mathematics

MDPI

Article

Sharp Coefficient Bounds for Analytic Functions Related to Bounded Turning Functions

Sudhansu Palei [1], Madan Mohan Soren [1,*], Luminiţa-Ioana Cotîrlă [2,*] and Daniel Breaz [3,*]

[1] Department of Mathematics, Berhampur University, Berhampur 760007, Odisha, India; sp.rs.math@buodisha.edu.in
[2] Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
[3] Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania
* Correspondence: mms.math@buodisha.edu.in (M.M.S.); luminita.cotirla@math.utcluj.ro (L.-I.C.); dbreaz@uab.ro (D.B.)

Abstract: Let $\mathfrak{B}$ denote the class of bounded turning functions $\mathcal{F}$ analytic in the open unit disk, where the image of $\mathcal{F}'(z)$ is contained in the domain $\Omega(z) = \cosh z + \frac{2z}{2-z^2}$. This article determines sharp coefficient bounds, a Fekete–Szegö-type inequality, and second- and third-order Hankel determinants for functions in the class $\mathfrak{B}$. Additionally, we obtain sharp Krushkal and Zalcman functional-type inequalities related to the logarithmic coefficient for functions belonging to $\mathfrak{B}$.

Keywords: analytic functions; bounded turning functions; coefficient bounds; Fekete–Szegö-type inequality; Hankel determinant; Krushkal and Zalcman functional

MSC: 30C45; 30C80

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of normalized analytic functions in the open unit disk

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$$

with the series expansion:

$$\mathcal{F}(z) := z + \sum_{j=2}^{\infty} b_j z^j. \tag{1}$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{E}$. For the functions $\mathcal{F}$ and $\mathcal{G}$ that are members of $\mathcal{A}$, we say that the function $\mathcal{F}$ is subordinate to $\mathcal{G}$ and write $\mathcal{F} \prec \mathcal{G}$ if there is a function $k(z)$, analytic in $\mathbb{E}$ with $k(0) = 0$ and $|k(z)| < 1$, such that $\mathcal{F}(z) = \mathcal{G}(k(z))$. In particular, if the function $\mathcal{G}$ is univalent, then $\mathcal{F} \prec \mathcal{G}$ if and only if $\mathcal{F}(0) = \mathcal{G}(0)$ and $\mathcal{F}(\mathbb{E}) \subset \mathcal{G}(\mathbb{E})$ (see [1]).

Among the various subclasses of $\mathcal{S}$, the classes of starlike, convex, and bounded turning functions have garnered the most attention. Ma and Minda [2] introduced several classes of convex and starlike functions, including

$$\mathcal{C}(\phi) := \left\{\mathcal{F} \in \mathcal{A} : 1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \prec \phi(z)\right\},$$
$$\mathcal{S}^*(\phi) := \left\{\mathcal{F} \in \mathcal{A} : \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \phi(z)\right\}$$
$$\text{and } \mathcal{R}(\phi) := \left\{\mathcal{F} \in \mathcal{A} : \mathcal{F}'(z) \prec \phi(z)\right\}.$$

check for updates

Academic Editor: Anna Kamińska

Received: 27 April 2025
Revised: 27 May 2025
Accepted: 30 May 2025
Published: 1 June 2025

Citation: Palei, S.; Soren, M.M.; Cotîrlă, L-I.; Breaz, D. Sharp Coefficient Bounds for Analytic Functions Related to Bounded Turning Functions. *Mathematics* **2025**, *13*, 1845. https://doi.org/10.3390/math13111845

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Also, they have investigated a more general analytic function ϕ, with the positive real part in $\mathbb{E}$, that maps the unit disc $\mathbb{E}$ onto starlike regions concerning 1, which are symmetric for the real axis and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. To obtain a suitable choice of image domains of ϕ, one can obtain the subfamilies of $\mathcal{C}$ and $\mathcal{S}^*$. By taking $\phi(z) = \sqrt{1+z}$, the class $\mathcal{S}^*$ that reduces, respectively, to the class $\mathcal{S}_L^* = \mathcal{S}^*(\sqrt{1+z})$ is related to the right half of the lemniscate of Bernoulli and was studied by Sokol and Stankiewicz [3]. By taking $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*$ reduces, respectively, to the class $\mathcal{S}^*[A, B]$, whereby the class of Janowski starlike and convex functions was studied by Janowski [4]. In 2016, Sharma et al. [5] considered the class of starlike functions bounded by the cardioid domain for $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$. In 2021, Wani and Swaminathan [6] studied the class of starlike and convex functions bounded by the nephroid domain. For the function $\phi(z)$, which maps the unit disk onto a cardioid domain, the class $\mathcal{S}^*(\phi)$ reduces to class $\mathcal{S}_C^*$, respectively, which was investigated by Sivaprasad Kumar and Kamaljeet [7]. In 2014, Mendiratta et al. [8] studied the subclass of $\mathcal{S}^*$ related to the interior of the left-half of the shifted lemniscate of Bernoulli, such as

$$\mathcal{S}_{RL}^* = \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2}-1)\left(\frac{1-z}{1+2(\sqrt{2}-1)z}\right)^{\frac{1}{2}}\right).$$

Similarly, in 2015, Mendiratta et al. [9] investigated the subclass of $\mathcal{S}^*$ associated with the exponential function, such as $\mathcal{S}_e^* = \mathcal{S}^*(e^z)$. Recently, many subclasses of $\mathcal{S}^*$ are extensively investigated in the literature (see [10–12]), such as the following:

1. $\mathcal{S}_M^* = \mathcal{S}^*((1+z)/(1-(M-1)/M)z))$ $(M > \frac{1}{2})$;
2. $\mathcal{M}(\beta) = \mathcal{S}^*((1+(1-2\beta)z)/(1-z))$ $(\beta > 1)$;
3. $\mathcal{S}^*[A, B] = \mathcal{S}^*\left[\frac{1+Az}{1+Bz}\right]$ $(-1 \leq B < A \leq 1)$ (see [13]).

Recently, in 2021, Sivaprasad Kumar and Kamaljeet [7] introduced and studied a class of starlike functions $\mathcal{S}^*$, where the image domain of $\phi(z)$ is bounded by a cardiod (see also [14]).

For this purpose, we consider the analytic function $\phi : \mathbb{E} \to \mathbb{C}$ satisfying the following:

1. $\phi(z)$ is univalent with $\Re(\phi) > 0$;
2. $\phi(\mathbb{E})$ is starlike with respect to $\phi(0) = 1$;
3. $\phi(\mathbb{E})$ is symmetric about the real axis;
4. $\phi'(0) > 0$.

Let $\Omega(z) : \mathbb{E} \to \mathbb{C}$ be defined by

$$\Omega(z) = \cosh z + \frac{2z}{2-z^2},$$

where $\Omega(z)$ satisfies all the aforementioned properties (Figure 1).

Remark 1. *The function $\Omega(z)$ is analytic in $\mathbb{E}$, and can be written as*

$$\Omega(z) = \cosh z + \frac{2z}{2-z^2} = 1 + z + \frac{z^2}{2} + \frac{z^3}{2} + \frac{z^4}{4!} + \cdots.$$

This power series with real and positive coefficients converges absolutely in $\mathbb{E}$. $\Re(\Omega(z))$ is harmonic on $\mathbb{E}$ since it is the real part of an analytic function. Moreover, $\Re(\Omega(0)) = \Omega(0) = 1 > 0$, and $\Omega(z)$ is not constant. By the minimum principle for harmonic functions, since $\Re(\Omega(0))$ is harmonic, positive at a point, and not constant, it follows that

$$\Re(\Omega(z)) > 0, z \in \mathbb{E}.$$

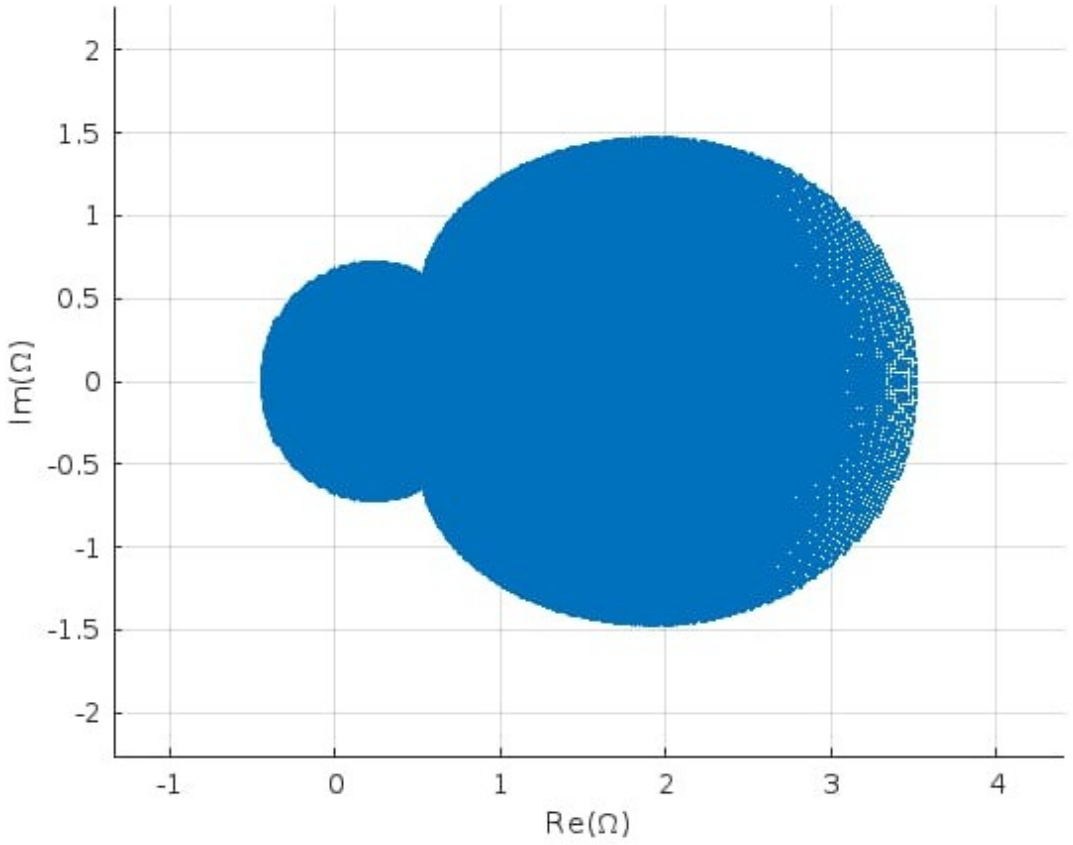


Figure 1. Image of $\Omega(z) = \cosh z + \frac{2z}{2-z^2}$ under a unit disk.

The study of coefficient problems in the fundamental class $\mathcal{S}$ is motivated by the geometric properties of their image domains. In the early 1970s, Lawrence Zalcman proposed a conjecture, now famously known as the Zalcman conjecture, which states that if $\mathcal{F} \in \mathcal{S}$ and is given by

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (n \geq 2)$$

with equality only for the Koebe function $K(z) = z/(1-z)^2$ or its rotations, then the Zalcman conjecture implies the Bieberbach conjecture, $|a_n| \leq n$. Bieberbach proved the Zalcman conjecture for $n = 2$. Recently, numerous researchers have investigated sharp bounds for the Zalcman functional, defined as $|b_2 b_3 - b_4|$, for various subclasses of functions, including convex, starlike, close-to-convex, and bounded turning functions (see [15–17]).

Pommerenke (see [18,19]) considered Hankel determinants $\mathfrak{H}_{k,d}$ of univalent functions, with coefficients of the function $\mathcal{F}$ in subclass $\mathcal{S}$ and introduced for $k, d \in \mathbb{N}$ as follows:

$$\mathfrak{H}_{k,d}(\mathcal{F}) = \begin{vmatrix} b_d & b_{d+1} & \cdots & b_{d+k-1} \\ b_{d+1} & b_{d+2} & \cdots & b_{d+1} \\ b_{d+2} & b_{d+3} & \cdots & b_{d+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d+k-1} & b_{d+k} & \cdots & b_{d+2(k-1)} \end{vmatrix}.$$

Hankel determinants play a crucial role in investigating the coefficients of analytic functions within various classes of functions, with far-reaching implications across multiple mathematical disciplines. For a comprehensive overview of their applications, see [20]. Numerous studies have explored sharp bounds for various function classes. Interested

readers can find a wealth of information in the existing literature and references cited therein (see [21–25]). Notably, we refer the reader to $k = 2$ and $d = 1$, and the functional

$$\mathfrak{H}_{2,1}(\mathcal{F}) = |b_3 - b_2^2|$$

is commonly referred to as the Fekete–Szegö functional. The maximum value of $|\mathfrak{H}_{2,1}(f)|$ was found for a class $\mathcal{S}$ in 1933. Many researchers looked into the maximum value of $|\mathfrak{H}_{2,1}(f)|$ for different subclasses of class $\mathcal{A}$; for more information, see [26–30]. Additionally, for $k = 2$ and $d = 2$, the second Hankel determinant is

$$\mathfrak{H}_{2,2}(\mathcal{F}) = |b_2 b_4 - b_3^2|.$$

Several authors have examined the upper bound of $\mathfrak{H}_{2,2}(\mathcal{F})$. For example, the works of Hayman [31], Noonan and Thomas [32], Janteng et al. [33], and Orhan et al. [34] are worthy of reading. The determinant

$$\mathfrak{H}_{3,1}(\mathcal{F}) = b_5(b_3 - b_2^2) - b_4(b_4 - b_2 b_3) + b_3(b_2 b_4 - b_3^2)$$

is known as the third-order Hankel determinant. The upper bound of $\mathfrak{H}_{3,1}(\mathcal{F})$ for a subclass of $\mathcal{S}$ was initially studied by Babalola [35]. Readers are encouraged to consult the work of various researchers, including Zaprawa [36], Raza et al. [37], Cho et al. [38], Lecko et al. [39], and Srivastava et al. [40], for additional information on this subject.

Motivated by the aforementioned works, we introduce the class of bounded turning functions and derive coefficient bounds for functions in the class $\mathfrak{B}$ that satisfy a given differential subordination implication. Furthermore, we identify extremal functions that render our results sharp for functions $\mathcal{F} \in \mathfrak{B}$. As applications of these findings, we establish the Fekete–Szegö inequality and determine the Zalcman functional.

Let $\mathcal{P}$ be the class analytic function $p \in \mathcal{P}$ with a positive real part and $p(0) = 1$. Thus, every $p \in \mathcal{P}$ can be represented by

$$p(z) := 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2}$$

which is also known as a Carathéodory function. This class of functions play a significant role in geometric function theory, particularly for the bound of Hankel determinants. In 1983, Duren [41] examined its coefficient bounds such as $c_n \leq 2$.

The following lemmas are essential for establishing the main results of this paper.

Lemma 1 (See [42,43]). *If $p \in \mathcal{P}$ has the form (2), then*

$$|c_{n+k} - \mu c_n c_k| \leq 2 \quad (0 < \mu \leq 1), \tag{3}$$

$$|c_n| \leq 2 \text{ for } n \geq 1, \tag{4}$$

$$|c_2 - \varsigma c_1^2| \leq 2 \max\{1, |2\varsigma - 1|\} \quad (\varsigma \in \mathbb{C}), \tag{5}$$

$$|\mathcal{J} c_1^3 - \mathcal{K} c_1 c_2 + \mathcal{L} c_3| \leq 2|\mathcal{J}| + |\mathcal{K} - 2\mathcal{J}| + 2|\mathcal{J} - \mathcal{K} + \mathcal{L}|, \tag{6}$$

where $\mathcal{J}, \mathcal{K}$ and $\mathcal{L}$ are real numbers.

Lemma 2 (See [44]). *If $p \in \mathcal{P}$ has the form (2) with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + \beta(4 - c_1^2), \tag{7}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1\beta - c_1(4 - c_1^2)\beta^2 + 2(4 - c_1^2)(1 - |\beta|^2)\eta, \tag{8}$$

$$\begin{aligned} 8c_4 =& c_1^4 + (4 - c_1^2)\beta[c_1^2(\beta^2 - 3\beta + 3) + 4\beta] - 4(4 - c_1^2)(1 - |\beta|^2) \\ & [c_1(\beta - 1)\eta + \overline{\beta}\eta^2 - (1 - |\eta|^2)k] \end{aligned} \tag{9}$$

for some $\beta, \eta, k \in \overline{\mathbb{E}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 3 (See [45]). *If* $p \in \mathcal{P}$ *has the form* (2) *and* $\vartheta \in \mathbb{C}$, *then*

$$|c_n - \vartheta c_k c_{n-k}| \leq 2\ max\{1, |2\ \vartheta - 1|\} \tag{10}$$

for all $1 \leq k \leq n - 1$.

Lemma 4 (See [46]). *Suppose that* $p \in \mathcal{P}$ *has the form* (2). *If* $\mathfrak{R}(2\mathfrak{M} - 1) \leq \mathfrak{S} \leq \mathfrak{R}$, *then*

$$|c_3 - 2\mathfrak{M}c_1c_2 + \mathfrak{S}c_1^3| \leq 2.$$

Lemma 5 (See [2]). *Suppose* $p \in \mathcal{P}$ *has the form* (2), *then*

$$|c_2 - \varkappa c_1^2| \leq \begin{cases} -4\varkappa + 2, & \text{if } \varkappa < 0 \\ 2, & \text{if } 0 \leq \varkappa \leq 1 \\ 4\varkappa - 2, & \text{if} \varkappa > 1. \end{cases}$$

The organization of this paper is as follows: Section 1 introduces fundamental definitions and lemmas crucial for proving the main results in subsequent sections. We define the class $\mathfrak{B}$ and estimate sharp coefficient bounds for b_2, b_3, b_4 and b_5, as well as sharp bounds for the Fekete–Szegö inequality and second and third Hankel determinants for the functions in $\mathfrak{B}$ in Section 2. In Section 3, we investigate the sharp bounds and logarithmic coefficient inequalities for the class $\mathfrak{B}$. This paper is finally concluded with closing remarks in Section 4.

2. Coefficient Bounds, Fekete–Szegö Inequality and Hankel Determinants

We now consider a subfamily of bounded turning functions defined by

$$\mathfrak{B} := \left\{ \mathcal{F} \in \mathcal{A} : \mathcal{F}'(z) \prec \Omega(z) = \cosh z + \frac{2z}{2 - z^2}, z \in \mathbb{E} \right\}.$$

Firstly, we determine the sharp coefficient bounds for functions in the class $\mathfrak{B}$.

Theorem 1. *Let* $\mathcal{F} \in \mathfrak{B}$. *Then,*

$$\begin{aligned} |b_2| &\leq \frac{1}{2}, \\ |b_3| &\leq \frac{1}{3}, \\ |b_4| &\leq \frac{7}{32}, \\ |b_5| &\leq \frac{2}{5}. \end{aligned}$$

The following functions are used to attain their respective sharp bounds:

$$\mathcal{F}_0(z) = \int_0^z \left(\cosh t + \frac{2t}{2 - t^2} \right) dt = z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots. \tag{11}$$

$$\mathcal{F}_1(z) = \int_0^z \left(\cosh t + \frac{t^2}{2-t^2}\right) dt = z + \frac{z^3}{3} + \frac{7}{120}z^5 + \cdots. \tag{12}$$

$$\mathcal{F}_2(z) = \int_0^z \left(\cosh t + \frac{11t^3}{8-4t^2} - \frac{t^2}{2}\right) dt = z + \frac{11}{32}z^4 + \frac{z^5}{120} + \cdots. \tag{13}$$

$$\mathcal{F}_3(z) = \int_0^z \left(\cosh t + \frac{23t^4}{24-12t^2} - \frac{t^2}{2}\right) dt = z + \frac{z^5}{5} + \frac{691}{720}z^6 + \cdots. \tag{14}$$

Proof. Let $\mathcal{F} \in \mathfrak{B}$. Then,

$$\mathcal{F}'(z) = \Omega(k(z)) = \cosh k(z) + \frac{2k(z)}{2-k(z)^2}, \tag{15}$$

where $k(z)$ is an analytic function with $k(0) = 0$, and $k(z) < 1$ for all $z \in \mathbb{E}$.

Let $p \in \mathcal{P}$. Then, by application of subordination, we have

$$p(z) = \frac{1+k(z)}{1-k(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

Hence, it is clear that

$$\begin{aligned} \kappa(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right)z^3 \\ &+ \frac{1}{2}\left(c_4 - c_1c_3 + \frac{3}{4}c_1^2c_3 - \frac{1}{2}c_2 - \frac{1}{8}c_1^4\right)z^4 \cdots, \quad (z \in \mathbb{E}). \end{aligned} \tag{16}$$

By using (16) in (15), we have

$$\begin{aligned} \Omega(k(z)) = 1 &+ \frac{1}{2}c_1 z + \frac{1}{8}(4c_2 - c_1^2)z^2 + \frac{1}{16}(c_1^3 - 4c_1c_2 + 8c_3)z^3 \\ &+ \frac{1}{384}(-23c_1^4 + 72c_1^2c_2 - 96c_1c_3 - 48c_2^2 + 192c_4)z^4 + \cdots. \end{aligned} \tag{17}$$

Further,

$$\mathcal{F}'(z) = 1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3 + 5b_5 z^4 + \cdots. \tag{18}$$

By comparing the coefficients of (17) and (18), we conclude that

$$b_2 = \frac{c_1}{4}, \tag{19}$$

$$b_3 = \frac{1}{6}\left(c_2 - \frac{c_1^2}{4}\right), \tag{20}$$

$$b_4 = \frac{1}{64}c_1^3 - \frac{1}{16}c_1c_2 + \frac{1}{8}c_3, \tag{21}$$

$$b_5 = \frac{-1}{10}\left(\frac{1}{2}c_1\left(c_3 - \frac{3}{4}c_1c_2 + \frac{23}{96}c_1^3\right) - \left(c_4 - \frac{1}{4}c_2^2\right)\right). \tag{22}$$

Using (4) in (19), we get

$$|b_2| \leq \frac{1}{2}.$$

Using (3) in (20), we get

$$|b_3| \leq \frac{1}{3}.$$

Using (6) in (21), we get

$$|b_4| \leq \frac{7}{32}.$$

Therefore, by using (3), (4) and Lemma 4 in (22), we get

$$|b_5| \leq \frac{2}{5}.$$

Hence, it completes the proof. □

Next, we estimate the Fekete–Szegö inequality and Hankel determinants using the previously established coefficient bounds for b_2, b_3, b_4, and b_5.

Theorem 2. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_3 - \mathcal{M}b_2^2| \leq \begin{cases} \frac{-(2-3\mathcal{M})}{12}, & \text{if } \mathcal{M} < \frac{-2}{3} \\ \frac{1}{3}, & \text{if } \frac{-2}{3} \leq \mathcal{M} \leq 2 \\ \frac{(2-3\mathcal{M})}{12}, & \text{if } \mathcal{M} > 2. \end{cases}$$

The above inequality is sharp.

Proof. Using (19) and (20), we have

$$\begin{aligned} |b_3 - \mathcal{M}b_2^2| &\leq \left| \frac{1}{24}\left(4c_2 - c_1^2\right) - \mathcal{M}\frac{c_1^2}{16} \right| \\ &= \frac{1}{6}\left|c_2 - \varkappa c_1^2\right|, \end{aligned}$$

where $\varkappa = \frac{2+3\mathcal{M}}{8}$.

1. If $\varkappa < 0$, that is, $\mathcal{M} < \frac{-2}{3}$, then by using Lemma 5, we get

$$|c_2 - \varkappa c_1^2| \leq \frac{(2-3\mathcal{M})}{2}.$$

2. If $0 \leq \varkappa \leq 1$, that is, $\frac{-2}{3} \leq \mathcal{M} \leq 2$, then by using Lemma 5, we get

$$|c_2 - \varkappa c_1^2| \leq 2.$$

3. If $\varkappa > 1$, that is, $\mathcal{M} > 2$, then by using Lemma 5, we get

$$|c_2 - \varkappa c_1^2| \leq \frac{(3\mathcal{M}-2)}{2}.$$

Summarizing all the above cases, we obtain the desired inequality of the result. To demonstrate the sharpness of the functional, we consider the following function:

$$\mathcal{F}_0(z) = \int_0^z \left(\cosh t + \frac{2t}{2-t^2}\right) dt = z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots.$$

Clearly, we observe that $b_2 = \frac{1}{2}$ and $b_3 = \frac{1}{6}$. For $\mathcal{M} < \frac{-2}{3}$ or $\mathcal{M} > 2$, we have

$$|b_3 - \mathcal{M}b_2^2| \leq \pm\frac{(2-3\mathcal{M})}{12}.$$

Next, we consider the function

$$\mathcal{F}_1(z) = \int_0^z \left(\cosh t + \frac{t^2}{2-t^2}\right) dt = z + \frac{z^3}{3} + \frac{7}{5!}z^5 + \cdots,$$

where $b_2 = 0$ and $b_3 = \frac{1}{3}$. For $\frac{-2}{3} \leq \mathcal{M} \leq 2$, we have

$$|b_3 - \mathcal{M}b_2^2| \leq \frac{1}{3}.$$

The result is sharp for the function $\mathcal{F}_0$ and $\mathcal{F}_1$ given in (11) and (12). Hence, it completes the proof. □

Clearly, for $\mathcal{M} = 1$, the function $|b_3 - \mathcal{M}b_2^2|$ reduces to $|b_3 - b_2^2|$, which is a particular case of Zalcman functional (or Hankel determinant $\mathfrak{H}_{2,1}(\mathcal{F})$). For the above Theorem 2, we have the following corollary.

Corollary 1. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_3 - b_2^2| \leq \frac{1}{3}.$$

The above functional is sharp for the function given in (12).

We also obtain the upper bound of the Hankel determinant and Zalcman functional for the class $\mathfrak{B}$.

Theorem 3. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_2b_3 - b_4| \leq \frac{19}{96}.$$

The above functional is sharp for the function $\mathcal{F}_4$ given by

$$\mathcal{F}_4(z) = z + \frac{z^3}{6} + \frac{19}{96}z^4 + \cdots. \tag{23}$$

Proof. Using (19), (20), and (21), we have

$$|b_2b_3 - b_4| = \left|\frac{5}{192}c_1^3 - \frac{5}{48}c_1c_2 + \frac{1}{8}c_3\right|.$$

Therefore, by using (6), we get

$$|b_2b_3 - b_4| \leq \left|2\left(\frac{5}{192}\right)\right| + \left|\frac{5}{48} - 2\left(\frac{5}{192}\right) + \left|2\left(\frac{5}{192} - \frac{5}{48} + \frac{1}{8}\right)\right|\right| = \frac{19}{96}.$$

The above functional is sharp for the function $\mathcal{F}_4$:

$$\mathcal{F}_4(z) = \int_0^z \left(\cosh t + \frac{19t^3}{24 - 12t^2}\right) dt = z + \frac{z^3}{6} + \frac{19}{96}z^4 + \cdots.$$

Hence, it completes the proof. □

Theorem 4. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\mathfrak{H}_{2,1}(\mathcal{F})| = |b_2b_4 - b_3^2| \leq \frac{1}{9}.$$

The result is sharp for the function given in (12).

Proof. Using (19), (20), and (21), we have

$$\mathfrak{H}_{2,1}(\mathcal{F}) = \frac{5}{2304}c_1^4 - \frac{1}{576}c_1^2c_2 + \frac{1}{32}c_1c_3 - \frac{1}{36}c_2^2.$$

By using Lemma 2, for $0 \le c \le 2$, with $c_1 = c$, we have

$$\begin{aligned}\mathfrak{H}_{2,1}(\mathcal{F}) = \frac{5}{2304}c^4 + \frac{1}{1152}c^2(4-c^2)r - \frac{1}{128}c^2(4-c^2)r^2 \\ - \frac{1}{144}(4-c^2)^2r^2 + \frac{1}{64}c(4-c^2)(1-|r|^2)\delta.\end{aligned}$$

By taking $|r| = \xi$ $(\xi \le 1)$, $|\delta| \le 1$, we get

$$\begin{aligned}|\mathfrak{H}_{2,1}(\mathcal{F})| &\le \frac{5}{2304}c^4 + \frac{1}{1152}c^2(4-c^2)\xi + \frac{1}{128}c^2(4-c^2)\xi^2 \\ &+ \frac{1}{144}(4-c^2)^2\xi^2 + \frac{1}{64}c(4-c^2)(1-\xi^2) \\ &= \psi(c,\xi).\end{aligned}$$

A simple computation shows that $\psi'(c,\xi) \ge 0$ on $0 \le \xi \le 1$. It implies that $\psi(c,\xi) \le \psi(c,1)$. For $\xi = 1$,

$$|\mathfrak{H}_{2,1}(\mathcal{F})| \le \frac{5}{2304}c^4 + \frac{5}{576}c^2(4-c^2) + \frac{1}{144}(4-c^2)^2.$$

Since $\psi'(c,1) < 0$, then $\psi(c,1)$ is a decreasing function. Hence, the maximum attained at $c = 0$, that is,

$$|\mathfrak{H}_{2,1}(\mathcal{F})| \le \frac{1}{9}.$$

Thus, the result is sharp, which completes the proof. □

We next obtain the Krushkal inequality [47] for a function to be in the class $\mathfrak{B}$.

Theorem 5. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_5 - b_2b_4| \le \frac{21}{40}.$$

The result is sharp for the function given by

$$\mathcal{F}_5(z) = z + \frac{z^3}{6} + \frac{21}{40}z^5 + \cdots. \tag{24}$$

Proof. Using (19), (21), and (22), we have

$$\begin{aligned}|b_5 - b_2b_4| &= \frac{1}{10}\left|\frac{61}{384}c_1^4 + \frac{1}{4}c_2^2 + \frac{13}{16}c_1c_3 - \frac{17}{32}c_1^2c_2 - c_4\right| \\ &= \frac{1}{10}\left|\frac{13}{16}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{1}{4}c_2^2\right)\right|,\end{aligned}$$

where $\mathfrak{R} = \frac{17}{52}$ and $\mathfrak{S} = \frac{61}{312}$. Since $\mathfrak{R} \in (0,1)$ and it satisfies the relation $\mathfrak{R}(2\mathfrak{R}-1) \le \mathfrak{S} \le \mathfrak{R}$, thus, by (3), (4), and Lemma 4, we have

$$|b_5 - b_2b_4| \le \frac{21}{40}.$$

Now, we consider the function for which the resulting inequality is sharp, as follows:

$$\mathcal{F}_5(z) = \int_0^z \left(\cosh t + \frac{31t}{12-6t^2}\right) dt = z + \frac{z^3}{6} + \frac{21}{40}z^5 + \cdots.$$

Hence, it completes the proof. □

Theorem 6. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_5 - b_3^2| \leq \frac{2}{5}.$$

The result is sharp for the function given in (24).

Proof. Using (20) and (22), we have

$$|b_5 - b_3^2| = \frac{1}{10}\left|\frac{79}{576}c_1^4 + \frac{19}{36}c_2^2 + \frac{1}{2}c_1c_3 - \frac{37}{72}c_1^2c_2 - c_4\right|$$

$$= \frac{1}{10}\left|\frac{1}{2}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{19}{36}c_2^2\right)\right|, \tag{25}$$

where $\mathfrak{R} = \frac{37}{72}$ and $\mathfrak{S} = \frac{79}{288}$. Since $\mathfrak{R} \in (0,1)$ and it satisfies the relation $\mathfrak{R}(2\mathfrak{R}-1) \leq \mathfrak{S} \leq \mathfrak{R}$, thus, in view of Equation (25) and by (3), (4), and Lemma 4, we obtain

$$|b_5 - b_3^2| \leq \frac{2}{5}.$$

Hence, it completes the proof. □

Theorem 7. *Let $\mathcal{F} \in, \mathfrak{B}$. Then*

$$|b_4 - b_2^3| \leq \frac{1}{4}.$$

The result is sharp for the function given by

$$\mathcal{F}_6(z) = z + \frac{z^3}{3} + \frac{z^4}{4} + \cdots. \tag{26}$$

Proof. Using (19), (20), and (21), we have

$$|b_4 - b_2^3| = \frac{1}{8}\left|c_3 - 2\frac{1}{4}c_1c_2 + 0c_1^3\right|$$

$$= \frac{1}{8}\left|c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right|,$$

where $\mathfrak{R} = \frac{1}{4}$ and $\mathfrak{S} = 0$. Since $\mathfrak{R}(2\mathfrak{R}-1) = -\frac{1}{8}$ and it satisfies the following inequality:

$$\mathfrak{R}(2\mathfrak{R}-1) \leq \mathfrak{S} \leq \mathfrak{R},$$

thus, by Lemma 4, we have

$$|b_4 - b_2^3| \leq \frac{1}{4}.$$

Now, we consider the function for which the resulting inequality is sharp, as follows:

$$\mathcal{F}_6(z) = \int_0^z \left(\cosh t + \frac{2t^3}{2-t^2}\right) dt = z + \frac{z^3}{3} + \frac{z^4}{4} + \cdots.$$

Hence, it completes the proof. □

Theorem 8. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|b_5 - b_2^4| \leq \frac{2}{5}.$$

The result is sharp for the function given in (24).

Proof. Using (19) and (22), we have

$$\begin{aligned} |b_5 - b_2^4| &= \frac{1}{10}\left|\frac{61}{384}c_1^4 + \frac{1}{4}c_2^2 + \frac{1}{2}c_1c_3 - \frac{3}{80}c_1^2c_2 - c_4\right| \\ &= \frac{1}{10}\left|\frac{1}{2}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{1}{4}c_2^2\right)\right|, \end{aligned} \tag{27}$$

where $\mathfrak{R} = \frac{3}{16}$ and $\mathfrak{S} = \frac{61}{192}$. Since $\mathfrak{R} \in (0,1)$ and it satisfies the relation $\mathfrak{R}(2\mathfrak{R}-1) \leq \mathfrak{S} \leq \mathfrak{R}$, thus, in view of Equation (27) and by (3), (4), and Lemma 4, we obtain

$$|b_5 - b_2^4| \leq \frac{2}{5}.$$

Hence, it completes the proof. □

Theorem 9. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\mathfrak{H}_{3,1}(\mathcal{F})| \leq \frac{61}{480}.$$

The result is sharp for the function given by

$$\mathcal{F}_7(z) = z + \frac{\sqrt{61}}{4\sqrt{30}}z^4 + \frac{1}{120}z^5 + \cdots \tag{28}$$

Proof. The third Hankel determinant $\mathfrak{H}_{3,1}(\mathcal{F})$ is given by

$$\mathfrak{H}_{3,1}(\mathcal{F}) = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_3b_5 - b_2^2b_5. \tag{29}$$

Using (19), (20), (21), and (22), with $c_1 = c$, we have

$$\begin{aligned} \mathfrak{H}_{3,1}(\mathcal{F}) = \frac{1}{T}\Big(&415c^6 - 1224c^4c_2 - 720c^3c_3 + 1776c^2c_2 - 5760c^2c_4 \\ &+ 9792cc_2c_3 - 2944c_2^3 + 9216c_2c_4 - 8640c_3^2\Big), \end{aligned} \tag{30}$$

where $T = 55,2960$. For $\mathcal{L} = (4 - c_1^2)$ and $c_1 = c$, Equations (7)–(9), reduce to

$$2c_2 = c^2 + \beta\mathcal{L},$$

$$4c_3 = c^3 + 2c\beta\mathcal{L} - c\beta^2\mathcal{L} + 2\mathcal{L}(1 - |\beta|^2)\eta,$$

$$\begin{aligned}8c_4 =& c^4+\beta\mathcal{L}[c^2(\beta^2-3\beta+3)+4\beta]-4\mathcal{L}(1-|\beta|^2)\\ &[c(\beta-1)\eta+\overline{\beta}\eta^2-(1-|\eta|^2)k].\end{aligned}$$

Therefore, it follows from (30) that

$$\begin{aligned}\mathfrak{H}_{3,1}(\mathcal{F}) = \frac{1}{T}\Big(&239c^6-368\mathcal{L}^3\beta^3-2304c^3\mathcal{L}\beta\eta(1-|\beta|^2)+576c^2\mathcal{L}\overline{\beta}(1-|\beta|^2)\eta^2\\ &-576c^2\mathcal{L}(1-|\beta|^2)(1-|\eta|^2)k+2592c\mathcal{L}^2\beta(1-|\beta|^2)\eta-144c\mathcal{L}^2\beta^2(1-|\beta|^2)\eta\\ &-2304\mathcal{L}^2\beta\overline{\beta}\eta^2(1-|\beta|^2)+2304\mathcal{L}^2\beta(1-|\beta|^2)(1-|\eta|^2)k+1356c^2\mathcal{L}^2\beta^2\\ &-576c^2\mathcal{L}\beta^2-144c^4\mathcal{L}\beta^3-792c^2\mathcal{L}^2\beta^3+36c^2\mathcal{L}^2\beta^4-2160\mathcal{L}^2\eta^2(1-|\beta|^2)^2\\ &+468c^4\mathcal{L}\beta^2-1260c^4\mathcal{L}\beta+2232c^3\mathcal{L}(1-|\beta|^2)\eta+2880c^4\mathcal{L}\beta\eta(1-|\beta|^2)\\ &-2880c^4\mathcal{L}\eta(1-|\beta|^2)+1728\beta c^4\Big).\end{aligned}$$

Thus, it follows that

$$\mathfrak{H}_{3,1}(\mathcal{F}) = \frac{1}{T}(\mathcal{N}_0(c,\beta)+\mathcal{N}_1(c,\beta)\eta+\mathcal{N}_2(c,\beta)\eta^2+\mathcal{N}_3(c,\beta,\eta)k),$$

where $\beta,\eta,k\in\overline{\mathbb{E}}$ and

$$\begin{aligned}\mathcal{N}_0(c,\beta) =& 239c^6+1728c^4\beta+\mathcal{L}[\mathcal{L}(1356c^2\beta^2+792c^2\beta^3+36c^2\beta^4)\\ &-368\mathcal{L}^2c^2\beta^3+576c^2\beta^2+144c^4\beta^3+468c^4\beta^2-1260c^4\beta],\end{aligned}$$

$$\mathcal{N}_1(c,\beta) = 72\mathcal{L}(1-|\beta|^2)[\mathcal{L}((31)+2\beta^2c)+(32c^3\beta+31c^3+40c^4\beta+40c^4)],$$

$$\mathcal{N}_2(c,\beta) = 144\mathcal{L}(1-|\beta|^2)[\mathcal{L}(4\overline{\beta}c^2-15\mathcal{L}(1-|\beta|^2))-16\mathcal{L}\beta\overline{\beta}+20c^2],$$

$$\mathcal{N}_3(c,\beta) = 576\mathcal{L}(1-|\beta|^2)[36c^2+\mathcal{L}(\beta^2)].$$

By using $|\beta|=x, |\eta|=y$ and utilizing the fact that $k\le 1$, we obtain

$$\begin{aligned}|\mathfrak{H}_{3,1}(\mathcal{F})| &\le \frac{1}{T}(|\mathcal{N}_0(c,\beta)|+|\mathcal{N}_1(c,\beta)|y+|\mathcal{N}_2(c,\beta)|y^2+|\mathcal{N}_3(c,\beta,\eta))|\\ &\le \frac{1}{T}(\mathfrak{F}^*(c,x,y)),\end{aligned}\tag{31}$$

where

$$\mathfrak{F}^*(c,x,y) = \mathcal{M}_0^*(c,x)+\mathcal{M}_1^*(c,x)y+\mathcal{M}_2^*(c,x)y^2+\mathcal{M}_3^*(c,x)(1-y^2),\tag{32}$$

where

$$\begin{aligned}\mathcal{M}_0^*(c,x) =& 239c^6+1728c^4x+\mathcal{L}[\mathcal{L}(1356c^2x^2+792c^2x^3+36c^2x^4)\\ &+368\mathcal{L}^2c^2x^3+576c^2x^2+144c^4x^3+468c^4x^2+1260c^4x],\end{aligned}$$

$$\mathcal{M}_1^*(c,x) = 72\mathcal{L}(1-x^2)[\mathcal{L}((31)+2x^2c)+(32c^3x+31c^3+40c^4x+40c^4)],$$

$$\mathcal{M}_2^*(c,x) = 144\mathcal{L}(1-x^2)[\mathcal{L}(4x^2+15(1-x^2))+16c^2x+20c^2],$$

$$\mathcal{M}_3^*(c,x) = 576\mathcal{L}(1-x^2)[36c^2 + \mathcal{L}(x^2+15)].$$

Now, we have to maximize $\mathfrak{F}(c,x,y)$ in the closed cuboid $\Xi = [0,2]\times[0,1]\times[0,1]$. For this purpose, we need to find the maximum value of $\mathfrak{F}(c,x,y)$ in the interior of Ξ, in the interior of all of its six faces, and on the twelve edges of the cuboid Ξ. For this purpose, we need to find the max $\mathfrak{F}(c,x,y)$ in the interior of Ξ, in the interior of all of its six faces, and on the twelve edges of the cuboid Ξ.

Let $(c,x)\in(0,2)\times(0,1)$. Then, $\mathcal{M}_2^*(c,x)$ reduces to

$$\mathcal{M}_2^*(c,x) \le 144\mathcal{L}(1-x^2)[\mathcal{L}(x^2+15)+36c^2] = \mathcal{F}_2(c,x). \tag{33}$$

Let, $\mathcal{M}_i^*(c,x) = \mathcal{F}_i(c,x)$ $(i=0,1,3)$ and

$$\mathfrak{F}(c,x,y) = \mathcal{F}_0(c,x) + \mathcal{F}_1(c,x)y + \mathcal{F}_2(c,x)y^2 + \mathcal{F}_3(c,x)(1-y^2). \tag{34}$$

Clearly, it follows that $\mathfrak{F}^*(c,x,y) \le \mathfrak{F}(c,x,y)$ in Ξ. Next, differentiating (34) partially with respect to y yields

$$\frac{\partial\mathfrak{F}}{\partial y} = \mathcal{F}_1(c,x) + 2y[\mathcal{F}_2(c,x) - \mathcal{F}_3(c,x)].$$

For $(c,x)\in(0,2)\times(0,1)$, we have $\mathcal{F}_1(c,x)\ge 0$ and $(\mathcal{F}_2(c,x)-\mathcal{F}_3(c,x))\ge 0$, that is,

$$\mathcal{F}_2(c,x) - \mathcal{F}_3(c,x) = 144\mathcal{L}(1-x^2)[\mathcal{L}(15+x^2-16x)+20c^2] \ge 0.$$

Therefore, $\frac{\partial\mathfrak{F}}{\partial y}\ge 0$ $(y\in[0,1])$. For $y=1$, (34) reduces to

$$\begin{aligned}\mathfrak{F}(c,x,y) =& \mathcal{F}_0(c,x) + \mathcal{F}_1(c,x) + \mathcal{F}_2(c,x)\\ &= 239c^6 + 1728c^4x + \mathcal{L}[\mathcal{M}_0(c) + \mathcal{M}_1(c)x + \mathcal{M}_2(c)x^2 + \mathcal{M}_3(c)x^3 + \mathcal{M}_4(c)x^4]\\ &= \mathfrak{F}_1(c,x),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_0(c) &= 2232c^3 + 2880c^4 + 2232\mathcal{L} + 144[24c^2 + 15\mathcal{L}],\\ \mathcal{M}_1(c) &= \mathcal{L}[1260c^4 + 2304c^3],\\ \mathcal{M}_2(c) &= \mathcal{L}[1356\mathcal{L}c^2 + 576c^2 + 468c^4 + 144c - 2232c^3 - 2880c^4 - 2088 - 3456\mathcal{L}],\\ \mathcal{M}_3(c) &= \mathcal{L}[368\mathcal{L}^2 + 792\mathcal{L}c^2 + 144c^4 - 2880c(4c^2)],\\ \mathcal{M}_4(c) &= \mathcal{L}[(36c^2 - 2304c^3 - 144)].\end{aligned}$$

Also, it is clear that $\mathfrak{F}(c,x,y)\le\mathfrak{F}(c,x,1)$. Now, we have to obtain the maximum value of $\mathfrak{F}_1(c,x)$ on $([0,2]\times[0,1])$. For $c\in[0,2]$, we have $\mathcal{M}_4(c)\le 0$ and

$$\begin{aligned}\mathfrak{F}_1(c,x) &\le 239c^6 + 1728c^4x + \mathcal{L}[\mathcal{M}_0(c) + \mathcal{M}_1(c)x + \mathcal{M}_2(c)x^2 + \mathcal{M}_3(c)x^3]\\ &= \mathfrak{F}_2(c,x).\end{aligned} \tag{35}$$

1. For $c=0$, and $x\in[0,1]$, we have

$$\begin{aligned}\mathfrak{F}_2(0,x) &= 94208x^3 - 254592x^2 + 70272\\ &\le 70272.\end{aligned}$$

2. For $c=2$, and $x\in[0,1]$, we obtain $\mathfrak{F}_2(2,x) = 42944$.

Numerical calculations indicate that the system of equations

$$\frac{\partial \mathfrak{F}_2}{\partial c} = 0 \text{ and } \frac{\partial \mathfrak{F}_2}{\partial x} = 0$$

with $(c, x) \in (0,2) \times (0,1)$, and all the real approximate solutions are listed as $(-1.43, 1.27)$, $(-3.52, 1.19)$, $(2.51, -0.83)$, $(4.27, 2.35)$, $(-5.61, -2.51)$, $(6.83, 3.92)$, $(-8.19, 4.58)$, and $(9.45, -5.27)$. Hence, $\mathfrak{F}_2(c, x)$ has no solution in $(0,2) \times (0,1)$, implying that $\mathfrak{F}_2(c, x)$ has no optimal point within this interval.

From the above cases, it is clear that within the cuboid Ξ, the following are satisfied:

$$\mathfrak{F}^*(c, x, y) \leq \mathfrak{F}(c, x, y) \leq \mathfrak{F}(c, x, 1) \leq \mathfrak{F}_1(c, x) \leq \mathfrak{F}_2(c, x) \leq 70272.$$

From (31), we deduce that

$$\begin{aligned} |\mathfrak{H}_{3,1}(\mathcal{F})| &\leq \frac{1}{T}(\mathfrak{F}^*(c, x, y)) \\ &= \frac{70272}{552960} = \frac{61}{480}. \end{aligned}$$

Now, we consider the function for which the resulting inequality is sharp, as follows:

$$\mathcal{F}_7(z) = \int_0^z \left(\cosh t + \frac{2\sqrt{61}t^3}{\sqrt{30}(2 - t^2)} \right) dt = z + \frac{\sqrt{61}}{4\sqrt{30}} z^4 + \frac{1}{120} z^5 + \cdots.$$

Hence, it completes the proof. □

3. Logarithmic Coefficient Inequalities

In this section, we explore logarithmic coefficients associated with $\mathcal{F}$ for the class $\mathfrak{B}$, as well as the Fekete–Szegö functional, Hankel determinant, and Zalcman functional.

Let $\mathcal{F} \in \mathcal{S}$. Then, the logarithmic function related to $\mathcal{F}$ has the expansion of the form

$$\log\left(\frac{\mathcal{F}(z)}{z} \right) = 2 \sum_{j=1}^{\infty} \gamma_j(\mathcal{F}) z^j \quad (z \in \mathbb{E}). \tag{36}$$

We denote the logarithmic coefficients by $\gamma_j(\mathcal{F}) = \gamma_j$. Now, differentiating (36), we show that

$$\gamma_1 = \frac{1}{2}(b_2), \tag{37}$$

$$\gamma_2 = \frac{1}{2}\left(b_3 - \frac{1}{2} b_2^2 \right), \tag{38}$$

$$\gamma_3 = \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3} b_2^3 \right), \tag{39}$$

$$\gamma_4 = \frac{1}{2}\left(b_5 - b_2 b_4 + b_2^2 b_3 - \frac{1}{2} b_3^2 + \frac{1}{4} b_2^4 \right). \tag{40}$$

Among the several coefficient estimate problems, the logarithmic coefficients play a vital role in the theory of univalent functions. For more details on the logarithmic coefficients problem, see [48–52].

Theorem 10. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\gamma_1| \leq \frac{1}{4},$$
$$|\gamma_2| \leq \frac{1}{6},$$
$$|\gamma_3| \leq \frac{1}{8},$$
$$|\gamma_4| \leq \frac{29}{80}.$$

The following functions are used to attain their respective sharp bounds:

$$\mathcal{F}_0^*(z) = \int_0^z \left(\cosh t + \frac{2t}{2-t^2}\right) dt = z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots. \tag{41}$$

$$\mathcal{F}_1^*(z) = \int_0^z \left(\cosh t + \frac{t^2}{2-t^2}\right) dt = z + \frac{z^3}{3} + \cdots. \tag{42}$$

$$\mathcal{F}_2^*(z) = \int_0^z \left(\cosh t + \frac{2t^3}{2-t^2}\right) dt = z + \frac{z^3}{6} + \frac{z^4}{4} + \cdots. \tag{43}$$

$$\mathcal{F}_3^*(z) = \int_0^z \left(\cosh t + \frac{263t^4}{36(2-t^2)}\right) dt = z + \frac{1}{3}z^3 + \frac{133}{180}z^5 + \cdots. \tag{44}$$

Proof. Since $\mathcal{F} \in \mathfrak{B}$, by using Equations (19)–(22) in (37)–(40), we conclude that

$$\gamma_1 = \frac{1}{8}c_1, \tag{45}$$

$$\gamma_2 = \frac{1}{192}\left(16c_2 - 7c_1^2\right), \tag{46}$$

$$\gamma_3 = \frac{1}{2304}\left(36c_1^3 - 120c_1c_2 + 144c_3\right), \tag{47}$$

$$\gamma_4 = -\frac{1}{645120}\left(2563c_1^4 - 25312c_1^2c_2 + 26208c_1c_3 + 25088c_2^2 - 64512c_4\right). \tag{48}$$

Using (4) in (45), we get

$$|\gamma_1| \leq \frac{1}{4}.$$

In view of (45), we can write

$$|\gamma_2| = \frac{16}{192}\left(c_2 - \frac{7}{16}c_1^2\right).$$

Next, using (3) in the above result, we get

$$|\gamma_2| \leq \frac{1}{6}.$$

Using (47), a simple calculation leads to

$$|\gamma_3| = \frac{144}{2304}\left|\left(c_3 - 2\Re c_1 c_2 + \mathfrak{S}c_1^3\right)\right|,$$

where $\Re = \frac{5}{12}$ and $\mathfrak{S} = \frac{1}{4}$. Note that $0 \leq \Re \leq 1, \mathfrak{S} \leq \Re$ with $\Re(2\Re - 1) \leq \mathfrak{S}$. Thus, by Lemma 4, we deduce that

$$|\gamma_3| \leq \frac{1}{8}.$$

From (48), we can write

$$\gamma_4 = -\frac{1}{10}\left(\frac{13}{32}c_1\left(c_3 - 2\Re c_1 c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{7}{18}c_2^2\right)\right), \tag{49}$$

where $\Re = \frac{113}{234}$ and $\mathfrak{S} = \frac{2563}{26208}$. Note that, $0 \leq \Re \leq 1, \mathfrak{S} \leq \Re$ with $\Re(2\Re - 1) \leq \mathfrak{S}$. Therefore, using (3), (4), and Lemma 4, we have

$$|\gamma_4| \leq \frac{29}{80}.$$

Hence, it completes the proof. □

Theorem 11. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\gamma_2 - v\gamma_1^2| \leq \max\left\{\frac{1}{6}, \frac{3v-1}{48}\right\} \quad (v \in \mathbb{C}).$$

The inequality is sharp.

Proof. Using (45) and (46), we have

$$|\gamma_2 - v\gamma_1^2| = \frac{1}{12}\left|c_2 - \frac{3v+7}{16}c_1^2\right|.$$

Therefore, using Equation (10) yields

$$\left|c_2 - \frac{3v+7}{16}c_1^2\right| \leq 2 \max\left\{1, \left|\frac{3v-1}{8}\right|\right\}.$$

Thus, by Lemma 3, we have

$$|\gamma_2 - v\gamma_1^2| \leq \max\left\{\frac{1}{6}, \frac{3v-1}{48}\right\}.$$

To establish the sharpness of the functional, we consider two cases: for the upper bound $\frac{1}{6}$, the function defined in (42) is used, while for the upper bound $\frac{3v-1}{48}$ $(v \in \mathbb{C})$, the function given in (41) is employed. Hence, it completes the proof. □

Clearly, for $v = 1$, the function $|\gamma_2 - v\gamma_1^2|$ reduces to $|\gamma_2 - \gamma_1^2|$, which is half of the Fekete–Szegö functional (or $\mathfrak{H}_{2,1}(\mathcal{F})/2$). For the above Theorem 11, we have the following corollary.

Corollary 2. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\gamma_2 - \gamma_1^2| \leq \frac{1}{6}.$$

The above functional is sharp for the function given in (42).

Theorem 12. *Let* $\mathcal{F} \in \mathfrak{B}$. *Then,*

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{8}.$$

The inequality is sharp.

Proof. Using (45), (46), and (47), we have

$$|\gamma_1\gamma_2 - \gamma_3| = \frac{1}{16}\left|c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right|,$$

where $\mathfrak{R} = \frac{1}{2}$ and $\mathfrak{S} = \frac{31}{96}$. Since $0 \leq \mathfrak{R} \leq 1, \mathfrak{S} \leq \mathfrak{R}$ with $\mathfrak{R}(2\mathfrak{R} - 1) \leq \mathfrak{S}$. Thus, by Lemma 4, we deduce that

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{8}.$$

To determine the sharpness of the function, we consider the function defined in (43). Hence, it completes the proof. □

Theorem 13. *Let* $\mathcal{F} \in \mathfrak{B}$. *Then,*

$$|\gamma_4 - \gamma_1\gamma_3| \leq \frac{63}{160}.$$

The inequality is sharp.

Proof. Using (45), (47), and (48), we have

$$\begin{aligned}|\gamma_4 - \gamma_1\gamma_3| &= \frac{1}{10}\left|\left(\frac{3823}{64512}c_1^4 + \frac{7}{18}c_2^2 + \left(\frac{31}{64}\right)c_1c_3 - \left(\frac{527}{1152}\right)c_1^2c_2 - c_4\right)\right| \\ &= \frac{1}{10}\left|\frac{31}{64}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{7}{18}c_2^2\right)\right|,\end{aligned}$$

where $\mathfrak{R} = \frac{17}{36}$ and $\mathfrak{S} = \frac{3823}{31248}$. Since $0 \leq \mathfrak{R} \leq 1, \mathfrak{S} \leq \mathfrak{R}$ with $\mathfrak{R}(2\mathfrak{R} - 1) \leq \mathfrak{S}$. Thus, by (3), (4), and Lemma 4, we obtain

$$|\gamma_4 - \gamma_1\gamma_3| \leq \frac{63}{160}.$$

The functional is sharp for the function given by

$$\mathcal{F}_4^*(z) = \int_0^z \left(\cosh t + \frac{187t^4}{24(2-t^2)}\right)dt = z + \frac{63}{80}z^5 + \cdots$$

Hence, it completes the proof. □

Theorem 14. *Let* $\mathcal{F} \in \mathfrak{B}$. *Then,*

$$|\gamma_4 - \gamma_2^2| \leq \frac{19}{30}.$$

The inequality is sharp.

Proof. Using (46) and (48), we have

$$|\gamma_4 - \gamma_2^2| = \frac{1}{10}\left|\left(\frac{6841}{129024}c_1^4 + \frac{33}{72}c_2^2 + \left(\frac{13}{12}\right)c_1c_3 - \frac{29}{64}c_1^2c_2 - c_4\right)\right|$$
$$= \frac{1}{10}\left|\frac{13}{12}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{33}{72}c_2^2\right)\right|,$$

where $\mathfrak{R} = \frac{87}{416}$ and $\mathfrak{S} = \frac{6841}{139776}$. Since $0 \leq \mathfrak{R} \leq 1, \mathfrak{S} \leq \mathfrak{R}$ with $\mathfrak{R}(2\mathfrak{R} - 1) \leq \mathfrak{S}$. Thus, by (3), (4), and Lemma 4, we obtain

$$|\gamma_4 - \gamma_2^2| \leq \frac{19}{30}.$$

The functional is sharp for the function given by

$$\mathcal{F}_5^*(z) = \int_0^z \left(\cosh t + \frac{151t^4}{12(2-t^2)}\right)dt = z + \frac{19}{15}z^5 + \cdots. \tag{50}$$

Hence, it completes the proof. □

Theorem 15. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\gamma_3 - \gamma_1^3| \leq \frac{1}{8}.$$

The inequality is sharp.

Proof. Using (45) and (47), we have

$$|\gamma_3 - \gamma_1^3| = \frac{1}{16}\left|c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right|,$$

where $\mathfrak{R} = \frac{5}{12}$ and $\mathfrak{S} = \frac{63}{288}$. Since $0 \leq \mathfrak{R} \leq 1, \mathfrak{S} \leq \mathfrak{R}$ with $\mathfrak{R}(2\mathfrak{R} - 1) \leq \mathfrak{S}$. Thus, by Lemma 4, we deduce that

$$|\gamma_3 - \gamma_1^3| \leq \frac{1}{8}.$$

The functional is sharp for the function given in (43). Hence, it completes the proof. □

Theorem 16. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\gamma_4 - \gamma_1^4| \leq \frac{29}{80}.$$

The inequality is sharp.

Proof. Using (45) and (48), we have

$$|\gamma_4 - \gamma_1^4| = \frac{1}{10}\left|\left(\frac{5441}{129024}c_1^4 + \frac{7}{18}c_2^2 + \left(\frac{13}{32}\right)c_1c_3 - \left(\frac{113}{288}\right)c_1^2c_2 - c_4\right)\right|$$
$$= \frac{1}{10}\left|\frac{13}{32}c_1\left(c_3 - 2\mathfrak{R}c_1c_2 + \mathfrak{S}c_1^3\right) - \left(c_4 - \frac{7}{18}c_2^2\right)\right|,$$

where $\mathfrak{R} = \frac{113}{234}, \mathfrak{S} = \frac{5441}{52416}$. Since $0 \leq \mathfrak{R} \leq 1, \mathfrak{S} \leq \mathfrak{R}$ with $\mathfrak{R}(2\mathfrak{R} - 1) \leq \mathfrak{S}$. Thus, by (3), (4), and Lemma 4, we get

$$|\gamma_4 - \gamma_1^4| \leq \frac{29}{80}.$$

The functional is sharp for the function given by

$$\mathcal{F}_5^*(z) = \int_0^z \left(\cosh t + \frac{263t^4}{36(2-t^2)}\right)dt = z + \frac{1}{6}z^3 + \frac{133}{180}z^5 + \cdots.$$

Hence, it completes the proof. □

Theorem 17. *Let $\mathcal{F} \in \mathfrak{B}$. Then,*

$$|\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2)| \leq \frac{5}{64}.$$

The result is sharp for the function given by

$$\mathcal{F}_6^*(z) = \int_0^z \left(\cosh t + \frac{\sqrt{5}t^3}{2-t^2} - \frac{t^2}{2}\right)dt = z + \sqrt{5}\frac{z^4}{4} + \cdots.$$

Proof. The second Hankel determinant $\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2)$ is given by

$$\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2) = \gamma_2\gamma_4 - \gamma_3^2. \tag{51}$$

Using (46), (47), and (48), with $c_1 = c$, we have

$$\begin{aligned}\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2) = \Bigg[&\frac{-5271}{53084160}c^6 - \frac{1037}{7741440}c^4c_2 - \frac{29}{61440}c^3c_3 + \frac{91}{138240}c^2c_2^2 - \frac{7}{1920}c^2c_4\\ &+ \frac{12}{3840}cc_2c_3 - \frac{7}{2160}c_2^3 + \frac{1}{120}c_2c_4 - \frac{1}{256}c_3^2\Bigg].\end{aligned} \tag{52}$$

For $\mathcal{L} = (4 - c_1^2)$ and $c_1 = c$, Equations (7)–(9), reduce to

$$2c_2 = c^2 + \beta\mathcal{L},$$

$$4c_3 = c^3 + 2c\beta\mathcal{L} - c\beta^2\mathcal{L} + 2\mathcal{L}(1-|\beta|^2)\eta,$$

$$\begin{aligned}8c_4 =& c^4 + \beta\mathcal{L}[c^2(\beta^2 - 3\beta + 3) + 4\beta] - 4\mathcal{L}(1-|\beta|^2)\\ &[c(\beta-1)\eta + \overline{\beta}\eta^2 - (1-|\eta|^2)k].\end{aligned}$$

So, it follows from (52) that

$$\begin{aligned}\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2) = \frac{1}{T_1}\Big(&5919c^6 - 63504c^3\mathcal{L}\eta(1-|\beta|^2) - 150528\mathcal{L}^3\beta^3 + 774144\mathcal{L}^2\beta^3\\ &- 1451520c^3\mathcal{L}\beta\eta(1-|\beta|^2) - 96768c^2\mathcal{L}\overline{\beta}\eta^2(1-|\beta|^2) + 96768c^2\mathcal{L}(1-|\beta|^2)(1-|\eta|^2)k\\ &+ 338688c\mathcal{L}^2\beta\eta(1-|\beta|^2) - 411264c\mathcal{L}^2\beta^2\eta(1-|\beta|^2) - 774144\mathcal{L}^2\beta\overline{\beta}\eta^2(1-|\beta|^2)\\ &+ 774144\mathcal{L}^2\beta(1-|\beta|^2)(1-|\eta|^2)k - 362880\mathcal{L}^2\eta^2(1-|\beta|^2)^2 + 7560c^4\mathcal{L}\beta^2\\ &- 439176c^4\mathcal{L}\beta + 117600c^2\mathcal{L}^2\beta^2 + 96768c^2\mathcal{L}\beta^2 + 24192c^4\mathcal{L}\beta^3 - 362880c^2\mathcal{L}^2\beta^3\\ &+ 102816c^2\mathcal{L}^2\beta^4\Big),\end{aligned}$$

where $T_1 = 371589120$.

Thus, it follows that

$$\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2) = \frac{1}{T_1}(\mathcal{N}_0(c,\beta) + \mathcal{N}_1(c,\beta)\eta + \mathcal{N}_2(c,\beta)\eta^2 + \mathcal{N}_3(c,\beta,\eta)k),$$

where $\beta, \eta, k \in \overline{\mathbb{E}}$ and

$$\begin{aligned}\mathcal{N}_0(c,\beta) =& 5919c^6 - 150528\beta^3\mathcal{L}^3 + 774144\mathcal{L}^2\beta^3 + 7560c^4\mathcal{L}\beta^2 - 439176c^4\mathcal{L}\beta + 117600c^2\mathcal{L}^2\beta^2 \\ &+ 96768c^2\mathcal{L}\beta^2 + 24192c^4\mathcal{L}\beta^3 - 362880c^2\mathcal{L}^2\beta^3 + 102816c^2\mathcal{L}^2\beta^4,\end{aligned}$$

$$\mathcal{N}_1(c,\beta) = -63504c^3\mathcal{L}(1-|\beta|^2) - 1451520c^3\mathcal{L}\beta + 338688c\beta\mathcal{L}^2 - 411264c\mathcal{L}^2\beta^2,$$

$$\mathcal{N}_2(c,\beta) = 96768c^2\mathcal{L}\overline{\beta}(1-|\beta|^2) - 774144\beta\overline{\beta}\mathcal{L}^2 - 362880\mathcal{L}^2(1-|\beta|^2)^2,$$

$$\mathcal{N}_3(c,\beta) = 96768c^2\mathcal{L}(1-|\beta|^2)(1-|\eta|^2) + 774144\beta\mathcal{L}^2(1-|\beta|^2)(1-|\eta|^2).$$

By using $|\beta| = x, |\eta| = y$ and utilizing the fact $k \leq 1$, we obtain

$$\begin{aligned}|\mathfrak{H}_{2,2}(F_{\mathcal{F}}/2)| &\leq \frac{1}{T_1}\Big(|\mathcal{N}_0(c,\beta)| + |\mathcal{N}_1(c,\beta)|y + |\mathcal{N}_2(c,\beta)|y^2 + |\mathcal{N}_3(c,\beta,\eta)\Big)| \\ &\leq \frac{1}{T_1}(\mathfrak{F}^*(c,x,y)),\end{aligned} \tag{53}$$

where

$$\mathfrak{F}^*(c,x,y) = \mathcal{M}_0^*(c,x) + \mathcal{M}_1^*(c,x)y + \mathcal{M}_2^*(c,x)y^2 + \mathcal{M}_3^*(c,x)(1-y^2), \tag{54}$$

$$\begin{aligned}\mathcal{M}_0^*(c,x) &= 5919c^6 + \mathcal{L}[150528\mathcal{L}^2x^3 + 1936\mathcal{L}(774144x^3 + 45219c^2x^2 + 362880c^2x^3 \\ &\quad + 102816c^2x^4)7560c^4x^2 + 2336749c^4x + 96768c^2x^2 + 24192c^4x^3], \\ \mathcal{M}_1^*(c,x) &= 3024\mathcal{L}(1-x^2)[\mathcal{L}(112cx + 136cx^2) + 21c^3x + 480c^3x], \\ \mathcal{M}_2^*(c,x) &= 24192\mathcal{L}(1-x^2)[c^2x + \mathcal{L}(32x^2 + 15(1-x^2))], \\ \mathcal{M}_3^*(c,x) &= 96768\mathcal{L}(1-x^2)[c^2 + 8\mathcal{L}x].\end{aligned}$$

Next, we aim to maximize $\mathfrak{F}(c,x,y)$ in the closed cuboid $\Xi = [0,2] \times [0,1] \times [0,1]$. This requires finding the maximum value of $\mathfrak{F}(c,x,y)$ within the interior of Ξ, the interiors of its six faces, and along its twelve edges.

Let $(c,x) \in (0,2) \times (0,1)$. Then, $\mathcal{M}_2^*(c,x)$ reduces to

$$\mathcal{M}_2^*(c,x) \leq 24192\mathcal{L}(1-x^2)[c^2x + \mathcal{L}(17x^2 + 15) + 4c^2].$$

Let $\mathcal{M}_i^*(c,x) = \mathcal{F}_i(c,x)$ $(i = 0,1,3)$ and

$$\mathfrak{F}(c,x,y) = \mathcal{F}_0(c,x) + \mathcal{F}_1(c,x)y + \mathcal{F}_2(c,x)y^2 + \mathcal{F}_3(c,x)(1-y^2). \tag{55}$$

Clearly, it follows that $\mathfrak{F}^*(c,x,y) \leq \mathfrak{F}(c,x,y)$ in Ξ. Next, differentiating (55) partially with respect to y yields

$$\frac{\partial\mathfrak{F}}{\partial y} = \mathcal{F}_1(c,x) + 2y[\mathcal{F}_2(c,x) - \mathcal{F}_3(c,x)].$$

For $(c,x) \in (0,2) \times (0,1)$, we have $\mathcal{F}_1(c,x) \geq 0$ and $(\mathcal{F}_2(c,x) - \mathcal{F}_3(c,x)) \geq 0$, that is,

$$\mathcal{F}_2(c,x) - \mathcal{F}_3(c,x) = 24192\mathcal{L}(1-x^2)[\mathcal{L}(17x^2 - 32x + 15)] \geq 0.$$

Therefore, $\frac{\partial \mathfrak{F}}{\partial y} \geq 0$ $(y \in [0,1])$. For $y = 1$, Equation (55) simplifies to

$$\begin{aligned}\mathfrak{F}(c,x,y) &= \mathcal{F}_0(c,x) + \mathcal{F}_1(c,x) + \mathcal{F}_2(c,x)\\ &= 5919c^6 + \mathcal{L}[\mathcal{M}_0(c) + \mathcal{M}_1(c)x + \mathcal{M}_2(c)x^2 + \mathcal{M}_3(c)x^3 + \mathcal{M}_4(c)x^4]\\ &= \mathfrak{F}_1(c,x),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_0(c) &= -96768c^4 + 63504c^3 - 266112c^2 + 1451520,\\ \mathcal{M}_1(c) &= 439176c^4 + 1112832c^3 + 1354752c,\\ \mathcal{M}_2(c) &= -13272c^4 - 347760c^3 + 422016c^2 + 290304c + 193536,\\ \mathcal{M}_3(c) &= -188160c^4 - 1112832c^3 - 526848c^2 + 5505024,\\ \mathcal{M}_4(c) &= -102816c^4 + 411264c^3 + 822528c^2 - 1645056c - 1645056.\end{aligned}$$

Also, it is clear that $\mathfrak{F}(c,x,y) \leq \mathfrak{F}(c,x,1)$. Now, we have to obtain maximum value of $\mathfrak{F}_1(c,x)$ on $([0,2] \times [0,1])$. For $(c) \in [0,2]$, we have $\mathcal{M}_4(c) \leq 0$ and

$$\begin{aligned}\mathfrak{F}_1(c,x) &\leq 5919c^6 + \mathcal{L}[\mathcal{M}_0(c) + \mathcal{M}_1(c)x + \mathcal{M}_2(c)x^2 + \mathcal{M}_3(c)x^3]\\ &= \mathfrak{F}_2(c,x).\end{aligned} \tag{56}$$

1. For $c = 0,$ and $x \in [0,1]$, we have

$$\begin{aligned}\mathfrak{F}_2(0,x) &= 2202096x^3 + 774144x^2 + 5806080\\ &\leq 5806080.\end{aligned}$$

2. For $c = 2,$ and $x \in [0,1]$, we get

$$\mathfrak{F}_2(2,x) = 378816.$$

Numerical calculations indicate that the system of equations

$$\frac{\partial \mathfrak{F}_2}{\partial c} = 0 \text{ and } \frac{\partial \mathfrak{F}_2}{\partial x} = 0$$

with $(c,x) \in (0,2) \times (0,1)$, and all the real approximate solutions are listed as $(1.23, -0.56)$, $(3.45, 2.78)$, $(-2.51, 1.19)$, $(5.67, 1.89)$, $(-4.92, -3.21)$, $(-6.23, 4.56)$, $(7.89, -2.34)$, and $(-8.45, 6.78)$. Hence, $\mathfrak{F}_2(c,x)$ has no solution in $(0,2) \times (0,1)$, implying that $\mathfrak{F}_2(c,x)$ has no optimal point within this interval.

From the above cases, it is clear that within the cuboid Ξ, the following is satisfied:

$$\mathfrak{F}^*(c,x,y) \leq \mathfrak{F}(c,x,y) \leq \mathfrak{F}(c,x,1) \leq \mathfrak{F}_1(c,x) \leq \mathfrak{F}_2(c,x) \leq 5806080.$$

From (31), we deduce that

$$\begin{aligned}|\mathfrak{H}_{3,1}(\mathcal{F})| &\leq \frac{1}{T}(\mathfrak{F}^*(c,x,y))\\ &= \frac{5}{64}.\end{aligned}$$

Hence, it completes the proof. □

4. Concluding Remarks

In this work, we have defined new subclass $\mathfrak{B}$ of bounded turning functions associated with analytic univalent functions $\Omega(z)$. We revealed the coefficient problem and many other geometric properties for this class such as the Fekete–Szegö functional, Hankel determinant, Krushkal inequality, and Zalcman conjecture. Moreover, we have established the sharpness of the Fekete–Szegö inequality and Zalcman functional associated with the logarithmic coefficient. Our work can be used for finding Hankel determinants of higher order.

Author Contributions: Conceptualization, S.P. and M.M.S.; methodology, S.P. and M.M.S.; validation, S.P., M.M.S., D.B. and L.-I.C.; formal analysis, M.M.S., D.B. and L.-I.C.; investigation, S.P., M.M.S., D.B. and L.-I.C.; writing—original draft preparation, S.P. and M.M.S.; writing—review and editing, S.P., M.M.S., D.B. and L.-I.C.; visualization, S.P., M.M.S., D.B. and L.-I.C.; supervision, M.M.S., D.B. and L.-I.C.; funding acquisition, D.B. and L.-I.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: We thank the referees for their valuable suggestions, which significantly improved this work.

Conflicts of Interest: The authors disclose no conflicts of interest.

References

1. Miller, S.S.; Mocanu, P.T. *Differential Subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*; Marcel Dekker Inc.: New York, NY, USA; Basel, Switzerland, 2000; Volume 225.
2. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; International Press: Cambridge, UK, 1994; pp. 157–169.
3. Sokól, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz. Mat.* **1996**, *19*, 101–105.
4. Janowski W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Polon. Math.* **1971**, *23*, 159–177. [CrossRef]
5. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Mat.* **2016**, *27*, 923–939. [CrossRef]
6. Wani, L.A.; Swaminathan, A. Starlike and convex functions associated with a nephroid domain. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 79–104. [CrossRef]
7. Sivaprasad Kumar, S.; Kamaljeet, G. A cardioid domain and starlike functions. *Anal. Math. Phys.* **2021**, *11*, 54.
8. Mendiratta, R.; Nagpal, S.; Ravichandran, V. A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli. *Int. J. Math.* **2014**, *25*, 1450090. [CrossRef]
9. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
10. Janowski, W. Some extremal problems for certain families of analytic functions. I. *Ann. Polon. Math.* **1973**, *28*, 297–326. [CrossRef]
11. Owa S.; Srivastava, H.M.: Some generalized convolution properties associated with certain subclasses of analytic functions. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 42.
12. Uralegaddi, B.A.; Ganigi, M.D.; Sarangi, S.M. Univalent functions with positive coefficients. *Tamkang J. Math.* **1994**, *25*, 225–230. [CrossRef]
13. Ravichandran, V.; Verma, S. Bound for the fifth coefficient of certain starlike functions. *C. R. Math. Acad. Sci. Paris* **2015**, *353*, 505–510. [CrossRef]
14. Palei, S.; Soren, M.M.; Cotîrlă, L.-I. Coefficient Bounds for Alpha-Convex Functions Involving the Linear q-Derivative Operator Connected with the Cardioid Domain. *Fractal Fract.* **2025**, *9*, 172. [CrossRef]
15. Krushkal, S.L. Proof of the Zalcman conjecture for initial coefficients. *Georgian Math. J.* **2010**, *17*, 663–681. [CrossRef]
16. Lecko, A.; Sim, Y.J. Coefficient Problems in the Subclasses of Close-to-Star Functions. *Results Math.* **2019**, *74*, 104. [CrossRef]
17. Ravichandran, V.; Verma, S. Generalized Zalcman conjecture for some classes of analytic functions. *J. Math. Anal. Appl.* **2017**, *450(1)*, 592–605. [CrossRef]

18. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* **1966**, *1*, 111–122. [CrossRef]
19. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
20. Ye, K.; Lim, L.H. Every matrix is a product of Toeplitz matrices. *Found. Comput. Math.* **2016**, *16*, 577–598. [CrossRef]
21. Kowalczyk, B.; Lecko, A. Second Hankel determinant of logarithmic coefficients of convex and starlike functions. *Bull. Aust. Math. Soc.* **2022**, *105*, 458–467. [CrossRef]
22. Kowalczyk, B.; Lecko, A. The second Hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2023**, *117*, 91. [CrossRef]
23. Murugusundaramoorthy, G.; Bulboacă, T. Hankel Determinants for New Subclasses of Analytic Functions Related to a Shell Shaped Region. *Mathematics* **2020**, *8*, 1041. [CrossRef]
24. Saliu, A.; Noor, K.I. On Coefficients Problems for Certain Classes of Analytic Functions. *J. Math. Anal.* **2021**, *12*, 13–22.
25. Zhang, H.-Y.; Huo, T. A study of fourth-order Hankel determinants for starlike functions connected with the sine function. *J. Funct. Spaces* **2021**, *2021*, 9991460. [CrossRef]
26. Shafiq, M.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Darus, M.; Kiran, S. An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers. *Symmetry* **2020**, *12*, 1043. [CrossRef]
27. Güney, H.Ö.; Murugusundaramoorthy, G.; Srivastava, H.M. The second hankel determinant for a certain class of bi-close-to-convex function. *Results Math.* **2019**, *74*, 13. [CrossRef]
28. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q-starlike functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
29. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. *Mathematics* **2019**, *7*, 181. [CrossRef]
30. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. *Bull. Sci. Math.* **2021**, *167*, 16. [CrossRef]
31. Hayman, W.K. On the second Hankel determinant of mean univalent functions. *Proc. Lond. Math. Soc.* **1968**, *18*, 77–94. [CrossRef]
32. Noonan, J.W.; Thomas, D.K. On the second Hankel determinant of areally mean p-valent functions. *Trans. Am. Math. Soc.* **1976**, *223*, 337–346. [CrossRef]
33. Jangteng, A.; Halim, S.A.; Darus, M. Coefficient inequality for a function whose derivative has a positive real part. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 1–5.
34. Orhan, H.; Magesh, N.; Yamini, J. Bounds for the second Hankel determinant of certain bi-univalent functions. *Turk. J. Math.* **2016**, *40*, 679–687. [CrossRef]
35. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *Inequal. Theory Appl.* **2007**, *6*, 1–7.
36. Zaprawa, P. Third Hankel determinants for subclasses of univalent functions. *Mediterr. J. Math.* **2017**, *14*, 10. [CrossRef]
37. Raza, M.; Malik, S.N. Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. *J. Inequal. Appl.* **2013**, *2013*, 8. [CrossRef]
38. Cho, N.E.; Kowalczyk, B.; Kwon, O.S.; Lecko, A.; Sim, J. Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha. *J. Math. Inequal.* **2017**, *11*, 429–439. [CrossRef]
39. Lecko, A.; Sim, Y.J.; Śmiarowska, B. The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. *Complex Anal. Oper. Theory* **2019**, *13*, 2231–2238. [CrossRef]
40. Srivastava, H.M.; Ahmad, Q.Z.; Darus, M.; Khan, N.; Khan, B.; Zaman, N.; Shah, H.H. Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli. *Mathematics* **2019**, *7*, 848. [CrossRef]
41. Duren, P.T. *Univalent Functions*; Springer: New York, NY, USA, 1983.
42. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
43. Pommerenke, C. *Univalent Functions*; Vandenhoeck and Ruprecht: Gottingen, Germany, 1975; Volume 1, p. 1.
44. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* **2018**, *18*, 307–314. [CrossRef]
45. Efraimidis, I. A generalization of Livingston's coefficient inequalities for functions with positive real part. *J. Math. Anal. Appl.* **2016**, *435*, 369–379. [CrossRef]
46. Libera, R.J.; Zlotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivatives in P. *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [CrossRef]
47. Krushkal, S.L. Univalent functions and holomorphic motions. *J. Anal. Math.* **1995**, *66*, 253–275. [CrossRef]
48. Ali, M.F.; Allu, V. On logarithmic coefficients of some close-to-convex functions. *Proc. Am. Math. Soc.* **2018**, *146*, 1131–1142. [CrossRef]

49. Ali, M.F.; Vasudevarao, A.; Thomas, D.K.; Lecko, A. On the third logarithmic coefficients of close-to- convex functions. In *Current Research in Mathematical and Computer Sciences II*; Publisher UWM: Olsztyn, Poland, 2018; pp. 271–278.
50. Girela, D. Logarithmic coefficients of univalent functions. *Ann. Acad. Sci. Fenn.* **2000**, *25*, 337–350.
51. Roth, O. A sharp inequality for the logarithmic coefficients of univalent functions. *Proc. Am. Math. Soc.* **2007**, *135*, 2051–2054. [CrossRef]
52. Thomas, D.K. On logarithmic coefficients of close to convex functions. *Proc. Am. Math. Soc.* **2016**, *144*, 1681–1687. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.