomogeneous extension of the 3/2 model, also known as the Ahn and Gao (AG) model [1] or inverse Feller process [2,3], which generalizes the standard 3/2 model by allowing its parameters to vary with time. The dynamics of the process $\{v_t\}_{t \geq 0}$ are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and given by the stochastic differential equation (SDE)

$$dv_t = \kappa(t) v_t (\theta(t) - v_t) dt + \sigma(t) v_t^{3/2} dW_t, \quad v_0 > 0, \quad (1)$$

where $\{W_t\}_{t \geq 0}$ is a standard Brownian motion and $\kappa(t)$, $\theta(t)$, and $\sigma(t)$ are positive real-valued deterministic functions. Consider the drift term $\kappa(t) v_t (\theta(t) - v_t)$, where $\theta(t)$ represents the threshold value of the process v_t at which the drift is zero. Under the assumption $\kappa(t) > 0$, if the process remains below $\theta(t)$, the drift is positive, pulling the process back toward its normal range. Once the process exceeds $\theta(t)$, the drift becomes negative for the same reason. Consequently, $\theta(t)$ determines the range of interest rates in which the drift is positive; a larger/smaller $\theta(t)$ expands/reduces this range. Meanwhile, $\kappa(t)$ governs the curvature of the drift; a higher $\kappa(t)$ produces a steeper drift curve, whereas a lower

$\kappa(t)$ yields a gentler one. Unlike linear drift models such as extended Cox–Ingersoll–Ross (ECIR) process [4], the mean-reversion speed in (1) depends linearly on the state variable, $\kappa(t) v_t$, meaning that it increases in tandem with the process. This feature is advantageous because it balances the model by imposing stronger mean reversion at higher process values. Initially, the drift rises until the process reaches $\theta(t)/2$; at that point, it attains its maximum value, $\kappa(t) \theta(t)^2/4$. Once the process exceeds this level, the drift begins to decrease, vanishing when the process equals $\theta(t)$. If the process climbs above $\theta(t)$, the drift becomes negative, driving the process back toward its equilibrium range. Moreover, the farther the process rises above $\theta(t)$, the faster the mean reversion occurs.

Ahn and Gao [1] showed that the time-homogeneous 3/2 model (constant parameters) effectively captures nonlinear volatility dynamics—its diffusion term, $\sigma v_t^{3/2}$, matches Chan et al.’s [5] parametric estimates and Aït-Sahalia and Stanton’s nonparametric findings [6,7], producing a heavy-tailed distribution for interest rates. Meanwhile, its drift $\kappa v_t(\theta - v_t)$ deviates from Chan et al. [5] but aligns with Aït-Sahalia and Stanton [6,8], who show that a sharply declining drift at high rates is needed to prevent divergence. Balancing this strong mean reversion with super-linear diffusion is crucial for realistically restoring interest rates to their long-term mean [8] (p. 545).

Although interest in the time-homogeneous 3/2 model has been substantial, spanning option pricing [9], bond pricing [10], swap pricing [11], and interest-rate dynamics [12], research on its time-inhomogeneous counterpart remains relatively limited. Time-varying parameters can capture evolving market regimes more flexibly, yet only a handful of recent works (e.g., [13–17]) have begun to explore such extensions. In particular, many modern studies now combine stochastic-volatility models with data-driven filtering or forecasting techniques. For instance, Mashamba et al. [18] propose a high-momentum equity-timing method that uses a Kalman filter and ARIMA forecasts to adapt model parameters dynamically. Similarly, Li et al. [19] develop a methodology for valuing financial data via analyst forecasts, demonstrating how modern data-valuation approaches can improve prediction accuracy for returns and volatility. Incorporating these data-driven perspectives into a time-inhomogeneous 3/2 model could enhance applicability in real-world risk management, option pricing, and volatility forecasting.

A substantial body of literature has focused on the ECIR process, particularly in trying to derive its transition probability density function (PDF) (see Carmona [20], Maghsoodi [15], Peng and Schellhorn [14], and Rogers [21]), and exploiting its analytical tractability for various financial applications. These studies have provided valuable insights into the behavior of state-dependent volatility models and have been instrumental in calibrating interest rate models. However, despite the extensive research on the ECIR process, there remains a notable gap in the literature regarding the time-inhomogeneous 3/2 model. The lack of a closed-form expression for the transition PDF of the time-inhomogeneous 3/2 model poses significant challenges for both theoretical analysis and practical implementation, such as model calibration and derivative pricing.

In this work, we derive an explicit expression for the transition PDF of the time-inhomogeneous 3/2 model. The main result is stated in the following theorem.

Theorem 1. Suppose that V_t follows the time-inhomogeneous 3/2 model (1). The transition PDF of V_t is given by

$$f_{V_t}(v, t \mid v_0) = \frac{e^{-\frac{1}{2v\Lambda(0,t)} v^{\frac{-\delta(t)}{2}-1}}}{(2\Lambda(0,t))^{\delta(t)/2}} \sum_{k \geq 0} \frac{k! c_k}{\Gamma(\delta(t)/2 + k)} \mathbf{L}_k^{\left(\frac{\delta(t)}{2}-1\right)}\left(\frac{1}{2v\Lambda(0,t)}\right), \quad (2)$$

for $v, v_0 > 0$ and $t > 0$, where $\Gamma(\cdot)$ is the gamma function and $\mathbf{L}_k^{(\cdot)}$ is the k -th generalized Laguerre polynomial [22],

$$\begin{aligned}\delta(t) &= \frac{4(\kappa(t) + \sigma^2(t))}{\sigma^2(t)} \in C^1(\mathbb{R}_0^+), \quad \Delta(s, t) = \int_s^t \kappa(\zeta) \theta(\zeta) d\zeta, \\ \Lambda(s, t) &= \frac{1}{4} \int_s^t e^{-\Delta(\zeta, t)} \sigma^2(\zeta) d\zeta, \quad c_k := c_k(t, v_0) = \frac{1}{k!} \mathbf{B}_k(d_1, 1! d_2, 2! d_3, \dots, (k-1)! d_k), \\ d_k &:= d_k(t, v_0), \quad d_1 = -\frac{e^{-\Delta(0, t)}}{2v_0 \Lambda(0, t)} + \frac{1}{2} \int_0^t \delta'(s) \left(1 - \frac{\Lambda(s, t)}{\Lambda(0, t)}\right) ds, \quad \text{and} \\ d_j &= \frac{1}{2} \int_0^t \delta'(s) \left(1 - \frac{\Lambda(s, t)}{\Lambda(0, t)}\right)^j ds,\end{aligned}$$

for all $j \geq 2$ and $k \geq 0$, where $\mathbf{B}_k$ is the complete Bell polynomials [23].

The derivation of Theorem 1 relies on two key connections. First, we establish a relationship between the characteristic function of a time-varying dimensional squared Bessel process and that of the ECIR($\delta(t)$) process, where the dimension is given by $\delta(t) = 4(\kappa(t) + \sigma^2(t))/\sigma^2(t)$; here, $\delta(t)$ is assumed to be continuously differentiable and finite over $[0, \infty)$, i.e., $\delta \in C^1(\mathbb{R}_0^+)$. Second, we link the ECIR process to the time-inhomogeneous 3/2 model via Itô's lemma [24] (p. 44). Together, these connections enable the derivation of the closed-form expression for the transition PDF presented above. It is important to emphasize that our derivation builds on the seminal observation of Maghsoodi [15] (p. 94), who demonstrated that the conditional distribution of the ECIR process can be characterized as a rescaled noncentral Chi-square distribution.

2. Proof of Theorem 1

Proof of Theorem 1. Let $\{Y_t^{(\eta)}\}_{t \geq 0}$ denote a squared Bessel process whose dimension varies in time according to $\eta(\cdot)$, and given the initial value $Y_0^{(\eta)} = y$. By invoking Proposition 3.4, proposed by Carmona [20], one shows that for any real ω , the characteristic function of $Y_t^{(\eta)}$ admits the form

$$\mathbb{E}[e^{i\omega Y_t^{(\eta)}}] = \exp\left(i\omega\left(\frac{y}{1-2i\omega t} + \int_0^t \frac{\eta(u)}{1-2i\omega(t-u)} du\right)\right). \quad (3)$$

Suppose that $\{x_t\}_{t \geq 0}$ follows the following ECIR process:

$$dx_t = (\kappa(t) + \sigma^2(t) - \kappa(t) \theta(t) x_t) dt + \sigma(t) \sqrt{x_t} dW_t, \quad x_0 > 0. \quad (4)$$

By applying Lemma 2.4 and Corollary 3.1, given by Shirakawa [16], the ECIR process $\{x_t\}_{t \geq 0}$ in (4) is a time-varying dimensional squared Bessel process with time and state changes, so we have

$$\{x_t\}_{t \geq 0} \sim \left\{e^{-\Delta(0, t)} Y_{\tau(t)}^{(\eta)}\right\}_{t \geq 0},$$

where $\Delta(s, t) = \int_s^t \kappa(\zeta) \theta(\zeta) d\zeta$ and $\tau(t) = \frac{1}{4} \int_0^t e^{\Delta(0, \zeta)} \sigma^2(\zeta) d\zeta$. According to the ECIR process (4), $\{Y_{\tau(t)}^{(\eta)}\}_{t \geq 0}$ is the squared Bessel process with time-varying dimension $\eta(t) = 4(\kappa(\tau^{-1}(t)) + \sigma^2(\tau^{-1}(t)))/\sigma^2(\tau^{-1}(t))$ with initial value $Y_0^{(\eta)} = x_0$. This indicates that the characteristic function of $\{x_t\}_{t \geq 0}$ is identical to $\{e^{-\Delta(0, t)} Y_{\tau(t)}^{(\eta)}\}_{t \geq 0}$. From (3), we have

$$\mathbb{E}[e^{i\omega x_t}] = \mathbb{E}[e^{i\omega e^{-\Delta(0, t)} Y_{\tau(t)}^{(\eta)}}]$$

$$\begin{aligned}
&= \exp \left(i\omega e^{-\Delta(0,t)} \left(\frac{x_0}{1-2i\omega e^{-\Delta(0,t)}\tau(t)} + \int_0^{\tau(t)} \frac{4(\kappa(\tau^{-1}(s)) + \sigma^2(\tau^{-1}(s))) / \sigma^2(\tau^{-1}(s))}{1-2i\omega e^{-\Delta(0,t)}(\tau(t)-s)} ds \right) \right) \\
&= \exp \left(i\omega e^{-\Delta(0,t)} \left(\frac{x_0}{1-2i\omega e^{-\Delta(0,t)}\tau(t)} + \int_0^t \frac{4(\kappa(u) + \sigma^2(u)) / \sigma^2(u)}{1-2i\omega e^{-\Delta(0,t)}(\tau(t)-\tau(u))} \tau'(u) du \right) \right) \\
&= \exp \left(i\omega \left(\frac{x_0 e^{-\Delta(0,t)}}{1-2i\omega e^{-\Delta(0,t)}\tau(t)} + \int_0^t \frac{(\kappa(u) + \sigma^2(u)) e^{-\Delta(u,t)}}{1-2i\omega e^{-\Delta(0,t)}(\tau(t)-\tau(u))} du \right) \right) \\
&= \exp \left(i\omega \left(\frac{x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} + \int_0^t \frac{(\kappa(u) + \sigma^2(u)) e^{-\Delta(u,t)}}{1-2i\omega \Lambda(u,t)} du \right) \right), \tag{5}
\end{aligned}$$

where $\Lambda(u, t) = \frac{1}{4} \int_u^t e^{-\Delta(\xi, t)} \sigma^2(\xi) d\xi$. Since $\delta \in C^1(\mathbb{R}_0^+)$ and $\kappa(u) + \sigma^2(u) = \delta(u) \sigma^2(u) / 4$, the characteristic function (5) of the special case of ECIR process (4) becomes

$$\begin{aligned}
\mathbb{E}[e^{i\omega x_t}] &= \exp \left(\frac{i\omega x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} + \frac{i\omega}{4} \int_0^t \frac{\delta(u) \sigma^2(u) e^{-\Delta(u,t)}}{1-2i\omega \Lambda(u,t)} du \right) \\
&= \exp \left(\frac{i\omega x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} - \frac{1}{2} \int_0^t \delta'(u) \ln(1-2i\omega \Lambda(u,t)) du - \frac{\delta(0)}{2} \ln(1-2i\omega \Lambda(0,t)) \right) \\
&= \frac{\exp \left(\frac{i\omega x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} \right)}{(1-2i\omega \Lambda(0,t))^{\delta(0)/2}} \exp \left(-\frac{1}{2} \int_0^t \delta'(u) \ln(1-2i\omega \Lambda(u,t)) du \right) \tag{6}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\exp \left(\frac{i\omega x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} \right)}{(1-2i\omega \Lambda(0,t))^{\delta(0)/2}} \left(\lim_{n \rightarrow \infty} \exp \left(-\frac{1}{2} \sum_{\ell=0}^{n-1} \frac{t}{n} \delta' \left(\frac{\ell t}{n} \right) \ln \left(1-2i\omega \Lambda \left(\frac{\ell t}{n}, t \right) \right) \right) \right) \\
&= \lim_{n \rightarrow \infty} \frac{\exp \left(\frac{i\omega x_0 e^{-\Delta(0,t)}}{1-2i\omega \Lambda(0,t)} \right)}{(1-2i\omega \Lambda(0,t))^{\delta(0)/2}} \prod_{\ell=0}^{n-1} \left(1-2i\omega \Lambda \left(\frac{\ell t}{n}, t \right) \right)^{-\frac{t}{2n} \delta' \left(\frac{\ell t}{n} \right)}. \tag{7}
\end{aligned}$$

We pause to look at the characteristic function (6). In distribution, by using the Lévy–Cramér continuity theorem, the random variable x_t can be represented as a continuous mixture of scaled central Chi-square distributions together with a single initial scaled noncentral chi-square component. Specifically, x_t is equal in distribution to

$$x_t \stackrel{d}{=} \underbrace{\Lambda(0, t) \chi^2 \left(\delta(0), \frac{x_0 e^{-\Delta(0,t)}}{\Lambda(0, t)} \right)}_{\text{scaled noncentral Chi-square}} + \int_0^t \Lambda(u, t) d\chi^2(\delta'(u) du, 0),$$

where $\delta(0)$ denotes the degrees of freedom of the noncentral Chi-square distribution with noncentrality parameter $x_0 e^{-\Delta(0,t)} / \Lambda(0, t)$, the parameter $\delta'(u) du$ is the infinitesimal degrees of freedom of the scaled central Chi-square distributions, and Λ is the time-dependent scaling parameter. We next turn to (7) to conclude, in distribution, that

$$x_t \stackrel{d}{=} \Lambda(0, t) \chi^2 \left(\delta(0), \frac{x_0 e^{-\Delta(0,t)}}{\Lambda(0, t)} \right) + \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} \Lambda \left(\frac{\ell t}{n}, t \right) \chi^2 \left(\frac{t}{n} \delta' \left(\frac{\ell t}{n} \right), 0 \right). \tag{8}$$

It follows from (8) that X_t can be written as a linear combination of independent non-central Chi-square random variables. Let X_t satisfy the SDE in (4). Then, by the result of Castaño-Martínez and López-Blázquez [22], its transition PDF is

$$f_{X_t}(x, t | x_0) = \frac{e^{-\frac{x}{2\Lambda(0,t)}} x^{\frac{\delta(t)}{2}-1}}{(2\Lambda(0,t))^{\delta(t)/2}} \sum_{k \geq 0} \frac{k! c_k}{\Gamma(\delta(t)/2 + k)} \mathbf{L}_k^{\left(\frac{\delta(t)}{2}-1\right)} \left(\frac{x}{2\Lambda(0,t)} \right),$$

for all $x, x_0 > 0$ and $t > 0$, where the notation is as in Theorem 1. To obtain the PDF of V_t , set X_t to V_t , define $X_t \equiv 1/V_t$. Applying Itô's lemma to (1) yields (3), and then the standard change-of-variable formula gives

$$f_{V_t}(v, t | v_0) = f_{X_t}(x, t | x_0) \left| \frac{\partial X}{\partial V} \right| = f_{X_t}\left(\frac{1}{v}, t \mid \frac{1}{v_0}\right) \frac{1}{v^2}.$$

This completes the proof of Theorem 1. $\square$

3. Consequences

In this section, we analyze the conditional moments of the variance process as described by the 3/2 model. Building on the integration of Laguerre series summation, we derive a closed-form expression for the conditional γ -moment which is expressed as a series involving complete Bell polynomial coefficients. We then discuss the necessary conditions on γ (and on the decay of the series coefficients) that guarantee convergence of the moment expression. In particular, for the case of constant model parameters, our results simplify and recover the known formulation presented by Ahn and Gao [1], thereby highlighting both the generality of the approach and its consistency with established results.

To compute the conditional γ -moment of the time-inhomogeneous 3/2 model (1) via the transition PDF (2), we begin by performing the change of variable $z = 1/2v\Lambda(0, t)$ which implies $dv = -dz/2\Lambda(0, t)z^2$. We have

$$\begin{aligned} \int_0^\infty v^{\gamma - \frac{\delta(t)}{2} - 1} e^{-\frac{1}{2v\Lambda(0,t)}} \mathbf{L}_k^{\left(\frac{\delta(t)}{2} - 1\right)} \left(\frac{1}{2v\Lambda(0,t)} \right) dv \\ = \left(\frac{1}{2\Lambda(0,t)} \right)^{\gamma - \frac{\delta(t)}{2}} \int_0^\infty z^{\frac{\delta(t)}{2} - \gamma - 1} e^{-z} \mathbf{L}_k^{\left(\frac{\delta(t)}{2} - 1\right)}(z) dz. \end{aligned}$$

Applying the standard formula provided by Gradstein and Ryzhik [25] (assertion 7, p. 817) yields

$$\mathbb{E}[V_t^\gamma | V_0 = v_0] = \int_0^\infty v^\gamma f_{V_t}(v, t | v_0) dv = \frac{\Gamma(\delta(t)/2 - \gamma)}{\Gamma(\gamma) (2\Lambda(0, t))^\gamma} \sum_{k \geq 0} \frac{c_k \Gamma(k + \gamma)}{\Gamma(\delta(t)/2 + k)}. \quad (9)$$

It is noteworthy that our final result is derived without the need to solve the recurrence relation of the underlying ordinary differential equations, in contrast to the approach proposed by Sutthimat et al. [2]. Moreover, for $-\gamma = n \in \mathbb{N}$, the infinite series (9) reduces to a finite sum running from $k = 0$ to n because all terms with $k > n$ vanish.

To ensure that (9) converges, we must impose appropriate conditions on γ as well as on the series coefficients. First, for positive γ , the condition $\gamma < 2(\kappa + \sigma^2)/\sigma^2$ (or equivalently, $\gamma < \delta(t)/2$) guarantees that the Gamma function $\Gamma(\delta(t)/2 - \gamma)$ remains finite. For negative γ , the derivation remains valid, provided that γ does not coincide with any of the poles of the Gamma functions (i.e., every Gamma function in the expression is well-defined). In addition, the convergence of series (9) is ensured if the coefficients c_k (which are defined via complete Bell polynomials in the parameters d_j) decay sufficiently fast. In practice, this decay is achieved by imposing suitable regularity conditions on the functions $\delta(t)$, $\kappa(t)$, $\theta(t)$, and $\sigma(t)$, and consequently on the derived expressions for the d_j .

In the context of the original 3/2 model, suppose that V_t follows the 3/2 model, which is the model (1), with constant parameters $\kappa(t) = \kappa$, $\theta(t) = \theta$, and $\sigma(t) = \sigma$. The transition PDF of V_t is given by

$$f_{V_t}(v, t | v_0) = \alpha^{-1} e^{-u-w} u^{\frac{\delta}{4} + \frac{3}{2}} w^{\frac{1}{2} - \frac{\delta}{4}} \mathbf{J}_{\frac{\delta}{2} - 1}(2(uw)^{\frac{1}{2}}), \quad (10)$$

where $J_q(\cdot)$ denotes the modified Bessel function of the first kind of order q (see Ahn and Gao [1]), and

$$\delta = \frac{4(\kappa + \sigma^2)}{\sigma^2}, \quad \alpha = \frac{2\kappa\theta}{\sigma^2(1 - e^{-\kappa\theta t})}, \quad u = \frac{\alpha}{v} \quad \text{and} \quad w = \frac{\alpha}{v_0} e^{-\kappa\theta t}.$$

Technically speaking, since $\delta(t) = \delta$ is constant, $\delta'(t) = 0$ for all $t \geq 0$, which implies that $d_j = 0$ for all $j \geq 2$. Consequently, the complete Bell polynomial coefficients are simplified to

$$c_k = \frac{1}{k!} B_k(d_1, 0, 0, \dots, 0) = \frac{d_1^k}{k!}, \quad \text{where} \quad d_1 = -\frac{e^{-\Delta(0,t)}}{2v_0\Lambda(0,t)} = -\frac{\alpha}{v_0} e^{-\kappa\theta t} = -w.$$

Also, the prefactor in the transition PDF (2) becomes

$$\frac{e^{-\frac{1}{2v\Lambda(0,t)}} v^{-\frac{\delta}{2}-1}}{(2\Lambda(0,t))^{\delta/2}} = e^{-u} v^{-\frac{\delta}{2}-1} \alpha^{\frac{\delta}{2}} = \alpha^{-1} e^{-u} u^{\frac{\delta}{2}+1}.$$

By applying a standard formula that relates the Laguerre polynomials to the modified Bessel function of the first kind of order $\delta/2 - 1$ (see Gradstein and Ryzhik [25], section 8.975, p. 1011), we obtain

$$\begin{aligned} f_{V_t}(v, t | v_0) &= \alpha^{-1} e^{-u} u^{\frac{\delta}{2}+1} \sum_{k \geq 0} \frac{(-w)^k}{\Gamma(\delta/2 + k)} L_k^{(\frac{\delta}{2}-1)}(u) \\ &= \alpha^{-1} e^{-u-w} u^{\frac{\delta}{4}+\frac{3}{2}} w^{\frac{1}{2}-\frac{\delta}{4}} J_{\frac{\delta}{2}-1}(2(uw)^{\frac{1}{2}}). \end{aligned}$$

This completes the brief proof of (10). It should be noted that the constant parameter $\delta(t)$ consistently produces $c_k = d_1^k/k!$, regardless of whether the parameters are constant or time-dependent, as illustrated by the example $\kappa(t) = (t+1)^2$ and $\sigma(t) = t+1$, which implies that $\delta(t) = 8$ and thus, $\delta'(t) = 0$. Moreover, the transition PDF reduced from (2) coincides with the result presented by Ahn and Gao [1]. Consequently, it is not complicated to show that the conditional γ -moment, where $\gamma \in \mathbb{R}$, admits the explicit formula

$$\mathbb{E}[V_t^\gamma | V_0 = v_0] = \int_0^\infty v^\gamma f_{V_t}(v, t | v_0) dv = \alpha^\gamma e^{-w} \frac{\Gamma(\delta/2 - \gamma)}{\Gamma(\delta/2)} {}_1F_1\left(\frac{\delta}{2} - \gamma; \frac{\delta}{2}; w\right), \quad (11)$$

where ${}_1F_1$ denotes the confluent hypergeometric function (Kummer's function). Note that (11) can be derived directly from (9) by simplifying it for the time-independent case (i.e., under constant parameters). Unlike the result proposed by Sutthimat et al. [2], which requires restrictive conditions on γ , our result in (11) has explicit convergence criteria; the formula converges for $\gamma < 2(\kappa + \sigma^2)/\sigma^2$, diverges for $\gamma > 2(\kappa + \sigma^2)/\sigma^2$ due to heavy-tailed behavior in the transition PDF, and becomes undefined at $\gamma = 2(\kappa + \sigma^2)/\sigma^2$, where the gamma function $\Gamma(0)$ is singular.

4. Discussion

One key contribution of this work is deriving the characteristic function and transition density as an infinite series of weighted noncentral Chi-square variables. By exploiting the connection between the ECIR process and the squared Bessel process with a time-varying dimension, our method provides both an analytical treatment of time-inhomogeneity and insights into the interplay between the drift and diffusion components. Using generalized Laguerre and complete Bell polynomials, our series yields a compact closed-form solution

that naturally reduces to classical results under constant parameters, in line with Ahn and Gao [1] and Maghsoodi [15].

Numerical examples demonstrate the robustness and practicality of our approach. Using time-varying parameters $\kappa(t) = 1 + 3t$, $\theta(t) = 1 + 2t$, and $\sigma(t) = 1 + t$ with $0 \leq t \leq 1$ and $v_0 = 1$. We found that the series partial sums with `maxOrder` values of 15–17 converge rapidly to those obtained with `maxOrder` = 100 (see Figure 1 left). In addition, Monte Carlo simulations via the Euler–Maruyama method yield a terminal distribution histogram that, when normalized, closely aligns with the theoretical curve (Figure 1 right), confirming both the convergence of the series expansion and the accuracy of our numerical scheme.

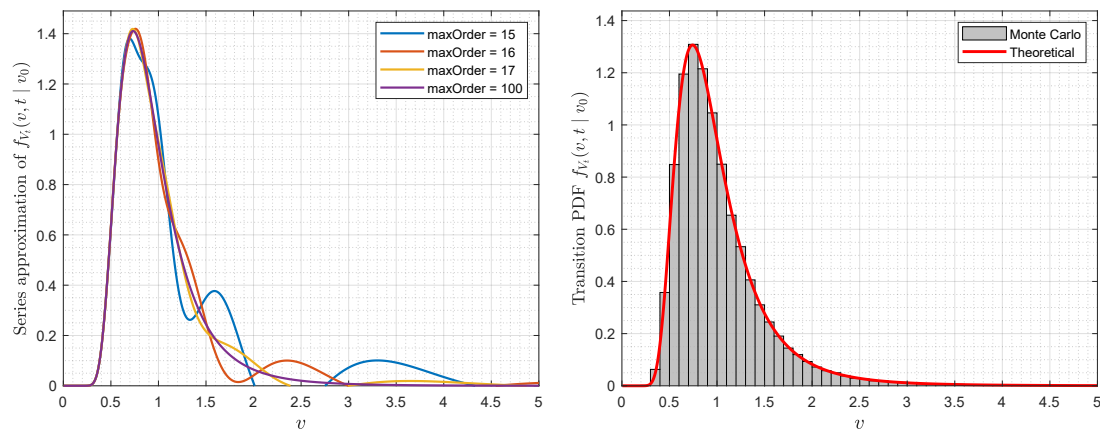


Figure 1. Illustration of the time-inhomogeneous 3/2 model transition PDF. Subfigure (left): Convergence of series partial sums for the transition PDF at various truncation orders. Subfigure (right): Comparison between the theoretical transition PDF and Monte Carlo histogram.

Extending this idea, our methodology is not confined to the time-inhomogeneous 3/2 model. By leveraging Itô's lemma, it can be used to derive transition PDFs for a wide class of processes. In particular, any process that can be transformed into an ECIR process or exhibits a structure similar to the time-inhomogeneous 3/2 model can be analyzed using our method. Such transformations typically map the original process into a squared Bessel process with time-varying parameters, for which closed-form transition densities are obtainable via series expansions. Thus, our approach broadens its applications in modeling interest rates and stochastic volatility, providing a systematic route for addressing the transition density problem in both affine and non-affine processes. For example, time-inhomogeneous processes such as the d -dimensional Ornstein–Uhlenbeck (OU) process,

$$dy_t^{(i)} = -\frac{1}{2}k(t)y_t^{(i)} dt + \frac{1}{2}\sigma(t)dw_t^{(i)}, \quad i = 1, 2, 3, \dots, d,$$

(see Maghsoodi [15] for further details). Another illustrative example is the generalized nonlinear drift constant elasticity of variance process,

$$dR_t = \kappa(t)(\theta(t)R_t^{-(1-\beta)} - R_t)dt + \sigma(t)R_t^{\beta/2}dW_t, \quad \beta \in [0, 2) \cup (2, \infty),$$

which contains several notable models, including the time-inhomogeneous OU and CIR processes, as well as the time-inhomogeneous 3/2 and 4/2 models (see Sutthimat et al. [2] for more details).

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Abbreviations

The following abbreviations are used in this manuscript:

CEV	constant elasticity of variance
CIR	Cox–Ingersoll–Ross
ECIR	extended Cox–Ingersoll–Ross
OU	Ornstein–Uhlenbeck
PDF	probability density function
SDE	stochastic differential equation

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