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Optimal Design of Variable-Stiffness Fiber-Reinforced Composites

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Abstract: The concept of variable-stiffness composites allows the stiffness properties to vary spatially in the material. In the case of fiber-reinforced composites, the mechanical properties of the composite can be improved by tailoring the fiber orientations in a spatially optimal manner. In this paper, the problem of optimal spatial orientation of fibers in a two-dimensional composite structure under in-plane loading is studied, using the strain energy-minimizing method. The fiber orientation is assumed to be constant within each element of the model but varies from element to element. The optimal design problem is solved numerically using a global optimization method based on a genetic algorithm. Some numerical examples illustrate the efficiency and applicability of the method.

Keywords: fiber-reinforced composites; variable-stiffness composites; curvilinear fiber composites; optimal design; structural optimization; FEM

MSC: 74-10; 74B05; 74K20; 74M25; 74E30; 74G65; 74P10; 74S05; 68W50

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1. Introduction

Fiber-reinforced composite materials are advanced materials that are engineered by combining high-strength, high-stiffness fibers with a matrix material, which is often a polymer but can also be a metal or ceramic. This combination results in materials with enhanced mechanical properties that surpass those of the individual components. The fibers provide reinforcement and bear the primary load, while the matrix binds them together, transfers stress between them, and shields them from environmental factors [1,2].

Common fiber types include glass fibers (with good strength-to-weight ratio and varying properties); carbon fibers (with exceptional stiffness and good fatigue resistance); aramid fibers (with high tensile strength, high impact resistance and good energy absorption); boron fibers (with high stiffness and strength); ceramic fibers (with high temperature resistance and stiffness); metal fibers (with various metallic properties); and natural fibers (e.g., flax, hemp and jute), which generally have lower strength and stiffness than synthetic fibers but are renewable and have potential for sustainable applications.

Common matrix types include a polymer matrix (thermoplastic and thermoset polymers); a metal matrix, which has high temperature strength, stiffness, and wear resistance; and a ceramic matrix, which provides high temperature strength and environmental resistance.

The interface between the fiber and matrix is crucial for load transfer. Good bonding at this interface is essential for optimal composite performance [3–5].

The two most common types of fiber reinforcement are continuous fibers and discontinuous, or short, fibers. Continuous fibers are long fibers that run throughout the part,

providing maximum reinforcement in the direction of the fibers. They can be unidirectional or woven into fabrics. Discontinuous fibers are short, randomly oriented or aligned fibers dispersed within the matrix. They offer more isotropic properties but are generally weaker than continuous fibers [6,7].

Variable-stiffness composites using reinforced fibers are advanced composite materials in which the stiffness properties are designed to vary spatially within the structure. This is usually achieved by directing the reinforcing fibers along curved paths within the material instead of using straight fibers at constant angles. By aligning fibers along the principal load paths, which can vary across a structure, variable-stiffness composites can distribute stresses more efficiently. This improves the structure's performance under complex loading conditions. This makes more efficient use of the material's anisotropic properties, leading to improved performance. Compared to constant stiffness composites with the same weight, variable-stiffness composites can offer higher buckling loads, increased strength, and improved fatigue life. Optimized material distribution and tailored stiffness can lead to lighter structures without compromising performance. Variable-stiffness composites give engineers greater freedom to design complex geometries and optimize structural behavior for specific applications [8–10].

Fiber-reinforced composite materials are characterized by very good mechanical properties. In addition, there are many advantages to be gained by optimizing the directional properties of these materials. For example, the material and geometrical properties of the reinforcing fibers influence the mechanical properties of the composite material and can be treated mathematically as design parameters in optimal design problems. In general, the optimal design of these structures often leads to complicated mathematical problems. The full advantages of composites can be realized when the reinforcing fibers in the matrix are optimally oriented or shaped with respect to the assumed objective behavior of the structure under actual loading conditions. In their book, Gürdal et al. [11] provide a thorough treatment of both the contemporary mechanics of composite laminates and design optimization techniques. Gürdal and Olmedo [12] also investigate variable-stiffness laminates using reinforced fibers with spatially varying orientations. Pedersen [13] investigates the problems involved in optimal 2D design of composites for stiffness and strength. For the optimization problem, there are several mathematical optimization methods based on minimizing a relevant function. The function can be the compliance of the structure, the weight of the structure, or another factor. Depending on the optimization problem, the function can include various parameters. Setoodeh et al. [14,15] study design problems of variable-stiffness laminates using lamination parameters and cellular automata. Wiśniewski [16] studies the optimal design of reinforcing fibers in laminated composites using genetic algorithms.

In general, most studies in the field of the optimal design of composite multilayer materials focus on improving their mechanical properties. This is typically achieved by combining simple micromechanical homogenization methods for fiber-reinforced layers with straight fibers using various macromechanical optimization methods. Additionally, when studying problems involving fiber-reinforced layers with curved fibers, restrictive conditions regarding the direction of the fibers are typically adopted to simplify the problem.

This paper studies the optimal micromechanical design of layers reinforced with curvilinear fibers, taking an element-by-element approach and without imposing restrictions on the possible directions of the fibers. The analysis is independent of the type of external loads. More specifically, this work deals with the problem of optimizing the fiber orientation angles in a two-dimensional elastic composite structure under in-plane loading using the strain energy minimization method. A finite element discretization is used, and the fiber orientations are assumed to be constant within each element of the

model but may vary from element to element. Two-dimensional linear elasticity theory and bilinear four-node rectangular elements are used for deformation analysis. However, such a formulation requires repeated transformation of material properties using classical trigonometric transformations. Furthermore, the local design problem based on strain energy minimization is non-convex when parameterized by fiber orientation angles. As a result, traditional gradient-based approaches are likely to be trapped in local optima, although these problems can be overcome to some extent in the numerical implementation. In this work, the optimal design problem is solved numerically using a genetic algorithm global optimization method. The efficiency and applicability of the method are illustrated by some numerical examples.

2. The Mathematical Model

The present study focuses on a thin fiber-reinforced composite panel. The fibre orientation angle, which is key to defining the stiffness and strength properties of the structure, can vary spatially throughout the structure. Such variation of the fibre orientation angle naturally produces curvilinear fibre paths and linear orthotropic elastic structures with "variable stiffness", as often referred to in the literature [11,12].

For the purpose of modelling, a uniform orthotropic elastic plate of rectangular cross-section with length L_x, width L_y and thickness h is considered, as shown in Figure 1.

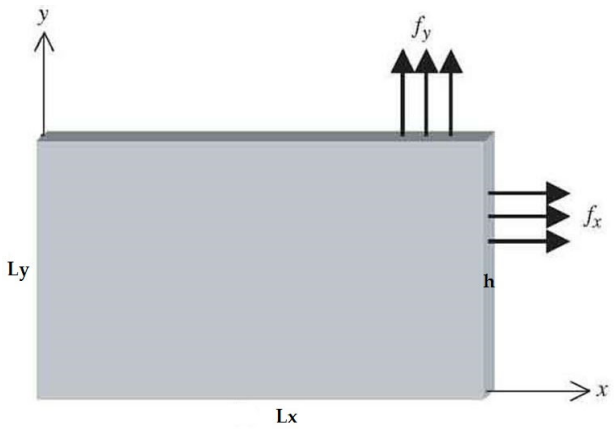


Figure 1. Uniform orthotropic elastic plate with a rectangular cross-section.

2.1. Kinematics and Constitutive Relationships

A rectangular Cartesian coordinate system (x, y, z) is defined on the mid-plane of the plate, where the x-axes, y-axes and z-axes coincide with the length, width and thickness of the structure, respectively. It is further assumed that the material principal directions of the orthotropic medium coincide with the coordinate axes used to describe the problem. The structure is considered to be under plane stress because its thickness is much smaller than the other two dimensions, and all the loads are contained in the central plane. Consequently, the analysis domain is the central section of the plate.

Plane elasticity problems in the xy-plane (z is the out-of-plane direction) are characterized by the displacement vector field $\mathbf{u} = (u_x, u_y, u_z)$ with components:

$$u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = 0, \tag{1}$$

and the following plane stress field in the (x, y, z) coordinate system:

$$\begin{aligned} &\sigma_{xx} = \sigma_{xx}(x, y), \quad \sigma_{xy} = \sigma_{xy}(x, y), \quad \sigma_{yy} = \sigma_{yy}(x, y), \\ &\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \end{aligned} \tag{2}$$

The plane strain field associated with the stress field of Equation (2) is given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}, \quad \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0. \tag{3}$$

where s_{ij} are the elastic compliance constants of the material:

$$\begin{aligned} &s_{11} = \tfrac{1}{E_1}, \quad s_{22} = \tfrac{1}{E_2}, \quad s_{66} = \tfrac{1}{G_{12}}, \\ &s_{12} = -\nu_{21} s_{22} = -\nu_{12} s_{11} \end{aligned} \tag{4}$$

The inverse form of Equation (3) is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{5}$$

where c_{ij} are the elastic stiffness constants of the material.

$$\begin{aligned} &c_{11} = \tfrac{E_1}{1-\nu_{12}\nu_{21}}, && c_{22} = \tfrac{E_2}{1-\nu_{12}\nu_{21}}, \\ &c_{12} = \nu_{21} c_{22} = \nu_{12} c_{11}, && c_{66} = G_{12}. \end{aligned} \tag{6}$$

The material constants E_1, E_2, ν_{12}, G_{12} are the engineering constants for an orthotropic material in plane elasticity theory.

2.2. Governing Equations and Boundary Conditions

The equations of motion and the associated boundary conditions of the plate are derived using Hamilton's principle:

$$\delta J(\mathbf{u}) = \delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V} + \mathcal{W}_{\mathfrak{e}})\, dt = 0\,, \tag{7}$$

where $\delta(\cdot)$ denotes the first variation operator, $\mathbf{u}$ is the displacement vector field, $\mathcal{T}$ is the kinetic energy of the structure, $\mathcal{V}$ is the strain energy of the structure a, $\mathcal{W}_{\mathfrak{e}}$ is the work done on the structure by the external body forces and surface tractions, and t is the time.

For a plane elasticity problem in xy-plane, the kinetic energy of the structure can be written as

$$\mathcal{T} = \int_V \frac{1}{2} \rho \{\dot{u}\}^T \{\dot{u}\} \; dV, \tag{8}$$

where ρ is the density of the material, $\{u\} = \begin{bmatrix} u_x & u_y \end{bmatrix}^T$ is the displacements field vector, and $\{\dot{u}\}$ is the first time derivative of the displacement field vector.

The strain energy of the structure can be written as

$$\mathcal{V} = \int_V \frac{1}{2} \{\sigma\}^T \{\varepsilon\} \; dV, \tag{9}$$

where $\{\sigma\} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T$ denotes the stress vector and $\{\varepsilon\} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & 2\varepsilon_{xy} \end{bmatrix}^T$ is the strain vector.

The external work can be written as

$$\mathcal{W}_{\mathfrak{e}} = \int_V \{u\}^T \{f\}\, dV + \int_S \{u\}^T \{\hat{\tau}\}\, dS\,, \tag{10}$$

where $\{f\} = \begin{bmatrix} f_x & f_y \end{bmatrix}^T$ is the body force vector, and $\{\hat{\tau}\} = \begin{bmatrix} \hat{\tau}_x & \hat{\tau}_y \end{bmatrix}^T$ is the traction vector applied on the boundary S of the structure.

Two differential equations of motion and the associated boundary conditions are obtained by substituting Equations (8)–(10) into Equation (7) and integrating by parts.

2.2.1. Equations of Motion and Boundary Conditions in Expanded Form

- Equations of motion:

$$\begin{aligned} \frac{\partial}{\partial x}\left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right] + f_x = \rho\frac{\partial^2 u_x}{\partial t^2}, \\ \frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right) + f_y = \rho\frac{\partial^2 u_y}{\partial t^2}. \end{aligned} \tag{11}$$

- Boundary conditions:

$$\begin{array}{ll} \text{Geometric conditions} & \text{Natural conditions} \\ u_x \text{ specified on } S_u \text{ or} & \left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right)n_x + c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)n_y = \hat{\tau}_x \text{ on } S_\sigma \\ u_y \text{ specified on } S_u \text{ or} & c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)n_x + \left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right)n_y = \hat{\tau}_y \text{ on } S_\sigma \end{array} \tag{12}$$

where S_u and S_σ are disjoint portions of the boundary S, and $\{n\} = \begin{bmatrix} n_x & n_y \end{bmatrix}^T$ denotes the unit normal vector on the boundary S.

2.2.2. Equations of Motion and Boundary Conditions in Vector Form

- Equations of motion:

$$[D]^T[C][D]\{u\} + \{f\} = \rho\{\ddot{u}\} \tag{13}$$

where $[D] = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}^T$ and $[C] = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix}$

- Boundary conditions:

$$\begin{array}{ll} \text{Geometric conditions} & \text{Natural conditions} \\ \{u\} \text{ specified on } S_u \text{ or} & [\overline{n}][C][D]\{u\} = \{\hat{\tau}\} \text{ on } S_\sigma \end{array} \tag{14}$$

where $[\overline{n}] = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$.

2.2.3. The Stiffness Matrix of a Fiber-Reinforced Composite Layer

For a fiber-reinforced composite layer, the in-plane orthotropic directions (r, s) of the material are inclined by an angle θ with respect to the global axes (x, y) of the structure, as shown in Figure 2.

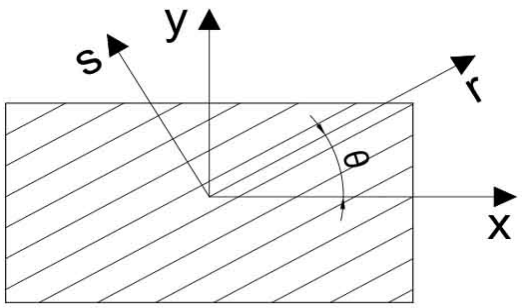


Figure 2. Fiber-reinforced layer with a local (r, s) and global (x, y) coordinate system.

The stiffness matrix $[C]$ of the material must be replaced in Equations (13) and (14) by the matrix $[\overline{C}(\theta)]$, according to the following transformation

$$[\overline{C}(\theta)] = [T(\theta)]^T [C][T(\theta)], \tag{15}$$

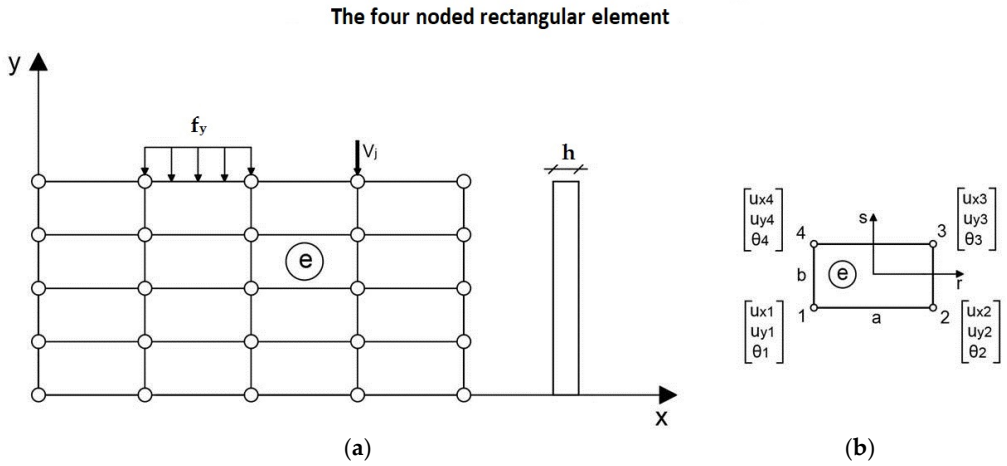
where $[T(\theta)]$ is the transformation matrix.

$$[T(\theta)] = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2 - s^2 \end{bmatrix} \tag{16}$$

with $s = \sin\theta$ and $c = \cos\theta$ [17].

3. Finite Element Formulation

A deep fiber-reinforced composite beam with curvilinear fibers is considered a thin, two-dimensional and linear orthotropic elastic structure. For the finite element formulation (see [18,19]), the structure is discretized in a mesh of bilinear four-node rectangular elements with three degrees of freedom per node, two translational degrees of freedom u_{xi} and u_{yi} along the x and y directions, and one rotational degree of freedom θ_i for the fiber orientation, as shown in Figure 3.



(**a**) (**b**)

Figure 3. (**a**) A two-dimensional composite structure reinforced with curvilinear fibers is under external loads f_y and V_j has been discretized with bilinear four-node rectangular elements. (**b**) The four-node bilinear rectangular element with three degrees of freedom per node.

The array of nodal displacements and rotations is defined as

$$\{\mathbf{d}^e\} = \begin{bmatrix} u_{x1} & u_{y1} & \theta_1 & u_{x2} & u_{y2} & \theta_2 & u_{x3} & u_{y3} & \theta_3 & u_{x4} & u_{y4} & \theta_4 \end{bmatrix}^{\mathrm{T}} \tag{17}$$

The displacements and rotation of the element are approximated by the following expressions:

$$\begin{aligned} u_x(x,y,t) &= \sum_{i=1}^{4} H_i^{u_x}(x,y)\, u_{xi}(t), \\ u_y(x,y,t) &= \sum_{i=1}^{4} H_i^{u_y}(x,y)\, u_{yi}(t), \\ \theta(x,y,t) &= \sum_{i=1}^{4} H_i^{\theta}(x,y)\, \theta_i(t). \end{aligned} \tag{18}$$

For the displacements in each element, the shape functions $H_i^{u_x}$ and $H_i^{u_y}$ in the local coordinate system (r, s) are quadratic functions of the following type:

$$\begin{array}{l} H_1^{u_x}(r,s) = H_1^{u_y}(r,s) = \frac{1}{4}\left(1-\frac{r}{a}\right)\left(1-\frac{s}{b}\right), \\ H_2^{u_x}(r,s) = H_2^{u_y}(r,s) = \frac{1}{4}\left(1+\frac{r}{a}\right)\left(1-\frac{s}{b}\right), \\ H_3^{u_x}(r,s) = H_3^{u_y}(r,s) = \frac{1}{4}\left(1+\frac{r}{a}\right)\left(1+\frac{s}{b}\right), \\ H_4^{u_x}(r,s) = H_4^{u_y}(r,s) = \frac{1}{4}\left(1-\frac{r}{a}\right)\left(1+\frac{s}{b}\right) \end{array} \tag{19}$$

The shape functions H_i^{θ} for the fiber rotation in each element are constant functions of the following type:

$$H_1^{\theta}(r,s) = H_2^{\theta}(r,s) = H_3^{\theta}(r,s) = H_4^{\theta}(r,s) = \frac{1}{4} \tag{20}$$

This is because each element is assumed to have constant stiffness properties and equal-to-average nodal stiffnesses.

Using Equations (19) and (20), Equation (18) can be written in matrix form as

$$\begin{Bmatrix} u_x \\ u_y \\ \theta^e \end{Bmatrix} = \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \\ [N_{\theta}] \end{bmatrix} \{\mathbf{d^e}\} \tag{21}$$

where

$$\begin{array}{l} [N_{u_x}] = \begin{bmatrix} H_1^{u_x} & 0 & 0 & H_2^{u_x} & 0 & 0 & H_3^{u_x} & 0 & 0 & H_4^{u_x} & 0 & 0 \end{bmatrix} \\ \left[N_{u_y}\right] = \begin{bmatrix} 0 & H_1^{u_y} & 0 & 0 & H_2^{u_y} & 0 & 0 & H_3^{u_y} & 0 & 0 & H_4^{u_y} & 0 \end{bmatrix} \\ [N_{\theta}] = \begin{bmatrix} 0 & 0 & H_1^{\theta} & 0 & 0 & H_2^{\theta} & 0 & 0 & H_3^{\theta} & 0 & 0 & H_4^{\theta} \end{bmatrix} \end{array} \tag{22}$$

Substituting the displacement distribution (Equation (21)) into the expression of Hamilton's principle (Equation (7)) and carrying out the integration over the element surface, the dynamic matrix equation of the plate element is developed:

$$[M^e]\left\{\ddot{\mathbf{X}}^e\right\} + [K^e(\theta^e)]\{\mathbf{X}^e\} = \{F^e\} + \left\{\hat{T}^e\right\} \tag{23}$$

The mass matrix, the stiffness matrix as a function of the fiber orientation, the body force vector, and the surface traction vector of each element are given as

$$\begin{array}{l} [M^e] = h\int_{S^e} \rho \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix}^T \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix} \, ds, \\ [K^e(\theta^e)] = h\int_{S^e} \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix}^T [D]^T \left[\overline{C}(\theta^e)\right][D] \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix} \, ds, \\ [F^e] = h\int_{S^e} \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix}^T \{f\} \, ds, \\ \left[\hat{T}^e\right] = h\int_{S^e} \begin{bmatrix} [N_{u_x}] \\ [N_{u_y}] \end{bmatrix}^T \left\{\hat{t}\right\} ds \end{array} \tag{24}$$

The vector $\{\mathbf{X}^e\} = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^{\mathrm{T}}$ is the nodal displacement vector.

By assembling all elemental equations, one obtains the global dynamic equation of the structure:

$$[M]\left\{\ddot{\mathbf{X}}\right\} + [K(\boldsymbol{\theta}_s)]\{\mathbf{X}\} = \{F\} + \left\{\hat{T}\right\} \tag{25}$$

where $[M]$ is the mass matrix, $[K(\boldsymbol{\theta}_s)]$ is the stiffness matrix as a function of the fiber orientation vector $\boldsymbol{\theta}_s$, $\{F\}$ is the body force vector, $\{\hat{T}\}$ is the surface traction vector, and $\{\mathbf{X}\}$ is the displacement vector of the structure.

For the static case, the above equation takes the following form:

$$[K(\boldsymbol{\theta}_s)]\{\mathbf{X}\} = \{F\} + \{\hat{T}\} \tag{26}$$

4. The Optimization Problem

In order to make the composite material as stiff as possible, an optimal design problem (ODP) for the curvilinear reinforcing fibers must be solved. More specifically, the aim of this work is the identification of the optimal values of the fiber orientation θ^e in each element.

4.1. The Optimal Design Problem

The optimization problem can be written in the following form:

Optimal Design Problem (ODP). *For an elastic fiber-reinforced composite plate subject to equilibrium constraint, find the value of the design vector variable* $\boldsymbol{\theta}_s$ *(values of fiber angle* θ^e *in each element), so that the strain energy* $W_s(\boldsymbol{\theta}_s)$ *of the structure to be minimized.*

$$(ODP)\begin{cases} \text{minimize} & W_s(\boldsymbol{\theta}_s) = \sum_{e=1}^{n} \frac{1}{2}\{\mathbf{X}^e\}^{\mathbf{T}}[K^e(\theta^e)]\{\mathbf{X}^e\} \text{ with respect to } \boldsymbol{\theta}_s \\ \text{subject to} & \begin{cases} [K(\boldsymbol{\theta}_s)]\{\mathbf{X}\} = \{F\} + \{\hat{T}\}, \\ 0 \le \theta^e \le \pi \end{cases} \end{cases} \tag{27}$$

The above optimization problem (ODP) is a highly nonlinear and non-convex problem due to the presence of trigonometric functions. To solve this problem, the finite element formulation described in Section 3 is used to solve the equilibrium equation of the structure (Equation (26)) each time, and a genetic algorithm (GA) method is used to solve the strain energy W_s minimization problem.

4.2. The Genetic Algorithm Optimization Method

The GA method is a global optimization method for solving both constrained and unconstrained optimization problems that is based on natural selection, the process that drives biological evolution. Briefly described, the GA iteratively modifies a population of individual solutions. At each step, the algorithm randomly selects individuals from the current population as parents and uses them to produce the children for the next generation. To create the next generation from the current population, the GA uses three main types of rules at each step:

- *Selection rules* select the individuals, called *parents*, who contribute to the population in the next generation.
- *Crossover rules* combine two parents to create children for the next generation.
- *Mutation rules* apply random changes to each parent to create children.

Over successive generations, the population "evolves" towards an optimal solution.

The following outline summarizes how the genetic algorithm works:

1. The algorithm starts by generating a random initial population.
2. The algorithm then generates a sequence of new populations. At each step, the algorithm uses the individuals in the current generation to create the next population. To create the new population, the algorithm
    a. Assesses each member of the current population by calculating their fitness value. These values are called the raw fitness scores;

b. Scales the raw fitness scores to convert them into a more usable range of values. These scaled values are referred to as expectation values;
c. Selects members, called parents, based on their expectations;
d. Selects some of the individuals in the current population that have lower fitness, called *elite*. These elite individuals are passed on to the next population;
e. Generates children from parents. Children are produced either by making random changes to a single parent—i.e., a *mutation*—or by combining the vector entries of a pair of parents—i.e., a *crossover*;
f. Replaces the current population with children to form the next generation.

3. The algorithm stops when one of the stopping criteria is met.

A detailed description of genetic algorithms and their functionality can be found in reference [20].

GAs can be used to solve a wide variety of optimization problems that are not well suited to standard optimization algorithms, including problems where the objective function is discontinuous, non-differentiable, stochastic, or highly nonlinear.

Applications of GAs to the problems of design and optimization of structures are described in the reference [21].

A genetic algorithm is used to optimize fiber angles in composite laminates in reference [22] and a coupling of genetic algorithms and finite element analysis for the solving of mechanical optimization problems is described in reference [23,24].

5. Numerical Examples and Discussion

In this section, three numerical examples will be presented in support of the above theoretical considerations.

5.1. Example 1

In the first example, a rectangular composite plate is considered with its left edge fixed and loaded by a uniformly distributed surface force on its upper edge, while the rest of the edges are stress free, as shown in Figure 4. The dimensions of the plate are $a = 2.25$ m, $b = 0.75$ m and the thickness is taken to be $h = 0.025$ m. The magnitude of the applied force is $q = 400$ kN/m.

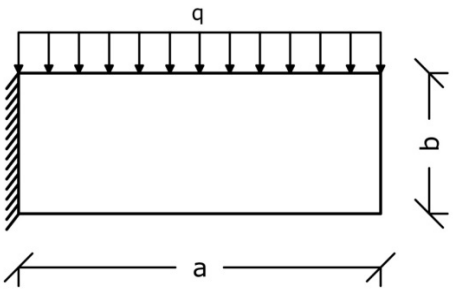


Figure 4. Example 1: A rectangular composite plate with length a and width b, the left edge fixed and the top edge subjected to a uniformly distributed surface force q, with the remaining edges free of stress.

The elastic constants corresponding to an initial design with horizontally oriented fibers are $E_{11} = 181$ GPa, $E_{22} = 10.3$ GPa, $G_{12} = 7.17$ GPa, $n_{12} = 0.28$, $n_{21} = 0.016$.

For the solution of the (ODP), the structure is discretized in a mesh of 27 bilinear rectangular elements and 40 nodes, as shown in Figure 5.

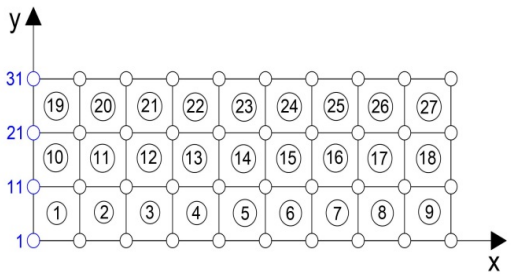


Figure 5. A discretized structure with 27 bilinear rectangular elements and 40 nodes.

The genetic algorithm was run with a population size of 50 and a maximum of 100 generations. The fiber angles fall within the range of [0, 180] degrees. Table 1 shows the resulting optimal fiber angle values for each element, and Figure 6 shows the resulting optimal design solution graphically.

Table 1. Example 1: Optimal fiber angle values for each element.

Elements	**1–4**	**5–7**	**8–9**	**10–13**	**14–16**	**17–18**	**19–22**	**23–25**	**26–27**
Optimal value of θ (degrees)	180	179.8385	30.1669	163.4943	24.7098	62.6838	161.9752	64.0427	91.3530

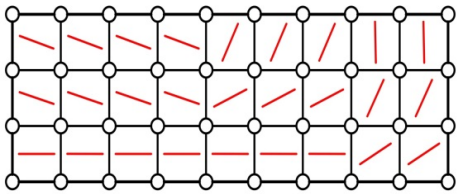


Figure 6. Example 1: The resulting optimal design solution. The red lines indicate the optimal fiber direction for each element.

The relationship between the mean value (red line) and the minimum value of the strain energy (blue line) of the structure for each generation is illustrated in Figure 7. The convergence between these two values of strain energy as a function of the number of generations is also presented. In this example, the mean and minimum values of strain energy in each generation converge to 2.7223 after about 50 generations. This is the global minimum of the strain energy (minW_s = 2.7223 Kj) corresponding to the structure with the optimal design of reinforcing fibers.

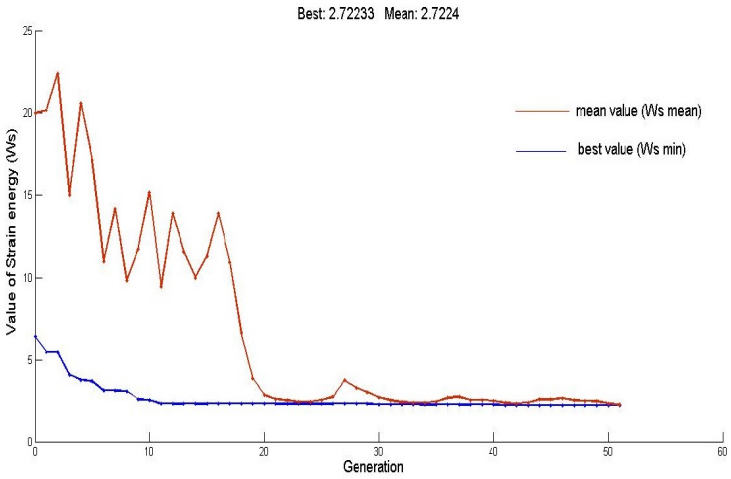


Figure 7. Example 1: Convergence of problem solutions to the optimal value.

5.2. Example 2

In the second example, different loading and support conditions are considered, while the geometry of the plate, its discretization, the parameters of the genetic algorithm, and the material coefficients are the same as in the first example. In particular, Figure 8 considers the case of a simply supported plate with a uniformly distributed load of magnitude $q = 444.44$ kN/m on its upper edge. The resulting optimal fiber angle values for each element are presented in Table 2, the optimal design solution in Figure 9, and the convergence of the problem solutions to the optimal value in Figure 10.

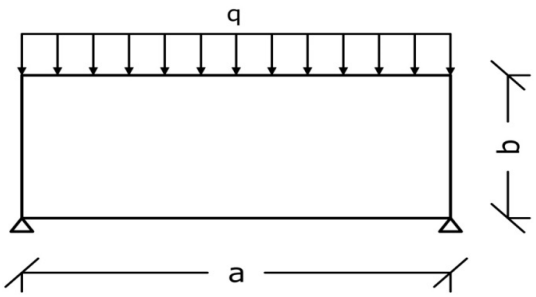


Figure 8. Example 2: A simply supported rectangular composite plate with length a and width b, loaded by a uniformly distributed surface force q.

Table 2. Example 2: Optimal fiber angle values for each element.

Elements	1–4	5	6–9	10–13	14	15–18	19–22	23	24–27
Optimal value of θ (degrees)	51.8807	36.5027	130.12024	43.5890	125.0221	136.2704	13.5154	1.7260	166.6832

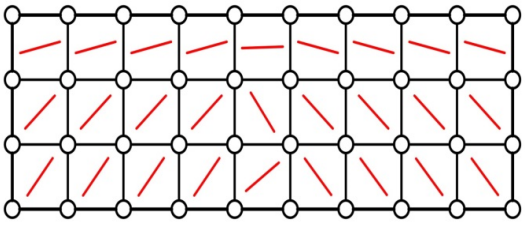


Figure 9. Example 2: The resulting optimal design solution indicates the optimal fiber direction for each element with red lines.

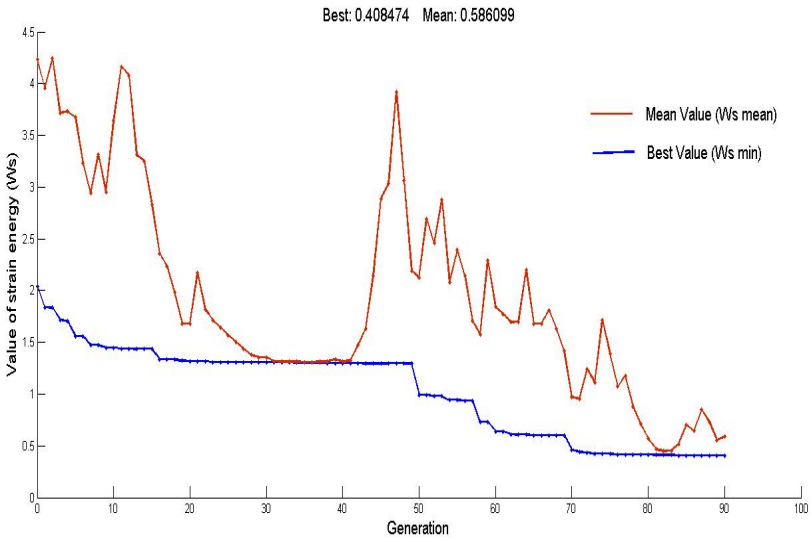


Figure 10. Example 2: Convergence of problem solutions to the optimal value.

In Example 2, the mean and minimum values of the strain energy in each generation converge to the value 0.4085 after about 80 generations, as shown in Figure 10. This value is the global minimum of the strain energy (minW_s = 0.4085 kJ), corresponding to the structure with the optimal design of reinforcing fibers, as shown in Figure 9.

5.3. Example 3

As shown in Figure 11, the last example concerns the tension of the plate under a load in the direction of the x-axis of magnitude $q = 533.33$ kN/m. The results of the proposed computational scheme are presented in Table 3 and Figures 12 and 13 below.

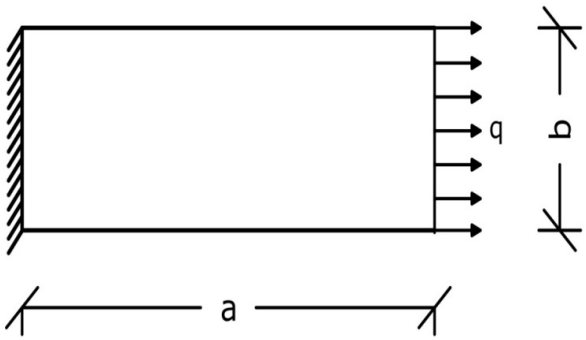


Figure 11. Example 3: A rectangular composite panel with length a and width b, in tension under the action of a uniformly distributed surface force q in the direction of the horizontal axis.

Table 3. Example 3: Optimal fiber angle values for each element.

Elements	1–4	5–7	8–9	10–13	14–16	17–18	19–22	23–25	26–27
Optimal value of θ (degrees)	4.7918	3.0911	4.3718	3.0246	6.1999	2.1623	6.0857	6.1747	22.4719

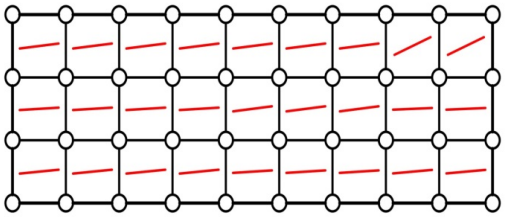


Figure 12. Example 3: Optimal fiber direction for each element is shown with red lines.

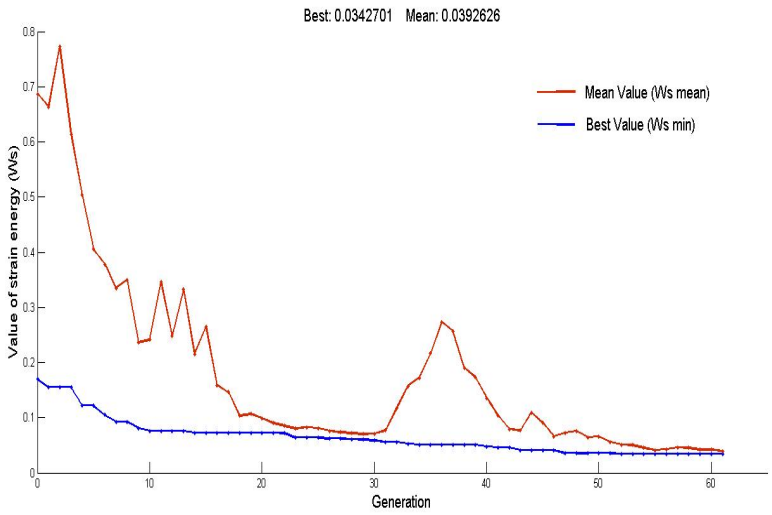


Figure 13. Example 3: Convergence of problem solutions to the optimal value.

In the last Example 3, the mean and minimum values of the strain energy in each generation converge to the value 0.0343 after about 60 generations, as shown in Figure 13. This value is the global minimum of the strain energy ($\min W_s$ = 0.0343 kJ) corresponding to the structure with the optimal design of reinforcing fibers, as shown in Figure 12.

All the numerical examples in this section demonstrate that the solutions correspond to the theoretical predictions of the principal stress trajectories within the structure. It is well known that reinforcement in composites is usually positioned along or across these tensile trajectories. This validation supports our methodological approach.

6. Conclusions

In the context of the optimal design of variable-stiffness fiber-reinforced composites, the problem of the optimal shape of curvilinear reinforcing fibers in a two-dimensional composite layer under in-plane loading is studied herein. More specifically,

1. In order to make the composite material as stiff as possible, an optimal design problem (ODP) for the curvilinear reinforcing fibers has been defined and solved numerically.
2. The optimization problem is highly non-linear and non-convex due to the presence of trigonometric functions. For these reasons, a global optimization method based on genetic algorithms is used.
3. Based on the strain energy-minimizing method, a finite element formulation is used to solve the equilibrium equations of the structure each time, and a genetic algorithm method is used to solve the strain energy W_s minimization problem.
4. Three numerical examples are presented in support of the proposed theoretical and numerical scheme.
5. In all examples, fast convergence of the proposed scheme was observed. The mean and minimum values of strain energy in each generation converged after about 50 generations in the first example, 80 generations in the second example, and about 60 generations in the third example, with a maximum number of generations equal to 100.
6. While many aspects of the proposed methodology require further study, the numerical examples in Section 5 show that the solutions obtained consistently align with the theoretical predictions of the principal stress trajectories within the structure. As is well known, reinforcement in composites is generally positioned along or across these tensile trajectories. The results obtained thus far validate the proposed methodological approach.
7. The proposed method can also be used to study reinforcement fibers of various shapes, including squares and ellipses. In such cases, it would be interesting to study the final shape of the curved fibers and their relationship with geometric characteristics.
8. The proposed method can also be improved by using more accurate elements to achieve better convergence and more precise results.
9. This study presents a straightforward method to optimize the shape of curvilinear fibers in fiber-reinforced composite materials. The proposed method is based on the appropriate arrangement of fiber angles, which change continuously and independently at design points—for example, at the centers of finite elements in the present work. The flexibility of independently optimizing fiber angles at different design points creates difficulties that must be carefully studied in the near future. For example, the resulting optimal design often has a discontinuous fiber path, leading in a structure that cannot be manufactured and the concentration of stress. Additionally, the non-convexity of the optimization problem and the large number of design variables make the solution sensitive to the initial design and potentially sub-optimal.

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C.A.P.; supervision, writing—review and editing, E.P.H. and V.K.K. All authors have read and agreed to the published version of the manuscript.

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