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An Efficient Numerical Method for the Fractional Bagley–Torvik Equation of Variable Coefficients with Robin Boundary Conditions

S. Joe Christin Mary [1], Sekar Elango [2,*], Muath Awadalla [3,*] and Rabab Alzahrani [4]

[1] Temporary Faculty, Department of Mathematics, National Institute of Technology, Tiruchirappalli 620015, Tamil Nadu, India; mary@nitt.edu
[2] Department of Mathematics, Amrita School of Physical Science, Amrita Vishwa Vidyapeetham, Coimbatore 641112, Tamil Nadu, India
[3] Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia
[4] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia; r.alzahrani@psau.edu.sa
* Correspondence: e_sekar@cb.amrita.edu (S.E.); mawadalla@kfu.edu.sa (M.A.)

Abstract: In this paper, we propose a numerical method for the fractional Bagley–Torvik equation of variable coefficients with Robin boundary conditions. The problem is approximated using a finite difference scheme on a uniform mesh that combines the L1 scheme with central differences. We prove that this numerical method is almost first-order convergent. The error bounds for the numerical approximation are derived. The numerical calculations carried out for the given examples validate the theoretical results.

Keywords: Bagley–Torvik equation; Caputo fractional derivative; Robin boundary conditions; minimum principle; finite difference scheme; L_1 scheme

MSC: 34A08; 26A33; 35B50; 65L12; 65L20

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Academic Editor: Mark Edelman

Received: 12 May 2025
Revised: 30 May 2025
Accepted: 4 June 2025
Published: 5 June 2025

Citation: Joe Christin Mary, S.; Elango, S.; Awadalla, M.; Alzahrani, R. An Efficient Numerical Method for the Fractional Bagley–Torvik Equation of Variable Coefficients with Robin Boundary Conditions. *Mathematics* **2025**, *13*, 1899. https://doi.org/10.3390/math13111899

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1. Introduction

Many physical processes of stochastic transport, cellular systems, diffusion waves, control theory [1], signal processing, and the oil industries [2] include fractional-order boundary value problems. In [3], the Bagley–Torvik equation was initially developed to investigate the behavior of real materials using fractional calculus. It may also be used to represent the motion of a rigid plate submerged in a Newtonian fluid, as well as the motion of a gas in a fluid [4].

In [5,6], the authors discussed the existence and uniqueness of a solution for the Bagley–Torvik equation with boundary conditions. H. M. Wei et al. [7] discussed the uniqueness of a solution for fractional Bagley–Torvik equations with variable coefficients.

Many authors have investigated the Bagley–Torvik fractional differential equation to develop methods for numerical solution. These methods are the Adomian decomposition method [4], collocation-shooting method [5], hybridizable discontinuous Galerkin methods [8,9], the wavelet method [10], and the fast multiscale algorithm [11]. Ref. [12] employed a direct piecewise polynomial collocation method to solve the Bagley–Torvik equation. The authors established global convergence results on graded meshes and derived pointwise error estimates on uniform meshes. The authors of [13–15] solved the Bagley–Torvik equations with various numerical methods.

S. Santra and J. Mohapatra [16] considered a time-fractional initial boundary value problem of mixed parabolic–elliptic type. Mainly, they used the classical L1 scheme to approximate the temporal derivatives on a uniform mesh and a second-order standard finite difference scheme to approximate the spatial derivatives. The L1 scheme is also used for the Caputo fractional derivative for solving fractional-order Volterra integro-differential equations in Mohapatra et al. [17]. Refs. [18–20] used the numerical methods to solve differential equations and integro-differential equations. Gracia et al. [21] investigated the convergence analysis of a time-fractional convection–diffusion problem, where they used the L1 scheme on a uniform mesh. J. Mohapatra et al. [15] utilized L1 discretization on a uniform mesh to approximate the differential operator, and the modified Newton–Raphson method is applied to convert the fractional model into a system of nonlinear algebraic equations. G. Saini et al. [22] discretize the non-local differential operator utilizing a recognized L1 technique, deriving a nonlinear difference equation that encapsulates the fundamental dynamics of the continuous problem. The associated nonlinear equation is resolved using the Daftardar-Gejji and Jafari (DGJ) method. Ghosh, B. and Mohapatra, J. [23] discretize the fractional differential operator employing the conventional L1 method on a uniform mesh, utilizing the composite trapezoidal rule for the integral component. The Daftardar–Gejji and Jafari approach is utilized to resolve the implicit algebraic equation.

Recently, the authors in [24,25] worked on an generalized piecewise Taylor-series expansion method for the generalized Bagley–Torvik equation with a fractional integral and three-point boundary conditions. These authors focused on the fractional Bagley–Torvik equation with variable coefficients. In this article, we propose a finite difference method for the fractional Bagley–Torvik equation of variable coefficients with Robin boundary conditions. The classical L1 scheme is used to approximate the Caputo fractional derivative. To approximate the second-order derivative, a second order central difference scheme is used. We examine the order of convergence on a uniform mesh of the proposed numerical method.

To highlight the novelty and relevance of our study, we summarize the main contributions as follows:

- We propose a finite difference method for solving the fractional Bagley–Torvik equation with variable coefficients subject to Robin boundary conditions. This type of equation arises in viscoelasticity and structural dynamics and poses computational challenges due to the presence of both integer and fractional derivatives.
- To approximate the Caputo fractional derivative, we employ the classical L1 scheme, which is known for its simplicity and effectiveness in handling weakly singular kernels.
- The second-order spatial derivative is discretized using a second-order central difference scheme on a uniform mesh, allowing the method to retain higher accuracy in space while remaining computationally efficient.
- We establish a rigorous convergence analysis of the proposed scheme and show that it achieves almost first-order accuracy in time under standard regularity assumptions on the exact solution.
- Priori error estimates are derived in a discrete norm, which provide theoretical justification for the reliability of the numerical method.
- Several numerical experiments are carried out to confirm the theoretical error bounds and to demonstrate the practical performance of the method. The results show good agreement with the predicted convergence rates, validating both the stability and accuracy of the scheme.

The outline of the paper is as follows: In Section 2, the existence and uniqueness of the solution are proved. Minimum principle, stability result and bounds of solution and its

integer derivative are also obtained. The central difference and L1 scheme are constructed for the proposed problem in Section 3. In Section 4, error analysis of this scheme is derived. Three numerical examples are illustrated in Section 5. Finally, the conclusion is presented.

Notations:

- $C^{n,\mu}(0,1]$ denotes the space of functions, where $n \in \mathbb{N}$, $\mu \in (-\infty, 1)$, $y \in C^0(\bar{D}) \cap C^n(0,1]$ such that

$$|y^{(j)}(x)| \leq C(1 + x^{1-\mu-j}) \quad \text{for } j = 1(1)n \text{ and } x \in (0,1].$$

- C is a positive constant independent of N. We write the maximum norm $||y||_D = \max_{x \in D} |y(x)|$.
- $\bar{D}^N = \{x_j : x_j = jh,\ 0 \leq j \leq N\}$.
- $||Y||_{D^N} = \max_{0 \leq j \leq N} |Y_j|$.

2. Continuous Problem

Consider the following fractional Bagley–Torvik equation of variable coefficients with Robin boundary conditions:

$$\begin{cases} \mathcal{L}y(x) = y''(x) + p(x)D_*^\nu y(x) - q(x)y(x) = f(x),\ x \in D := (0,1), \\ \mathcal{B}_0 y(0) = \beta_1 y(0) - \beta_2 y'(0) = A, \\ \mathcal{B}_1 y(1) = \gamma_1 y(1) + \gamma_2 y'(1) = B, \end{cases} \tag{1}$$

where $\mathcal{L}$ is the differential operator, $p(x), q(x)$ and $f(x)$ are sufficiently smooth functions on $\bar{D} = [0,1]$ and

$$\begin{cases} p(x) < 0,\ \ q(x) \geq 0,\ \ \forall x \in \bar{D} \\ \forall x \in \bar{D},\ \ \beta_1, \beta_2 \geq 0,\ \beta_1 - \beta_2 > 0,\ \ \gamma_1, \gamma_2 \geq 0. \end{cases} \tag{2}$$

The Caputo fractional derivative $D_*^\nu y(x)$ of order $0 < \nu < 1$ is defined by

$$D_*^\nu y(x) := \frac{1}{\Gamma(1-\nu)} \int_0^x (x-s)^{-\nu} y'(s)\, ds.$$

The assumptions (2) guarantee that problem (1) has a unique solution.

Theorem 1 (Existence and uniqueness)**.** *Assume that $p(x)$, $q(x)$, and $f(x)$ belong to the space $C^{n,\mu}(0,1]$ for some $n \in \mathbb{N}$ and $\mu \leq 1 - \nu$. Suppose further that the homogeneous problem associated with (1), that is, when $f(x) = 0$ and the boundary data $A = B = 0$, admits only the trivial solution $y(x) = 0$. Then, the boundary value problem (1) has a unique solution $y \in C^0(\bar{D})$ with $D_*^\nu y \in C^{n,\lambda}(0,1]$, where $\lambda = \max\{\mu, 1-\nu\}$.*

Proof. This theorem can be proved by adopting the techniques of the proof of Theorem 2.1 of [26]. □

The operator $\mathcal{L}$ in (1) satisfies the following minimum principle.

Theorem 2 (Minimum Principle)**.** *Let $\mathcal{L}$ in (1) with $\phi(x) \in C^0(\bar{D}) \cap C^{1,\mu}(0,1] \cap C^2(D)$, for some $\mu \leq 1 - \nu$. If $\mathcal{B}_0\phi(0) \geq 0$, $\mathcal{B}_1\phi(1) \geq 0$ and $\mathcal{L}\phi(x) \leq 0$, $\forall x \in D$, then $\phi(x) \geq 0$, $\forall x \in \bar{D}$.*

Proof. Define a test function $t(x)$ as

$$t(x) = 2 + x.$$

Then $t(x) > 0, \forall x \in D, \mathcal{B}_0 t(0), \mathcal{B}_1 t(1) > 0$, and $\mathcal{L}t(x) < 0, \forall x \in D$. Further, we define

$$\xi = \max_{x \in \bar{D}} \left(\frac{-\phi}{t} \right)(x).$$

Suppose the theorem is not true. Then $\xi > 0$ and there exists a point x_0 such that $(\phi + \xi t)(x_0) = 0$ and $(\phi + \xi t)(x) \geq 0, \forall x \in \bar{D}$.

Case 1: $(\phi + \xi t)(x_0) = 0$ and $x_0 = 0$. Then,

$$\begin{aligned} 0 &< \mathcal{B}_0(\phi + \xi t)(x_0) \\ &= \beta_1(\phi + \xi t)(x_0) - \beta_2(\phi + \xi t)'(x_0) \\ &\leq 0. \end{aligned}$$

It is a contradiction.

Case 2: $(\phi + \xi t)(x_0) = 0$ and $x_0 \in D$. Then,

$$\begin{aligned} 0 &> \mathcal{L}(\phi + \xi t)(x_0) \\ &= (\phi + \xi t)''(x_0) + p(x_0) D_*^{\nu}(\phi + \xi t)(x_0) - q(x_0)(\phi + \xi t)(x_0) \\ &\geq 0, \end{aligned}$$

by using Theorem 1 of [27].

It is a contradiction.

Case 3: $(\phi + \xi t)(x_0) = 0$ and $x_0 = 1$. Then,

$$\begin{aligned} 0 &< \mathcal{B}_1(\phi + \xi t)(x_0) \\ &= \gamma_1(\phi + \xi t)(x_0) + \gamma_2(\phi + \xi t)'(x_0) \\ &\leq 0. \end{aligned}$$

It is a contradiction. Hence, $\phi(x) \geq 0, \ \forall x \in \bar{D}$.

□

Theorem 3 (Stability Result)**.** *The solution $y(x)$ satisfies the bound*

$$|y(x)| \leq C \max\{|\mathcal{B}_0 y(0)|, |\mathcal{B}_1 y(1)|, ||\mathcal{L}y(x)||_D\}, \ \ \forall x \in \bar{D}.$$

Proof. Define the barrier functions $\psi^{\pm}(x) = Ct(x) \pm y(x)$, where $C_1 = C \max\{|\mathcal{B}_0 y(0)|, |\mathcal{B}_1 y(1)|, ||\mathcal{L}y(x)||_D\}$.

Then,

$$\begin{aligned} \mathcal{B}_0 \psi^{\pm}(0) &= \beta_1 \psi^{\pm}(0) - \beta_2 \psi^{\pm'}(0) \\ &= \beta_1(2C_1 \pm y(0)) - \beta_2(C_1 \pm y'(0)) \\ &= C_1(2\beta_1 - \beta_2) \pm \mathcal{B}_0 y(0) \\ &\geq 0. \end{aligned}$$

$$\begin{aligned} \mathcal{B}_1 \psi^{\pm}(1) &= \gamma_1 \psi^{\pm}(1) + \gamma_2 \psi^{\pm'}(1) \\ &= C_1(3\gamma_1 + \gamma_2) \pm \mathcal{B}_1 y(1) \\ &\geq 0. \end{aligned}$$

$$\begin{aligned}\mathcal{L}\psi^{\pm}(x) &= \psi^{\pm'}(x) + p(x)D_{*}^{\nu}\psi^{\pm}(x) - q(x)\psi^{\pm}(x)\\ &= C_1\left[p(x)\frac{x^{1-\nu}}{\Gamma(2-\nu)} - q(x)(2+x)\right] \pm \mathcal{L}y(x)\\ &\leq 0.\end{aligned}$$

Then by the Theorem 2, we get

$$|y(x)| \leq C\max\{|\mathcal{B}_0 y(0)|, |\mathcal{B}_1 y(1)|, ||\mathcal{L}y(x)||_D\}, \ \ \forall x \in \bar{D}.$$

□

Corollary 1. *Let $p(x),\ q(x),\ f(x) \in C^{n,\mu}(0,1]$ for some $n \in \mathbb{N}$ and $\mu \in (-\infty, 1)$ with $\mu \in 1-\nu$. Then the problem (1) possesses atmost one solution $y(x) \in C^0(\bar{D}) \cap C^n(0,1]$ such that $\forall x \in D$,*

$$|y^{(j)}(x)| \leq \begin{cases} C & \text{if } j = 0,\\ Cx^{\nu-j} & \text{if } j = 1, 2, \cdots, n.\end{cases} \tag{3}$$

where C is a constant.

Proof. From Theorem 2 it follows that the problem (1) with $f = 0,\ A = 0$ and $B = 0$ has only one solution. Also from Theorem 1 it follows that $y(x) \in C^0(\bar{D})$ is a unique solution for (1). Hence the result follows from Theorem 3.4 of [28]. □

3. Discrete Problem

The uniform mesh is constructed by dividing $\bar{D}$ into N subintervals. We define the uniform mesh as:

$$\begin{aligned}\bar{D}^N &= \{x_j : x_j = jh,\ 0 \leq j \leq N\},\\ D^N &:= \bar{D}^N \setminus \{x_0, x_N\}.\end{aligned}$$

The mesh width is given by $h = \frac{1}{N}$.

Consider the following finite difference scheme for the problem (1):

$$\begin{cases}\mathcal{L}^N = \delta^2 Y(x_j) + p(x_j)D_{*,L1}^{\nu}Y(x_j) - q(x_j)Y(x_j) = f(x_j), \ \ \forall x_j \in D^N,\\ \mathcal{B}_0^N Y(x_0) = \beta_1 Y(x_0) - \beta_2 D^+ Y(x_0) = A,\\ \mathcal{B}_1^N Y(x_N) = \gamma_1 Y(x_N) + \gamma_2 D^- Y(x_N) = B.\end{cases} \tag{4}$$

where,

$$D^+Y(x_j) = \frac{Y(x_{j+1}) - Y(x_j)}{h}, \ \ D^-Y(x_j) = \frac{Y(x_j) - Y(x_{j-1})}{h},$$

$$\delta^2 Y(x_j) = \frac{Y(x_{j+1}) - 2Y(x_j) + Y(x_{j-1})}{h^2},$$

$D_{*,L_1}^{\nu}Y(x_j)$ is the L1 discretization of the Caputo fractional derivative [29] defined as

$$D_{*,L1}^{\nu}Y(x_j) = \frac{1}{h^{\nu}\Gamma(2-\nu)}\sum_{k=0}^{j-1}\big(Y(x_{k+1}) - Y(x_k)\big)w_{j-k}$$

with $w_{j-k} = (j-k)^{1-\nu} - (j-k-1)^{1-\nu}$.

Note: Using the mean value theorem, we can prove

$$(1-\nu)(j-1)^{-\nu} \leq w_j \leq (1-\nu)j^{-\nu} \ \text{ and } \ w_j < w_{j-1} \text{ for } j \geq 2. \tag{5}$$

Theorem 4 (Discrete Minimum Principle)**.** *Suppose a mesh function* $\Phi(x_j)$ *satisfies* $\mathcal{B}_0^N\Phi(x_0) \geq 0,\ \mathcal{B}_1^N\Phi(x_N) \geq 0,\ \mathcal{L}^N\Phi(x_j) \leq 0,\ \forall x_j \in D^N$. *Then* $\Phi(x_j) \geq 0,\ \forall x_j \in \bar{D}^N$.

Proof. The test function $T(x_j)$ is defined as

$$T(x_j) = 2 + x_j.$$

Note that $\mathcal{B}_0^N\Phi(x_0) > 0,\ \mathcal{B}_1^N\Phi(x_N) > 0,\ \mathcal{L}^N\Phi(x_j) < 0,\ \forall x_j \in D^N$.

Define

$$\chi = \max_{x_j \in \bar{D}^N} \left(\frac{-\Phi}{T}\right)(x_j).$$

Assume the theorem is not true. Then $\chi > 0$ and there exists x_k such that $(\Phi + \chi T)(x_k) = 0$ and $(\Phi + \chi T)(x_j) \geq 0,\ \forall x_j \in \bar{D}^N$.

Case 1: Assume that $(\Phi + \chi T)(x_k) = 0$ for $x_k = x_0$.

Then,

$$\begin{aligned} 0 &< \mathcal{B}_0^N(\Phi + \chi T)(x_k) \\ &= \beta_1(\Phi + \chi T)(x_k) - \beta_2 D^+(\Phi + \chi T)(x_k) \\ &\leq 0. \end{aligned}$$

This is a contradiction.

Case 2: Assume that $(\Phi + \chi T)(x_k) = 0$ for $x_k \in D^N$.

Then,

$$\begin{aligned} 0 \quad &> \quad \mathcal{L}^N(\Phi + \chi T)(x_k) \\ &= \quad \delta^2(\Phi + \chi T)(x_k) + p(x_k)(\Phi + \chi T)(x_k) - q(x_k)(\Phi + \chi T)(x_k) \end{aligned}$$

The discrete Caputo fractional operator can be written as

$$\begin{aligned} D^{\nu}_{*,L1}(\Phi + \chi T)(x_k) &= \frac{1}{h^{\nu}\Gamma(2-\nu)}\Bigg((\Phi + \chi T)(x_k) - w_j(\Phi + \chi T)(x_0) \\ &\quad + \sum_{l=1}^{k-1}(\Phi + \chi T)(x_l)(w_{k-l+1} - w_{k-l})\Bigg) \\ &= \frac{1}{h^{\nu}\Gamma(2-\nu)}\Bigg(- w_j(\Phi + \chi T)(x_0) \\ &\quad + \sum_{l=1}^{k-1}(\Phi + \chi T)(x_l)(w_{k-l+1} - w_{k-l})\Bigg) \\ &\leq 0, \end{aligned}$$

because (5) implies that $w_j > 0$ and $w_{k-l+1} - w_{k-l} < 0$.

Therefore,

$$\mathcal{L}^N(\Phi + \chi T)(x_k) \geq 0.$$

This is a contradiction.

Case 3: Assume that $(\Phi + \chi T)(x_k) = 0$ for $x_k = x_N$.

Then,

$$\begin{aligned} 0 &< \mathcal{B}_1^N(\Phi + \chi T)(x_k) \\ &= \gamma_1(\Phi + \chi T)(x_k) + \gamma_2 D^-(\Phi + \chi T)(x_k) \\ &\leq 0. \end{aligned}$$

This is a contradiction. Hence, $\Phi(x_j) \geq 0,\ \forall x_j \in \bar{D}^N$. □

Theorem 5 (Discrete Stability Result). *If $Y(x_j)$ is a solution of the problem (4) then*

$$|Y(x_j)| \leq C \max\Big\{|\mathcal{B}_0^N Y(x_0)|, |\mathcal{B}_1^N Y(x_N)|, ||\mathcal{L}^N Y(x_j)||_{D^N}\Big\}, \quad \forall x_j \in \bar{D}^N.$$

Proof. Define the mesh functions

$$\Psi^{\pm}(x_j) = C_1(2 + x_j) \pm Y(x_j).$$

We have $\mathcal{B}_0^N \Psi^{\pm}(x_0) \geq 0,\ \mathcal{B}_1^N \Psi^{\pm}(x_N) \geq 0$ and $\mathcal{L}^N \Psi^{\pm}(x_0) \leq 0$. Using Theorem 4, the result can be proved. □

4. Error Analysis

4.1. Local Truncation Error

Let us consider the local truncation error of the problem (4):

$$\begin{aligned}
\mathcal{B}_0^N(y - Y)(x_0) &= \mathcal{B}_0^N y(x_0) - \mathcal{B}_0^N Y(x_0) \\
&= \beta_1 y(x_0) - \beta_2 y'(x_0) - \beta_1 y(x_0) + \beta_2 D^+ y(x_0) \\
&= \beta_2 \left(D^+ - \frac{d}{dx} \right) y(x_0). \\
\mathcal{B}_1^N(y - Y)(x_N) &= \mathcal{B}_1^N y(x_N) - \mathcal{B}_1^N Y(x_N) \\
&= \gamma_2 \left(\frac{d}{dx} - D^- \right) y(x_N). \\
\mathcal{L}^N(y - Y)(x_j) &= (\mathcal{L} - \mathcal{L}^N) y(x_j) \\
&= \left(\frac{d^2}{dx^2} - \delta^2 \right) y(x_j) + p(x_j)\big(D_*^{\nu} - D_{*,L1}^{\nu}\big) y(x_j).
\end{aligned}$$

Lemma 1. *Suppose $p(x),\ q(x),\ f(x) \in C^{n,\nu}(0,1]$ for some $n \in \mathbb{N}$ and $\mu \leq 1 - \nu$. Then the truncation error bound satisfies*

$$|\mathcal{B}_0^N(y - Y)(x_0)| \leq C\beta_2 h^{\nu-1},\ \ |\mathcal{B}_1^N(y - Y)(x_N)| \leq C\gamma_2 h, \tag{6}$$

$$|\mathcal{L}^N(y - Y)(x_j)| \leq C h x_j^{-3} \ \text{ for } \ j = 1(1)N. \tag{7}$$

Proof. Using Corollary 1 yields

$$\begin{aligned}
|\mathcal{B}_0^N(y - Y)(x_0)| &= \beta_2 \left| \frac{y(x_1) - y(x_0)}{h} - y' x_0 \right| \\
&= \frac{\beta_2}{h} \left| \int_{\zeta_1 = x_0}^{x_1} \left(\int_{\zeta_2 = x_0}^{\zeta_1} y''(\zeta_2) d\zeta_2 \right) d\zeta_1 \right| \\
&\leq C \frac{\beta_2}{h} \int_{\zeta_1 = 0}^{h} \left(\int_{\zeta_2 = 0}^{\zeta_1} \zeta_2^{\nu-2} d\zeta_2 \right) d\zeta_1 \\
&\leq C\beta_2 h^{\nu-1}.
\end{aligned}$$

$$\begin{aligned}
|\mathcal{B}_1^N(y - Y)(x_N)| &= \gamma_2 \left| \frac{y(x_N) - y(x_{N-1})}{h} - y'(x_N) \right| \\
&\leq \frac{\gamma_2}{h} \int_{\zeta_1 = x_{N-1}}^{x_N} \left(\int_{\zeta_2 = \zeta_1}^{x_N} |y''(\zeta_2)| d\zeta_2 \right) d\zeta_1 \\
&\leq C\gamma_2 h x_{N-1}^{\nu-2} \\
&\leq C\gamma_2 h.
\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{L}^N(y-Y)(x_j) &= \left(\frac{d^2}{dx^2}-\delta^2\right)y(x_j) + p(x_j)(D_*^\nu - D_{*,L1}^\nu)y(x_j)\\ &= I_1(x_j) + I_2(x_j).\end{aligned}$$

Consider,

$$I_1(x_j) := \left(\frac{d^2}{dx^2}-\delta^2\right)y(x_j).$$

Case (i): If $j = 1$

$$\begin{aligned}|I_1(x_1)| &= \left|\frac{y(x_2)-2y(x_1)+y(x_0)}{h^2} - y''(x_1)\right|\\ &= \left|\frac{1}{h^2}\left(\int_{\zeta_1=h}^{2h}\int_{\zeta_2=h}^{\zeta_1}\int_{\zeta_3}^{\zeta_2} y'''(\zeta_3)d\zeta_3\, d\zeta_2\, d\zeta_1 - \int_0^h\int_{\zeta_1}^{\zeta_2=h}\int_{\zeta_2}^{\zeta_3=h} y'''(\zeta_3)d\zeta_3\, d\zeta_2\, d\zeta_1\right)\right|\\ &\le \frac{C}{h^2}\left(\int_{\zeta_1=h}^{2h}\int_{\zeta_2=h}^{\zeta_1}\int_{\zeta_3}^{\zeta_2}\zeta_3^{\nu-3}d\zeta_3\, d\zeta_2\, d\zeta_1 - \int_0^h\int_{\zeta_1}^{\zeta_2=h}\int_{\zeta_2}^{\zeta_3=h}\zeta_3^{\nu-3}d\zeta_3\, d\zeta_2\, d\zeta_1\right)\\ &\le Ch^{\nu-2},\end{aligned}$$

from Corollary 1.

Case (ii): If $j \ge 2$,

$$\begin{aligned}|I_1(x_j)| &= \left|\frac{Y(x_{j+1})-2Y(x_j)+Y(x_{j-1})}{h^2} - y''(x_j)\right|\\ &= \frac{1}{h^2}\left|\int_{x_j}^{x_{j+1}}\int_{\zeta_2=x_j}^{\zeta_1}\int_{\zeta_3=x_j}^{\zeta_2} y'''(\zeta_3)d\zeta_3\, d\zeta_2\, \zeta_1 - \int_{x_{j-1}}^{x_j}\int_{\zeta_1}^{\zeta_2=x_j}\int_{\zeta_2}^{\zeta_1=x_j} y'''(\zeta_3)d\zeta_3\, d\zeta_2\, \zeta_1\right|\\ &\le \frac{1}{h^2}\left(\int_{x_j}^{x_{j+1}}\int_{\zeta_2=x_j}^{\zeta_1}\int_{\zeta_3=x_j}^{\zeta_2}|y'''(\zeta_3)|d\zeta_3\, d\zeta_2\, \zeta_1 + \int_{x_{j-1}}^{x_j}\int_{\zeta_1}^{\zeta_2=x_j}\int_{\zeta_2}^{\zeta_1=x_j}|y'''(\zeta_3)|d\zeta_3\, d\zeta_2\, \zeta_1\right)\\ &\le \frac{C}{h^2}\left(\int_{x_j}^{x_{j+1}}\int_{\zeta_2=x_j}^{\zeta_1}\int_{\zeta_3=x_j}^{\zeta_2}\zeta_3^{\nu-3}d\zeta_3\, d\zeta_2\, \zeta_1 + \int_{x_{j-1}}^{x_j}\int_{\zeta_1}^{\zeta_2=x_j}\int_{\zeta_2}^{\zeta_1=x_j}\zeta_3^{\nu-3}d\zeta_3\, d\zeta_2\, \zeta_1\right)\\ &\le Chx_j^{\nu-3},\end{aligned}$$

where we used Corollary 1 and $2x_{j-1} \ge x_j$.

Using remark 2 of [30], we get

$$\begin{aligned}|I_2(x_j)| &= |p(x_j)(D_{*,L1}^\nu - D_*^\nu)y(x_j)|\\ &\le ||p||_\infty|(D_{*,L1}^\nu - D_*^\nu)y(x_j)|\\ &\le Chx_j^{-1}.\end{aligned}$$

From the bound of $I_1(x_j)$ and $I_2(x_j)$, we get the desired result. □

4.2. Error Estimate

Theorem 6. *Let $y(x_j), Y(x_j)$ be the solutions of (1) and (4), respectively. Suppose the hypotheses of Lemma 1 and (2) are satisfied. Then, we have*

$$|y(x_j) - Y(x_j)| \le CN^{-1}|\ln N|, \quad \forall x_j \in \bar{D}^N. \tag{8}$$

where C is a constant.

Proof. Define the mesh functions

$$\Phi^{\pm}(x_j) = CN^{-1}|\ln N|T(x_j) \pm (y - Y)(x_j), \quad x_j \in \bar{D}^N.$$

Then, we have

$$\begin{aligned}
\mathcal{B}_0^N\Phi^{\pm}(x_0) &= \beta_1\Phi^{\pm}(x_0) - \beta_2 D^+\Phi^{\pm}(x_0) \\
&= CN^{-1}|\ln N|(2\beta_1 - \beta_2) \pm \mathcal{B}_0^N(y - Y)(x_0) \\
&\geq CN^{-1}|\ln N|(2\beta_1 - \beta_2) \pm C\beta_2 h^{\nu-1} \\
&\geq 0. \\
\mathcal{B}_1^N\Phi^{\pm}(x_N) &= \gamma_1\Phi^{\pm}(x_N) + \gamma_2 D^-\Phi^{\pm}(x_N) \\
&= CN^{-1}|\ln N|(3\gamma_1 + \gamma_2) \pm \mathcal{B}_1^N(y - Y)(x_N) \\
&\geq CN^{-1}|\ln N|(3\gamma_1 + \gamma_2) \pm C\gamma_2 h \\
&\geq 0. \\
\mathcal{L}^N\Phi^{\pm}(x_j) &= \delta^2\Phi^{\pm}(x_j) + p(x_j)D^{\nu}_{*,L1}\Phi^{\pm}(x_j) - q(x_j)\Phi^{\pm}(x_j) \\
&= CN^{-1}|\ln N|\left[p(x_j)D^{\nu}_{*,L1}T(x_j) - q(x_j)T(x_j)\right] \pm \mathcal{L}^N(y - Y)(x_j).
\end{aligned}$$

By the L1 discretization of Caputo fractional derivative and (5), we get

$$\begin{aligned}
D^{\nu}_{*,L1}T(x_j) &= \frac{1}{h^{\nu}\Gamma(2-\nu)}\sum_{l=0}^{j-1}(x_{l+1} - x_l)w_{j-l} \\
&\geq \frac{w_j}{h^{\nu}\Gamma(2-\nu)}\sum_{l=0}^{j-1}(x_{l+1} - x_l) \\
&\geq \frac{x_j^{1-\nu}}{\Gamma(1-\nu)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}^N\Phi^{\pm}(x_j) &= CN^{-1}|\ln N|\left[p(x_j)D^{\nu}_{*,L1}T(x_j) - q(x_j)T(x_j)\right] \pm Chx_j^{-3} \\
&\leq 0 \ \ \forall x_j \in D^N.
\end{aligned}$$

Applying Theorem 4, we get

$$\Phi^{\pm}(x_j) \geq 0, \ \ \forall x_j \in \bar{D}^N.$$

Hence, it is proved. □

5. Numerical Exemplifications

Example 1.

$$\begin{cases}
y''(x) + (x^2 - 1)D^{\nu}_{*}y(x) - \sin(2\pi x)y(x) = f(x), & x \in D, \\
y(0) = A, \\
y(1) + y'(1) = B.
\end{cases}$$

The exact solution is $y_e(x) = x^{(2+\nu)}(1 - x) + 2$, where the function $f(x)$ and the constants A and B are chosen using $y_e(x)$. We evaluate the maximum pointwise error D_{ν}^N and rate of convergence p_{ν}^N by

$$D_\nu^N := \max_{0\le i\le N} |y_e(x_i) - Y(x_i)| \text{ and } p_\nu^N = \log_2\left(\frac{D_\nu^N}{D_\nu^{2N}}\right).$$

Uniform errors for the various values of ν and the corresponding rate of convergence are obtained by

$$D^N = \max_\nu D_\nu^N \text{ and } p^N = \log_2\left(\frac{D^N}{D^{2N}}\right).$$

The numerical solution and exact solution of Example 1 are plotted in Figure 1.

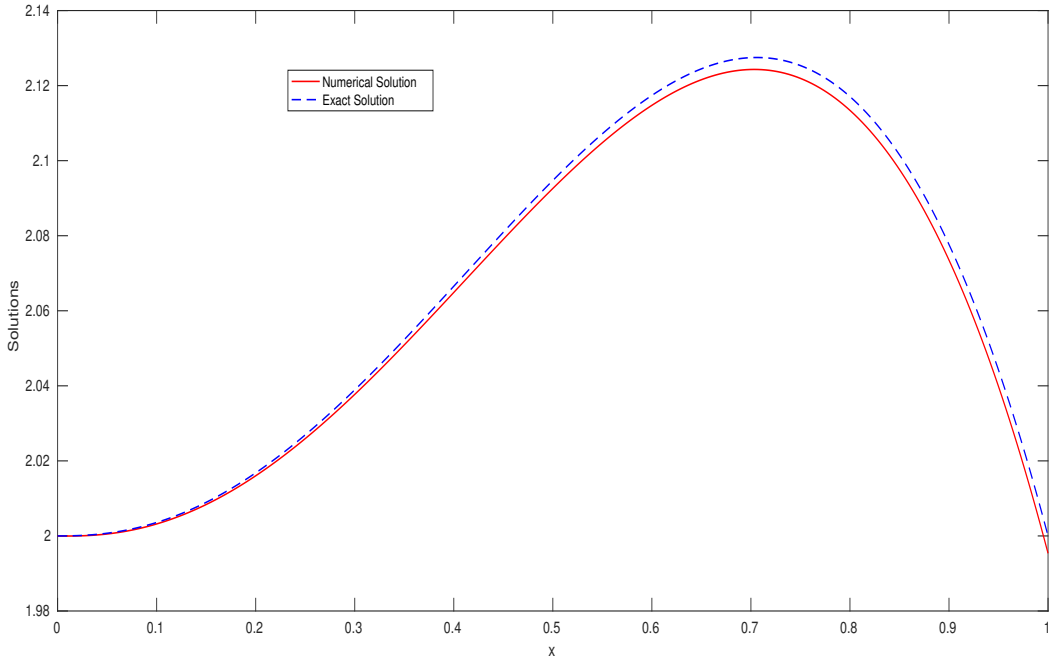


Figure 1. Exact and approximate solutions of Example 1 for $N = 2^8$ and $\nu = 0.4$.

Example 2.

$$\begin{cases} y''(x) - (1+x)D_*^\nu y(x) - xy(x) = \frac{1}{2} - x, \quad x \in D, \\ 2y(0) - y'(0) = 0, \\ 3y(1) + y'(1) = 1. \end{cases}$$

Example 3.

$$\begin{cases} y''(x) - (1+x^2)D_*^\nu y(x) - \cos(3x)y(x) = x^2, \quad x \in D, \\ 3y(0) - y'(0) = 1, \\ 2y(1) + y'(1) = 0. \end{cases}$$

The exact solutions to Examples 2 and 3 are unknown. Tables 1 and 2 show the maximum pointwise errors E^N and uniform rates of convergence r^N which are obtained by using the double mesh principle [31] defined as follows:

$$E_\nu^N := \max_{0\le i\le N} |Y_i^N - Y_i^{2N}| \text{ and } r_\nu^N := \log_2\left(\frac{E_\nu^N}{E_\nu^{2N}}\right).$$
$$E^N := \max_\nu E_\nu^N \text{ and } r^N := \log_2\left(\frac{E^N}{E^{2N}}\right).$$

The numerical solution of Examples 2 and 3 are plotted in Figures 2 and 3. The error plot in Figures 4–6 indicates that as the value of N increases as the maximum errors decreases. Tables 1–3 show the uniform errors and convergence orders of Examples 1, 2,

and 3, respectively. The loglog plot in Figures 7–9 validates our theoritical error bound which is of $O(N^{-1}(\ln N))$.

Table 1. Enumerated maximum errors and uniform errors E_ν^N, E^N and orders of convergence r_ν^N, r^N of Example 2 for the values of ν and N.

ν/N	64	128	256	512	1024	2048	4096
0.01	1.866e-03 0.9917	9.385e-04 0.9958	4.706e-04 0.9979	2.356e-04 0.9989	1.179e-04 0.9994	5.897e-05 0.9997	2.949e-05 -
0.1	1.894e-03 0.9910	9.528e-04 0.9955	4.779e-04 0.9977	2.393e-04 0.9988	1.197e-04 0.9994	5.990e-05 0.9997	2.995e-05 -
0.2	1.926e-03 0.9901	9.698e-04 0.9950	4.866e-04 0.9974	2.437e-04 0.9987	1.219e-04 0.9993	6.101e-05 0.9996	3.051e-05 -
0.3	1.960e-03 0.9889	9.877e-04 0.9943	4.958e-04 0.9970	2.484e-04 0.9984	1.243e-04 0.9991	6.220e-05 0.9995	3.111e-05 -
0.4	1.994e-03 0.9872	1.006e-03 0.9932	5.054e-04 0.9963	2.533e-04 0.9979	1.268e-04 0.9988	6.347e-05 0.9993	3.175e-05 -
0.5	2.027e-03 0.9850	1.024e-03 0.9916	5.151e-04 0.9952	2.584e-04 0.9971	1.294e-04 0.9982	6.480e-05 0.9989	3.242e-05 -
0.6	2.055e-03 0.9819	1.040e-03 0.9894	5.241e-04 0.9934	2.632e-04 0.9958	1.320e-04 0.9971	6.613e-05 0.9980	3.311e-05 -
0.7	2.072e-03 0.9782	1.051e-03 0.9864	5.308e-04 0.9909	2.671e-04 0.9936	1.341e-04 0.9953	6.728e-05 0.9964	3.372e-05 -
0.8	2.066e-03 0.9744	1.051e-03 0.9831	5.320e-04 0.9879	2.682e-04 0.9908	1.349e-04 0.9926	6.782e-05 0.9939	3.405e-05 -
0.9	2.018e-03 0.9729	1.028e-03 0.9818	5.206e-04 0.9866	2.627e-04 0.9892	1.323e-04 0.9908	6.659e-05 0.9919	3.348e-05 -
0.99	1.911e-03 0.9797	9.693e-04 0.9888	4.884e-04 0.9934	2.453e-04 0.9957	1.230e-04 0.9968	6.165e-05 0.9974	3.087e-05 -
E^N r^N	2.072e-03 0.9782	1.051e-03 0.9833	5.320e-04 0.9879	2.682e-04 0.9908	1.349e-04 0.9926	6.782e-05 0.9939	3.405e-05 -

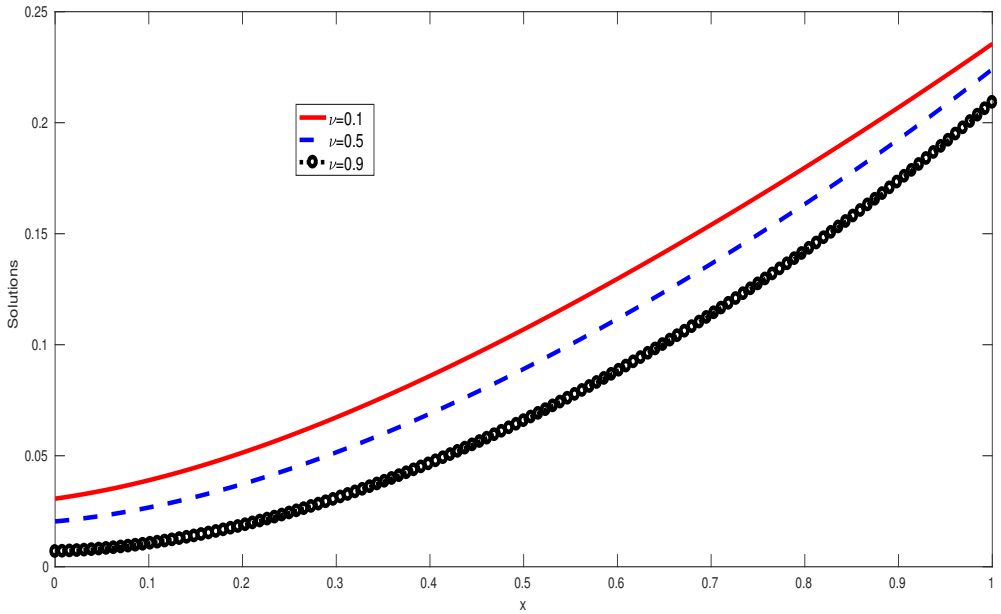


Figure 2. The numerical solution for $N = 2^7$ and $\nu \in \{0.1, 0.5, 0.9\}$.

Table 2. Enumerated maximum errors and uniform errors E_ν^N, E^N and orders of convergence r_ν^N, r^N of Example 3 for the values of ν and N.

ν/N	64	128	256	512	1024	2048	4096
0.01	2.594e-03 0.9967	1.299e-03 0.9983	6.506e-04 0.9992	3.255e-04 0.9996	1.628e-04 0.9998	8.141e-05 0.9999	4.071e-05 -
0.1	2.551e-03 0.9966	1.278e-03 0.9983	6.401e-04 0.9991	3.202e-04 0.9995	1.601e-04 0.9997	8.010e-05 0.9998	4.005e-05 -
0.2	2.503e-03 0.9963	1.255e-03 0.9981	6.283e-04 0.9990	3.143e-04 0.9995	1.572e-04 0.9997	7.863e-05 0.9998	3.931e-05 -
0.3	2.453e-03 0.9958	1.230e-03 0.9978	6.161e-04 0.9988	3.083e-04 0.9993	1.542e-04 0.9996	7.713e-05 0.9998	3.857e-05 -
0.4	2.402e-03 0.9951	1.205e-03 0.9973	6.036e-04 0.9985	3.021e-04 0.9991	1.511e-04 0.9995	7.561e-05 0.9997	3.781e-05 -
0.5	2.347e-03 0.9942	1.178e-03 0.9966	5.906e-04 0.9979	2.957e-04 0.9987	1.480e-04 0.9991	7.404e-05 0.9994	3.703e-05 -
0.6	2.287e-03 0.9929	1.149e-03 0.9955	5.765e-04 0.9970	2.888e-04 0.9979	1.446e-04 0.9985	7.239e-05 0.9989	3.622e-05 -
0.7	2.240e-03 0.9965	1.123e-03 0.9985	5.621e-04 0.9993	2.811e-04 0.9969	1.408e-04 0.9975	7.056e-05 0.9981	3.532e-05 -
0.8	2.280e-03 0.9955	1.143e-03 0.9980	5.725e-04 0.9992	2.864e-04 0.9997	1.432e-04 1.0000	7.162e-05 1.0000	3.580e-05 -
0.9	2.334e-03 0.9935	1.172e-03 0.9971	5.875e-04 0.9988	2.939e-04 0.9996	1.470e-04 1.0000	7.351e-05 1.0000	3.675e-05 -
0.99	2.403e-03 0.9900	1.210e-03 0.9950	6.070e-04 0.9975	3.040e-04 0.9988	1.521e-04 0.9995	7.610e-05 0.9998	3.805e-05 -
E^N r^N	2.594e-03 0.9967	1.299e-03 0.9983	6.506e-04 0.9992	3.255e-04 0.9996	1.628e-04 0.9998	8.141e-05 0.9999	4.071e-05 -

Table 3. Enumerated maximum errors and uniform errors D_ν^N, D^N and orders of convergence p_ν^N, p^N of Example 1 for the values of ν and N.

ν/N	64	128	256	512	1024	2048	4096
0.01	1.581e-02 0.9943	7.936e-03 0.9971	3.976e-03 0.9985	1.990e-03 0.9993	9.954e-04 0.9996	4.978e-04 0.9998	2.489e-04 -
0.1	1.642e-02 0.9943	8.245e-03 0.9972	4.130e-03 0.9986	2.067e-03 0.9993	1.034e-03 0.9997	5.171e-04 0.9998	2.586e-04 -
0.2	1.709e-02 0.9944	8.581e-03 0.9972	4.298e-03 0.9986	2.151e-03 0.9992	1.076e-03 0.9997	5.382e-04 0.9998	2.691e-04 -
0.3	1.775e-02 0.9944	8.911e-03 0.9973	4.464e-03 0.9987	2.234e-03 0.9993	1.117e-03 0.9997	5.588e-04 0.9998	2.794e-04 -
0.4	1.840e-02 0.9945	9.238e-03 0.9974	4.627e-03 0.9987	2.315e-03 0.9995	1.158e-03 0.9997	5.792e-04 0.9999	2.896e-04 -
0.5	1.905e-02 0.9946	9.562e-03 0.9976	4.789e-03 0.9989	2.396e-03 0.9996	1.198e-03 0.9999	5.992e-04 0.9999	2.996e-04 -
0.6	1.970e-02 0.9947	9.889e-03 0.9978	4.952e-03 0.9992	2.477e-03 0.9998	1.238e-03 1.000	6.193e-04 1.000	3.096e-04 -
0.7	2.037e-02 0.9949	1.022e-02 0.9982	5.118e-03 0.9996	2.560e-03 1.000	1.279e-03 1.000	6.397e-04 1.000	3.197e-04 -
0.8	2.108e-02 0.9949	1.057e-02 0.9983	5.295e-03 0.9999	2.648e-03 1.000	1.323e-03 1.000	6.613e-04 1.000	3.304e-04 -
0.9	2.185e-02 0.9942	1.097e-02 0.9979	5.493e-03 0.9997	2.747e-03 1.000	1.372e-03 1.001	6.859e-04 1.001	3.427e-04 -
0.99	2.263e-02 0.9923	1.137e-02 0.9963	5.704e-03 0.9983	2.855e-03 0.9992	1.428e-03 0.9998	7.142e-04 1.000	3.571e-04 -
D^N	2.263e-02	1.137e-02	5.704e-03	2.855e-03	1.428e-03	7.142e-04	3.571e-04
p^N	0.9923	0.9963	0.9983	0.9992	0.9998	1.000	-

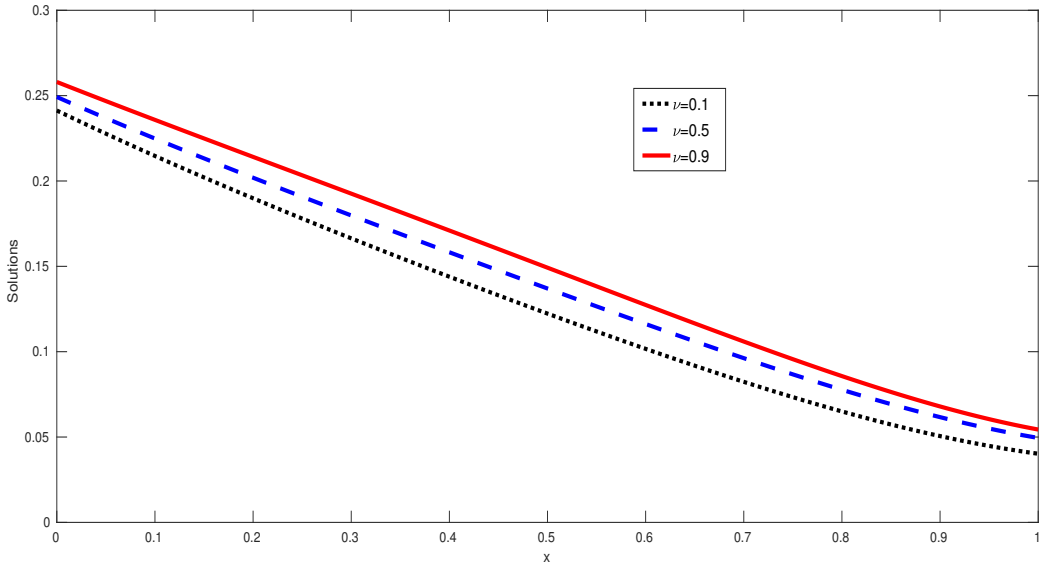


Figure 3. The numerical solution for $N = 2^7$ and $\nu \in \{0.1, 0.5, 0.9\}$.

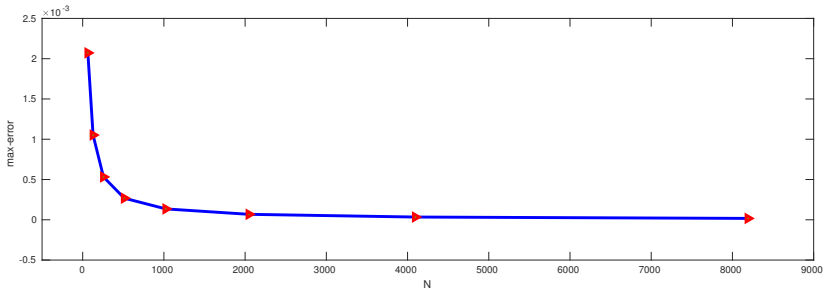


Figure 4. Error plot for Example 1.

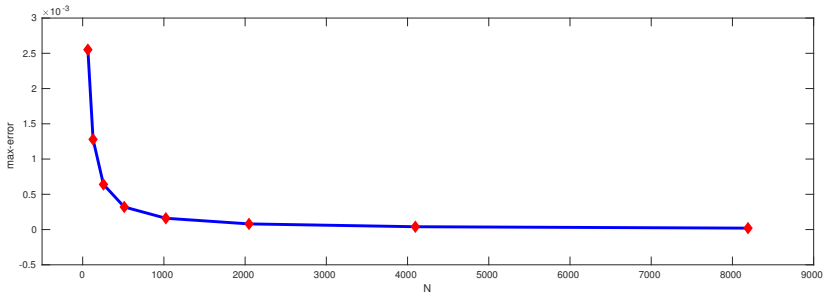


Figure 5. Error plot for Example 2.

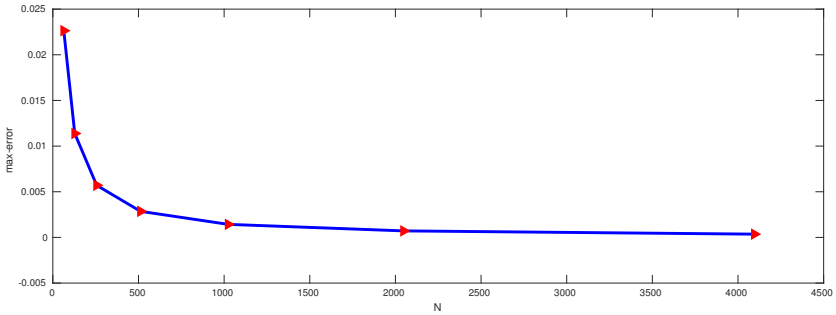


Figure 6. Error plot for Example 3.

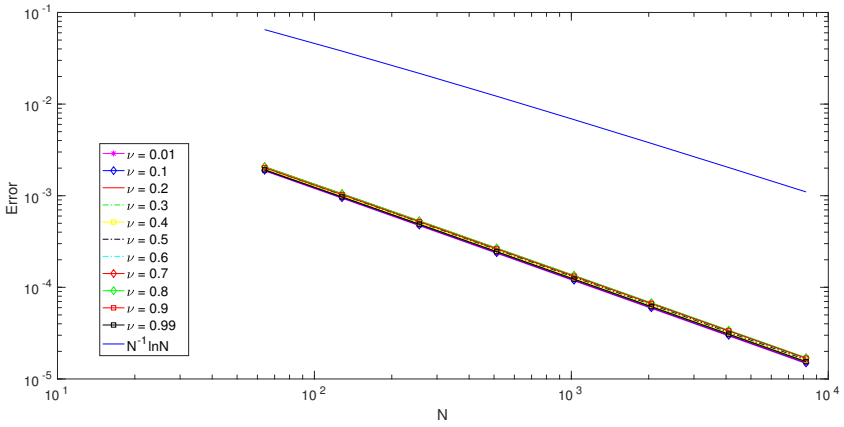


Figure 7. Log-log plot for Example 1.

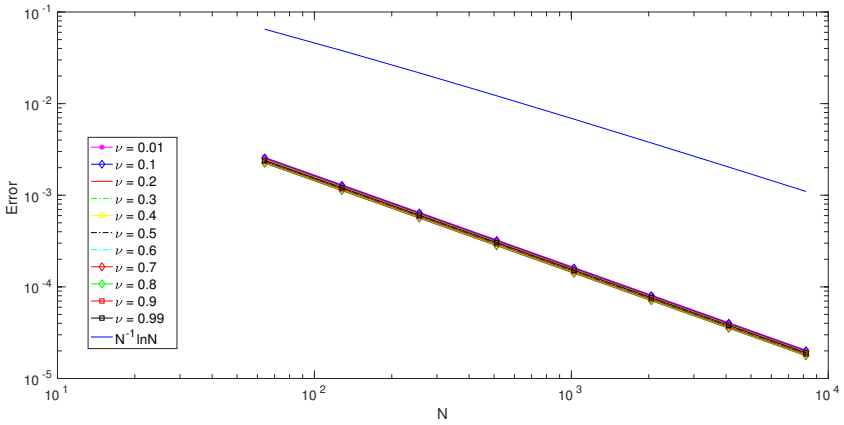


Figure 8. Log-log plot for Example 2.

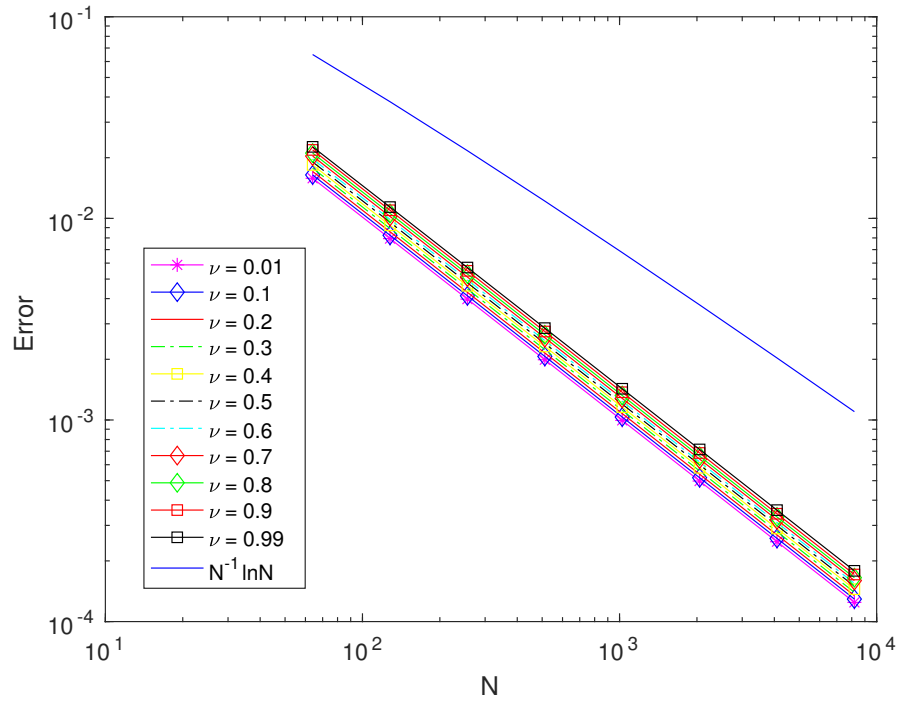


Figure 9. Log-log plot for Example 3.

6. Conclusions

The proposed finite difference scheme provides an efficient and accurate numerical method for solving the fractional Bagley–Torvik equation with variable coefficients and Robin boundary conditions. The combination of the L1 scheme for the fractional derivative with central difference approximations ensures simplicity in implementation while maintaining stability. Theoretical analysis confirms that the method achieves nearly first-order convergence, and rigorous error bounds are derived. Numerical experiments further validate the analytical results, demonstrating the reliability, accuracy, and practical applicability of the scheme for solving fractional differential equations arising in engineering and applied sciences.

Author Contributions: Validation, R.A.; Formal analysis, M.A.; Writing—original draft, S.J.C.M.; Writing—review & editing, S.E. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia. [Grant No.KFU252194]. This study is supported via funding from Prince Sattam bin Abdulaziz University, Saudi Arabia, project number [PSAU/2025/R/1446].

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare that they have no competing interests concerning the publication of the manuscript.

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