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Problems with Missing Tricomi Condition and Analog of Frankl Condition for One Class of Mixed Type Equations

Assel Makulbay [1], Mirakhmat Mirsaburov [2,*], Abdumauvlen Berdyshev [1,*] and Gulbakhor Mirsaburova [3]

[1] Department of Mathematics and Mathematical Modelling, Abai Kazakh National Pedagogical University, Almaty 050010, Kazakhstan; aseka_tynybekova@mail.ru
[2] Department of Mathematical Analysis, Termez State University, Termez 190111, Uzbekistan
[3] Department of General Mathematics, Tashkent State Pedagogical University, Tashkent 100185, Uzbekistan; gulbaxor.mirsaburova@mail.ru
* Correspondence: mirsaburov@mail.ru (M.M.); berdyshev@mail.ru (A.B.)

Abstract: In this paper, for a mixed elliptic-hyperbolic type equation with various degeneration orders and singular coefficients, theorems of uniqueness and existence of the solution to the problem with a missing Tricomi condition on boundary characteristic and with an analog of Frankl condition on different parts of the cut boundary along the degeneration segment in the mixed domain are proved. On the degeneration line segment, a general conjugation condition is set, and on the boundary of the elliptic domain and degeneration segment, the Bitsadze–Samarskii condition is posed. The considered problem, based on integral representations of the solution to the Dirichlet problem (in elliptic part of the domain) and a modified Cauchy problem (in hyperbolic part of the domain), is reduced to solving a non-standard singular Tricomi integral equation with a non-Fredholm integral operator (featuring an isolated first-order singularity in the kernel) in non-characteristic part of the equation. Non-standard approaches are applied here in constructing the solution algorithm. Through successive applications of the theory of singular integral equations and then the Wiener–Hopf equation theory, the non-standard singular Tricomi integral equation is reduced to a Fredholm integral equation of the second kind, the unique solvability of which follows from the uniqueness theorem for the problem.

Keywords: mixed-type equation with various degeneration orders; missing Tricomi condition; Bitsadze-Samarskii condition; Frankl condition analogue; Wiener–Hopf equation

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1. Introduction

The theory of mixed-type equations is currently one of the main branches of the modern theory of partial differential equations due to its important applications in solving many practical problems. The foundations of this theory were laid in the fundamental works of F. Tricomi, S. Gellerstedt, A.V. Bitsadze, and F.I. Frankl. There is a whole range of works by mathematician scholars that study the main mixed problems (the Tricomi problem, the Gellerstedt problem, the general mixed problem of A.V. Bitsadze, the Frankl problem) and propose new well-posed non-classical problems for mixed type equations. The main bibliography on these issues is contained in books by scientists, such as A.V. Bitsadze, L. Bers, F.I. Frankl, I.N. Vekua, M.S. Salakhitdinov, M.M. Smirnov, and A.M. Nakhushev [1–11].

A significant contribution to the study of mixed-type equations was made by the research of M. Protter, K. Friedrichs, P. Lax and N. Phillips, K. Morawetz, A.K. Aziz and M. Schneider, Sh.A. Alimov, and E.I. Moiseev [12–21].

F. Tricomi [22] studied the main boundary value problem for a model equation of mixed type, which is now known as the Tricomi problem. The existence of a solution to this problem was investigated using the theory of singular integral equations. S.G. Mikhlin [23], using Carleman's results, developed an algorithm for solving the singular integral equation of F. Tricomi, and A.V. Bitsadze formulated and proved the extremum principle for the Tricomi problem, which is now widely used in proving the uniqueness and stability of solutions to problems for mixed type equations.

I. Frankl [24] formulated a problem with a nonlocal condition on the boundary of ellipticity and hyperbolicity for the Chaplygin equation. The well-posedness of this problem was proven by K. Moravetz [25] and Yu.V. Devingtal [26]. G. Karatoprakliev [27] studied a problem for mixed-type equations with general conjugation conditions.

After these works, the theory of boundary value problems for degenerate equations and equations of mixed type developed in many directions: S. Gellerstedt [28], K.I. Babenko [29], A.I. Kozhanov [30], S.P. Pulkin [31], A.V. Bitsadze, M.S. Salakhiddinov [32], and others have studied the Tricomi problem for more general equations of mixed type; V.A.Bitsadze [33], A.M.Nakhushev [34], M.M.Smirnov [10], S.Gellerstedt [28,35], F.I.Frankl [36], A.V. Bitsadze, A.A. Samarsky [37] and others investigated various modifications of the Tricomi problem (including nonlocal problems); Sh.A. Alimov [38], M.S. Salakhitdinov [39], M.S. Salakhitdinov, A.K. Urinov [40], A.K. Urinov [41], A.S. Berdyshev [42–46], and many other researchers have studied spectral problems for equations of mixed type.

The most significant results and a comprehensive bibliography on such problems are presented in the monographs by A.V. Bitsadze [2], M.S. Salakhitdinov [8,9], A.M. Nakhushev [11], E.I. Moiseev [47], A.P. Soldatov [48], A.S. Berdyshev [49], A.I. Kozhanov [30], and others.

By the early 1970s, many issues in the theory of boundary value problems for degenerate equations and equations of mixed type had already acquired a mathematically complete form, and further progress in this direction was largely determined by qualitatively new problems.

In recent years, there has been intensive development of the theory of boundary value problems for mixed-type equations with singular coefficients in lower-order terms [50]. A nonlocal problem for a mixed-type equation with a singular coefficient in an unbounded domain was studied in the work of M. Ruziyev and M. Reissig [51], while the works of G. Feng [52], Z. Feng and J. Kuang [53] are devoted to the study of boundary value problems for nonlinear mixed-type equations.

The present work is devoted to the study of issues related to the unique solvability of a Tricomi-type problem with a missing boundary condition on the characteristic boundary and an analog of the Frankl condition on the degeneration segment for a certain class of mixed-type equations with varying orders of degeneration and singular coefficients. The degenerate equations with singular coefficients considered in this paper differ from the well-known classical problems in that the well-posedness of the standard Cauchy problem (in the hyperbolic part of the domain) and the Holmgren problem (in the elliptic part) does not always hold. In the usual formulation, these problems may turn out to be unsolvable in the domains under consideration if the mixed-type equation degenerates along a line that is simultaneously a characteristic (an envelope of a family of characteristics) or if the coefficients of the equation in the lower-order terms are singular. Therefore, in such cases,

it is natural to consider modified Cauchy and Holmgren problems, where the conditions on the degeneration line are given with weight functions.

The problem under investigation differs from known problems in that, on one part of the boundary characteristic, the values of the sought function and a Frankl-type condition are specified on different edges of the cut along the segment where the equation degenerates. On the segment of the degeneration line, a general conjugation condition is given, and in the elliptic region of the equation, the Bitsadze–Samarskii condition is imposed, which pointwise links the boundary of the ellipticity region of the equation with the lines of degeneration. At the same time, part of the boundary of the hyperbolic region is freed from boundary conditions.

By modifying the extremum principle of A.V. Bitsadze for an equation of mixed type with singular coefficients, the uniqueness of the formulated problem has been proven, and the solvability of the problem is reduced to the study of non-standard singular integral equations of Tricomi type with a non-singular part of the kernel and a non-Fredholm operator in the equation.

2. Preliminaries

Let Ω be a finite simply connected domain of the complex plane $C = \{z = x + iy\}$, bounded for $y > 0$ by a smooth curve $\sigma_0(y = \sigma_0(x))$ with endpoints $A(-1, 0)$ and $B(1, 0)$, defined by the equation $x^2 + 4(l+2)^{-2}y^{l+2} = 1,$, and for $y < 0$ by the characteristics AC and BC of the equation

$$G(u) = 0,\ G(u) = \begin{cases} y^l u_{xx} + u_{yy} + \frac{\delta_0}{y} u_y & \text{for } y > 0 \\ -(-y)^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y & \text{for } y < 0, \end{cases} \tag{1}$$

where l, m are positive constants, $\delta_0 \in \left(-\frac{l}{2}, 1\right)$, $\beta_0 \in \left(-\frac{m}{2}, 1\right)$, and $l \neq m$.

Let Ω_0 and Ω_1 be parts of the domain Ω lying in the half-planes $y > 0$ and $y < 0$, respectively, and let C_0 and C_1 be the points where the boundary characteristics AC and BC intersect with the characteristics of Equation (1), emanating from the point $E(c,\ 0)$, where c is some number belonging to the interval $(-1, 1)$ on the axis $y = 0$.

Let $p(x) \in C^1[-1, c]$ be a diffeomorphism from the set of points of the segment $[-1, c]$ to the set of points of the segment $[c, 1]$ such that $p'(x) < 0,\ p(-1) = 1,\ p(c) = c$. As an example of such a function, consider the linear function $p(x) = d - kx$, where $k = \frac{1-c}{1+c},\ d = \frac{2c}{1+c}$.

Note that Equation (1) in the upper half-plane $y > 0$ is an elliptic-type equation, and in the lower half-plane $y < 0$ it is a hyperbolic-type equation, and the orders of parabolic degeneration of these equations on the axis $y = 0$ are different [54] (p. 75).

In the Tricomi problem [22] (p. 29) the value of the desired function is given at all points of the boundary characteristic AC. In the present work, within a mixed domain Ω for Equation (1) the well-posedness of the problem is investigated when the boundary characteristic AC is arbitrarily divided into two parts AC_0 and C_0C and on the first part $AC_0 \subset AC$ the value of sought function is apecified (incomplete Tricomi condition), while the second part C_0C is free from boundary conditions and is instead replaced by an analog of the Frankl condition [24,26,55] on different parts of the cut edges along the degeneration segment AB. On the curve σ_0 and the degeneration segment AB the Bitsadze–Samarskii condition [37,56] is imposed.

3. Formulation of Problem MA

Problem MA. It is required to find a function $u(x, y)$, in the domain Ω satisfying the following conditions:

(1) $u(x,y)$ is continuous in each of the closed subdomains $\overline{\Omega}_0$ and $\overline{\Omega}_1$;

(2) the function $u(x,y)$ belongs to the class $C^2(\Omega_0)$ and satisfies Equation (1) in the domain Ω_0;

(3) the function $u(x,y)$ is a generalized solution of class R_1 in the domain Ω_1 [7] (p. 35), [10] (p. 104);

(4) on the segment AB the line of parabolic degeneration of Equation (1) the general conjugation condition is fulfilled [27],

$$u(x,-0) = a(x)u(x,+0) + a_0(x),\ x \in [-1,1] \tag{2}$$

$$\lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = b(x)\lim_{y\to+0} y^{\delta_0}\frac{\partial u}{\partial y} + b_0(x),\ x \in (-1,1)/\{c\}, \tag{3}$$

where $a(x)$, $b(x)$, $a_0(x)$, $b_0(x)$ are given continuously differentiable functions on $[-1,1]$, such that $a(x) \neq 0$, $b(x) \neq 0$, $\forall x \in [-1,1]$, $a_0(-1) = 0$, and limits $\lim\limits_{y\to-0}(-y)^{\beta_0}\dfrac{\partial u}{\partial y}$ $\left(\lim\limits_{y\to+0} y^{\delta_0}\dfrac{\partial u}{\partial y}\right)$ for $x = \pm 1$, $x = c$, may have singularities of order lower than $1-2\beta$ $(1-2\delta)$, where $\beta = \frac{m+2\beta_0}{2(m+2)}$, $\delta = \frac{l+2\delta_0}{2(l+2)}$;

(5) the function $u(x,y)$ satisfies the following conditions:

$$u(x,\sigma_0(x)) = \rho(x)u(x,+0) + \varphi_0(x),\ \ x \in [-1,1], \tag{4}$$

$$u(x,y)\big|_{AC_0} = \psi(x),\ x \in [-1,(c-1)/2], \tag{5}$$

$$u(p(x),-0) = \mu(x)u(x,+0) + f(x),\ \ x \in [-1,c], \tag{6}$$

where $\rho(x)$, $\varphi_0(x)$, $\mu(x)$, $f(x)$ are continuously differentiable functions on their domains, and $a(x) = (1-x^2)\widetilde{a}(x)$, $a_0(x) = (1-x^2)\widetilde{a}_0(x)$, $\psi(-1) = 0$, $\mu(-1) = 0$, $f(-1) = 0$,

$$\rho(x) = (1-x^2)\widetilde{\rho}(x),\ \varphi_0(x) = (1-x^2)\widetilde{\varphi}_0(x),\ x \in [-1,1]. \tag{7}$$

Note that conditions (2) and (3) are general conjugation conditions, (4) is the Bitsadze–Samarsky condition connecting the values of the desired function $u(x,y)$ on the curve σ_0 and on the degeneracy segment AB, (5) is an incomplete Tricomi condition on a part of the boundary characteristic $AC_0 \subset AC$, (6) is an analog of the Frankl condition on different edges of the section along the degeneracy segment AB.

Let us introduce notations:

$$u(x,-0) = \tau^-(x),\ x \in [-1,1];\ \lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = \nu^-(x),\ x \in (-1,1), \tag{8}$$

$$u(x,+0) = \tau(x),\ x \in [-1,1];\ \lim_{y\to+0} y^{\delta_0}\frac{\partial u}{\partial y} = \nu(x),\ x \in (-1,1). \tag{9}$$

By virtue of the notations (8) and (9), the general conjugation conditions (2) and (3), as well as the analogue of the Frankl condition (6), respectively, take the form

$$\tau^-(x) = a(x)\tau(x) + a_0(x),\ x \in [-1,1], \tag{10}$$

$$\nu^-(x) = b(x)\nu(x) + b_0(x),\ \ x \in (-1,1)/\{c\}, \tag{11}$$

and

$$\tau^-(p(x)) = \mu(x)\tau(x) + f(x),\ \ x \in [-1,c]. \tag{12}$$

Taking into account (10), the condition (12) can be represented as

$$\tau(p(x)) = \mu_1(x)\tau(x) + f_1(x), \quad x \in [-1, c] \tag{13}$$

where

$$\mu_1(x) = \mu(x)/a(p(x)),\ f_1(x) = (f(x) - a_0(p(x)))/a(p(x)).$$

If in Equation (1) $l = m,\ \beta_0 = \delta_0$, then (1) becomes the Gellerstedt equation with a singular coefficient for which problems with analogues of the Frankl condition (13) have been studied in [57–60].

4. Results

4.1. Uniqueness of the Solution to Problem MA

The Darboux formula, which provides in the domain Ω_1 the solution to the modified Cauhcy problem with initial data (8) has the form [7] (p. 34), [61].

$$\begin{aligned} u(x,y) = \gamma_1 \int_{-1}^{1} \tau^- \left[x + \frac{2t}{m+2}(-y)^{(m+2)/2} \right] (1+t)^{\beta-1}(1-t)^{\beta-1} dt + \\ + \gamma_2 (-y)^{1-\beta_0} \int_{-1}^{1} \nu^- \left[x + \frac{2t}{m+2}(-y)^{(m+2)/2} \right] (1+t)^{-\beta}(1-t)^{-\beta} dt, \end{aligned} \tag{14}$$

where

$$\gamma_1 = \frac{2^{1-2\beta}\Gamma(2\beta)}{\Gamma^2(\beta)}, \quad \gamma_2 = -\frac{2^{2\beta-1}\Gamma(2-2\beta)}{(1-\beta_0)\Gamma^2(1-\beta)},$$

$\Gamma(x)$ is Euler's gamma function [10] (p. 4).

By virtue of Darboux's formula, it is not difficult to obtain the following equality from boundary condition (5):

$$\nu^-(x) = \gamma D_{-1,x}^{1-2\beta}\tau^-(x) + \Psi(x),\ x \in (-1, c), \tag{15}$$

where

$$\gamma = \frac{2\Gamma(2\beta)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-2\beta)} \left(\frac{m+2}{4} \right)^{2\beta} > 0,$$

$$\Psi(x) = -\gamma \frac{\Gamma(\beta)}{\Gamma(2\beta)} (1+x)^{\beta} D_{-1,x}^{1-\beta} \psi((x-1)/2),\ x \in (-1, c).$$

Due to the conjugation conditions (11) and (12), equality (15) is written as

$$b(x)\nu(x) = \gamma D_{-1,x}^{1-2\beta} a(x)\tau(x) + \Psi_1(x),\ x \in (-1, c), \tag{16}$$

where

$$\Psi_1(x) = \Psi(x) + \gamma D_{-1,x}^{1-2\beta} a_0(x) - b_0(x),$$

here

$$D_{-1\ x}^{r} f(x) = \begin{cases} \dfrac{1}{\Gamma(-r)} \displaystyle\int_{-1}^{1} \dfrac{f(t)dt}{(x-t)^{1+l}} & \text{for } -1 < r < 0 \\ \dfrac{d}{dx} D_{-1\ x}^{r-1} f(x) & \text{for } 0 < r < 1 \end{cases}$$

are fractional-order integro-differential operators in the sense of Riemann–Liouville [7]. Equality (17) is the first functional relation between unknown functions $\tau(x)$ and $\nu(x)$, introduced into the interval $(-1, c)$ from the domain Ω_1. The following theorem holds.

Theorem 1. Problem MA *when the following conditions are met*

$$\begin{gathered} a_0(x) \equiv 0, b_0(x) \equiv 0, \varphi_0(x) \equiv 0,\ \psi(x) \equiv 0,\ f(x) \equiv 0, \\ a(x) > 0,\ b(x) > 0,\ 0 < \rho(x) < 1,\ 0 < \mu(x) < a(p(x)), \end{gathered} \tag{17}$$

has only a trivial (identically zero) solution.

Proof. Assume the contrary, that under the conditions of Theorem 1, problem MA has nontrivial solution. Then in the closed domain $\overline{\Omega}_0$ the desired solution $u(x,y) \neq 0$, and therefore it reaches its greatest positive value (GPV) and least negative value (LNV). Suppose that (x_0, y_0) is the point of positive maximum of function $u(x,y)$ in domain $\overline{\Omega}_0$. According to Hopf's principle [2] (p. 25) $(x_0, y_0) \notin \Omega_0$. Let $(x_0, y_0) = (x_0, 0) \in AB$. Due to (17), $0 < \mu_1(x) < 1$, where $\mu_1(x) = \mu(x)/a(p(x))$, then from the corresponding homogeneous condition (13) $(c\ f_1(x) \equiv 0)$ it is followed that $\tau(p(x)) = \mu_1(x)\tau(x) < \tau(x)$, where $x \in (-1, c),\ p(x) \in (c, 1)$, i.e., the value of function $\tau(p(x))$ on EB is less than the value of $\tau(x)$ on AE, thus the function $u(x,y)$ does not reach its GPV on the interval EB., i.e., $x_0 \notin (c, 1)$.

Taking into account $p(c) = c$ from (13) $(c\ f_1(x) \equiv 0)$, we have $\tau(c) = 0$, and therefore $x_0 \neq c$. Then the extremum point $(x_0, 0) \in AE\ (x_0 \in (-1, c))$. In this case, by the extremum principle for fractional-order differential operators $D^{\alpha}_{a,x}$ and $D^{\alpha}_{x,b}$ $(0 < \alpha < 1)$ [10] (p. 19) at the point x_0, the point of positive maximum of the function $\tau(x)$ we have $D^{1-2\beta}_{-1,x}\tau(x)|_{x=x_0} > 0$, hence from (15) (with $\Psi_1(x) \equiv 0,\ b(x) > 0$) we get $\nu(x_0) > 0$. On the other hand, at this point, by the known analogue of the Zaremba–Giraud extremum principle, we have $\nu(x_0) < 0$ [7] (p. 74), [2] (p. 26). The resulting contradiction shows that the point of the positive maximum $(x_0, 0) \notin AE\ (x_0 \notin (-1, c))$.

Thus, the point (x_0, y_0), the point of positive maximum of desired function $u(x,y)$ does not belong to the degenerate segment AB, i.e., $(x_0, y_0) \notin AB$.

From this, due to the inequality $0 < \rho(x) < 1$ from the boundary condition (4) (with $\varphi_0(x) \equiv 0$) and taking inequality (17), into account, we similarly conclude that the desired function $u(x,y)$ also does not reach its GPV at points of the curve σ_0. Thus, the solution to problem MA under the conditions of Theorem 1 reaches its GPV in domain $\overline{\Omega}_0$ at points $A(-1, 0)$ or $B(1, 0)$.

Similarly, as shown above, the solution of problem MA under the conditions of Theorem 1 reaches its LNV in domain $\overline{\Omega}_0$ at points $A(-1, 0)$ or $B(1, 0)$. By virtue of (17): $0 < \rho(x) < 1$ from boundary condition (4) (with $\varphi_0(x) = 0$) it is followed that $u(A) = u(B) = 0$, and so $u(x,y) \equiv 0$ in domain $\overline{\Omega}_0$.

Hence, it follows that

$$\tau(x) = u(x, +0) \equiv 0,\ \forall x \in [-1, 1];\ \nu(x) = \lim_{y \to +0} y^{\delta_0} \frac{\partial u}{\partial y} \equiv 0,\ \forall x \in (-1, 1).$$

Therefore, from the conditions of conjugation (2) and (3) (with $a_0(x) \equiv 0,\ b_0(x) \equiv 0$) it is followed that

$$\tau^-(x) = u(x, -0) \equiv 0,\ \forall x \in [-1, 1];\ \nu^-(x) = \lim_{y \to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} \equiv 0,\ \forall x \in (-1, 1). \tag{18}$$

Now the solution of problem MA in domain $\overline{\Omega}_1$ is restored as the solution of a modified Cauchy problem with null data (18) using Darboux's Formula (14) and we obtain that $u(x,y) \equiv 0$, also in domain $\overline{\Omega}_1$. Theorem 1 is proved. □

4.2. Solvability of Problem MA for the Case $\delta > \beta$

Theorem 2. *The solution to problem MA exists, when the following conditions are met:*

$$\begin{gathered} a(x),\ a_0(x),\ b(x),\ b_0(x),\ \rho(x),\ \varphi_0(x) \in C[-1,1] \cap C^1(-1,1), \\ \psi(x) \in C[-1,(c-1)/2] \cap C^1(-1,(c-1)/2),\ f(x) \in C[-1,c] \cap C^1(-1,c), \\ \tfrac{l+2\delta_0}{2(l+2)} > \tfrac{m+2\beta_0}{2(m+2)},\ a(x) > 0,\ b(x) > 0. \end{gathered} \tag{19}$$

Proof. The solution of Dirichlet problem in domain Ω_0 for Equation (1), which satisfies the boundary conditions

$$u(x,0) = \tau(x),\ \ x \in [-1,1];\ \ u(x,y)|_{\sigma_0} = \varphi(x),\ \ x \in [-1,1],$$

has the form [7]

$$\begin{aligned} u(x,y) = k_2(1-\delta_0)y^{1-\delta_0} \int\limits_{-1}^{1} \tau(t) \Bigg\{ \left[(x-t)^2 + \frac{4}{(l+2)^2} y^{l+2} \right]^{\delta-1} - \\ - \left[(1-xt)^2 + \frac{4t^2}{(l+2)^2} y^{l+2} \right]^{\delta-1} \Bigg\} dt - k_2(1-\delta)(l+2)\left(1-R^2\right) y^{1-\delta_0} \times \\ \times \int\limits_0^{l_0} \varphi(\xi(s)) \left(r_1^2\right)^{\delta-2} F(1-\delta, 2-\delta, 2-2\delta; 1-\sigma) d\xi(s), \quad (x,y) \in \overline{\Omega}_0, \end{aligned} \tag{20}$$

where S is the length of the curve σ_0 measured from point $B(1,0)$ to point $M(x(s),y(s)) \in \sigma_0$, l_0 is the length of the whole curve σ_0,

$$\sigma = \frac{r^2}{r_1^2}, \qquad \left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = (x-\xi(s))^2 + \frac{4}{(l+2)^2}\left(y^{\frac{l+2}{2}} \mp \eta(s)^{\frac{l+2}{2}}\right)^2,$$

$$k_2 = \frac{1}{4\pi}\left(\frac{4}{l+2}\right)^{2-2\delta} \frac{\Gamma^2(1-\delta)}{\Gamma(2-2\delta)},\ R^2 = x^2 + \frac{4}{(l+2)^2} y^{l+2},\ (\xi(s),\ \eta(s)) \in \sigma_0\,.$$

From (20), assuming $\xi(s) = t$, $\eta(s) = \left[\frac{(l+2)^2}{4}(1-t^2)\right]^{\frac{1}{l+2}}$, after some transformations [7], Ref. [10] (p. 55) taking into account the boundary condition (4), it is easy to calculate that

$$\begin{aligned} \nu(x) = -k_2(1-\delta_0)\frac{l+2}{2}\left[\int\limits_{-1}^{1} \frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\delta}} - (2\delta-1)\int\limits_{-1}^{1} \frac{\tau(t)dt}{(1-xt)^{2-2\delta}} \right] + \\ + k_2(1-\delta_0)(1-\delta(l+2)(1-x^2) \int\limits_{-1}^{1} (x^2-2xt+1)^{\delta-2} \rho(t)\tau(t)dt + \Phi(x),\ x \in (-1,1), \end{aligned} \tag{21}$$

where $\nu(x) = \lim\limits_{y \to +0} y^{\delta_0} \dfrac{\partial u}{\partial y},\ \ \tau(x) = u(x,0),$

$$\Phi(x) = k_2(1-\delta_0)(1-\delta)(l+2)\left(1-x^2\right) \int\limits_{-1}^{1} \left(x^2-2xt+1\right)^{\delta-2} \varphi_0(t)dt \in C[-1,\,1] \cap C^1(-1,\,1).$$

By virtue of (21), from relation (16), excluding $\nu(x)$, we obtain

$$D_{-1,x}^{1-2\beta}a(x)\tau(x) = -\frac{k_2(1-\delta_0)(l+2)}{2\gamma}b(x)\left[\int\limits_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\delta}} - (2\delta-1)\int\limits_{-1}^{1}\frac{\tau(t)dt}{|1-xt|^{2-2\delta}}\right]+ \\ +\frac{k_2(1-\delta_0)(1-\delta)(l+2)(1-x^2)b(x)}{\gamma}\int\limits_{-1}^{1}(x^2-2xt+1)^{\delta-2}\rho(t)\tau(t)dt + F(x), \tag{22}$$

where

$$F(x) = \frac{b(x)\Phi(x)}{\gamma} - \frac{\Psi_1(x)}{\gamma}, \quad x \in (-1,c).$$

Note that relation (22) is an integro-differential equation with respect to unknown function $\tau(x)$ in the interval $(-1,c)$. Let us continue to study Equation (22). Applying the fractional order integration operator $D_{-1,x}^{2\beta-1}$ [10] (p. 16) to Equality (22), where $2\beta-1<0$, we obtain

$$a(x)\tau(x) = -\frac{k_2(1-\delta_0)(l+2)}{2\gamma}\left\{D_{-1,x}^{2\beta-1}\left(b(x)\int\limits_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\delta}}\right) - (2\delta-1)D_{-1,x}^{2\beta-1}\left(b(x)\int\limits_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\delta}}\right)\right\}+ \\ +\frac{k_2(1-\delta_0)(1-\delta)(l+2)}{2\gamma}D_{-1,x}^{2\beta-1}\left\{(1-x^2)b(x)\int\limits_{-1}^{1}(x^2-2xt+1)^{\delta-2}\rho(t)\tau(t)dt\right\} + F_1(x), \quad x\in(-1,c), \tag{23}$$

where $F_1(x) = D_{-1,x}^{2\beta-1}F(x)$. □

Lemma 1. *The following identities are valid:*

$$D_{-1,x}^{2\beta-1}\left\{b(x)\int\limits_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\delta}}\right\} = \\ = \frac{b(x)}{\Gamma(1-2\beta)}\left[\frac{\beta}{\delta}\int\limits_{-1}^{x}\left(\frac{1+t}{1+x}\right)^{2\delta}\frac{F(-2\delta,2\beta-2\delta,1+2\delta;(1+t)/(1+x))}{(x-t)^{1+2\beta-2\delta}}\tau(t)dt - \right. \\ -\int\limits_{x}^{1}\left(\frac{1+x}{1+t}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+t))}{(t-x)^{1+2\beta-2\delta}}\tau(t)dt + \\ \left.+(2\beta-2\delta)B(1-2\beta,2\delta)\int\limits_{-1}^{x}\frac{\tau(t)dt}{(x-t)^{1+2\beta-2\delta}} + (1+x)^{-2\beta}\int\limits_{-1}^{1}\frac{\tau(t)dt}{(1+t)^{1-2\delta}}\right] - \\ -\frac{1}{\Gamma(1-2\beta)}\left\{\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds - \right. \\ -\int_{-1}^{1}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}}ds + \int_{x}^{1}\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds + \\ \left.+\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{1}\frac{\tau(t)dt}{(1+t)^{1-2\delta}}\right\}, \quad x\in(-1,c); \tag{24}$$

$$D_{-1,x}^{2\beta-1}\left\{b(x)\int\limits_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\delta}}\right\} = \\ = \frac{b(x)}{\Gamma(1-2\beta)}\int\limits_{-1}^{1}\frac{(1+x)^{1-2\beta}}{(1+t)^{1-2\delta}}\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)t/(xt-1))}{1-xt}\tau(t)dt + \\ +\frac{1}{\Gamma(1-2\beta)}\int\limits_{-1}^{1}\tau(t)dt\int\limits_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-st)^{2-2\delta}}; \tag{25}$$

$$D_{-1,x}^{2\beta-1}\left\{(1-x^2)b(x)\int\limits_{-1}^{1}(x^2-2xt+1)^{\delta-2}\rho(t)\tau(t)dt\right\} \\ = \frac{1}{\Gamma(1-2\beta)}\int\limits_{-1}^{1}\tau(t)dt\int\limits_{-1}^{x}\frac{(1-t^2)(1-s^2)\widetilde{\rho}(t)b(s)ds}{(x-s)^{2\beta}(s^2-2st+1)^{2-\delta}}. \tag{26}$$

Proof. Let us prove identity (24)

$$\begin{aligned} \mathrm{I}(x) &= D_{-1,x}^{2\beta-1}\left\{ b(x)\int\limits_{-1}^{1}\frac{(x-t)\tau'(t)dt}{\left|x-t\right|^{2-2\delta}}\right\} \\ &= \frac{1}{\Gamma(1-2\beta)}\int\limits_{-1}^{x}\frac{(b(x)+(b(s)-b(x)))ds}{(x-s)^{2\beta}}\int\limits_{-1}^{1}\frac{(s-t)\tau'(t)dt}{\left|s-t\right|^{2-2\delta}} = \mathrm{I}_1(x)-\mathrm{I}_2(x),\ x\in(-1,c), \end{aligned} \tag{27}$$

where

$$\mathrm{I}_1(x) = \frac{b(x)}{\Gamma(1-2\beta)}\int\limits_{-1}^{x}\frac{ds}{(x-s)^{2\beta}}\int\limits_{-1}^{1}\frac{(s-t)\tau'(t)dt}{\left|s-t\right|^{2-2\delta}}; \tag{28}$$

$$\mathrm{I}_2(x) = \frac{1}{\Gamma(1-2\beta)}\int\limits_{-1}^{x}\frac{(b(x)-b(s))ds}{(x-s)^{2\beta}}\int\limits_{-1}^{1}\frac{(s-t)\tau'(t)dt}{\left|s-t\right|^{2-2\delta}}. \tag{29}$$

Calculating the inner integral from (28) and (29), we have

$$J(s) = \int\limits_{-1}^{1}\frac{(s-t)\tau'(t)dt}{\left|s-t\right|^{2-2\delta}} = \lim_{\varepsilon\to 0}\left[\int\limits_{-1}^{s-\varepsilon}(s-t)^{2\delta-1}d(\tau(t)) - \int\limits_{s+\varepsilon}^{1}(t-s)^{2\delta-1}d(\tau(t))\right].$$

Here, by performing the integration operation by parts, we obtain

$$J(s) = \lim_{\varepsilon\to 0}\left[(s-t)^{2\delta-1}\tau(t)|_{-1}^{s-\varepsilon}+(2\delta-1)\int\limits_{-1}^{s-\varepsilon}(s-t)^{2\delta-2}\tau(t)dt-(t-s)^{2\delta-1}\tau(x)|_{s+\varepsilon}^{1}+(2\delta-1)\int\limits_{s+\varepsilon}^{1}(t-s)^{2\delta-2}\tau(t)dt\right]. \tag{30}$$

Due to the identities

$$(2\delta-1)\int\limits_{-1}^{s-\varepsilon}(s-t)^{2\delta-2}\tau(t)dt = \frac{d}{ds}\int\limits_{-1}^{s-\varepsilon}(s-t)^{2\delta-1}\tau(t)dt-\varepsilon^{2\delta-1}\tau(s-\varepsilon),$$

$$(2\delta-1)\int\limits_{s+\varepsilon}^{1}(t-s)^{2\delta-2}\tau(t)dt = -\frac{d}{ds}\int\limits_{s+\varepsilon}^{1}(t-s)^{2\delta-1}\tau(t)dt-\varepsilon^{2\delta-1}\tau(s+\varepsilon)$$

equality (30) is written in the form

$$J(s) = \lim_{\varepsilon\to 0}\left[\frac{d}{ds}\int\limits_{-1}^{s-\varepsilon}(s-t)^{2\delta-1}\tau(t)dt-\frac{d}{ds}\int\limits_{s+\varepsilon}^{1}(t-s)^{2\delta-1}\tau(t)dt\right].$$

Now, passing to the limit for $\varepsilon\to 0$, we have

$$J(s) = \frac{d}{ds}\int\limits_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\delta}}-\frac{d}{ds}\int\limits_{s}^{1}\frac{\tau(t)dt}{(t-s)^{1-2\delta}} \tag{31}$$

Now, taking into account the representation (31) for the inner integral (28), we deduce that

$$\begin{aligned} \mathrm{I}_1(x) &= \frac{b(x)}{\Gamma(1-2\beta)} \int\limits_{-1}^{x} \frac{ds}{(x-s)^{2\beta}} \left[\frac{d}{ds} \int\limits_{-1}^{s} \frac{\tau(t),dt}{(s-t)^{1-2\delta}} - \frac{d}{ds} \int\limits_{s}^{1} \frac{\tau(t),dt}{(t-s)^{1-2\delta}} \right] \\ &= \frac{b(x)}{\Gamma(1-2\beta)} \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \left[\int\limits_{-1}^{x-\varepsilon_1} (x-s)^{-2\beta} d\left(\int\limits_{-1}^{s} \frac{\tau(t),dt}{(s-t)^{1-2\delta}} \right) - \int\limits_{-1}^{x-\varepsilon_2} (x-s)^{-2\beta} d\left(\int\limits_{s}^{1} \frac{\tau(t),dt}{(t-s)^{1-2\delta}} \right) \right]. \end{aligned} \tag{32}$$

Here, by performing the integration operation by parts, then using the identities, we have

$$-2\beta \int\limits_{-1}^{x-\varepsilon_1} ds \int\limits_{-1}^{s} \frac{\tau(t)dt}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}} = \frac{d}{dx} \int\limits_{-1}^{x-\varepsilon_1} (x-s)^{-2\beta} ds \int\limits_{-1}^{s} \frac{\tau(t)dt}{(s-t)^{1-2\delta}} - \varepsilon_1^{-2\beta} \int\limits_{-1}^{x-\varepsilon_1} \frac{\tau(t)dt}{(x-\varepsilon_1-t)^{1-2\delta}};$$

$$2\beta \int\limits_{-1}^{x-\varepsilon_2} ds \int\limits_{s}^{1} \frac{\tau(t)dt}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}} = -\frac{d}{dx} \int\limits_{-1}^{x-\varepsilon_2} (x-s)^{-2\beta} ds \int\limits_{s}^{1} \frac{\tau(t)dt}{(t-s)^{1-2\delta}} - \varepsilon_2^{-2\beta} \int\limits_{x-\varepsilon_2}^{1} \frac{\tau(t)dt}{(t-x+\varepsilon_2)^{1-2\delta}}.$$

Equality (32) is written as follows:

$$\begin{aligned} \mathrm{I}_1(x) = \frac{b(x)}{\Gamma(1-2\beta)} \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \Bigg[& \frac{d}{dx} \int\limits_{-1}^{x-\varepsilon_1} \frac{ds}{(x-s)^{2\beta}} \int\limits_{-1}^{s} \frac{\tau(t)dt}{(s-t)^{1-2\delta}} - \\ & - \frac{d}{dx} \int\limits_{-1}^{x-\varepsilon_2} \frac{ds}{(x-s)^{2\beta}} \int\limits_{s}^{1} \frac{\tau(t)dt}{(t-s)^{1-2\delta}} + (1+x)^{-2\beta} \int\limits_{-1}^{1} \frac{\tau(t)dt}{(1+t)^{1-2\delta}} \Bigg]. \end{aligned} \tag{33}$$

Now, in (33) passing to limit (33) for $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, we obtain

$$\begin{aligned} \mathrm{I}_1(x) = \frac{b(x)}{\Gamma(1-2\beta)} \Bigg[& \frac{d}{dx} \int\limits_{-1}^{x} \frac{ds}{(x-s)^{2\beta}} \int\limits_{-1}^{s} \frac{\tau(t)dt}{(s-t)^{1-2\delta}} - \\ & - \frac{d}{dx} \int\limits_{-1}^{x} \frac{ds}{(x-s)^{2\beta}} \int\limits_{s}^{1} \frac{\tau(t)dt}{(t-s)^{1-2\delta}} + (1+x)^{-2\beta} \int\limits_{-1}^{1} \frac{\tau(t)dt}{(1+t)^{1-2\delta}} \Bigg]. \end{aligned} \tag{34}$$

In Equation (34), changing the order of integration, we obtain

$$\mathrm{I}_1(x) = \frac{b(x)}{\Gamma(1-2\beta)} \left[\mathrm{I}_{11}(x) - \mathrm{I}_{12}(x) - \mathrm{I}_{13}(x) + (1+x)^{-2\beta} \int\limits_{-1}^{1} \frac{\tau(t)dt}{(1+t)^{1-2\delta}} \right], \tag{35}$$

where

$$\mathrm{I}_{11}(x) = \frac{d}{dx} \int\limits_{-1}^{x} \tau(t)dt \int\limits_{t}^{x} \frac{ds}{(x-s)^{2\beta}(s-t)^{1-2\delta}} = B(1-2\beta, 2\delta)(2\delta-2\beta) \int\limits_{-1}^{x} \frac{\tau(t)dt}{(x-t)^{1+2\beta-2\delta}}; \tag{36}$$

$$\begin{aligned} \mathrm{I}_{12}(x) &= \frac{d}{dx} \int\limits_{-1}^{x} \tau(t)dt \int\limits_{-1}^{t} \frac{ds}{(x-s)^{2\beta}(t-s)^{1-2\delta}} \\ &= \frac{(1+x)^{2\delta-2\beta}}{2\delta-2\beta} \tau(x) - \frac{\beta}{\delta} \int\limits_{-1}^{x} \left(\frac{1+t}{1+x} \right)^{2\delta} \frac{F(-2\delta, 2\beta-2\delta, 1+2\delta; (1+t)/(1+x))}{(x-t)^{1+2\beta-2\delta}} \tau(t)dt; \end{aligned} \tag{37}$$

$$\mathrm{I}_{13}(x) = -\frac{(1+x)^{2\delta-2\beta}}{2\delta-2\beta}\tau(x) + \int_x^1 \left(\frac{1+t}{1+x}\right)^{2\beta} \frac{F(-2\beta, 2\delta-2\beta, 1-2\beta; (1+x)/(1+t))}{(t-x)^{1+2\beta-2\delta}}\tau(t)dt, \tag{38}$$

where $B(\alpha,\beta)$ is Euler's beta function $F(a,b,c;x)$ is Gauss hypergeometric function [10] (p. 6).

In the proof of Formula (36) in the inner integral $\mathrm{I}_{11}(x)$ change of the integration variable $s = t + (x-t)\sigma$ is made, by which the integral is expressed through Euler's beta function. Then, by performing the differentiation operation, we obtain Formula (36). In the proof of Formula (37), in the inner integral $\mathrm{I}_{12}(x)$, change of the integration variable $s = -1 + (1+t)\sigma$ is made, by which the integral is expressed through the Gaussian hypergeometric function [10] (p. 5), and after applying the differentiation formula

$$\frac{d}{dz}[z^b F(a,b,c;z)] = bz^{b-1}F(a,b+1,c;z),$$

by performing the differentiation operation, and using the autotransformation formula

$$F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z), \tag{39}$$

we obtain Formula (37). In the proof of Formula (38) in the inner integral $\mathrm{I}_{13}(x)$ change of the integration variable $s = -1 + (1+x)\sigma$ is made, due to this it is expressed through the Gaussian hypergeometric function [10] (p. 8), then apply differentiation formula

$$\frac{d}{dz}[z^{c-1}F(a,b,c;z)] = (c-1)z^{c-2}F(a,b,c-1;z),$$

and autotransformation Formula (39) and get (38).

Thus, due to (36)–(38) from (35) for $\mathrm{I}_1(x)$, we obtain the formula

$$\begin{aligned}\mathrm{I}_1(x) = \frac{B(x)}{\Gamma(1-2\beta)}\Bigg[&\frac{\beta}{\delta}\int_{-1}^{x}\left(\frac{1+t}{1+x}\right)^{2\delta}\frac{F(-2\delta,2\beta-2\delta,1+2\delta;(1+t)/(1+x))}{(x-t)^{1+2\beta-2\delta}}\tau(t)dt-\\ &-\int_x^1\left(\frac{1+x}{1+t}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+t))}{(t-x)^{1+2\beta-2\delta}}\tau(t)dt+\\ &+(2\beta-2\delta)B(1-2\beta,2\delta)\int_{-1}^{x}\frac{\tau(t)dt}{(x-t)^{1+2\beta-2\delta}}+(1+x)^{-2\beta}\int_{-1}^{1}\frac{\tau(t)dt}{(1+t)^{1-2\delta}}\Bigg]\end{aligned} \tag{40}$$

Now, let us calculate $\mathrm{I}_2(x)$ from (29). Taking into account the representation (31) for inner integral in (29), $\mathrm{I}_2(x)$ will be written in the form

$$\mathrm{I}_2(x) = \frac{1}{\Gamma(1-2\beta)}\int_{-1}^{x}\frac{b(x)-b(s)}{(x-s)^{2\beta}}\left\{\frac{d}{ds}\int_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\delta}} - \frac{d}{ds}\int_s^1\frac{\tau(t)dt}{(t-s)^{1-2\delta}}\right\}ds. \tag{41}$$

In (41), performing the integration operation by parts, we have

$$\begin{aligned}\mathrm{I}_2(x) = \frac{1}{\Gamma(1-2\beta)}\Bigg\{&\left[\frac{b(x)-b(s)}{(x-s)^{2\beta}}\int_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\delta}}\right]\Bigg|_{-1}^{x} - \int_{-1}^{x}\left[\left(\frac{b(x)-b(s)}{(x-s)^{2\beta}}\right)'_s\int_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\delta}}\right]ds-\\ &-\left[\frac{b(x)-b(s)}{(x-s)^{2\beta}}\int_s^1\frac{\tau(t)dt}{(t-s)^{1-2\delta}}\right]\Bigg|_{-1}^{x} + \int_{-1}^{x}\left(\frac{b(x)-b(s)}{(x-s)^{2\beta}}\right)'_s\int_s^1\frac{\tau(t)dt}{(t-s)^{1-2\delta}}\Bigg]ds\Bigg\}\end{aligned} \tag{42}$$

Since the function $b(x)$ has continuous derivative of the first order in segment $[-1,1]$, then the difference $b(x)-b(s)$ satisfies Lipschitz's condition: $|b(x)-b(s)|\leq k|x-s|$, where $k=const$. Therefore

$$\left.\frac{b(x)-b(s)}{(x-s)^{2\beta}}\right|_{x=s}=0,$$

Taking this remark into account, equality (42) can be written as

$$\begin{aligned}\mathrm{I}_2(x)=\frac{1}{\Gamma(1-2\beta)}\Bigg\{&-\int_{-1}^{x}\left[\left(\frac{b(x)-b(s)}{(x-s)^{2\beta}}\right)'_s\int_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\delta}}\right]ds-\frac{b(x)-b(-1)}{(x+1)^{2\beta}}\int_{-1}^{1}\frac{\tau(t)dt}{(1+t)^{1-2\delta}}+\\&+\int_{-1}^{x}\left[\left(\frac{b(x)-b(s)}{(x-s)^{2\beta}}\right)'_s\int_{s}^{1}\frac{\tau(t)dt}{(t-s)^{1-2\delta}}\right]ds\Bigg\}.\end{aligned}\tag{43}$$

In the integrand expressions (43), after performing the specified differentiation operations and then changing the order of integration, we obtain

$$\begin{aligned}\mathrm{I}_2(x)=\frac{1}{\Gamma(1-2\beta)}\Bigg\{&\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds-\\&-\int_{-1}^{1}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}}ds+\\+\int_{x}^{1}\tau(t)dt\int_{-1}^{x}&\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds+\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{1}\frac{\tau(t)dt}{(1+x)^{1-2\delta}}\Bigg\},\quad x\in(-1,c).\end{aligned}\tag{44}$$

Now, expressions for $\mathrm{I}_1(x)$ and $\mathrm{I}_2(x)$ from (40) and (44) substituting in (27) we obtain equality (24). The equality (24) is proved.

Proceeding similarly as in the case of identity (24), identity (25) is also proven. Taking into account that the function $\frac{(1+t)(1+s)}{(s^2-2st+1)^{2-\delta}}$ is continuous in the square $-1\leq s,t\leq c$, it is not difficult to establish the validity of identity (26) through direct calculation. Lemma 1 is proven. □

Now, by virtue of Lemma 1 (24)–(26) substituting into Equation (23) and with the aim of reducing the limits of integration to the main interval $(-1,c)$, we divide the integration interval $(x,1)$ into the subintervals (x,c) and $(c,1)$, and the interval $(-1,1)$ into the subintervals $(-1,c)$ and $(c,1)$. Then, in the integrals over the interval $(c,1)$ we perform a change of the integration variable $t=p(z)$, where $t\in(c,1)$, $z\in(-1,c)$, $p(-1)=1$, $p(c)=c$ and taking into account condition (13), we write Equation (23) in the form

$$
\begin{aligned}
a(x)\tau(x) = -\frac{k_2(1-\delta_0)(l+2)}{2\gamma}\Bigg\{\Bigg\{\frac{b(x)}{\Gamma(1-2\beta)}\Bigg[\frac{\beta}{\delta}\int_{-1}^{x}\left(\frac{1+t}{1+x}\right)^{2\delta}\frac{F(-2\delta,2\beta-2\delta,1+2\delta;(1+t)/(1+x))}{(x-t)^{1+2\beta-2\delta}}\tau(t)dt- \\
-\int_{x}^{c}\left(\frac{1+x}{1+t}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+t))}{(t-x)^{1+2\beta-2\delta}}\tau(t)dt- \\
-\int_{-1}^{c}\left(\frac{1+x}{1+p(t)}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+p(t)))}{(p(t)-x)^{1+2\beta-2\delta}}\mu_1(t)p\prime(t)\tau(t)dt- \\
-\int_{-1}^{c}\left(\frac{1+x}{1+p(t)}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+p(t)))}{(p(t)-x)^{1+2\beta-2\delta}}f_1(t)p\prime(t)dt- \\
-\frac{\Gamma(1-2\beta)\Gamma(2\delta)}{\Gamma(2\delta-2\beta)}\int_{-1}^{x}\frac{\tau(t)dt}{(x-t)^{1+2\beta-2\delta}}+(1+x)^{-2\beta}\int_{-1}^{c}\left(\frac{1}{(1+t)^{1-2\delta}}+\frac{\mu_1(t)p\prime(t)}{(1+p(t))^{1-2\delta}}\right)\tau(t)dt+ \\
+(1+x)^{-2\beta}\int_{-1}^{c}\frac{f_1(t)p\prime(t)dt}{(1+p(t))^{1-2\delta}}\Bigg]-\frac{1}{\Gamma(1-2\beta)}\Bigg[\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds- \\
-\int_{-1}^{x}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}}ds+\int_{x}^{c}\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}}ds+ \\
+\int_{-1}^{c}\mu_1(t)p'(t)\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\delta}}ds+ \\
+\int_{-1}^{c}f_1(t)p'(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\delta}}ds+ \\
+\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{c}\tau(t)\left(\frac{1}{(1+t)^{1-2\delta}}+\frac{\mu_1(t)p'(t)}{(1+p(t))^{1-2\delta}}\right)dt-\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{c}\frac{f_1(t)p'(t)dt}{(1+p(t))^{1-2\delta}}\Bigg]\Bigg\}- \\
-(2\delta-1)\Bigg[\frac{b(x)(1+x)^{1-2\beta}}{\Gamma(1-2\beta)}\int_{-1}^{c}\Bigg(\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)t/(xt-1))}{(1+t)^{1-2\delta}(1-xt)}+ \\
+\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)p(t)/(xp(t)-1))}{(1+p(t))^{1-2\delta}(1-xp(t))}\mu_1(t)p'(t)\Bigg)\tau(t)dt+ \\
+\frac{b(x)}{\Gamma(1-2\beta)}\int_{-1}^{c}\frac{(1+x)^{1-2\beta}}{(1+p(t))^{1-2\delta}}\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)p(t)/(xp(t)-1))}{1-xp(t)}f_1(t)p'(t)dt+ \\
+\frac{1}{\Gamma(1-2\beta)}\int_{-1}^{c}\tau(t)dt\int_{-1}^{x}\frac{(b(s)-b(x))}{(x-s)^{2\beta}}\left(\frac{1}{(1-st)^{2-2\delta}}+\frac{\mu_1(t)p'(t)}{(1-sp(t))^{2-2\delta}}\right)ds+ \\
+\frac{1}{\Gamma(1-2\beta)}\int_{-1}^{c}f_1(t)p\prime(t)dt\int_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-sp(t))^{2-2\delta}}\Bigg]\Bigg\}+ \\
+\frac{k_2(1-\delta_0)(1-\delta)(l+2)}{\gamma\Gamma(1-2\beta)}\Bigg\{\int_{-1}^{c}\tau(t)dt\int_{-1}^{x}\frac{(1-s^2)b(s)}{(x-s)^{2\beta}}\left(\frac{(1-t^2)\widetilde{\rho}(t)}{(s^2-2st+1)^{2-\delta}}+\frac{\mu_1(t)p\prime(t)(1-p^2(t))\widetilde{\rho}(p(t))}{(s^2-2sp(t)+1)^{2-\delta}}\right)ds+ \\
+\frac{k_2(1-\delta_0)(1-\delta)(l+2)}{\gamma\Gamma(1-2\beta)}\int_{-1}^{c}f_1(t)p\prime(t)dt\int_{-1}^{x}\frac{(1-p^2(t))(1-s^2)\widetilde{\rho}(p(t))b(s)ds}{(x-s)^{2\beta}(s^2-2sp(t)+1)^{2-\delta}}\Bigg\}+F_1(x),\ x\in(-1,c).
\end{aligned}
\tag{45}
$$

After some transformations, Equation (46) is written as follows:

$$
\tau(x)+\lambda\int_{-1}^{c}\frac{K(x,t)\tau(t)dt}{|x-t|^{1+2\beta-2\delta}}=F_2(x),\quad x\in(-1,c),
\tag{46}
$$

where $\lambda=\frac{k_2(1-\delta_0(l+2)}{2\gamma\Gamma(1-2\beta)}$,

$$K(x,s)=\frac{b(x)}{a(x)}\left\{\theta(x-t)\left[\frac{\beta}{\delta}\left(\frac{1+t}{1+x}\right)^{2\delta}F\left(-2\delta,2\beta-2\delta,1+2\delta;\frac{1+t}{1+x}\right)+\frac{\Gamma(1-2\beta)\Gamma(2\delta)}{\Gamma(2\delta-2\beta)}\right]-\right.$$
$$\left.-\theta(t-x)\left(\frac{1+x}{1+t}\right)^{2\beta}F\left(-2\beta,2\delta-2\beta,1+2\beta;\frac{1+x}{1+t}\right)\right\}+$$
$$+\frac{\lambda\left|x-t\right|^{1+2\beta-2\delta}}{a(x)}\left\{\left\{b(x)\left[-\left(\frac{1+x}{1+p(t)}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+p(t)))}{(p(t)-x)^{1+2\beta-2\delta}}\mu_1(t)p\prime(t)+\right.\right.$$
$$\left.+(1+x)^{-2\beta}\left(\frac{1}{(1+t)^{1-2\delta}}+\frac{\mu_1(t)p\prime(t)}{(1+p(t))^{1-2\delta}}\right)\right]+\theta(x-t)\left[\int_t^x\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds+\right.$$
$$\left.+\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\delta}}ds\right]-\theta(t-x)\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\delta}}ds-$$
$$-\mu_1(t)p'(t)\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\delta}}ds-\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\left(\frac{1}{(1+t)^{1-2\delta}}+\frac{\mu_1(t)p'(t)}{(1+p(t))^{1-2\delta}}\right)+$$
$$+(2\delta-1)\left[b(x)(1+x)^{1-2\beta}\left(\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)t/(xt-1))}{(1+t)^{1-2\delta}(1-xt)}+\right.\right.$$
$$\left.+\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)p(t)/(xp(t)-1))}{(1+p(t))^{1-2\delta}(1-xp(t))}\mu_1(t)p'(t)\right)+$$
$$\left.\left.+\int_{-1}^{x}\frac{(b(s)-b(x))}{(x-s)^{2\beta}}\left(\frac{1}{(1-st)^{2-2\delta}}+\frac{\mu_1(t)p'(t)}{(1-sp(t))^{2-2\delta}}\right)ds\right]\right\}-$$
$$\left.-(1-\delta)\left[\int_{-1}^{x}\frac{(1-s^2)b(s)}{(x-s)^{2\beta}}\left(\frac{(1-t^2)\widetilde{\rho}(t)}{(s^2-2st+1)^{2-\delta}}+\frac{\mu_1(t)p'(t)(1-p^2(t))\widetilde{\rho}(p(t))}{(s^2-2tp(s)+1)^{2-\delta}}\right)\right]ds\right\}. \tag{47}$$

Here, $\theta(x)=1$, if $x>0$ and $\theta(x)=0$, if $x<0$.

$$F_2(x)=\frac{F_1(x)}{a(x)}-\frac{\lambda}{a(x)}\left\{b(x)\left[\int_{-1}^{c}\left(\frac{1+x}{1+p(z)}\right)^{2\beta}\frac{F(-2\beta,2\delta-2\beta,1-2\beta;(1+x)/(1+p(z)))}{(p(z)-x)^{1+2\beta-2\delta}}f_1(z)p'(z)dz+\right.\right.$$
$$\left.+\frac{1}{(1+x)^{2\beta}}\int_{-1}^{c}\frac{f_1(t)p'(t)dt}{(1+p(t))^{1-2\delta}}\right]-\int_{-1}^{c}f_1(t)p'(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\delta}}ds-$$
$$-\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{c}\frac{f_1(t)p'(t)dt}{(1+p(t))^{1-2\delta}}-$$
$$-(2\delta-1)\left[b(x)\int_{-1}^{c}\frac{(1+x)^{1-2\beta}}{(1+p(t))^{1-2\delta}}\frac{F(1,2\delta-2\beta,2-2\beta;(1+x)p(t)/(xp(t)-1))}{1-xp(t)}f_1(t)p'(t)dt+\right.$$
$$\left.\left.+\int_{-1}^{c}f_1(t)p\prime(t)dt\int_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-sp(t))^{2-2\delta}}\right]+(1-\delta)\int_{-1}^{c}f_1(t)p'(t)dt\int_{-1}^{x}\frac{(1-p^2(t))(1-s^2)\widetilde{\rho}(p(t))b(s)ds}{(x-s)^{2\beta}(s^2-2sp(t)+1)^{2-\delta}}\right\} \tag{48}$$

is known function.

Thus, in the case $\delta>\beta$ the solvability of problem MA is reduced to proving the unique solvability of the integral Equation (46).

Lemma 2. The function $(1-x)^{2\delta}K(x,s)$, *defined by relation (47), is continuous in the square $-1\le x,t\le c$ except on the diagonal $x=t$, where it has a discontinuity of the first kind.*

The Proof of Lemma 2: Due to condition (19) (smoothness of the given functions), follows from the form (47) of the definition of the function $K(x,s)$. Also, due to (19) and (48), it is easy to establish that $F_2(x)\in C(-1,c)$.

Due to condition (19), i.e., $0<1+2\beta-2\delta<1$, so Equation (46) is an integral equation with a weak singularity [62] (p. 481), with respect to the unknown function $\tau(x)$, the unique solvability of which in the class of functions $C[-1,\ c]$ follows from the uniqueness of the solution to problem MA. Theorem 2 is proven. □

4.3. Existence of a Solution to Problem MA for the Case $\delta = \beta$

4.3.1. Derivation of a Singular Integral Equation with a Non-Fredholm Integral Operator Having an Isolated First-Order Singularity at the Point

Now, consider the case $\delta = \beta,\ \delta \neq 0,\ \beta \neq 0$. In this case in Equation (23) changing δ to β and taking into account identities [7] (pp. 107–108)

$$D_{-1,x}^{2\beta-1}\left\{b(x)\int\limits_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|2-2\beta}\right\}=\frac{b(x)}{\Gamma(1-2\beta)}tg(\beta\pi)\tau(x)-\frac{b(x)}{\Gamma(1-2\beta)}\int\limits_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{2\beta}\frac{\tau(t)dt}{t-x}-$$
$$-\frac{1}{\Gamma(1-2\beta)}\left\{\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\beta}}ds-\int_{-1}^{x}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds+\right.$$
$$\left.+\int_{x}^{1}\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds+\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{1}\frac{\tau(t)dt}{(1+x)^{1-2\beta}}\right\},\quad x\in(-1,c);$$
$$D_{-1,x}^{2\beta-1}\left\{b(x)\int\limits_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right\}=\frac{b(x)}{\Gamma(2-2\beta)}\int\limits_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\frac{\tau(t)dt}{1-xt}+$$
$$+\frac{1}{\Gamma(1-2\beta)}\int\limits_{-1}^{1}\tau(t)dt\int\limits_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-st)^{2-2\beta}};$$

we obtain

$$\begin{aligned}&\tau(x)+A_0(x)\int\limits_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt\\&=R[\tau(x)]+\frac{F_1(x)}{\Gamma(1-2\beta)a(x)-k_0tg(\beta\pi)b(x)},\quad x\in(-1,c),\end{aligned}\tag{49}$$

where

$$A_0(x)=\frac{k_0b(x)}{\Gamma(1-2\beta)a(x)-k_0tg(\beta\pi)b(x)},\quad k_0=\frac{k_2(1-\delta_0)(1-\beta)(\updownarrow+2)}{2\gamma}.$$
$$R[\tau(x)]=\frac{k_0}{(1-\beta)(\Gamma(1-2\beta)a(x)-k_0tg(\beta\pi)b(x))}\left[2\int\limits_{-1}^{1}\tau(t)dt\int\limits_{-1}^{x}\frac{(1-t^2)(1-s^2)\widetilde{\rho}(t)b(s)ds}{(x-s)^{2\beta}(s^2-2st+1)^{2-\delta}}+\right.$$
$$+\int\limits_{-1}^{1}\tau(t)dt\int\limits_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-st)^{2-2\beta}}-\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\beta}}ds+$$
$$+\int_{-1}^{x}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds-$$
$$\left.-\int_{x}^{1}\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds-\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{1}\frac{\tau(t)dt}{(1+t)^{1-2\beta}}\right],\quad x\in(-1,c);$$

here $R[\tau(x)]$ is regular operator.

Note that Equation (49) is considered in the interval $(-1,c)$, therefore the kernel of Equation (49) has a singular feature only for $t\in(-1,c)$. In order to isolate the singular part of Equation (49) integrals over the interval $(-1,1)$ decomposed into two integrals over the intervals $(-1,c)$ and $(c,1)$, then in integrals over the interval $(c,1)$ replacing $t=p(s)=d-ks$, where $t\in[c,1],\ s\in[-1,c]$ and using condition (13): $\tau(p(x))=\mu_1(x)\tau(x)+f_1(x),\ x\in[-1,c]$. Equation (50) is written as

$$\begin{aligned}&\tau(x)+A_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt\\&=-kA_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+p(t)}\right)^{1-2\beta}\frac{\mu_1(t)\tau(t)dt}{p(t)-x}+R_1[\tau(x)]+F_2(x),\end{aligned}\tag{50}$$

where

$$R_1[\tau(x)] = kA_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+p(t)}\right)^{1-2\beta}\frac{\mu_1(t)\tau(t)dt}{1-xp(t)}+$$
$$+\frac{k_0}{(\Gamma(1-2\beta)a(x)-k_0tg(\beta\pi)b(x))}\left[\frac{2}{1-\beta}\int\limits_{-1}^{c}\tau(t)\,dt\int\limits_{-1}^{x}\frac{(1-t^2)(1-s^2)\widetilde{\rho}(t)b(s)ds}{(x-s)^{2\beta}(s^2-2st+1)^{2-\delta}}+\right.$$
$$+\frac{2k}{1-\beta}\int\limits_{-1}^{c}\mu_1(t)\tau(t)dt\int\limits_{-1}^{x}\frac{(1-p^2(t))(1-s^2)\widetilde{\rho}(p(t))b(s)ds}{(x-s)^{2\beta}(s^2-2sp(t)+1)^{2-\delta}}+\int\limits_{-1}^{c}\tau(t)dt\int\limits_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-st)^{2-2\beta}}-$$
$$-k\int\limits_{-1}^{c}\mu_1(t)\tau(t)dt\int\limits_{-1}^{x}\frac{(b(s)-b(x))ds}{(x-s)^{2\beta}(1-sp(t))^{2-2\beta}}-\int_{-1}^{x}\tau(t)dt\int_{t}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(s-t)^{1-2\beta}}ds+$$
$$+\int_{-1}^{x}\tau(t)dt\int_{-1}^{t}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds-\int_{x}^{c}\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(t-s)^{1-2\beta}}ds+$$
$$+k\int_{-1}^{c}\mu_1(t)\tau(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\beta}}ds-\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{c}\frac{\tau(t)dt}{(1+t)^{1-2\beta}}+$$
$$\left.+k\frac{b(x)-b(-1)}{(1+x)^{2\beta}}\int_{-1}^{c}\frac{\mu_1(t)\tau(t)dt}{(1+p(t))^{1-2\beta}}\right]\quad x\in(-1,c);$$
$$F_2(x)=\frac{F_1(x)}{a(x)-k_0(\beta\pi)}+$$
$$+\frac{kk_0}{(\Gamma(1-2\beta)a(x)-k_0tg(\beta\pi)b(x))}\left[\frac{2}{1-\beta}\int\limits_{-1}^{c}f_1(t)dt\int\limits_{-1}^{x}\frac{(1-p^2(t))(1-s^2)\widetilde{\rho}(p(t))b(s)ds}{(x-s)^{2\beta}(s^2-2sp(t)+1)^{2-\delta}}-\right.$$
$$\left.-\int\limits_{-1}^{c}f_1(t)dt\int\limits_{-1}^{x}\frac{(b(x)-b(s))ds}{(x-s)^{2\beta}(1-sp(t))^{2-2\beta}}\right]+k\int_{-1}^{c}f_1(t)dt\int_{-1}^{x}\frac{(x-s)b'(s)-2\beta(b(x)-b(s))}{(x-s)^{1+2\beta}(p(t)-s)^{1-2\beta}}ds,\quad x\in(-1,c);$$

$F_2(x)$ is a known function.

Equation (50) is a non-standard singular integral equation because the kernel of the first integral operator on the right-hand side of (50) contains an isolated first-order singularity at the point $(x,t)=(c,c)$ and therefore this operator is separated out.

Following Carleman's idea, considering the right-hand side of Equation (50) as a known function, we write it in the form

$$\tau(x)+A_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt=g_0(x),\ x\in(-1,c),\qquad(51)$$

where

$$g_0(x)=-kA_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+p(t)}\right)^{1-2\beta}\frac{\mu_1(t)\tau(t)dt}{p(t)-x}+R_1[\tau(x)]+F_2(x).\qquad(52)$$

From (52), due to the conditions of Theorem 2, it is not difficult to establish that the function $g_0(x)$ satisfies the Hölder condition on $(-1,\,c)$ and $g_0(x)\in L_p(-1,\,c),\ p>1$. Here $L_p(-1,\,c)$ denotes the Lebesgue space with the norm $\|\varphi(x)\|_p=\left(\int\limits_{-1}^{c}|\varphi(x)|^p dx\right)^{\frac{1}{p}}$.

4.3.2. Regularization of the Singular Integral Equation

Theorem 3. *If $g_0(x)$ satisfies the Hölder condition for $x\in(-1,c)$ and $g_0(x)\in L_p(-1,c),\ p>1$, then the solution of Equation (51) in the class of Hölder functions $H(-1,\,c)$ in which the func-*

tion $(1+x)^{2\beta-1}\tau(x)$ *is bounded at the right endpoint and may be unbounded at the left endpoint of the interval* $(-1,c)$ *i.e., in the class* $h(c)$ [10] *(p. 43) is expressed by the formula*

$$\tau(x)=\frac{g_0(x)}{1+\pi^2A_0^2(x)}+\frac{A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\times \\ \times\left(\frac{1+x}{1+t}\right)^{1-2\beta-2\alpha_0}\frac{\omega(x)}{\omega(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)g_0(t)dt,\ x\in(-1,c), \tag{53}$$

where

$$\alpha_0=-\tfrac{1}{\pi}arctg(\pi A_0(-1)),\ \alpha_1=-\tfrac{1}{\pi}arctg(\pi A_0(c)),$$
$$\omega(x)=(1+i\pi A_0(x))W(x),\ W(z)=\exp\{\Psi_0(z)\},$$

$$\Psi_0(z)=-\frac{1}{2\pi i}\int\limits_{-1}^{c}[\ln(t-z)+\ln(1-zt)]d\ln G(t),$$

$$G(x)=\begin{cases}\frac{1-i\pi A_0(x)}{1+i\pi A_0(x)}\ \text{if}\ x\in(-1,c)\\ \frac{1+i\pi A_0(1/x)}{1-i\pi A_0(1/x)}\ \text{if}\ x\in\Delta\\ 1\ \text{if}\ x\notin(-1,c)\cup\Delta,\end{cases}\quad\text{and}\ \Delta=\begin{cases}\left(\frac{1}{c},-1\right)\ \text{if}\ c<0\\ (-\infty,-1)\ \text{if}\ c=0\\ (-\infty,-1)\cup\left(\frac{1}{c},+\infty\right)\ \text{if}\ c>0.\end{cases}$$

Proof. Introducing notations $\rho(x)=(1+x)^{2\beta-1}\tau(x)$, $g(x)=(1+x)^{2\beta-1}g_0(x)$, Equation (51) is written in the form

$$\rho(x)-A_0(x)\int\limits_{-1}^{c}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt=g(x),\ x\in(-1,c) \tag{54}$$

To regularize the Tricomi singular integral Equation (54), we apply the Carleman–Vecua method developed by S.G.Mikhlin [23].

Let z be an arbitrary point of complex plane $C=\{z=x+iy\}$. In this plane, we introduce the function

$$\Phi(z)=\frac{1}{2\pi i}\int\limits_{-1}^{c}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)dt.$$

Let us denote by Δ the following infinite intervals of the real axis $y=0$.

$$\Delta=\begin{cases}\left(\frac{1}{c},-1\right)\ \text{if}\ c<0\\ (-\infty,-1)\ \text{if}\ c=0\\ (-\infty,-1)\cup\left(\frac{1}{c},+\infty\right)\ \text{if}\ c>0.\end{cases}$$

The function $\Phi(z)$ is holomorphic in the entire complex plane C except for the points of the set $(-1,c)\cup\Delta$ of the real axis $y>0$, and $\Phi(z)\to 0$ if $|z|\to\infty$. According to the Sokhotski–Plemel formulas [63], on the interval $-1<x<c$ we have

$$\Phi^+(x)-\Phi^-(x)=\rho(x), \tag{55}$$

$$\Phi^+(x)+\Phi^-(x)=\frac{1}{\pi i}\int\limits_{-1}^{c}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt. \tag{56}$$

Here $\Phi^+(x)$ and $\Phi^-(x)$ are the boundary values of the function $\Phi(z)$, as the point z approaches the point $(x,0)$ on the real axis from the upper or lower half-plane, respectively.

Using Formulas (55) and (56), Equation (54) can be written in the form

$$(1 + i\pi A_0(x))\Phi^+(x) - (1 - i\pi A_0(x))\Phi^-(x) = g(x), x \in (-1, c). \tag{57}$$

It is easy to verify that the function $\Phi(z)$ satisfies the relation

$$\Phi\left(\frac{1}{z}\right) = z\Phi(z). \tag{58}$$

The transformation $w = \frac{1}{z}$ maps the upper half-plane to the lower half-plane and vice versa. At the same time, the interval $(-1, c)$ is mapped to the infinite intervals Δ in the reverse direction. In (57), replacing x with $\frac{1}{x}$, and taking (58) into account, we get

$$\Phi^+\left(\frac{1}{x}\right) = x\Phi^-(x),\ \Phi^-\left(\frac{1}{x}\right) = x\Phi^+(x),\ x \in \Delta. \tag{59}$$

Due to (59) the equality (57) is written as

$$\left(1 - i\pi A_0\left(\frac{1}{x}\right)\right)\Phi^+(x) - \left(1 + i\pi A_0\left(\frac{1}{x}\right)\right)\Phi^-(x) = -\frac{1}{x}g\left(\frac{1}{x}\right),\ x \in \Delta. \tag{60}$$

Let us introduce the following functions

$$G(x) = \begin{cases} \frac{1-i\pi A_0(x)}{1+i\pi A_0(x)} \text{ if } x \in (-1, c) \\ \frac{1+i\pi A_0(1/x)}{1-i\pi A_0(1/x)} \text{ if } x \in \Delta \\ 1 \text{ if } x \notin (-1, c) \cup \Delta, \end{cases} \tag{61}$$

$$h(x) = \begin{cases} \frac{g(x)}{1+i\pi A_0(x)} \text{ if } x \in (-1, c) \\ -\frac{g(1/x)}{x(1-i\pi A_0(1/x))} \text{ if } x \in \Delta \\ 0 \text{ if } x \notin (-1, c) \cup \Delta. \end{cases} \tag{62}$$

Using the introduced functions (61) and (62), Equations (57) and (60) can be combined into a single equation:

$$\Phi^+(x) - G(x)\Phi^-(x) = h(x),\ -\infty < x < +\infty. \tag{63}$$

Thus, the problem of finding the solution to the singular integral Equation (54) is reduced to a Riemann problem in the theory of functions of a complex variable: to find a function $\Phi(z)$, vanishing at infinity, holomorphic both in the upper and lower half-planes and on the real axis $y = 0$, satisfying condition (63).

We reduce the Riemann problem (63) to a jump problem. To do this, the function $G(x)$ is represented in the form of the ratio of boundary values on the axis $y = 0$ of a holomorphic function $X(z)$ which is bounded at infinity.

Let us solve the following problem: to find a function $X(z)$, holomorphic in both the upper and lower half-planes and on the real axis $y = 0$, that is bounded at infinity and satisfies the condition

$$X^+(x) - G(x)X^-(x) = 0,\ -\infty < x < +\infty,$$

or

$$\ln X^+(x) - \ln X^-(x) = \ln G(x). \tag{64}$$

One of the partial solutions to problem (64) has the form

$$X(z) = \exp\left\{ \frac{1}{2\pi i} \int\limits_{-1}^{c} \left(\frac{1}{t-z} - \frac{z}{1-zt} \right) G(t)dt \right\},\ x \in (-1, c). \tag{65}$$

Note that

$$X\left(\frac{1}{z}\right) = X(z).$$

Equality (65) is written as

$$X(z) = \exp\left\{ \frac{1}{2\pi i} \int\limits_{-1}^{c} \ln G(t) d[\ln(t-z) + \ln(1-zt)] \right\}.$$

In the latter case, by performing the integration operation by parts, we obtain

$$\begin{array}{c} X(z) = \exp\left\{ \frac{1}{2\pi i} [\ln G(c)(\ln(c-z) + \ln(1-cz)) - \right. \\ \left. - \ln G(-1)(\ln(-1-z) + \ln(1+z))] + \Psi_0(z) \right\}, \end{array} \tag{66}$$

where

$$\Psi_0(z) = -\frac{1}{2\pi i} \int\limits_{-1}^{c} [\ln(t-z) + \ln(1-zt)] d \ln G(t)$$

is a holomorphic function in both the upper and lower half-planes and has a finite limit for $z \to -1,\ z \to c$.

From (61), we calculate $\ln G(-1)$ and $\ln G(c)$.

$$\ln G(-1) = \ln \frac{1 - i\pi A_0(-1)}{1 + i\pi A_0(-1)} = [-2arctg(\pi A_0(-1)) + 2k\pi]i = -2iarctg(\pi A_0(-1)) = 2\pi\alpha_0 i,$$

where $\alpha_0 = -\frac{1}{\pi} arctg(\pi A_0(-1)),\ \ k = 0.$

$$\ln G(c) = \ln \frac{1 - i\pi A_0(c)}{1 + i\pi A_0(c)} = [-2arctg(\pi A_0(c)) + 2k\pi]i = -2iarctg(\pi A_0(c)) = 2\pi\alpha_1 i,$$

where $\alpha_1 = -\frac{1}{\pi} arctg(\pi A_0(c)),\ \ k > 0.$

Therefore, taking into account the expressions for $\ln G(-1)$ and $\ln G(c)$ from (66), we have

$$X^+(z) = \exp\{\alpha_1(\ln(c-z) + \ln(1-cz)) - \alpha_0(\ln(-1-z) + \ln(1+z))\} W(z),$$

where $W(z) = \exp\{\Psi_0(z)\}$ is a holomorphic function in both the upper and lower half-planes. Hence

$$X^+(x) = \exp\left\{ \alpha_1 \ln(c-x)(1-cx) - \alpha_0 \ln(1+x)^2 + \alpha_0 \pi i \right\} W(x) = R_0(x) e^{\alpha_0 \pi i}, \tag{67}$$

$$X^-(x) = \exp\left\{ \alpha_1 \ln(c-x)(1-cx) - \alpha_0 \ln(1+x)^2 - \alpha_0 \pi i \right\} W(x) = R_0(x) e^{-\alpha_0 \pi i}, \tag{68}$$

where $W^+(x) = W^-(x) = W(x),\ R_0(x) = \frac{(c-x)^{\alpha_1}(1-cx)^{\alpha_1}}{(1+x)^{\alpha_0}} W(x).$

Thus the function $G(x)$ can be represented in the form $G(x) = X^+(x)/X^-(x)$—the factorization of the function $G(x)$ [64] (p. 33). Taking into account the last representation, the boundary condition (63) is written as

$$\frac{\Phi^+(x)}{\Phi^-(x)} - \frac{\Phi^-(x)}{X^-(x)} = \frac{h(x)}{X^+(x)},\ x \in (-\infty, +\infty). \tag{69}$$

Thus for the function $F(z) = \frac{\Phi(z)}{X(z)}$, we arrive at a jump problem [64] (p. 30) in the theory of analytic functions. One of the particular solutions of Equation (69) has the form

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{h(t)dt}{X^+(t)(t-z)}.$$

Hence, taking into account definition (62) for the function $h(t)$, we write

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\left\{\int\limits_{-1}^{1} \frac{1}{X^+(t)}\frac{g(t)}{1+i\pi A_0(t)}\frac{dt}{t-z} - \int\limits_{\Delta} \frac{1}{X^+(s)}\frac{g(1/s)}{s(1-i\pi A_0(1/s))}\frac{ds}{s-z}\right\}. \tag{70}$$

In the second integral on the right-hand side of (70), making the change of variable $s = \frac{1}{t}$ we have

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int\limits_{-1}^{1} \frac{1}{X^+(t)(1+i\pi A_0(t))}\left(\frac{1}{t-z} - \frac{1}{1-zt}\right)g(t)dt. \tag{71}$$

Formula (71) gives one of the particular solutions of Equation (69). Now, let us find the general solution of the corresponding homogeneous Equation (69) (with $h(x) \equiv 0$).

To do this, consider the homogeneous equation

$$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = 0,\ -\infty < x < +\infty.$$

It follows that the function $\chi(z) = \frac{\Phi(z)}{X(z)}$ is holomorphic in the entire complex plane except for the points $z = -1,\ z = c$ which can only be poles. By the well-known generalized Liouville theorem on analytic continuation, it follows that $\chi(z)$ is a rational function of the form

$$\chi(z) = \frac{c_0}{1+z} + \frac{c_1}{c-z}. \tag{72}$$

Thus, due to (71) and (72), the general solution of the Riemann problem has the form

$$\Phi(z) = \frac{X(z)}{2\pi i}\int\limits_{-1}^{c} \frac{g(t)}{X^+(t)(1+i\pi A_0(t))}\left(\frac{1}{t-z} - \frac{1}{1-zt}\right)dt + \frac{c_0X(z)}{1+z} + \frac{c_1X(z)}{c-z}.$$

Hence, by virtue of the Sokhotsky–Plemel Formula (55), we find that

$$\begin{aligned}\rho(x) = \frac{g(x)}{2(1+i\pi A_0(x))}\left(1+\frac{X^-(x)}{X^+(x)}\right) + X^+(x)\left(1-\frac{X^-(x)}{X^+(x)}\right)\times\\ \times\frac{1}{2\pi i}\int\limits_{-1}^{c} \frac{g(t)}{X^+(t)(1+i\pi A(t))}\left(\frac{1}{t-x} - \frac{1}{1-xt}\right)dt + X^+(x)\left(1-\frac{X^-(x)}{X^+(x)}\right)\left(\frac{c_0}{1+x} + \frac{c_1}{c-x}\right).\end{aligned} \tag{73}$$

Here, taking into account the equalities

$$1+\frac{X^-(x)}{X^+(x)}=1+\frac{1}{G(x)}=1+\frac{1+i\pi A_0(x)}{1-i\pi A_0(x)}=\frac{2}{1-i\pi A_0(x)},$$

$$1-\frac{X^-(x)}{X^+(x)}=1-\frac{1}{G(x)}=1-\frac{1+i\pi A_0(x)}{1-i\pi A_0(x)}=\frac{-2i\pi A_0(x)}{1-i\pi A_0(x)},$$

and (67) and (68) equality (73) is written as

$$\rho(x)=\frac{g(x)}{1+\pi^2A_0^2(x)}-\frac{A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{R_0(x)}{R_0(t)}\frac{1+i\pi A_0(x)}{1+i\pi A_0(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)g(t)dt+ \\ +2i\sin(\alpha_0\pi)R_0(x)\left(\frac{c_0}{1+x}+\frac{c_1}{c-x}\right).$$

Hence, considering that $R_0(x)=\frac{(c-x)^{\alpha_1}(1-cx)^{\alpha_1}}{(1+x)^{2\alpha_0}}W(x)$, we have

$$\rho(x)=\frac{g(x)}{1+\pi^2A_0^2(x)}-\frac{A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\left(\frac{1+t}{1+x}\right)^{2\alpha_0}\frac{\omega(x)}{\omega(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)g(t)dt+ \\ +2i\sin(\alpha_0\pi)\frac{(c-x)^{\alpha_1}(1-cx)^{\alpha_1}}{(1+x)^{2\alpha_0}}W(x)\left(\frac{c_0}{1+x}+\frac{c_1}{c-x}\right),\ x\in(-1,c), \tag{74}$$

where $\omega(x)=(1+i\pi A_0(x))W(x)$. Since the solution $\rho(x)$ is sought in the class of Hölder functions $H(-1,c)$ that are bounded at point $x=c$ and may tend to infinity of order less than one, then in (74), it is necessary to set $c_0=0,\ c_1>0$. Now taking into account that

$$\rho(x)=(1+x)^{2\beta-1}\tau(x),\ g(x)=(1+x)^{2\beta-1}g_0(x),$$

from the solution (74) we obtain Formula (53). Theorem 3 is proved. □

4.3.3. Derivation and Analysis of the Wiener–Hopf Integral Equation

Let us continue the investigation of solution (53).

Note that in relation (52) $g_0(x)$ is linear dependent from unknownn function $\tau(x)$. The expression for $g_0(x)$ from (52) substituting in (53), we obtain

$$\tau(x)-\frac{1}{1+\pi^2A_0^2(x)}\left[-kA_0(x)\int\limits_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}\frac{\mu_1(s)\tau(s)ds}{p(s)-x}+R_1[\tau]+F_2(x)\right]- \\ -\frac{A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\frac{\omega(t)}{\omega(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\times \\ \times\left[-kA_0(t)\int\limits_{-1}^{c}\left(\frac{1+t}{1+p(s)}\right)^{1-2\beta}\frac{\mu_1(s)\tau(s)ds}{p(s)-t}+R_1[\tau]+F_2(t)\right]dt, \tag{75}$$

where $\beta_1=1-2\beta-2\alpha_0$.

By extracting the characteristic part in Equation (75), we transform it into the form

$$\tau(x)=-\frac{k\mu_1(c)A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{\tau(s)ds}{p(s)-x}+\frac{kA_0(c)\mu_1(c)A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\tau(s)ds\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\times \\ \times\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\frac{ds}{p(s)-t}+R_2[\tau(x)]+F_3(x),\ x\in(-1,c), \tag{76}$$

where

$$R_2[\tau(x)] = \frac{R_1[\tau(x)]}{1+\pi^2A_0^2(x)} - \frac{k\mu_1(c)A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\left[\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}\mu_1(s)-\mu_1(c)\right]\frac{\tau(s)ds}{p(s)-x}+$$
$$+\frac{kA_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\tau(s)ds\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\times$$
$$\times\left[A_0(t)\mu_1(s)\left(\frac{1+t}{1+p(s)}\right)^{1-2\beta}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\frac{\omega(x)}{\omega(t)}-A_0(c)\mu_1(c)\right]\frac{1}{p(s)-t}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt-$$
$$-\frac{A_0(x)}{1+\pi^2A_0^2(x)})\int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\frac{\omega(x)}{\omega(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)R[\tau(x)]dt$$

is regular operator, and

$$F_3(x) = \frac{F_2(x)}{1+\pi^2A_0^2(x)} - \frac{A_0(x)}{1+\pi^2A_0^2(x)})\int\limits_{-1}^{1}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1-cx}{1-ct}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\frac{\omega(x)}{\omega(t)}\times\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)F_2(t)dt$$

is known function. Let us calculate inner integral in (76).

$$A(x,s) = \int\limits_{-1}^{c}\left(\frac{c-x}{c-t}\right)^{\alpha_1}\left(\frac{1+x}{1+t}\right)^{\beta_1}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\frac{ds}{p(s)-t}. \tag{77}$$

The rational factor of the integrand is decomposed into partial fractions

$$\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\frac{1}{d-ks-t} = \frac{1}{d-ks-x}\left(\frac{1}{t-x}-\frac{1}{d-ks-t}\right)+\frac{1}{1-dx+kxs}\left(\frac{x}{1-xt}-\frac{1}{d-kx-t}\right).$$

By virtue of the latter decomposition, equality (77) is written as

$$A(x,s) = \frac{(c-x)^{\alpha_1}(1+x)^{\beta_1}}{d-ks-x}(T_1(x)+T_2(s))+\frac{(c-x)^{\alpha_1}(1+x)^{\beta_1}}{1-dx+kxs}(xT_3(x)-T_2(s)), \tag{78}$$

where

$$T_1(x) = \int\limits_{-1}^{c}\frac{dt}{(c-t)^{\alpha_1}(1+t)^{\beta_1}(t-x)},\ x\in(-1,c),$$
$$T_2(s) = \int\limits_{-1}^{c}\frac{dt}{(c-t)^{\alpha_1}(1+t)^{\beta_1}(d-ks-t)},\ x\in(-1,c),$$
$$T_3(x) = \int\limits_{-1}^{c}\frac{dt}{(c-t)^{\alpha_1}(1+t)^{\beta_1}(1-xt)},\ x\in(-1,c).$$

We calculate the improper integrals $T_1(x),\ T_2(s)$, and $T_3(x)$.

Let us compute $T_1(x)$:

$$T_1(x) = \lim_{\varepsilon\to0}\left[-\int\limits_{-1}^{x}\frac{dt}{(c-t)^{\alpha_1}(1+t)^{\beta_1}(x-t)^{1-\varepsilon}}+\int\limits_{x}^{c}\frac{dt}{(c-t)^{\alpha_1}(1+t)^{\beta_1}(t-x)^{1-\varepsilon}}\right]. \tag{79}$$

In the first and second integrals on the right-hand side of (79), performing a change of variables, respectively, $t=-1+(1+x)\sigma$ and $t=c-(c-x)\sigma$, we get

$$T_1(x) = \lim_{\varepsilon\to0}\left[-\frac{(1+c)^{-\alpha_1}}{(1+x)^{\beta_1-\varepsilon}}\frac{\Gamma(1-\beta_1)\Gamma(\varepsilon)}{\Gamma(1-\beta_1+\varepsilon)}\cdot F\left(1-\beta_1,\alpha_1,1-\beta_1+\varepsilon;\frac{1+x}{1+c}\right)+\right.$$
$$\left.+\frac{(1+c)^{-\beta_1}}{(c-x)^{\alpha_1-\varepsilon}}\frac{\Gamma(1-\alpha_1)\Gamma(\varepsilon)}{\Gamma(1-\alpha_1+\varepsilon)}F\left(1-\alpha_1,\beta_1,1-\alpha_1+\varepsilon;\frac{c-x}{1+c}\right)\right]. \tag{80}$$

To both hypergeometric functions on the right-hand side of (80), applying the autotransformation Formula (39), we get

$$T_1(x)=\lim_{\varepsilon\to 0}\left[-\frac{(1+c)^{-\alpha_1}}{(1+x)^{\beta_1-\varepsilon}}\frac{\Gamma(1-\beta_1)\Gamma(\varepsilon)}{\Gamma(1-\beta_1+\varepsilon)}\left(\frac{c-x}{1+c}\right)^{\varepsilon-\alpha_1}F\left(\varepsilon,1-\beta_1-\alpha_1+\varepsilon,1-\beta_1+\varepsilon;\frac{1+x}{1+c}\right)+\right.$$
$$\left.+\frac{(1+c)^{-\beta_1}}{(c-x)^{\alpha_1-\varepsilon}}\frac{\Gamma(1-\alpha_1)\Gamma(\varepsilon)}{\Gamma(1-\alpha_1+\varepsilon)}\left(\frac{1+x}{1+c}\right)^{\varepsilon-\beta_1}F\left(\varepsilon,1-\alpha_1-\beta_1+\varepsilon,1-\alpha_1+\varepsilon;\frac{c-x}{1+c}\right)\right]. \tag{81}$$

Applying the Boltz formula [10] (p. 11)

$$F(a,b,c;z)=\tfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)+\tfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\times$$
$$\times F(c-a,c-b,c-a-b-1;1-z),\quad c-a-b\neq 0,\pm1,\pm2,\dots$$

to the first term of (81), we transform (81) into the form

$$T_1(x)=\lim_{\varepsilon\to 0}\left\{-\frac{(1+c)^{-\alpha_1}}{(1+x)^{\beta_1-\varepsilon}}\frac{\Gamma(1-\beta_1)\Gamma(\varepsilon)}{\Gamma(1-\beta_1+\varepsilon)}\left(\frac{c-x}{1+c}\right)^{\varepsilon-\alpha_1}\left[\frac{\Gamma(1-\beta_1+\varepsilon)\Gamma(\alpha_1-\varepsilon)}{\Gamma(1-\beta_1)\Gamma(\alpha_1)}\times\right.\right.$$
$$\times F\left(\varepsilon,1-\beta_1-\alpha_1+\varepsilon,1-\alpha_1+\varepsilon;\frac{c-x}{1+c}\right)+\frac{\Gamma(1-\beta_1+\varepsilon)\Gamma(-\alpha_1+\varepsilon)}{\Gamma(\varepsilon)\Gamma(1-\beta_1-\alpha_1+\varepsilon)}\left(\frac{c-x}{1+c}\right)^{\alpha_1-\varepsilon}\times$$
$$\left.\times F\left(1-\beta_1,\alpha_1,1+\alpha_1-\varepsilon;\frac{c-x}{1+c}\right)\right]+\frac{(1+c)^{-\beta_1}}{(c-x)^{\alpha_1-\varepsilon}}\frac{\Gamma(1-\alpha_1)\Gamma(\varepsilon)}{\Gamma(1-\alpha_1+\varepsilon)}\left(\frac{1+x}{1+c}\right)^{\varepsilon-\beta_1}\times$$
$$\left.\times F\left(\varepsilon,1-\alpha_1-\beta_1+\varepsilon,1-\alpha_1+\varepsilon;\frac{c-x}{1+c}\right)\right\}$$

or

$$T_1(x)=\lim_{\varepsilon\to 0}\left\{-\frac{(1+c)^{-\varepsilon}(c-x)^{\varepsilon-\alpha_1}}{(1+x)^{\beta_1-\varepsilon}}F\left(\varepsilon,1-\alpha_1-\beta_1+\varepsilon,1-\alpha_1+\varepsilon;\frac{c-x}{1+c}\right)-\right.$$
$$-\Gamma(\varepsilon)\left(\frac{\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_1+\varepsilon)}-\frac{\Gamma(\alpha_1-\varepsilon)}{\Gamma(\alpha_1)}\right)-\frac{(1+c)^{-\alpha_1}}{(1+x)^{\beta_1-\varepsilon}}\frac{\Gamma(\varepsilon-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(1-\alpha_1-\beta_1+\varepsilon)}\times \tag{82}$$
$$\left.\times F\left(1-\beta_1,\alpha_1,1+\alpha_1-\varepsilon;\frac{c-x}{1+c}\right)\right\},\ x\in(-1,c).$$

Applying equality $\Gamma(\alpha_1)\Gamma(1-\alpha_1)=\pi/(\sin\alpha_1\pi)$, it is easy to prove that

$$\lim_{\varepsilon\to 0}\Gamma(\varepsilon)\left(\frac{\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_1+\varepsilon)}-\frac{\Gamma(\alpha_1-\varepsilon)}{\Gamma(\alpha_1)}\right)=-\pi ctg(\alpha_1\pi). \tag{83}$$

Now in (82) passing to limit for $\varepsilon\to 0$, taking into account (83), we obtain

$$T_1(x)=-\frac{\pi ctg(\alpha_1\pi)}{(c-x)^{\alpha_1}(1+x)^{\beta_1}}-\frac{(1+c)^{-\alpha_1}}{(1+x)^{\beta_1}}\frac{\Gamma(-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(1-\alpha_1-\beta_1)}F\left(1-\beta_1,\alpha_1,1+\alpha_1;\frac{c-x}{1+c}\right). \tag{84}$$

In the integrals $T_2(s)$, $T_3(x)$ after making a change of the integration variable, respectively, $t=-1+(1+c)\sigma$ and $t=c-(1+c)\sigma$, and then using the integral representation of the hypergeometric function along with the autotransformation formula, we deduce

$$T_2(s)=\frac{(1+c)^{1-\alpha_1-\beta_1}\Gamma(1-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(2-\alpha_1-\beta_1)(d-ks+1)^{1-\alpha_1}(d-ks-c)^{\alpha_1}}F\left(1-\alpha_1,1-\alpha_1-\beta_1,2-\alpha_1-\beta_1;\frac{1+c}{1+d-ks}\right), \tag{85}$$

$$T_3(x)=\frac{(1+c)^{1-\alpha_1-\beta_1}\Gamma(1-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(2-\alpha_1-\beta_1)(1-xc)^{1-\beta_1}(1+x)^{\beta_1}}F\left(1-\beta_1,1-\alpha_1-\beta_1,2-\alpha_1-\beta_1;\frac{(1+c)x}{xc-1}\right). \tag{86}$$

Now, expressions for $T_1(x),\ T_2(s),\ T_3(x)$ from (84), (85), and (86) substituting in (78) we obtain

$$\begin{gathered} A(x,s)=\frac{(c-x)^{\alpha_1}(1+x)^{\beta_1}}{d-ks-x}\left[-\frac{\pi ctg(\alpha_1\pi)}{(c-x)^{\alpha_1}(1+x)^{\beta_1}}+\frac{\Gamma(1-\alpha_1)\Gamma(1-\beta_1)(1+c)^{-\alpha_1}}{\alpha_1\Gamma(1-\alpha_1-\beta_1)(1+x)^{\beta_1}}\times\right. \\ \times F\big(1-\beta_1,\alpha_1,1+\alpha_1;\tfrac{c-x}{1+c}\big)+\frac{(1+c)^{1-\alpha_1-\beta_1}\Gamma(1-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(2-\alpha_1-\beta_1)(1+p(s))^{1-\alpha_1}(p(s)-c)^{\alpha_1}}\times \\ \left.\times F\Big(1-\alpha_1,1-\alpha_1-\beta_1,2-\alpha_1-\beta_1;\tfrac{1+c}{1+p(s)}\Big)\right]+\frac{(c-x)^{\alpha_1}(1+x)^{\beta_1}}{1-xp(s)}(xT_3(x)-T_2(s)). \end{gathered} \tag{87}$$

Taking into account the equalities $p(c)=c,\ F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, we have

$$\begin{gathered} \Big(\tfrac{1+c}{1+p(s)}\Big)^{1-\alpha_1}\Big(\tfrac{1+x}{1+c}\Big)^{\beta_1}\frac{\Gamma(1-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(2-\alpha_1-\beta_1)}\times \\ \times F\Big(1-\alpha_1,\ 1-\alpha_1-\beta_1,\ 2-\alpha_1-\beta_1;\ \tfrac{1+c}{1+p(s)}\Big)\Big|_{s=c}=\frac{\pi}{sin(\alpha_1\pi)}. \end{gathered} \tag{88}$$

Due to (88) from (87) isolating terms with a multiplier $\frac{1}{d-ks-x}$, we have

$$A(x,s)=\frac{1}{d-ks-x}\left[-\pi ctg(\alpha_1\pi)+\frac{\pi}{k^{\alpha_1}sin(\alpha_1\pi)}\left(\frac{c-x}{c-s}\right)^{\alpha_1}\right]+B(x,s), \tag{89}$$

where

$$\begin{gathered} B(x,s)=\tfrac{1}{d-ks-x}\Big\{\tfrac{\Gamma(1-\alpha_1)\Gamma(1-\beta_1)(c-x)^{\alpha_1}}{\alpha_1\Gamma(1-\alpha_1-\beta_1)}F\big(1-\beta_1,\alpha_1,1+\alpha_1;\tfrac{c-x}{1+c}\big)+ \\ +\Big[\tfrac{\Gamma(1-\alpha_1)\Gamma(1-\beta_1)}{\Gamma(2-\alpha_1-\beta_1)}\Big(\tfrac{1+c}{1+p(s)}\Big)^{1-\alpha_1}\Big(\tfrac{1+x}{1+c}\Big)^{\beta}F\Big(1-\alpha_1,1-\alpha_1-\beta_1,2-\alpha_1-\beta_1;\tfrac{1+c}{1+p(s)}\Big)- \\ -\tfrac{\pi}{\sin(\alpha_1\pi)}\Big]\cdot\Big(\tfrac{c-x}{p(s)-c}\Big)^{\alpha}\Big\}+\tfrac{(c-x)^{\alpha_1}(1+x)^{\beta_1}}{1-xp(s)}(xT_3(x)-T_2(s)). \end{gathered}$$

Here the square bracket at the point $(x,s)=(c,c)$ is equal to zero, and so $B(x,s)$ is regular kernel.

Due to (89), the Equation (76) is written in the form

$$\begin{gathered} \tau(x)=-\frac{kA_0(x)\mu_1(c)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{\tau(s)ds}{p(s)-x}+\frac{kA_0(c)\mu_1(c)A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{\tau(s)ds}{d-ks-x}\times \\ \times\left(-\pi ctg(\alpha_1\pi)+\frac{\pi}{k^{\alpha_1}\sin(\alpha_1\pi)}\left(\frac{c-x}{c-s}\right)^{\alpha_1}+B(x,s)\right)ds. \end{gathered}$$

The last equation is written as

$$\begin{gathered} \tau(x)=-\frac{kA_0(x)\mu_1(c)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{1}{d-ks-x}\left(1+A_0(c)\pi ctg(\alpha_1\pi)-\frac{\pi}{k^{\alpha_1}\sin(\alpha_1\pi)}\left(\frac{c-x}{c-s}\right)^{\alpha_1}\right)\tau(s)ds= \\ =R_3[\tau(x)]+F_3(x),\ x\in(-1,c), \end{gathered} \tag{90}$$

where

$$R_3[\tau(x)]=R_2[\tau(x)]+\frac{kA_0(c)A_0(x)}{1+\pi^2A_0^2(x)}\int\limits_{-1}^{c}\frac{B(x,s)\tau(s)ds}{d-ks-x}.$$

Due to the equality $B(c,c)=0,\ R_3[\tau(x)]$ is regular operator.

Here taking into account equalities $\alpha_1=-\frac{1}{\pi}arctg(\pi A_0(c))$, we have

$$1+A_0(c)\pi ctg(\alpha_1\pi)=0. \tag{91}$$

Due to equality (91), the Equation (90) is written in the form

$$\tau(x) = \frac{k^{1-\alpha_1}\pi A_0(x)\mu_1(c)}{\sin(\alpha_1\pi)\left(1+\pi^2A_0^2(x)\right)}\int\limits_{-1}^{c}\left(\frac{c-x}{c-s}\right)^{\alpha_1}\frac{\tau(s)ds}{d-ks-x} + R_2[\tau(x)] + F_3(x).$$

In the last equation, focusing on the characteristic part of the equation, we have

$$\tau(x) = \frac{k^{1-\alpha_1}\pi A_0(c)\mu_1(c)}{\sin(\alpha_1\pi)\left(1+\pi^2A_0^2(c)\right)}\int\limits_{-1}^{c}\left(\frac{c-x}{c-s}\right)^{\alpha_1}\frac{\tau(s)ds}{d-ks-x} + R_4[\tau(x)] + F_3(x),\ x\in(-1,c), \tag{92}$$

where

$$R_4[\tau(x)] = R_3[\tau(x)] + \frac{k^{1-\alpha_1}\pi}{\sin(\alpha_1\pi)}\left(\frac{A_0(x)}{1+\pi^2A_0^2(x)} - \frac{A_0(c)}{1+\pi^2A_0^2(c)}\right)\int\limits_{-1}^{c}\left(\frac{c-x}{c-s}\right)^{\alpha_1}\frac{\tau(s)ds}{d-ks-x}$$

is regular operator.

In Equation (92) making replacement $x = c-(1+c)e^{-y},\ s = c-(1+c)e^{-t}$ taking into account the identity $d-ks-x = k(c-s)+c-x$, Equation (92) is written as

$$\begin{aligned}\tau(c-(1+c)e^{-y})e^{(\alpha-\frac{1}{2})y} &= \frac{k^{1-\alpha_1}\pi A_0(c)\mu_1(c)}{\sin(\alpha_1\pi)\left(1+\pi^2A_0^2(c)\right)}\int\limits_{0}^{+\infty}\frac{e^{(\alpha-\frac{1}{2})t}\tau(c-(1+c)e^{-y})dt}{ke^{\frac{y-t}{2}}+e^{-\frac{y-t}{2}}}\\ &= e^{(\alpha-\frac{1}{2})y}R_5[\rho(y)] + e^{(\alpha-\frac{1}{2})y}F_4(y).\end{aligned} \tag{93}$$

Introducing notations

$$\rho(y) = \tau(c-(1+c)e^{-y})e^{(\alpha_1-\frac{1}{2})y},\ \ R_6[\rho(y)] = e^{(\alpha_1-\frac{1}{2})y}R_5[\rho(y)],\ \ F_5(y) = e^{(\alpha_1-\frac{1}{2})y}F_4(y),$$
$$K(x) = \tfrac{1}{ke^{x/2}+e^{-x/2}},$$

we write Equation (93) in the form

$$\rho(y) = \lambda_0\int\limits_{0}^{+\infty}K(y-t)\rho(t)dt = R_6[\rho(y)] + F_5(y), \tag{94}$$

where $\lambda_0 = \frac{k^{1-\alpha_1}\pi A_0(c)\mu_1(c)}{\sin(\alpha_1\pi)\left(1+\pi^2A_0^2(c)\right)}$. Function $K(x)$ and its derivative $K'(x)$ are continuous and have exponential order of decay at infinity, and therefore $K(x)\in L_2\cap H_\alpha = \{0\}$ [64] (p. 12).

Equation (94) is a Wiener–Hopf integral equation [64] (p. 55), this equation, by means of the Fourier transform, similarly to the well-known characteristic singular integral equation with a Cauchy kernel is reduced to a Riemann boundary value problem and thus is solved in quadratures.

If the singularity of the Cauchy kernel $\frac{1}{t-x}$ lies in the discontinuity at $t=x$, then the singularity of the kernel $K(x-t)$ of the convolution-type integral equation is caused by the unboundedness of the integration interval. This kernel, unlike a regular one, does not decrease as one moves to infinity along the straight line $t = x+c$, where $c = const$.

Fredholm's theorems for convolution-type integral equations holds only in the special case when the index of such equations is zero. The index χ of Equation (94) will be the index of the expression

$$1-\lambda_0K^{\wedge}(x),\ \ -\infty < x < +\infty$$

taken with the opposite sign: $\chi \equiv -Ind(1-\lambda_0 K^{\wedge}(x))$ [65] (p. 56), where

$$K^{\wedge}(x) = \int\limits_{-\infty}^{+\infty} \frac{e^{ixt}dt}{ke^{t/2}+e^{-t/2}} = \frac{\pi e^{ix\ln k}}{\sqrt{k}ch(\pi x)}.$$

Now let us compute the index of the expression:

$$1-\lambda_0 K^{\wedge}(x).$$

Let

$$\left|\left(\frac{1-c}{1+c}\right)^{\frac{1}{2}-\alpha_1} \frac{\pi A_0(c)\mu(c)}{a(c)\sin(\alpha_1\pi)(1+\pi^2 A_0 2(c))}\right| < 1, \qquad \left(\left|\frac{\lambda_0\pi}{\sqrt{k}}\right| < 1\right), \tag{95}$$

where

$$\alpha_1 = -\frac{1}{\pi} arctg(\pi A_0(c)), A_0(c) = \frac{k_2(1-\delta_0)(1-\beta)(l+2)b(c)}{2\gamma\Gamma(1-2\beta)a(c) - k_2(1-\delta_0)(1-\beta)(l+2)tg(\beta\pi)b(c)},$$

$$2\gamma\Gamma(1-2\beta)a(c) - k_2(1-\delta_0)(1-\beta)(l+2)tg(\beta\pi)b(c) \neq 0.$$

Then

$$\left|Re(\lambda_0 K^{\wedge}(x))\right| = \left|\lambda_0 \frac{\pi\cos(x\ln k)}{\sqrt{k}(\pi x)}\right| = \left|\frac{\lambda_0\pi}{\sqrt{k}}\right| < 1,$$

and $|Re(\lambda_0 K^{\wedge}(x))| \leq o\left(\frac{1}{ch(\pi x)}\right)$ for sufficiently large $|x|$.

Hence, we get

$$Re(1-\lambda_0 K^{\wedge}(x)) > 0.$$

Therefore

$$\chi = -Ind(1-\lambda_0 K^{\wedge}(x)) = -\frac{1}{2\pi}\left[\frac{Im(1-\lambda_0 K^{\wedge}(x))}{Re(1-\lambda_0 K^{\wedge}(x))}\right]_{-\infty}^{+\infty} = -\frac{1}{2\pi}\left[\frac{0}{1}-\frac{0}{1}\right] = 0,$$

i.e., changing the argument $1-\lambda_0 K^{\wedge}(x)$ along the real axis, expressed in full turns is equal to zero [64] (p. 28), from this and from the uniqueness of the solution to problem MA, it follows that Equation (94) is uniquely solvable, and therefore so is problem MA. Thus, the following Theorem is proved.

Theorem 4. *Problem MA under the conditions*

$$a(x),\ a_0(x),\ b(x),\ b_0(x),\ \rho(x),\ \varphi_0(x) \in C[-1,1]\cap C^1(-1,1),$$
$$\psi(x) \in C[-1,(c-1)/2]\cap C^1(-1,(c-1)/2),\ f(x) \in C[-1,c]\cap C^1(-1,c),$$
$$\tfrac{l+2\delta_0}{2(l+2)} = \tfrac{m+2\beta_0}{2(m+2)}, a(x) > 0,\ b(x) > 0,\ \beta = \delta,$$

and inequality (95) is uniquely solvable.

Let us note that the set of values of numerical parameters of problem MA satisfying inequality (95) is non-empty. Indeed, since $k = (1-c)/(1+c)$, then for values of the parameter c sufficiently close to -1, the value of k becomes sufficiently large, and so inequality (95) is satisfied for such values of the parameters.

5. Conclusions

(1) The paper investigates issues of existence and uniqueness of the solution to a problem with local missing Tricomi condition on one boundary characteristic and Frankl-type

condition on the degeneration segment for a certain class of mixed elliptic-hyperbolic type equations with various orders of degeneration and singular coefficients.

(2) The considered problem differs from known problems in that, on one part of the boundary characteristic, the values of the sought function and Frankl-type condition are specified at different ends of the cut along the degeneration segment of the equation. On the segment of the degeneration line, a general conjugation condition is specified, and in the elliptic region of the equation, the Bitsadze–Samarskii condition is imposed, which pointwise connects the boundary of the ellipticity region with the lines of degeneration. At the same time, a portion of the hyperbolic region's boundary is freed from boundary conditions.

(3) The uniqueness of the solution to the formulated problem is proved using an analog of A.V. Bitsadze's extremum principle for mixed-type equations with singular coefficients.

(4) The proof of the existence of a solution to the problem, depending on the numerical values of the ratio of the orders of degeneracy and the numerical parameters in the lower-order terms of the equations, is reduced either to solving Fredholm integral equation with a polar singularity or to non-standard singular Tricomi-type integral equation with a non-Fredholm operator, whose kernel contains an isolated singularity of the first order at only one point. Using the Carleman–Vekua regularization method, the resulting singular integral equation with a non-Fredholm operator is reduced to a Wiener–Hopf integral equation. It is proven that the index of the Wiener–Hopf equation is zero. Therefore, the Wiener–Hopf equation is reduced to a second-kind Fredholm integral equation, the unique solvability of which follows from the uniqueness of the solution to problem MA.

(5) An algorithm has been developed for solving non-standard singular Tricomi-type integral equations with a non-Fredholm operator whose kernel contains an isolated first-order singularity.

(6) It has been established that the well-posedness of the formulated problem, posed on a part of the characteristic and involving Frankl-type conditions on the degeneration line, essentially depends on the relationship between the coefficients of the given conditions at the junction point of the local condition and the Frankl-type condition, which lies on the degeneration line of the equation.

In conclusion, it should be noted that the developed methods for studying non-standard singular integral equations can be applied to the investigation of a broader class of partial differential equations with singular coefficients, including for other values of the parameter β_0 in Equation (1).

In conclusion, we note the works [65–70] are closely related to the topics considered in this paper and were published in recent years.

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Abbreviations

The following abbreviations are used in this manuscript:

GPV	Greatest positive value
LNV	Least negative value

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