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The Estimation of a Signal Generated by a Dynamical System Modeled by McKean–Vlasov Stochastic Differential Equations Under Sampled Measurements

Vasile Dragan ^{1,2} and Samir Aberkane ^{3,4,*} 

¹ Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania; vasile.dragan@imar.ro

² Academy of Romanian Scientists, 3 Ilfov, 050044 Bucharest, Romania

³ Université de Lorraine, CRAN, UMR 7039, Campus Sciences, BP 70239, 54506 Vandoeuvre-les-Nancy Cedex, France

⁴ CNRS, CRAN, UMR 7039, 54506 Vandoeuvre-les-Nancy Cedex, France

* Correspondence: samir.aberkane@univ-lorraine.fr

Abstract: This paper addresses the problem of optimal H_2 -filtering for a class of continuous-time linear McKean–Vlasov stochastic differential equations under sampled measurements. The main tool used to solve the filtering problem is a forward jump matrix linear differential equation with a Riccati-type jumping operator. More specifically, the stabilizing solution of such a jump Riccati-type equation plays a key role.

Keywords: H_2 -filtering; stochastic systems; sampled measurements; McKean–Vlasov equations; jump Riccati equations.

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1. Introduction

Estimating a remote signal from the measurements of another observed signal is a classical problem in the systems theory with numerous applications in different real life problems. Such a problem is usually named in the literature a signal filtering problem or signal estimation/observation problem. The problem of filtering is fundamental in control theory. Since the pioneering works of Kalman and Bucy derived in the 1960s (see [1,2]), a huge amount of contributions has been made in this field stimulated by successful implementation in many applications including aerospace, signal processing, geophysics, etc.

On the other hand, control of McKean–Vlasov stochastic differential equations (SDEs) has attracted significant attention in the last decade. This is partly due to the fact that such SDEs can effectively characterize dynamical systems of large populations subject to a mean-field interaction. McKean–Vlasov SDEs appeared initially in the works of [3,4]. There are already several important contributions related to this type of stochastic systems from the control perspective [5–10]. More recently, LQ control problems as well as LQ games for such systems have been investigated by [11–16]. To the best of the authors’ knowledge, it seems that the filtering problem for McKean–Vlasov SDEs has been addressed in the literature very marginally when compared to the control counterpart. Very recently, we addressed in [17] a filtering problem for a class of linear periodic McKean–Vlasov SDEs where the metric used as an optimality measure of the proposed estimation scheme belongs to the H_2 -type norm setting. In [17], the proposed filter is fed at its input with continuous-time measurements in order to generate the *optimal* state estimation. Note,

however, that in modern control systems like digital control as well as networked control, the assumption of measurement continuity is no longer valid and one has to deal with a sampled-data setting. For deterministic systems and constant sampled-data rate, stability as well as optimal control problems have been successfully solved in the literature thanks to a discrete-time equivalent representation of the original system where such a representation is obtained without any kind of approximation. In the stochastic framework, the discrete-time equivalent representation cannot be obtained. This explains why sampled-data control for stochastic systems did not receive the attention it deserves until the last few years [18–23].

In the present paper, we will consider the problem of H_2 -optimal filtering for a class of linear McKean–Vlasov SDEs under sampled measurements. Such a problem is addressed under an impulsive SDE setting. The class of admissible filters consists of deterministic impulsive linear systems with arbitrary dimension of the state space. We show that the dimension of the optimal filter is equal to two times the dimension of the system under consideration. The corresponding feedback gain of the optimal filter is constructed based on the stabilizing solution of an adequately defined system of forward jump matrix linear differential equation with a Riccati-type jumping operator. The result obtained in this article can be viewed as an extension of the author's results from [17] to the sampled-data case.

This paper is organized as follows: Section 2 gives the mathematical model of the considered class of systems and describes the filtering problem. Section 3 gives several preliminary results related to the stability of McKean–Vlasov SDEs that will be used in Section 4 in order to obtain the formulae for the computation of the performance value. The main results are given in Section 5. Some numerical experiments are included in Section 6.

2. Problem Formulation

2.1. Notations

Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be a given probability space. In this work, $\{w(t)\}_{t \geq 0}$, $w(t) = \begin{pmatrix} w_1(t) & w_2(t) & \cdots & w_{r_1}(t) \end{pmatrix}^\top$ is an r_1 -dimensional standard Wiener process defined on $(\Omega, \mathfrak{F}, \mathcal{P})$. For precise definitions and useful properties of the stochastic processes, we refer for example to [24,25].

For each $t > 0$, $\mathfrak{F}_t \subset \mathfrak{F}$ is the σ -algebra generated by the random vectors $w(s)$, $0 \leq s \leq t$. For $t = 0$, the σ -algebra $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. We assume that for each $t \geq 0$, the σ -algebra $\mathfrak{F}_t$ is augmented with all subsets $\mathfrak{M} \in \mathfrak{F}$ with $\mathcal{P}(\mathfrak{M}) = 0$. For each $t_0 \geq 0$, $\mathbb{E}[\cdot | \mathfrak{F}_{t_0}]$ denotes the conditional expectation with respect to the σ -algebra $\mathfrak{F}_{t_0}$, while $\mathbb{E}[\cdot]$ denotes the mathematical expectation. Let $\{w_d(k)\}_{k \in \mathbb{Z}_+}$, $w_d(k) = \begin{pmatrix} w_{d_1}(k) & w_{d_2}(k) & \cdots & w_{d_{r_2}}(k) \end{pmatrix}^\top$ be a sequence of r_2 -dimensional independent random vectors. $\mathcal{G}_t \subset \mathfrak{F}$ stands for the smallest σ -algebra generated by the random vectors $w(s)$, $w_d(k)$, $0 \leq s \leq t$, $0 \leq k < \lceil \frac{t}{h} \rceil$, where $h > 0$ is given. The superscript $\top$ denotes the transposition of a vector or a matrix. If $t_0 \geq 0$, we denote $\mathcal{X}_{t_0}$ the set of random vectors $x_0 : \Omega \rightarrow \mathbb{R}^n$ which are $\mathfrak{F}_{t_0}$ -measurable and $\mathbb{E}[|x_0|^2] < \infty$.

2.2. Model Description

We consider the dynamical system $\mathbb{G}$ having the state space representation described by the following mean-field linear stochastic differential equations (MF-SLDE):

$$dx(t) = (A_0 x(t) + \bar{A}_0 \mathbb{E}_{t_0}[x(t)])dt + \sum_{l=1}^{r_1} (A_l x(t) + \bar{A}_l \mathbb{E}_{t_0}[x(t)])dw_l(t) + Bdv(t) \quad (1a)$$

$$z(t) = C_z x(t) + \bar{C}_z \mathbb{E}_{t_0}[x(t)] \quad (1b)$$

$$t > t_0, x(t_0) = x_0 \quad (1c)$$

$$y(kh) = C_0 x(kh) + \bar{C}_0 \mathbb{E}_{t_0}[x(kh)] + \sum_{p=1}^{r_2} w_{d_p}(k) (C_p x(kh) + \bar{C}_p \mathbb{E}_{t_0}[x(kh)]) + D v_d(k), \quad k = 0, 1, \dots \quad (2)$$

where $x(t) \in \mathbb{R}^n$ are the state parameters of the dynamical system $\mathbb{G}$, while $z(t) \in \mathbb{R}^{n_z}$ is the remote signal which must be estimated. In (2), $y(kh) \in \mathbb{R}^{n_y}$ are the measurements at the instance time $t_k = kh$, $k = 0, 1, \dots$, $h > 0$ being the sampling period. In (1) and (2), $\{w(t), t \geq 0\}$, $\{v(t), t \geq 0\}$, $\{w_d(k), k \in \mathbb{Z}_+\}$, $\{v_d(k), k \in \mathbb{Z}_+\}$ are stochastic processes which satisfy the following assumption:

Hypothesis 1 (H₁). (i) $\{w(t)\}_{t \geq 0}$ is a r_1 -dimensional standard Wiener process, i.e., $w(0) = 0$, $\mathbb{E}[w(t)] = 0$, $\mathbb{E}[(w(t) - w(s))(w(t) - w(s))^T] = I_{r_1}(t - s)$, for all $t \geq s \geq 0$, I_{r_1} being the identity matrix of size r_1 ;

(ii) $\{v(t)\}_{t \geq 0}$ is a m_v -dimensional standard Wiener process with zeros mean and $\mathbb{E}[(v(t) - v(s))(v(t) - v(s))^T] = V(t - s)$, for all $t \geq s \geq 0$, $V \geq 0$ being a known matrix;

(iii) $\{w_d(k)\}_{k \geq 0}$ is a sequence of r_2 -dimensional independent random vectors with zero mean and $\mathbb{E}[w_d(k)w_d^T(k)] = I_{r_2}$, for all $k \geq 0$;

(iv) $\{v_d(k)\}_{k \geq 0}$ is a sequence of m_{v_d} -dimensional independent random vectors with zero mean and $\mathbb{E}[v_d(k)v_d^T(k)] = V_d$, for all $k \geq 0$, $V_d \geq 0$ is a known matrix of appropriate dimension;

(v) $\{w(t)\}_{t \geq 0}$, $\{v(t)\}_{t \geq 0}$, $\{w_d(k)\}_{k \geq 0}$, $\{v_d(k)\}_{k \geq 0}$ are independent stochastic processes.

The differential equations of the form (1a) are known in the literature as mean-field stochastic linear differential equations (MF-SLDE) (see [5–10]).

2.3. The Filtering Problem

Our aim is to design a dynamic linear system $\mathbb{G}_F$ (named filter) which, fed with the measured values $y(kh)$, $k = 0, 1, \dots$, generates at its output a signal $z_F(t)$ which must be the best estimate of the output signal $z(t)$ of the dynamical system $\mathbb{G}$. Since a filter $\mathbb{G}_F$ is activated by a discrete-time signal $\{y(kh)\}_{k \geq 0}$ and it must provide at its output a continuous-time signal $z_F(\cdot)$, we consider that the most appropriate state-space representation of such a filter is of the following form:

$$\dot{x}_F(t) = A_F x_F(t), \quad kh < t \leq (k+1)h \quad (3a)$$

$$x_F(kh^+) = A_{F_d} x_F(kh) + B_{F_d} y(kh), \quad k \in \mathbb{Z}_+ \quad (3b)$$

$$z_F(t) = C_F x_F(t), \quad t \in \mathbb{R}_+ \quad (3c)$$

where $x_F(t) \in \mathbb{R}^{n_F}$ are the state parameters of the filter $\mathbb{G}_F$. In our approach, the dimension n_F of the state space of the filters $\mathbb{G}_F$ is not prefixed. In (3), $A_F \in \mathbb{R}^{n_F \times n_F}$, $A_{F_d} \in \mathbb{R}^{n_F \times n_F}$, $B_{F_d} \in \mathbb{R}^{n_F \times n_y}$, $C_F \in \mathbb{R}^{n_z \times n_F}$ are arbitrary matrices subject only to the constraint that the corresponding jump linear differential equation (JLDE)

$$\dot{x}_F(t) = A_F x_F(t), \quad kh < t \leq (k+1)h \quad (4a)$$

$$x_F(kh^+) = A_{F_d} x_F(kh), \quad k \in \mathbb{Z}_+ \quad (4b)$$

is exponentially stable (ES) (see [26]).

We denote $\mathfrak{F}_{\text{adm}}$ the set of all filters $\mathbb{G}_F$ having the state-space representation of the form (3) with arbitrary dimension $n_F \geq 1$, with the property that the corresponding JLDE (4) is ES. The elements of $\mathfrak{F}_{\text{adm}}$ are called admissible filters.

In order to measure the quality of the estimation achieved by an admissible filter $\mathbb{G}_F \in \mathfrak{F}_{\text{adm}}$, we introduce the following performance criterion:

$$\mathcal{J}(\mathbb{G}_F) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \mathbb{E} \left[|z(t) - z_F(t)|^2 \right] dt. \quad (5)$$

The problem we want to address in this work requires us to find an admissible filter $\tilde{\mathbb{G}}_F$ satisfying the following optimality condition:

$$\mathcal{J}(\tilde{\mathbb{G}}_F) = \min_{\mathbb{G}_F \in \mathfrak{F}_{\text{adm}}} \mathcal{J}(\mathbb{G}_F). \quad (6)$$

In Section 4, we shall compute the value of the function (5) achieved by an admissible filter $\mathbb{G}_F$. We shall see that the value of the right-hand side of (5) does not depend on the initial time t_0 and nor the initial states x_0 of the dynamical system $\mathbb{G}$ and x_{F_0} of the filter. In Section 5, we shall provide a set of conditions which allow us to construct the optimal filter $\tilde{\mathbb{G}}_F$.

Before doing so, we first discuss in Section 3 some aspects related to the mean-square stability of a system obtained when a filter (3) verifying (4) is coupled to an MF-SLDE (1a).

3. Several Preliminary Issues

3.1. Mean-Square Stability of a MF-SLDE

Let $x(t; t_0, x_0)$ be a solution of the MF-SLDE (1). We set

$$x^1(t) \triangleq x(t) - \mathbb{E}[x(t)] \quad (7a)$$

$$x^2(t) \triangleq \mathbb{E}[x(t)] \quad (7b)$$

$t \geq t_0$. Bearing in mind the properties of the stochastic integrals of Itô-type, we obtain that $(x^1(\cdot), x^2(\cdot))$ is the solution of the following initial value problem (IVP):

$$d \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} = \mathbb{A}_0 \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} dt + \sum_{l=1}^{r_1} \mathbb{A}_l \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} dw_l(t) + \mathbb{B} dv(t), \quad t > t_0 \quad (8a)$$

$$x^1(t_0) = 0, \quad x^2(t_0) = x_0, \quad (8b)$$

$$z(t) = C_z x^1(t) + \tilde{C}_z x^2(t), \quad t \geq t_0 \quad (8c)$$

In this case, the measured output (2) becomes

$$y(kh) = C_0 x^1(kh) + \tilde{C}_0 x^2(kh) + \sum_{p=1}^{r_2} w_{d_p}(k) \left(C_p x^1(kh) + \tilde{C}_p x^2(kh) \right) + D v_d(k) \quad (9)$$

where

$$\mathbb{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \tilde{A}_0 \end{pmatrix} \quad (10a)$$

$$\mathbb{A}_l = \begin{pmatrix} A_l & \tilde{A}_l \\ 0 & 0 \end{pmatrix}, \quad 1 \leq l \leq r_1 \quad (10b)$$

$$\tilde{A}_l = A_l + \tilde{A}_l, \quad 0 \leq l \leq r_1 \quad (10c)$$

$$\tilde{C}_z = C_z + \tilde{C}_z \quad (10d)$$

$$\tilde{C}_p = C_p + \tilde{C}_p, \quad 0 \leq p \leq r_2 \quad (10e)$$

$$\mathbb{B} = \begin{pmatrix} B \\ 0 \end{pmatrix} \quad (10f)$$

From (7), it follows that

$$\mathbb{E}_{t_0} \left[x^1(t) \left(x^2(t) \right)^\top \right] = 0 \quad (11a)$$

$$\mathbb{E} \left[x^1(t) \left(x^2(t) \right)^\top \right] = 0 \quad (11b)$$

$t \geq t_0 \geq 0$.

If $(x^1(\cdot), x^2(\cdot))$ is a solution of the IVP (8a)–(8b), we set $X(t) = \mathbb{E} \left[\begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix}^\top \right]$. Employing (11), we obtain that $X(t) = \begin{pmatrix} X_1(t) & 0 \\ 0 & X_2(t) \end{pmatrix}$ with

$$X_j = \mathbb{E} \left[x^j(t) \left(x^j(t) \right)^\top \right], \quad j = 1, 2 \quad (12)$$

By direct calculation based on Itô's formula, we obtain via (8a)–(8b) and (12) that $t \rightarrow (X_1(t), X_2(t)) : [t_0, \infty) \rightarrow \mathcal{S}_n \times \mathcal{S}_n$ is the solution of the following IVP on the space $\mathcal{H} \triangleq \mathcal{S}_n \times \mathcal{S}_n$:

$$\frac{d}{dt}(X_1(t), X_2(t)) = \mathcal{L}[X_1(t), X_2(t)], \quad t > t_0 \quad (13a)$$

$$(X_1(t_0), X_2(t_0)) = (0, \mathbb{E}[x_0 x_0^\top]) \quad (13b)$$

where $(X_1, X_2) \rightarrow \mathcal{L}[X_1, X_2] = (\mathcal{L}_1[X_1, X_2], \mathcal{L}_2[X_1, X_2]) : \mathcal{H} \rightarrow \mathcal{H}$ is described by

$$\mathcal{L}_1[X_1, X_2] = A_0 X_1 + X_1 A_0^\top + \sum_{l=1}^{r_1} (A_l X_1 A_l^\top + \tilde{A}_l X_2 \tilde{A}_l^\top) \quad (14a)$$

$$\mathcal{L}_2[X_1, X_2] = \tilde{A}_0 X_2 + X_2 \tilde{A}_0^\top \quad (14b)$$

Equipped with the inner product

$$([X_1, X_2], [Y_1, Y_2]) \triangleq \text{Tr}[X_1 Y_1] + \text{Tr}[X_2 Y_2] \quad (15)$$

$\mathcal{H}$ becomes a Hilbert space. Moreover, $\mathcal{H}$ is an ordered Hilbert space with respect to the ordering relation $\succsim$ induced by the convex cone $\mathcal{H}^+ = \{(X_1, X_2) \in \mathcal{H} | X_i \geq 0, i = 1, 2\}$. Here, $X_i \geq 0$ means that X_i is a positive semi-definite matrix.

The adjoint operator with respect to the inner product (15) of $\mathcal{L}[\cdot]$ defined in (14) is $\mathcal{L}^*[X_1, X_2] = (\mathcal{L}_1^*[X_1, X_2], \mathcal{L}_2^*[X_1, X_2])$ with

$$\mathcal{L}_1^*[X_1, X_2] = A_0^\top X_1 + X_1 A_0 + \sum_{k=1}^{r_1} A_k^\top X_1 A_k \quad (16a)$$

$$\mathcal{L}_2^*[X_1, X_2] = \tilde{A}_0^\top X_2 + X_2 \tilde{A}_0 + \sum_{k=1}^{r_1} \tilde{A}_k^\top X_1 \tilde{A}_k \quad (16b)$$

Based on the operator $\mathcal{L}[\cdot]$, we may consider the following IVP on the Hilbert space $\mathcal{H}$:

$$\frac{d}{dt}(X_1(t), X_2(t)) = \mathcal{L}[X_1(t), X_2(t)], \quad t > t_0 \quad (17a)$$

$$[X_1(t_0), X_2(t_0)] = [H_1, H_2], \quad t_0 \in \mathbb{R}_+, \quad H_i \in \mathcal{S}_n, \quad i = 1, 2 \quad (17b)$$

The IVP (13) is a particular case of (17) when $H_1 = 0, H_2 = \mathbb{E}[x_0 x_0^\top]$. According to Theorem 2.6.1 from [27], we may infer that the linear differential Equation (17a) defined a positive evolution on the ordered Hilbert space $(\mathcal{H}, \mathcal{H}^+)$. This means that if $(X_1(\cdot; t_0, \mathbf{H}), X_2(\cdot; t_0, \mathbf{H}))$

is a solution of the IVP (17), then $(X_1(t; t_0, \mathbf{H}), X_2(t; t_0, \mathbf{H})) \in \mathcal{H}^+, \forall t \geq t_0 \geq 0$, whenever $\mathbf{H} = (H_1, H_2) \in \mathcal{H}^+$.

Definition 1. We say that the zero solution of the MF-SLDE (1a) is

(i) exponentially stable in the mean-square sense (ESMS) if its solutions $x(\cdot; t_0, x_0)$ are satisfying

$$\mathbb{E}[|x(t; t_0, x_0)|^2] \leq \beta e^{-\alpha(t-t_0)} |x_0|^2 \quad (18)$$

$\forall t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n$, where $\beta \geq 1, \alpha > 0$ are constants not depending upon t, t_0, x_0 ,

(ii) MF-strong exponentially stable (MF-SES) if the linear differential equation (17a) is exponentially stable, which means that the solutions of the IVP (17) are satisfying

$$\|X_1(t; t_0, H_1, H_2)\|_2 + \|X_2(t; t_0, H_1, H_2)\|_2 \leq \tilde{\beta} e^{-\tilde{\alpha}(t-t_0)} (\|H_1\|_2 + \|H_2\|_2) \quad (19)$$

$\forall t \geq t_0 \geq 0, (H_1, H_2) \in \mathcal{H}, \tilde{\beta} \geq 1, \tilde{\alpha} > 0$ being constants not depending upon t, t_0, H_1, H_2 .

Here and in the sequel, $\|\cdot\|_2$ denotes the euclidian norm of a symmetric matrix.

Proposition 1. Under the assumption $\mathbf{H}_1$, the zero solution of the MF-SLDE (1a) is ESMS if it is MF-SES.

Proof. Let us assume that the zero solution of the MF-SLDE (1a) is MF-SES. Bearing in mind that the IVP (13) is a special case of (17), we obtain via (12) that

$$\left\| \mathbb{E} \left[x^1(t) (x^1(t))^T \right] \right\|_2 + \left\| \mathbb{E} \left[x^2(t) (x^2(t))^T \right] \right\|_2 \leq \tilde{\beta} e^{-\tilde{\alpha}(t-t_0)} \|x_0 x_0^T\|_2 \quad (20)$$

for all $t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n$. Taking into account that $\left\| \mathbb{E} \left[x^j(t) (x^j(t))^T \right] \right\|_2 = \mathbb{E} [|x^j(t)|^2]$, $j = 1, 2$, we obtain from (20) that $\mathbb{E} [|x^j(t)|^2] \leq \tilde{\beta} e^{-\tilde{\alpha}(t-t_0)} |x_0|^2, j = 1, 2$. This allows us to conclude that the solution $x(\cdot; t_0, x_0)$ is satisfying (18) because from (7), we have $x(t; t_0, x_0) = x^1(t) + x^2(t)$. Thus, the proof is complete. $\square$

The next developments from this paper are performed under the following assumption:

Hypothesis 2 ($\mathbf{H}_2$). The zero solution of the MF-SLDE (1a) is MF-SES.

Following step by step the proof of Proposition 3.2 from [17], we obtain a set of conditions equivalent to the assumption $\mathbf{H}_2$.

Proposition 2. Under the assumption $\mathbf{H}_1$ (i), the following are equivalent:

- (i) $\mathbf{H}_2$ holds;
- (ii) the zero solution of the SLDE

$$dx(t) = A_0 x(t) dt + \sum_{l=1}^{r_1} A_l x(t) dw_l(t)$$

is ESMS and $\tilde{A}_0$ is a Hurwitz matrix;

- (iii) the eigenvalues of the linear operator $\mathcal{L}[\cdot]$ defined in (14) are located in the half-plane $\mathbb{C}^- = \{\lambda \in \mathbb{C}; \operatorname{Re}\{\lambda\} < 0\}$.

3.2. Resulting Jump Linear Stochastic System

Plugging (9) in (3), we obtain the following resulting (closed-loop) jump linear system (JLSS):

$$dx_{cl}(t) = A_{0,cl}x_{cl}(t) + \sum_{l=1}^{r_1} A_{l,cl}x_{cl}(t)dw_l(t) + B_{cl}dv(t), \quad kh < t \leq (k+1)h, \quad (21a)$$

$$x_{cl}(kh^+) = A_{d,0,cl}x_{cl}(kh) + \sum_{p=1}^{r_2} w_{d,p}(k)A_{d,p,cl}x_{cl}(kh) + D_{cl}v_d(k) \quad (21b)$$

$$z_{cl}(t) = z(t) - z_F(t) = C_{cl}x_{cl}(t), \quad t \in \mathbb{R}_+ \quad (21c)$$

where $x_{cl} = \left((x^1(t))^\top \quad (x^2(t))^\top \quad (x_F(t))^\top \right)^\top \in \mathbb{R}^{\hat{n}}, \hat{n} = 2n + n_F$ and

$$A_{0,cl} = \begin{pmatrix} \mathbb{A}_0 & 0 \\ 0 & A_F \end{pmatrix}, \quad A_{l,cl} = \begin{pmatrix} \mathbb{A}_l & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 \leq l \leq r_1 \quad (22a)$$

$$A_{d,0,cl} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ B_{F,d}C_0 & B_{F,d}\tilde{C}_0 & A_{F,d} \end{pmatrix}, \quad A_{d,p,cl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{F,d}C_p & B_{F,d}\tilde{C}_p & 0 \end{pmatrix} \quad (22b)$$

$$C_{cl} = \begin{pmatrix} C_z & \tilde{C}_z & -C_F \end{pmatrix}, \quad B_{cl} = \begin{pmatrix} \mathbb{B} \\ 0 \end{pmatrix}, \quad D_{cl} = \begin{pmatrix} 0 \\ 0 \\ B_{F,d}D \end{pmatrix} \quad (22c)$$

Let $t_0 \geq 0$ and $x_{cl,0} = \left(0 \quad x_0^\top \quad x_{F,0}^\top \right)^\top \in \mathbb{R}^{\hat{n}}, \hat{n} = 2n + n_F$ be arbitrary. Let $x_{cl}(t) = x_{cl}(t; t_0, x_{cl,0})$ be the solution of the resulting JLSS (21) corresponding to the initial par $(t_0, x_{cl,0})$. Starting from (21c), we may write successively

$$\begin{aligned} \mathbb{E}[|z(t) - z_F(t)|^2] &= \text{Tr}[\mathbb{E}[z_{cl}(t)z_{cl}^\top(t)]] = \text{Tr}[C_{cl}\mathbb{E}[x_{cl}(t)x_{cl}^\top(t)]C_{cl}^\top] \\ &= \text{Tr}[C_{cl}Y_{cl}(t)C_{cl}^\top] \end{aligned} \quad (23)$$

for all $t \geq t_0$, where we denote

$$Y_{cl}(t) \triangleq \mathbb{E}[x_{cl}(t)x_{cl}^\top(t)], \quad t \geq t_0 \quad (24)$$

According to (23), we may rewrite the performance criterion (5) in the following equivalent form:

$$\mathcal{J}(\mathbb{G}_F) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr}[C_{cl}Y_{cl}(t)C_{cl}^\top]dt \quad (25)$$

Employing Itô's formula on each interval $[kh, (k+1)h], k \geq \kappa(t_0)$ and $[t_0, \kappa(t_0)h]$, respectively, we obtain the following:

Proposition 3. Under the assumption $\mathbf{H}_1$, the function $t \rightarrow Y_{cl}(t) : [t_0, \infty) \rightarrow \mathcal{S}_{\hat{n}}$ defined by (24) is the solution of the following IVP:

$$\dot{Y}_{cl}(t) = \mathcal{L}_{c,cl}[Y_{cl}(t)] + B_{cl}VB_{cl}^\top, \quad kh < t \leq (k+1)h \quad (26a)$$

$$Y_{cl}(kh^+) = \mathcal{L}_{d,cl}[Y_{cl}(kh)] + D_{cl}V_dD_{cl}^\top, \quad k \geq \kappa(t_0) \quad (26b)$$

$$Y_{cl}(t_0) = \mathbb{E}[x_{cl,0}(x_{cl,0})^\top] \quad (26c)$$

where

$$\mathcal{L}_{c,cl}[Y] = A_{0,cl}Y + YA_{0,cl}^\top + \sum_{l=1}^{r_1} A_{l,cl}YA_{l,cl}^\top \quad (27a)$$

$$\mathcal{L}_{d,cl}[Y] = \sum_{p=0}^{r_2} A_{d,p,cl}YA_{d,p,cl}^\top \quad (27b)$$

for all $Y \in \mathcal{S}_{\hat{n}}$, $\kappa(t_0)$ being the integer with the property that $\kappa(t_0 - 1)h < t_0 \leq \kappa(t_0)h$.

Let $\mathbb{S}_{\hat{n}} \subset \mathcal{S}_{\hat{n}}$ be the linear subspace of symmetric matrices of size $\hat{n} \times \hat{n}$ having the structure

$$\mathbf{Y} = \begin{pmatrix} Y_1 & 0 & Y_{13} \\ 0 & Y_2 & Y_{23} \\ Y_{13}^\top & Y_{23}^\top & Y_3 \end{pmatrix} \quad (28)$$

where Y_1, Y_2 are arbitrary symmetric matrices of size $n \times n$ and Y_3 is an arbitrary matrix from $\mathcal{S}_{n_F}$, while $Y_{j3}, j = 1, 2$ are arbitrary matrices from $\mathbb{R}^{n \times n_F}$. One checks that $\mathbb{S}_{\hat{n}}$ is a closed linear subspace. Hence, it is a Hilbert space with respect to the restriction of the inner product (15).

Remark 1. Based on (10), (11), and (27), one can show that if $Y_{cl}(\cdot; t_0, \mathbf{Y}_0)$ is the solution of the jump matrix linear differential Equations (26a) and (26b) with the initial condition, $Y_{cl}(t_0; t_0, \mathbf{Y}_0) = \mathbf{Y}_0$ lies in $\mathbb{S}_{\hat{n}}$ for any $t > t_0$ whenever $\mathbf{Y}_0 \in \mathbb{S}_{\hat{n}}$. Particularly, the solution $Y_{cl}(t)$ of the IVP (26) is in $\mathbb{S}_{\hat{n}}$ for all $t \geq t_0$.

Further, we denote $\mathbb{S}_{2n}$ the linear subspace of $\mathcal{S}_{2n}$ formed by the matrices

$$\mathbf{Y} = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \quad (29)$$

where $Y_j, j = 1, 2$, are arbitrary matrices from $\mathcal{S}_n$.

Remark 2. From (29), one sees that $\mathbf{Y} \rightarrow (Y_1, Y_2)$ is an isomorphism between the subspaces $\mathbb{S}_{2n}$ and $\mathcal{H}$.

Let $Y \rightarrow \mathbb{L}[Y] : \mathbb{S}_{2n} \rightarrow \mathbb{S}_{2n}$ be the linear operator defined by

$$\mathbb{L}[Y] = \mathbb{A}_0Y + Y\mathbb{A}_0^\top + \sum_{l=1}^{r_1} \mathbb{A}_lY\mathbb{A}_l^\top \quad (30)$$

for all $Y \in \mathbb{S}_{2n}$.

Remark 3. By direct calculation involving (10a), (10b), (29), and (30), we obtain the equality

$$\mathbb{L}[\mathbf{Y}] = (\mathcal{L}_1[Y_1, Y_2], \mathcal{L}_2[Y_1, Y_2],) \quad (31)$$

for all $\mathbf{Y} \in \mathbb{S}_{2n}$, Y_1, Y_2 being related by $\mathbf{Y}$ via (29).

From (31), we may infer that the spectrum of $\mathbb{L}[\cdot]$ defined by (30) coincides with the spectrum of the operator $\mathcal{L}[\cdot]$ introduced in (14).

The next result will play an important role in the computation of the limit from the right-hand side of (25).

Theorem 1. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, for each admissible filter $\mathbb{G}_F$, the zero solution of the corresponding jump matrix linear differential equation (JMLDE)

$$\dot{\mathbf{Y}}(t) = \mathcal{L}_{c,cl}[\mathbf{Y}(t)], \quad kh < t \leq (k+1)h \quad (32a)$$

$$\mathbf{Y}(kh^+) = \mathcal{L}_{d,cl}[\mathbf{Y}(kh)], \quad k \in \mathbb{Z} \quad (32b)$$

is exponentially stable with respect to the invariant subspace $\mathbb{S}_{\hat{n}}$. This means that

$$\|\mathbf{Y}(t; t_0, \mathbf{Y}_0)\| \leq \beta e^{-\alpha(t-t_0)} \|\mathbf{Y}_0\| \quad (33)$$

for all $t \geq t_0 \geq 0$, $\mathbf{Y}_0 \in \mathbb{S}_{\hat{n}}$, $\beta \geq 1$, $\alpha > 0$ being constants which are not depending upon t , t_0 , $\mathbf{Y}_0$ and $\|\cdot\|$ being a norm on $\mathbb{S}_{\hat{n}}$.

Proof. See Appendix A. $\square$

Corollary 1. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, for each admissible filter $\mathbb{G}_F$, the eigenvalues of the linear operator $\mathbf{Y} \rightarrow e^{\mathcal{L}_{c,cl}h} \mathcal{L}_{d,cl}[\mathbf{Y}] : \mathbb{S}_{\hat{n}} \rightarrow \mathbb{S}_{\hat{n}}$ are located in the disk $|\lambda| < 1$.

Proof. Based on the one-to-one correspondence between the set of the solutions of the JMLDE (32) and the set of the solutions of the discrete-time linear equation on $\mathbb{S}_{\hat{n}}$

$$\mathbf{X}(k+1) = e^{\mathcal{L}_{c,cl}h} \mathcal{L}_{d,cl}[\mathbf{X}(k)] \quad (34)$$

one shows that under the considered assumptions, the zero solution of (34) is ES, which leads to the desired conclusion. $\square$

4. Computation of the Performance Value Achieved by an Admissible Filter

In this section, we show how one computes the value of the functional (5) which measures the performance achieved by an admissible filter $\mathbb{G}_F$.

Proposition 4. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, for each admissible filter $\mathbb{G}_F$, the corresponding non-homogeneous forward jump matrix linear differential equation (FJMLDE)

$$\dot{\mathbf{Y}}_{cl}(t) = \mathcal{L}_{c,cl}[\mathbf{Y}_{cl}(t)] + B_{cl} V B_{cl}^\top, \quad kh < t \leq (k+1)h \quad (35a)$$

$$\mathbf{Y}_{cl}(kh^+) = \mathcal{L}_{d,cl}[\mathbf{Y}_{cl}(kh)] + D_{cl} V_d D_{cl}^\top, \quad k \in \mathbb{Z} \quad (35b)$$

has a unique bounded solution $t \rightarrow \tilde{\mathbf{Y}}_{cl}(t) : \mathbb{R} \rightarrow \mathbb{S}_{\hat{n}}$. This solution has the following properties:

- (i) is a continuously differentiable function on each interval $(kh, (k+1)h)$, left-continuous in $t_k = kh, k \in \mathbb{Z}$;
- (ii) $\tilde{\mathbf{Y}}_{cl}(t) \geq 0, t \in \mathbb{R}$;
- (iii) $\tilde{\mathbf{Y}}_{cl}(\cdot)$ is a periodic function of period h .

Proof. Let us consider the linear affine equation on the Hilbert space $\mathbb{S}_{\hat{n}}$

$$\mathbf{X} = \mathcal{L}_{d,cl} \left[e^{\mathcal{L}_{c,cl}h} [\mathbf{X}] \right] + \Theta \quad (36)$$

where

$$\Theta = D_{cl} V_d D_{cl}^\top + \mathcal{L}_{d,cl} \left[\int_0^h e^{\mathcal{L}_{c,cl}s} \left[B_{cl} V B_{cl}^\top \right] ds \right] \quad (37)$$

Bearing in mind that $\lambda \neq 0$ is an eigenvalue of the linear operator $\mathbf{X} \rightarrow \mathcal{L}_{d,cl} [e^{\mathcal{L}_{c,cl}h}[\mathbf{X}]]$ if and only if it is an eigenvalue of the linear operator $\mathbf{X} \rightarrow e^{\mathcal{L}_{c,cl}h}[\mathcal{L}_{d,cl}[\mathbf{X}]]$, we obtain via Corollary 1 that, under the considered assumptions, the eigenvalues of the operator $\mathbf{X} \rightarrow \mathcal{L}_{d,cl} [e^{\mathcal{L}_{c,cl}h}[\mathbf{X}]]$ are located in the disk $|\lambda| < 1$. Based on (iii) and (iv) from Theorem 2.6 in [28] we may conclude that Equation (36) has a unique solution $\tilde{\mathbf{X}} \in \mathbb{S}_{\hat{n}}$. Moreover, $\tilde{\mathbf{X}} \geq 0$ because from (37), it follows that $\Theta \geq 0$.

We set

$$\begin{aligned} \tilde{\mathbf{Y}}_{cl}(t) &:= e^{\mathcal{L}_{c,cl}(t-kh)} [\tilde{\mathbf{X}}] \\ &+ \int_{kh}^t e^{\mathcal{L}_{c,cl}(t-s)} [B_{cl} V B_{cl}^\top] ds, \end{aligned} \quad (38)$$

if $t \in (kh, (k+1)h]$, $k \in \mathbb{Z}$. From (38), we deduce that $\tilde{\mathbf{Y}}_{cl}(kh^+) := \lim_{\substack{t \rightarrow kh \\ t > kh}} \tilde{\mathbf{Y}}_{cl}(t) = \tilde{\mathbf{X}}$. With this equality, (38) becomes

$$\begin{aligned} \tilde{\mathbf{Y}}_{cl}(t) &:= e^{\mathcal{L}_{c,cl}(t-kh)} [\tilde{\mathbf{Y}}_{cl}(kh^+)] \\ &+ \int_{kh}^t e^{\mathcal{L}_{c,cl}(t-s)} [B_{cl} V B_{cl}^\top] ds, \text{ if } t \in (kh, (k+1)h], k \in \mathbb{Z} \end{aligned} \quad (39)$$

for all $kh < t \leq (k+1)h$, $k \in \mathbb{Z}$. From (36) and (39), written for k replaced by $k-1$, we obtain

$$\tilde{\mathbf{Y}}_{cl}(kh^+) = \mathcal{L}_{d,cl} [\tilde{\mathbf{Y}}_{cl}(kh)] + D_{cl} V_d D_{cl}^\top$$

Hence, the function $\tilde{\mathbf{Y}}_{cl}(\cdot)$ defined by (38) satisfies (35b). Also, from (38), one sees that $\tilde{\mathbf{Y}}_{cl}(\cdot)$ is differentiable on each interval $(kh, (k+1)h)$ and it satisfies (35a).

Since $\tilde{\mathbf{X}} \geq 0$ and $e^{\mathcal{L}_{c,cl}(t-s)}$ is a positive operator for all $t \geq s$, we may conclude via (38) that $\tilde{\mathbf{Y}}_{cl}(t) \geq 0$, $\forall t \in \mathbb{R}_+$. Finally, (38) allows us to deduce by direct calculations that $\tilde{\mathbf{Y}}_{cl}(t+h) = \tilde{\mathbf{Y}}_{cl}(t)$ for all $t \in \mathbb{R}$. Thus, we have shown that $\tilde{\mathbf{Y}}_{cl}(\cdot)$ is a bounded solution of the FJMLDE (35). $\square$

The next result provides a formula for the computation of the value of the performance achieved by an admissible filter.

Theorem 2. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the value of the performance criterion (5) achieved by an admissible filter $\mathbb{G}_F$ is given by

$$\mathcal{J}(\mathbb{G}_F) = \frac{1}{h} \int_0^h \text{Tr} [C_{cl} \tilde{\mathbf{Y}}_{cl}(t) C_{cl}^\top] dt \quad (40)$$

$\tilde{\mathbf{Y}}_{cl}(\cdot)$ being the unique bounded $\mathbb{R}$ solution of the corresponding non-homogeneous FJMLDE (26a)–(26b).

Proof. Let $\mathbb{G}_F \in \mathfrak{F}_{\text{adm}}$ be arbitrary but fixed. According to Proposition 4, the corresponding non-homogeneous FJMLDE (26a)–(26b) has a unique solution $\tilde{\mathbf{Y}}_{cl}(\cdot)$ which is bounded on $\mathbb{R}$. Using the equivalent version of (5) given in (25) we obtain that

$$\begin{aligned} \mathcal{J}(\mathbb{G}_F) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr} [C_{cl} \tilde{\mathbf{Y}}_{cl}(t) C_{cl}^\top] dt + \\ &+ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr} [C_{cl} (\mathbf{Y}_{cl}(t) - \tilde{\mathbf{Y}}_{cl}(t)) C_{cl}^\top] dt \end{aligned} \quad (41)$$

Since $t \rightarrow Y_{cl}(t) - \tilde{Y}_{cl}(t)$ is a solution of (32), we may infer via (33) that $\|Y_{cl}(t) - \tilde{Y}_{cl}(t)\| \leq \beta e^{-\alpha(t-t_0)} \|x_{cl,0} x_{cl,0}^\top - \tilde{Y}_{cl}(t_0)\|$, for all $t \geq t_0$. Hence, we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr}[C_{cl}(Y_{cl}(t) - \tilde{Y}_{cl}(t))C_{cl}^\top] dt = 0, \text{ for all } t_0 \in \mathbb{R}.$$

Substituting the above equality in (41), we obtain that

$$\mathcal{J}(\mathbb{G}_F) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top] dt \quad (42)$$

Since $t \rightarrow \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top] : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function, we may obtain that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top] dt &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top] dt \\ &= \frac{1}{h} \int_0^h \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top] dt \end{aligned} \quad (43)$$

for all $t_0 \in \mathbb{R}$. For the last equality from (43) we took into account that $t \rightarrow \text{Tr}[C_{cl}\tilde{Y}_{cl}(t)C_{cl}^\top]$ is a periodic function of period h . Finally, from (42) and (43), we obtain that (40) holds. Thus, the proof is complete. $\square$

Let $\begin{pmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^\top(t) & Y_{22}(t) \end{pmatrix}, t \in \mathbb{R}$ be the partition of the matrix $\tilde{Y}_{cl}(t)$ such that $Y_{11}(t) \in \mathbb{S}_{2n}$ and $Y_{22}(t) \in \mathbb{S}_{n_F}$. Employing (22), (27), we obtain the following partition of the FJMLDE (26a)–(26b) written for its bounded on $\mathbb{R}$ solution:

$$\dot{Y}_{11}(t) = \mathbb{A}_0 Y_{11}(t) + Y_{11}(t) \mathbb{A}_0^\top + \sum_{l=1}^{r_1} \mathbb{A}_l Y_{11}(t) \mathbb{A}_l^\top + \mathbb{B} V \mathbb{B}^\top, \quad kh < t \leq (k+1)h \quad (44a)$$

$$Y_{11}(kh^+) = Y_{11}(kh), \quad k \in \mathbb{Z} \quad (44b)$$

$$\dot{Y}_{12}(t) = \mathbb{A}_0 Y_{12}(t) + Y_{12}(t) A_F^\top, \quad kh < t \leq (k+1)h \quad (45a)$$

$$Y_{12}(kh^+) = Y_{11}(kh) \mathbb{C}_0^\top B_{F,d}^\top + Y_{12}(kh) A_{F,d}^\top, \quad k \in \mathbb{Z} \quad (45b)$$

$$\dot{Y}_{22}(t) = A_F Y_{22}(t) + Y_{22}(t) A_F^\top, \quad kh < t \leq (k+1)h \quad (46a)$$

$$\begin{aligned} Y_{22}(kh^+) &= \sum_{p=1}^{r_2} B_{F,d} \mathbb{C}_p Y_{11}(kh) \mathbb{C}_p^\top B_{F,d}^\top + B_{F,d} \mathbb{C}_0 Y_{12}(kh) A_{F,d}^\top \\ &\quad + A_{F,d} Y_{12}^\top(kh) \mathbb{C}_0^\top B_{F,d}^\top + A_{F,d} Y_{22}(kh) A_{F,d}^\top + B_{F,d} D V_d D^\top B_{F,d}^\top, \quad k \in \mathbb{Z} \end{aligned} \quad (46b)$$

where

$$\mathbb{C}_p = \begin{pmatrix} C_p & \tilde{C}_p \end{pmatrix} \quad (47)$$

$\tilde{C}_p$ being introduced in (10e).

From (44), it follows that $Y_{11}(\cdot)$ is continuously differentiable and solves the linear ordinary differential equation on the Hilbert space $\mathbb{S}_{2n}$

$$\dot{Y}_{11}(t) = \mathbb{L}[Y_{11}(t)] + \mathbb{B} V \mathbb{B}^\top, \quad t \in \mathbb{R} \quad (48)$$

$\mathbb{L}[\cdot]$ being the linear operator defined by (30).

Proposition 5. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the ordinary differential Equation (48) has a unique bounded $\mathbb{R}$ solution $t \rightarrow \tilde{Y}_{11}(t) : \mathbb{R} \rightarrow \mathbb{S}_{2n}$. Moreover, $\tilde{Y}_{11}(\cdot)$ is a constant function that solves the algebraic equation

$$\mathbb{L}[\tilde{Y}_{11}] + \mathbb{B} V \mathbb{B}^\top = 0 \quad (49)$$

and has the structure

$$\tilde{Y}_{11} = \begin{pmatrix} \Pi_c & 0 \\ 0 & 0 \end{pmatrix} \quad (50)$$

where $\Pi_c = \Pi_c^\top \geq 0$ is the unique solution of the linear algebraic equation

$$A_0 \Pi_c + \Pi_c A_0^\top + \sum_{l=1}^{r_1} A_l \Pi_c A_l^\top + BVB^\top = 0 \quad (51)$$

Proof. First, let us remark that Proposition 2 together with Remark 3, allows us to deduce that under the assumption H_2 , the eigenvalues of the linear operator $\mathbb{L}[\cdot]$ are located in the semi-plane $\mathbb{C}^-$. Invoking Theorem 2.3.7 (ii)–(iii) from [27], we may conclude that under these conditions, the ordinary differential Equation (48) has a unique bounded $\mathbb{R}$ solution and, additionally, that solution is a constant one. Taking $\begin{pmatrix} \tilde{Y}_1 & \tilde{Y}_{12} \\ \tilde{Y}_{12}^\top & \tilde{Y}_{22} \end{pmatrix}$ as the partition of the matrix $\tilde{Y}_{11}$ such that $\tilde{Y}_1, \tilde{Y}_{12} \in \mathcal{S}_n$, we obtain via (10a), (10b), (10f) that $\tilde{Y}_{22} = 0$, $\tilde{Y}_{12} = 0$ and $\tilde{Y}_1$ is a solution of Equation (51).

From Proposition 2 (ii), we may conclude that (51) has a unique solution; this means that $\tilde{Y}_1 = \Pi_c$. Hence, the proof is complete. $\square$

5. State Space Representation of the Optimal Filter

5.1. Forward Jump Matrix Linear Differential Equation with Riccati-Type Jumping Operator

We consider the following FJMLDE on the Hilbert space $\mathcal{S}_{2n}$:

$$\dot{Y}(t) = A_0 Y(t) + Y(t) A_0^\top + \tilde{M}, \quad kh < t \leq (k+1)h \quad (52a)$$

$$Y(kh^+) = Y(kh) - Y(kh) C_0^\top \left(\tilde{R}_d + C_0 Y(kh) C_0^\top \right)^{-1} C_0 Y(kh), \quad k \in Z \quad (52b)$$

where

$$\tilde{M} := BVB^\top + \sum_{l=1}^{r_1} A_l \tilde{Y}_{11} A_l^\top \quad (53a)$$

$$\tilde{R}_d := DV_d D^\top + \sum_{p=1}^{r_2} C_p \tilde{Y}_{11} C_p^\top \quad (53b)$$

with A_l , $0 \leq l \leq r_1$ are defined in (10a)–(10b), while C_p , $0 \leq p \leq r_2$ are defined in (47). In (53), $\tilde{Y}_{11}$ is the unique solution of Equation (49). Employing Proposition 5, we obtain that under the assumption H_2 , we have

$$\tilde{M} = \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & 0 \end{pmatrix} \quad (54a)$$

$$\tilde{M}_1 = BVB^\top + \sum_{l=1}^{r_1} A_l \Pi_c A_l^\top \quad (54b)$$

$$\tilde{R}_d = DVD^\top + \sum_{p=1}^{r_2} C_p \Pi_c C_p^\top \quad (54c)$$

Since the right-hand side of (52b) looks like a Riccati-type operator arising in connection with a discrete-time linear quadratic (LQ) control problem, we shall call Equation (52) an FJMLDE with Riccati-type jumping operator.

Definition 2. A solution $t \rightarrow Y_s(t) : R \rightarrow \mathcal{S}_{2n}$ of the FJMLDE (52) is a stabilizing solution if the following hold:

- (a) $\tilde{R}_d + \mathbb{C}_0 \mathbf{Y}_s(kh) \mathbb{C}_0^\top, k \in \mathbb{Z}$ are invertible matrices;
 (b) the zero solution of the closed-loop JLDE on $\mathbb{R}^{2n}$

$$\dot{\mathbf{x}}(t) = \mathbb{A}_0 \mathbf{x}(t), kh < t \leq (k+1)h \quad (55a)$$

$$\mathbf{x}(kh^+) = (I_{2n} + \mathbf{K}_s(kh) \mathbb{C}_0) \mathbf{x}(kh), k \in \mathbb{Z} \quad (55b)$$

is ES, where

$$\mathbf{K}_s(kh) = -\mathbf{Y}_s(kh) \mathbb{C}_0^\top \left(\tilde{R}_d + \mathbb{C}_0 \mathbf{Y}_s(kh) \mathbb{C}_0^\top \right)^{-1} \quad (56)$$

By direct calculation, the FJMLDE (52) written for $\mathbf{Y}_s(\cdot)$ instead of $\mathbf{Y}(\cdot)$ may be rewritten as

$$\dot{\mathbf{Y}}_s(t) = \mathbb{A}_0 \mathbf{Y}_s(t) + \mathbf{Y}_s(t) \mathbb{A}_0^\top + \tilde{M}, kh < t \leq (k+1)h \quad (57a)$$

$$\begin{aligned} \mathbf{Y}_s(kh^+) &= (I_{2n} + \mathbf{K}_s(kh) \mathbb{C}_0) \mathbf{Y}_s(kh) (I_{2n} + \mathbf{K}_s(kh) \mathbb{C}_0)^\top \\ &+ \mathbf{K}_s(kh) \tilde{R}_d \mathbf{K}_s^\top(kh), k \in \mathbb{Z} \end{aligned} \quad (57b)$$

- Remark 4.** (a) From (57), we may infer that under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the bounded and stabilizing solution $\mathbf{Y}_s(\cdot)$ of the FJMLDE (52) satisfies $\mathbf{Y}_s(t) \geq 0, \forall t \in \mathbb{R}$.
 (b) From Proposition 2 (ii), we deduce that under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the eigenvalues of the matrices A_0 and $\tilde{A}_0$ are located in the half plane $\mathbb{C}^-$. Hence, $\mathbb{A}_0$ is a Hurwitz matrix.
 (c) Subtracting (52) (written for $\mathbf{Y}(\cdot)$ replaced by $\mathbf{Y}_s(\cdot)$) from (48), we obtain that $\Delta(\cdot) \triangleq \tilde{Y}_{11}(\cdot) - \mathbf{Y}_s(\cdot)$ is a bounded $\mathbb{R}$ solution of the following FJMLDE:

$$\dot{\Delta}(t) = \mathbb{A}_0 \Delta(t) + \Delta(t) \mathbb{A}_0^\top, kh < t \leq (k+1)h \quad (58a)$$

$$\Delta(kh^+) = \Delta(kh) + G(kh), k \in \mathbb{Z} \quad (58b)$$

where $G(kh) \triangleq \mathbf{K}_s(kh) (\tilde{R}_d + \mathbb{C}_0 \mathbf{Y}_s(kh) \mathbb{C}_0^\top) \mathbf{K}_s^\top(kh)$.

Since $G(kh) \geq 0$ and $\mathbb{A}_0$ is a Hurwitz matrix, we can show that (58) has a unique bounded $\mathbb{R}$ solution and, additionally, this solution is positive semi-definite. So, under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the bounded and stabilizing solution $\mathbf{Y}_s(\cdot)$ of the FJMLDE (52), if any, satisfies

$$0 \leq \mathbf{Y}_s(t) \leq \tilde{Y}_{11}(t), t \in \mathbb{R} \quad (59)$$

- (d) Let $\begin{pmatrix} Y_s^1(t) & Y_s^2(t) \\ (Y_s^2(t))^\top & Y_s^3(t) \end{pmatrix}$ be the partition of the matrix $\mathbf{Y}_s(t)$, such that $Y_s^1(t), Y_s^3(t) \in \mathcal{S}_n$. Employing (50) and (59), we may conclude that, necessarily, $Y_s^3(t) \equiv 0, Y_s^2(t) \equiv 0$ for all $t \in \mathbb{R}$. Hence, under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the bounded and stabilizing solution of the FJMLDE with Riccati-type jumping operator (52), if any, has the structure

$$\mathbf{Y}_s(t) = \begin{pmatrix} Y_s(t) & 0 \\ 0 & 0 \end{pmatrix}, t \in \mathbb{R} \quad (60)$$

where $Y_s(\cdot)$ solves the FJMLDE with Riccati-type jumping operator of dimension n

$$\dot{Y}(t) = A_0 Y(t) + Y(t) A_0^\top + \tilde{M}_1, kh < t \leq (k+1)h \quad (61a)$$

$$Y(kh^+) = Y(kh) - Y(kh) \mathbb{C}_0^\top \left(\tilde{R}_d + \mathbb{C}_0 Y(kh) \mathbb{C}_0^\top \right)^{-1} \mathbb{C}_0 Y(kh), k \in \mathbb{Z} \quad (61b)$$

Adapting Definition 2 to the case of the FJMLDE (61), we say that $t \rightarrow Y_s(t) : \mathbb{R} \rightarrow \mathcal{S}_n$ is a stabilizing solution of the FJMLDE with Riccati-type jumping operator (61) if

$\tilde{R}_d + C_0 Y(kh) C_0^\top$, $k \in \mathbb{Z}$ are invertible matrices and the zero solution of the closed-loop jump linear differential equation on $\mathbb{R}^n$

$$\dot{x}(t) = A_0 x(t), \quad kh < t \leq (k+1)h \quad (62a)$$

$$x(kh^+) = (I_n + \tilde{K}_s(kh) C_0) x(kh), \quad k \in \mathbb{Z} \quad (62b)$$

is ES, where

$$\tilde{K}_s(kh) = -Y_s(kh) C_0^\top \left(\tilde{R}_d + C_0 Y_s(kh) C_0^\top \right)^{-1} \quad (63)$$

The next result establishes a relationship between the stabilizing solution of the FJMLDE with Riccati-type jumping operator (61) and the stabilizing solution of the FJMLDE with Riccati-type jumping operator (52).

Lemma 1. Assume that the assumptions $\mathbf{H}_1$, $\mathbf{H}_2$ hold. Under these conditions, if $Y_s(\cdot)$ is the bounded and stabilizing solution of the FJMLDE (61), then $\mathbf{Y}_s(\cdot)$ constructed as in (60) is the bounded and stabilizing solution of the FJMLDE (52). In this case, (56) becomes

$$\mathbf{K}_s(kh) = \begin{pmatrix} \tilde{K}_s(kh) \\ 0 \end{pmatrix} \quad (64)$$

$\tilde{K}_s(kh)$ being defined in (63).

Proof. Employing (10a), (47), one shows that the zero solution of (55) is ES where $\mathbf{K}_s(kh)$ is computed as in (64). The details are omitted. $\square$

The result stated in the above Lemma leads us to pay special attention to the problem of existence of the bounded and stabilizing solution of the FJMLDE (61) in order to obtain informations about the existence of the bounded and stabilizing solution of the FJMLDE (52).

From (61a), we obtain

$$Y((k+1)h) = e^{A_0 h} Y(kh^+) e^{A_0^\top h} + \int_0^h e^{A_0 s} \tilde{M}_1 e^{A_0^\top s} ds, \quad k \in \mathbb{Z} \quad (65)$$

Substituting (61b) in (65), we obtain that the sequence $\{Y(kh)\}_{k \in \mathbb{Z}}$ of the values of the globally defined solution of the FJMLDE (61) solves the forward discrete-time Riccati equation (FDTRE)

$$X_{k+1} = \mathcal{A} X_k \mathcal{A}^\top - \mathcal{A} X_k C_0^\top \left(\tilde{R}_d + C_0 X_k C_0^\top \right)^{-1} C_0 X_k \mathcal{A}^\top + M_d, \quad k \in \mathbb{Z} \quad (66)$$

where

$$\mathcal{A} = e^{A_0 h}, \quad M_d = \int_0^h e^{A_0 s} \tilde{M}_1 e^{A_0^\top s} ds \quad (67)$$

Since (66) is a time-invariant discrete-time Riccati equation, its bounded and stabilizing solution, if any, is a constant sequence and satisfies the discrete-time algebraic Riccati equation (DTARE)

$$X = \mathcal{A} X \mathcal{A}^\top - \mathcal{A} X C_0^\top \left(\tilde{R}_d + C_0 X C_0^\top \right)^{-1} C_0 X \mathcal{A}^\top + M_d, \quad k \in \mathbb{Z} \quad (68)$$

We recall that $\tilde{X}$ is a stabilizing solution of the DTARE (68) if the eigenvalues of the matrix $\mathcal{A} + \mathcal{K}_s C_0$ lie in the disk $|\lambda| < 1$, where

$$\mathcal{K}_s \triangleq -\mathcal{A}\tilde{X}C_0^\top (\tilde{R}_d + C_0\tilde{X}C_0^\top)^{-1} \quad (69)$$

We set

$$\begin{aligned} Y_s(t) = & e^{A_0(t-kh)} \left(\tilde{X} - \tilde{X}C_0^\top (\tilde{R}_d + C_0\tilde{X}C_0^\top)^{-1} C_0\tilde{X} \right) e^{A_0^\top(t-kh)} \\ & + \int_{kh}^t e^{A_0(t-s)} \tilde{M}_1 e^{A_0^\top(t-s)} ds, \quad t \in (kh, (k+1)h], \quad k \in \mathbb{Z} \end{aligned} \quad (70)$$

From (68) and (70) (written for $t \rightarrow (k+1)h$), one obtains that

$$Y_s((k+1)h) = \tilde{X} \quad (71)$$

for all $k \in \mathbb{Z}$.

Letting $t \rightarrow kh, t > kh$ in (70), we obtain via (71) that

$$Y_s(kh^+) = Y_s(kh) - Y_s(kh)C_0^\top (\tilde{R}_d + C_0Y_s(kh)C_0^\top)^{-1} C_0Y_s(kh), \quad k \in \mathbb{Z} \quad (72)$$

Further, differentiating (70) on the interval $(kh, (k+1)h)$, we obtain that

$$\dot{Y}_s(t) = A_0Y_s(t) + Y_s(t)A_0^\top + \tilde{M}_1, \quad kh < t \leq (k+1)h \quad (73)$$

From (72) and (73) one sees that $Y_s(\cdot)$ defined in (70) is a solution of the FJMLDE (61). Furthermore, $Y_s(\cdot)$ is just the unique bounded and stabilizing solution of (61). Summarizing the previous developments, we can state the following:

Proposition 6. Under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, if the DTARE (68) has a stabilizing solution $\tilde{X}$ which satisfies the condition $\tilde{R}_d + C_0\tilde{X}C_0^\top > 0$, then the FJMLDE (61) has a bounded and stabilizing solution $Y_s(\cdot)$ which is a periodic function of period h and satisfies the condition $\tilde{R}_d + C_0Y_s(kh)C_0^\top > 0, \forall k \in \mathbb{Z}$.

Remark 5. From (63) and (71), we may infer that the stabilizing gain (63) does not depend upon k and it can be computed as

$$\tilde{K}_s = -\tilde{X}C_0^\top (\tilde{R}_d + C_0\tilde{X}C_0^\top)^{-1} \quad (74)$$

$\tilde{X}$ being the unique stabilizing solution of the DTARE (68). In this case, (64) becomes

$$\mathbf{K}_s = \begin{pmatrix} \tilde{K}_s \\ 0 \end{pmatrix} \quad (75)$$

5.2. Optimal Filter

Theorem 3. Assume the following:

- (a) the assumptions $\mathbf{H}_1, \mathbf{H}_2$ hold;
- (b) the DTARE (68) has a stabilizing solution $\tilde{X}$ which satisfies the condition

$$\tilde{R}_d + C_0\tilde{X}C_0^\top > 0 \quad (76)$$

Let $\tilde{K}_s$ be associated to the stabilizing solution $\tilde{X}$ via (74) and $\mathbf{K}_s \in \mathbb{R}^{2n \times n_y}$ be defined as in (75). We consider the filter $\tilde{\mathbb{G}}_F$ with $n_F = 2n$ being the dimension of its state-space and with a state-space representation described as follows:

$$\dot{\tilde{x}}_F(t) = \mathbb{A}_0 \tilde{x}_F(t), \quad kh < t \leq (k+1)h \quad (77a)$$

$$\tilde{x}_F(kh^+) = (I_{2n} + \mathbf{K}_s \mathbb{C}_0) \tilde{x}_F(kh) - \mathbf{K}_s y(kh), \quad k \in \mathbb{Z}_+ \quad (77b)$$

$$\tilde{z}_F(t) = \mathbb{C}_z \tilde{x}_F(t), \quad t \in \mathbb{R}_+ \quad (77c)$$

where $\mathbb{C}_z = \begin{pmatrix} C_z & \tilde{C}_z \end{pmatrix}$. Under the considered assumptions, the filter $\tilde{\mathbb{G}}_F$ lies in $\mathfrak{F}_{adm}$ and minimizes the cost (5), of which the minimal value is given by

$$\mathcal{J}(\tilde{\mathbb{G}}_F) = \frac{1}{h} \int_0^h \text{Tr} \left[C_z Y_s(t) C_z^\top \right] dt \quad (78)$$

where $Y_s(t)$ is defined in (70).

Proof. See Appendix B. $\square$

6. Numerical Experiments

From Theorem 3, it appears that in order to construct the optimal H_2 -filter, one has to compute the stabilizing solution of the DTARE (68) which verifies the sign condition (76). The coefficients of DTARE (68) depend on the unique solution of the non-homogeneous Lyapunov-type algebraic Equation (51). Hence, in order to synthesize the optimal filter, one has first to solve Equations (51) and (68), respectively. In order to solve (51), we use the same method as the one described in [29]. Equation (68) being a standard DTARE, we use existing numerical solvers in MATLAB 2020b ©. More specifically, we use the function `idare`.

For the numerical application, we generate the following system matrices:

$$A_0 = \begin{pmatrix} -0.841 & 2 & -0.019 & -1 \\ 0 & -1.752 & -0.041 & 1 \\ 0 & 0 & -2.1 & 1.9 \\ 0 & 0 & 0 & -1.2 \end{pmatrix},$$

$$\bar{A}_0 = \begin{pmatrix} -0.1962 & -0.1771 & 0.3177 & 0.5917 \\ -0.6382 & -0.1586 & 0.0326 & 0.4438 \\ -0.6483 & -0.7518 & -0.5198 & 0.0840 \\ -0.6440 & -0.2957 & -0.6482 & 0.7039 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.05 & -0.1 & 0 & -0.05 \\ 0.1 & 0.5 & -0.2 & 0.5 \\ 0 & 0.25 & 0.5 & 0.05 \\ -0.25 & 0 & 0 & -0.5 \end{pmatrix},$$

$$\bar{A}_1 = \begin{pmatrix} 0.5920 & -0.4634 & 0.2457 & -0.3596 \\ 0.0579 & -0.7923 & -0.7624 & 0.0846 \\ 0.6176 & -0.1988 & 0.2355 & 0.0122 \\ 0.6534 & 0.2137 & -0.3608 & 0.8589 \end{pmatrix},$$

$$\begin{aligned}
 B &= \begin{pmatrix} -0.1 & 0.05 \\ 0.15 & 0.2 \\ -0.1 & -0.15 \\ 0.1 & 0.05 \end{pmatrix}, C_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
 \bar{C}_0 &= \begin{pmatrix} -0.1594 & -0.0743 & 0.0561 & 0.0801 \\ 0.0422 & -0.034 & -0.143 & -0.02 \end{pmatrix}, \\
 C_1 &= \begin{pmatrix} 0.2 & 0.3 & 0 & -0.1 \\ 0 & 0 & 0.5 & -0.2 \end{pmatrix}, \bar{C}_1 = \begin{pmatrix} 0.5178 & -0.1987 & 0.3356 & 0.3652 \\ 1.1127 & 0.0116 & -0.1453 & -0.7048 \end{pmatrix}, \\
 D &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, C_z = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{pmatrix}, \\
 \bar{C}_z &= \begin{pmatrix} -1.2733 & -0.1428 & -0.1467 & 0.4082 \\ -0.5428 & 0.3743 & -1.9668 & -0.1116 \end{pmatrix}.
 \end{aligned}$$

For the above parameters, the assumption $\mathbf{H}_2$ is satisfied. The unique solution of the Lyapunov-type algebraic Equation (51) as well as the unique stabilizing solution of the DTARE (68) are given by

$$\begin{aligned}
 \Pi_c &= \begin{pmatrix} 0.0445 & 0.0161 & -0.0030 & 0.0012 \\ 0.0161 & 0.0277 & -0.0073 & 0.0084 \\ -0.0030 & -0.0073 & 0.0067 & -0.0016 \\ 0.0012 & 0.0084 & -0.0016 & 0.0072 \end{pmatrix} \\
 \tilde{X} &= \begin{pmatrix} 2.0957 & 1.0543 & -0.7931 & -0.3107 \\ 1.0543 & 4.2084 & -1.5616 & 1.2019 \\ -0.7931 & -1.5616 & 1.2417 & -0.3890 \\ -0.3107 & 1.2019 & -0.3890 & 1.2238 \end{pmatrix}
 \end{aligned}$$

respectively. The associated optimal filter gain is given by

$$\mathbf{K}_s = \begin{pmatrix} -0.7313 & 0.3196 \\ 0.0149 & 0.9421 \\ -0.1394 & -0.3344 \\ 0.2895 & 0.2366 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Figure 1 shows the evolution of the estimation error $e(t) = z(t) - z_F(t)$ for one realization of the stochastic processes.

We have also performed Monte-Carlo simulations (corresponding to different realizations of the stochastic processes). We have considered 1000 realizations. The corresponding results are summarized in Figure 2.

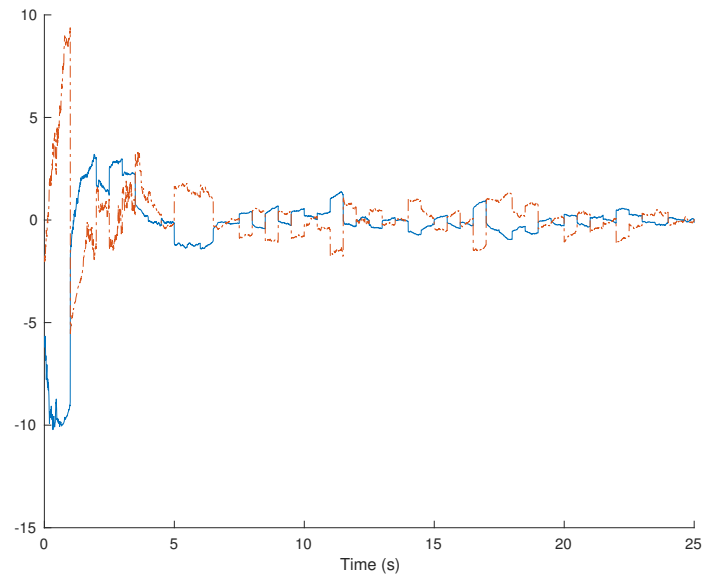


Figure 1. Estimation error: $e(t) = z(t) - z_F(t)$.

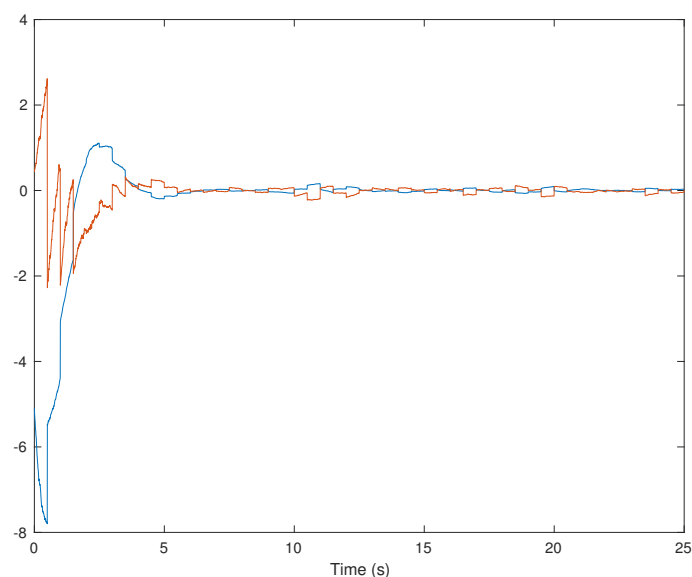


Figure 2. Monte-Carlo simulation.

7. Conclusions

In this work, we dealt with the problem of the optimal estimation of a remote signal generated by a dynamical system modeled by a McKean–Vlasov type system of linear stochastic differential equations subject to additive white noise perturbations. We assumed that only measurements (of another signal generated by the dynamical system under consideration) at some discrete-time instances are available in order to activate the dynamical filter which has to provide the best estimation of the remote signal.

Since in the considered filtering/estimation problem, an admissible filter is fed to its input by a discrete-time signal and has to provide to its output a continuous-time signal, we assumed that the more appropriate class of admissible filters are those having the state-space representation in the form of a jump linear system, as in (3).

In our approach, the dimension of the state-space of an admissible filter (often named the order of the filter) is not prefixed. In Theorem 3, we have shown that among all

admissible filters of arbitrary order $n_F \geq 1$, the best estimation of the remote signal generated by the given dynamical system is achieved by a filter having the order equal to twice the dimension of the state-space of the dynamical system under consideration. We have provided explicit formulae of the optimal filter. To this end, we have used the stabilizing solution of an adequately defined forward jump matrix linear differential equation (FJMLDE) with a Riccati-type jumping operator. We have shown how we can reduce the computation of such a solution to the computation of the stabilizing solution of a discrete-time algebraic Riccati equation (DTARE) for which there exist efficient numerical procedures. It is worth mentioning that the state-space representation of the optimal filter is computed offline, which facilitates its implementation. It remains a challenge for future research directions to compare the performances of the optimal filter of the form (77) with the performances of a filter whose gain matrices are computed online. Another research direction would be to consider the case when the time between two consecutive measurement samples is not constant but modeled by a stochastic process.

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Appendix A. Proof of Theorem 1

Let $\mathbf{Y}(\cdot) = \mathbf{Y}(\cdot; t_0, \mathbf{Y}_0)$ be the solution of the JMLDE (32) corresponding to the initial pair $(t_0, \mathbf{Y}_0) \in \mathbb{R}_+ \times \mathbb{S}_{\hat{n}}$. According to Remark 1, we know that $\mathbf{Y}(t) \in \mathbb{S}_{\hat{n}}$ for all $t \geq t_0$. Based on (28), we have the partition

$$\mathbf{Y}(t) = \begin{pmatrix} Y_1(t) & Y_{12}(t) \\ Y_{12}^\top(t) & Y_F(t) \end{pmatrix}, \quad t \geq t_0$$

where $Y_1(t) \in \mathbb{S}_{2n}$, $Y_F(t) \in \mathbb{S}_{n_F}$ and $Y_{12}(t) \in \mathbb{R}^{2n \times n_F}$. Employing (22), (27), (30), we obtain the following partition of the JMLDE (32):

$$\dot{Y}_1(t) = \mathbb{L}[Y_1(t)], \quad kh < t \leq (k+1)h \quad (\text{A1a})$$

$$Y_1(kh^+) = Y_1(kh), \quad k \geq \kappa(t_0) \quad (\text{A1b})$$

$$\dot{Y}_{12}(t) = \mathbb{A}_0 Y_{12}(t) + Y_{12}(t) A_F^\top, \quad kh < t \leq (k+1)h \quad (\text{A2a})$$

$$Y_{12}(kh^+) = Y_1(kh) \mathbb{C}_0^\top B_{F,d}^\top + Y_{12}(kh) A_{F,d}^\top, \quad k \geq \kappa(t_0) \quad (\text{A2b})$$

$$\dot{Y}_F(t) = A_F Y_F(t) + Y_F(t) A_F^\top, \quad kh < t \leq (k+1)h \quad (\text{A3a})$$

$$\begin{aligned} Y_F(kh^+) &= A_{F,d} Y_F(kh) A_{F,d}^\top + \sum_{p=0}^{r_2} B_{F,d} \mathbb{C}_p Y_1(kh) \mathbb{C}_p^\top B_{F,d}^\top \\ &\quad + B_{F,d} \mathbb{C}_0 Y_{12}(kh) A_{F,d}^\top + A_{F,d} Y_{12}(kh) \mathbb{C}_0^\top B_{F,d}^\top, \quad k \geq \kappa(t_0) \end{aligned} \quad (\text{A3b})$$

where $\kappa(t_0)$ is the integer with the property that $(\kappa(t_0) - 1)h < t_0 \leq \kappa(t_0)h$.

From (A1), we deduce that $Y_1(\cdot)$ is a solution of the linear ordinary differential equation

$$\dot{Y}_1(t) = \mathbb{L}[Y_1(t)], \quad t \geq t_0$$

Bearing in mind $\mathbf{H}_2$, Proposition 2 (iii), and Remark 3 we may infer that

$$\|Y_1(t)\|_2 \leq \beta_1 e^{-\alpha_1(t-t_0)} \|Y_1(t_0)\|_2 \quad (\text{A4})$$

for all $t \geq t_0$, where $\beta_1 \geq 1$, $\alpha_1 > 0$ do not depend upon t , t_0 , $Y_1(t_0)$. From (10a), we obtain via Proposition 2 (ii) that $\mathbb{A}_0$ is a Hurwitz matrix if the assumption $\mathbf{H}_2$ holds. Hence, the eigenvalues of the matrix $e^{\mathbb{A}_0 h}$ are located in the disk $|\lambda| < 1$. On the other hand, the exponential stability of the zero solution of the JLDE (4) is equivalent to the fact that the eigenvalues of the matrix $e^{A_F h} A_{F,d}^\top$ are also in the disk $|\lambda| < 1$. So, the eigenvalues of the operator $Y_{12} \rightarrow e^{\mathbb{A}_0 h} Y_{12} A_{F,d}^\top e^{A_F^\top h} : \mathbb{R}^{2n \times n_F} \rightarrow \mathbb{R}^{2n \times n_F}$ are in the disk $|\lambda| < 1$. From (A2), we deduce that the sequence $\{Y_{12}(kh)\}_{k \geq \kappa(t_0)}$ solves the following discrete-time linear equation:

$$Z(k+1) = e^{\mathbb{A}_0 h} Z(k) A_{F,d}^\top e^{A_F^\top h} + H_k, \quad k \geq \kappa(t_0)$$

where $H_k = Y_1(kh) \mathbb{C}_0^\top B_{F,d}^\top$. By considering (A4), we may infer that $\|H_k\|_2 \leq \gamma_1 \delta_1^{k-\kappa(t_0)} \|Y_1(t_0)\|_2$, $\forall k \geq \kappa(t_0)$, $\gamma_1 > 0$, $\delta_1 = e^{-\alpha_1 h} \in (0, 1)$. This allows us to deduce via (A2b) that

$$\|Y_{12}(kh^+)\|_2 \leq \gamma_2 e^{-\alpha_2(k-\kappa(t_0))h} (\|Y_1(t_0)\|_2 + \|Y_{12}(\kappa(t_0)h)\|_2) \quad (\text{A5})$$

for all $k \geq \kappa(t_0)$. Since $Y_{12}(t) = e^{\mathbb{A}_0(t-kh)} Y_{12}(kh^+) e^{A_F^\top(t-kh)}$, $t \in (kh, (k+1)h]$, $k \geq \kappa(t_0)$, we may conclude via (A5) that

$$\|Y_{12}(t)\|_2 \leq \beta_2 e^{-\alpha_2(t-t_0)} (\|Y_1(t_0)\|_2 + \|Y_{12}(t_0)\|_2) \quad (\text{A6})$$

where $\beta_2 \geq 1$, $\alpha_2 > 0$ do not depend upon t , t_0 , $Y_1(t_0)$, $Y_{12}(t_0)$. Further, from (A3), we deduce that $\{Y_F(kh)\}_{k \geq \kappa(t_0)}$ solves the following DTLE on the space $\mathcal{S}_{n_F}$:

$$Z_F(k+1) = e^{A_F h} A_{F,d} Z_F(k) A_{F,d}^\top e^{A_F^\top h} + H_{F,k}, \quad k \geq \kappa(t_0)$$

with

$$\begin{aligned} H_{F,k} = & \sum_{p=0}^{r_2} e^{A_F h} B_{F,d} \mathbb{C}_p Y_1(kh) \mathbb{C}_p^\top B_{F,d}^\top e^{A_F^\top h} \\ & + e^{A_F h} \left(B_{F,d} \mathbb{C}_0 Y_{12}(kh) A_{F,d}^\top + A_{F,d} Y_{12}^\top(kh) \mathbb{C}_0^\top B_{F,d}^\top \right) e^{A_F^\top h} \end{aligned}$$

Invoking again (A4) together with (A6), we obtain

$$\|H_{F,k}\|_2 \leq \hat{\gamma}_2 \delta_2^{k-\kappa(t_0)} (\|Y_1(t_0)\|_2 + \|Y_{12}(t_0)\|_2), \quad \forall k \geq \kappa(t_0)$$

Hence,

$$\|Y_F(kh)\|_2 \leq \gamma_3 e^{-\alpha_3(k-\kappa(t_0))h} (\|Y_1(t_0)\|_2 + \|Y_{12}(t_0)\|_2 + \|Y_F(\kappa(t_0)h)\|_2), \quad (\text{A7})$$

for all $k \geq \kappa(t_0)$, with $\gamma_3 \geq 1$, $\alpha_3 > 0$ do not depend upon $\kappa(t_0)$, $Y_1(t_0)$, $Y_{12}(t_0)$, $Y_F(\kappa(t_0)h)$. Since $Y_F(t) = e^{A_F(t-kh)} Y_F(kh^+) e^{A_F^\top(t-kh)}$ if $t \in (kh, (k+1)h]$, $k \geq \kappa(t_0)$, we may conclude via (A7) that

$$\|Y_F(t)\|_2 \leq \beta_3 e^{-\alpha_3(t-t_0)} (\|Y_1(t_0)\|_2 + \|Y_{12}(t_0)\|_2 + \|Y_F(t_0)\|_2), \quad \forall t \geq t_0 \quad (\text{A8})$$

where $\beta_3 \geq 1$, $\alpha_3 > 0$ do not depend upon $t, t_0, Y_1(t_0), Y_{12}(t_0), Y_F(t_0)$. Taking into account that $\|Y(t)\|_2 \leq \|Y_1(t)\|_2 + 2\|Y_{12}(t)\|_2 + \|Y_F(t)\|_2$, we may conclude via (A4), (A6), and (A8) that (33) holds. Hence, the proof is completed.

Appendix B. Proof of Theorem 3

The fact that the filter $\tilde{\mathbb{G}}_F$ lies in $\mathfrak{F}_{\text{adm}}$ is obvious because in this case, the corresponding JLDE (4) coincides with (55) and it is ES. Further, we show that the filter (77) achieves the minimal value of the cost (4).

First, let us remark that under the assumption (b) from the statement, we obtain via Proposition 6 that the FJMLDE with Riccati-type jumping operator (61) has a bounded and stabilizing solution $Y_s(\cdot)$. Further, Lemma 1 guarantees the existence of the bounded and stabilizing solution $Y_s(\cdot)$ of the FJMLDE (52). The corresponding stabilizing matrix gain is given by (75).

Let $\mathbb{G}_F$ be an arbitrary but fixed admissible filter. The value of the performance (5) achieved by the filter $\mathbb{G}_F$ is given by (40). Let $\begin{pmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^\top(t) & Y_{22}(t) \end{pmatrix}$ be the partition of the unique bounded $\mathbb{R}$ solution $\tilde{Y}_{cl}(t)$, $t \in \mathbb{R}$ of the corresponding non-homogeneous FJMLDE of type (26a)–(26b) with $Y_{11}(t) \in \mathcal{S}_{2n}$, $Y_{22}(t) \in \mathcal{S}_{n_F}$.

We define $\mathcal{U}(t) = \begin{pmatrix} Y_{11}(t) - Y_s(t) & Y_{12}(t) \\ Y_{12}^\top(t) & Y_{22}(t) \end{pmatrix}$, $t \in \mathbb{R}$. By direct calculations, involving (44)–(46), (52) and taking into account (50), (53), (54), we obtain that $t \rightarrow \mathcal{U}(t) : \mathbb{R} \rightarrow \mathcal{S}_{\hat{n}}$ is a bounded $\mathbb{R}$ solution of the following non-homogeneous FJMLDE:

$$\dot{\mathcal{U}}(t) = A_{0,cl}\mathcal{U}(t) + \mathcal{U}(t)A_{0,cl}^\top, \quad kh < t \leq (k+1)h \quad (\text{A9a})$$

$$\mathcal{U}(kh^+) = A_{d,0,cl}\mathcal{U}(kh)A_{d,0,cl}^\top + \Gamma(\tilde{R}_d + C_0\tilde{X}C_0^\top)\Gamma^\top, \quad k \in \mathbb{Z} \quad (\text{A9b})$$

where

$$\Gamma = \begin{pmatrix} \mathbf{K}_s \\ -B_{F,d} \end{pmatrix} \quad (\text{A10})$$

From (75) and (A10), we deduce via (28) that $\Gamma(\tilde{R}_d + C_0\tilde{X}C_0^\top)\Gamma^\top \in \mathbb{S}_{\hat{n}}$. Hence, $\mathbb{S}_{\hat{n}}$ is an invariant subspace of the FJMLDE (A9). Invoking Theorem 1, we may infer that under the assumptions $\mathbf{H}_1, \mathbf{H}_2$, the zero solution of the corresponding JMLDE of the form (32) is ES with respect to the invariant subspace $\mathbb{S}_{\hat{n}}$. Employing Proposition 3 and Theorem 3 from [30], we deduce that there exists a function $t \rightarrow Z(t) : \mathbb{R} \rightarrow \mathbb{S}_{\hat{n}}$ satisfying

$$\dot{Z}(t) + \mathcal{L}_{c,cl}^*[Z(t)] + I_{\hat{n}}, \quad kh \leq t < (k+1)h \quad (\text{A11a})$$

$$Z(kh^-) = \mathcal{L}_{d,cl}^*[Z(kh)] + I_{\hat{n}}, \quad k \in \mathbb{Z} \quad (\text{A11b})$$

$$0 < c_1 I_{\hat{n}} \leq Z(t) \leq c_2 I_{\hat{n}}, \quad t \in \mathbb{R} \quad (\text{A11c})$$

$\mathcal{L}_{c,cl}^*[\cdot]$ and $\mathcal{L}_{d,cl}^*[\cdot]$ being the adjoint operators of the linear operators defined via (27). Bearing in mind the formulae of the adjoint operators of type (27), we obtain from (A11) that $Z(\cdot)$ also solves

$$\dot{Z}(t) + A_{0,cl}^\top Z(t) + Z(t)A_{0,cl} + H_{c,cl}(t) = 0, \quad kh \leq t < (k+1)h \quad (\text{A12a})$$

$$Z(kh^-) = A_{d,0,cl}^\top Z(kh)A_{d,0,cl} + H_{d,cl}(kh), \quad k \in \mathbb{Z} \quad (\text{A12b})$$

where $I_{\hat{h}} \leq H_{c,cl}(t) \leq \nu_1 I_{\hat{h}}, \forall t \in \mathbb{R}$ and $I_{\hat{h}} \leq H_{d,cl}(kh) \leq \nu_2 I_{\hat{h}}$ for all $k \in \mathbb{Z}$. Based on Proposition 4.2 from [26], we may conclude that the zero solution of the following JLDE on the space $\mathbb{R}^{\hat{n}}$

$$\dot{\xi}(t) = A_{0,cl}\xi(t), \quad kh < t \leq (k+1)h \quad (\text{A13a})$$

$$\xi(kh^+) = A_{d,0,cl}\xi(kh), \quad k \in \mathbb{Z} \quad (\text{A13b})$$

is ES. Applying Proposition 3.2 from [19], we may conclude that the exponential stability of the zero solution of (A13) guarantees that the non-homogeneous FJMLDE (A9)–(A10) has a unique bounded $\mathbb{R}$ solution and, additionally, this solution satisfies

$$\mathcal{U}(t) \geq 0, \quad t \in \mathbb{R} \quad (\text{A14})$$

Further, we rewrite (40) in the form

$$\mathcal{J}(\mathbb{G}_F) = \frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_z \mathbf{Y}_s(t) \mathbb{C}_z^\top] dt + \frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_{cl} \mathcal{U}(t) \mathbb{C}_{cl}^\top] dt \quad (\text{A15})$$

According to (A14) we may infer that

$$\mathcal{J}(\mathbb{G}_F) \geq \frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_z \mathbf{Y}_s(t) \mathbb{C}_z^\top] dt \quad (\text{A16})$$

for any admissible filter $\mathbb{G}_F$. Now, we show that in the special case of the filter $\tilde{\mathbb{G}}_F$ described by (77), the inequality (A16) becomes equality. Let us remark that in the special case of the filter $\tilde{\mathbb{G}}_F$, we have

$$\mathbb{C}_{cl} \mathcal{U}(t) \mathbb{C}_{cl}^\top = \mathbb{C}_z \begin{pmatrix} I_{2n} & -I_{2n} \end{pmatrix} \mathcal{U}(t) \begin{pmatrix} I_{2n} & -I_{2n} \end{pmatrix}^\top \mathbb{C}_z^\top = \mathbb{C}_z \hat{\mathcal{U}}_{11}(t) \mathbb{C}_z^\top \quad (\text{A17})$$

where $\hat{\mathcal{U}}_{11}(t)$ is the 11-block of

$$\begin{pmatrix} \hat{\mathcal{U}}_{11}(t) & \hat{\mathcal{U}}_{12}(t) \\ \hat{\mathcal{U}}_{12}^\top(t) & \hat{\mathcal{U}}_{22}(t) \end{pmatrix} = \begin{pmatrix} I_{2n} & -I_{2n} \\ 0 & I_{2n} \end{pmatrix} \mathcal{U}(t) \begin{pmatrix} I_{2n} & 0 \\ -I_{2n} & I_{2n} \end{pmatrix}$$

By direct calculation, involving (22), (52), (77), (A9), (A14), we obtain that $\hat{\mathcal{U}}_{11}(\cdot)$ is a bounded solution of the following non-homogeneous FJMLDE:

$$\dot{\hat{\mathcal{U}}}_{11}(t) = \mathbb{A}_0 \hat{\mathcal{U}}_{11}(t) + \hat{\mathcal{U}}_{11}(t) \mathbb{A}_0^\top, \quad kh < t \leq (k+1)h \quad (\text{A18a})$$

$$\hat{\mathcal{U}}_{11}(kh^+) = (I_{2n} + \mathbf{K}_s \mathbb{C}_0) \hat{\mathcal{U}}_{11}(kh) (I_{2n} + \mathbf{K}_s \mathbb{C}_0)^\top, \quad k \in \mathbb{Z} \quad (\text{A18b})$$

where $\mathbf{K}_s$ is the stabilizing gain matrix defined by (75). Invoking again Proposition 3.2 from [19], we deduce that the FJMLDE (A18) has a unique bounded on $\mathbb{R}$ solution. So, $\hat{\mathcal{U}}_{11}(t) = 0, t \in \mathbb{R}$. Substituting this in (A17), we obtain that in the special case of the filter described by (77), we have $\frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_{cl} \mathcal{U}(t) \mathbb{C}_{cl}^\top] dt = 0$. Hence, in the case of the filter $\tilde{\mathbb{G}}_F$ described by (77), the equality (A15) becomes

$$\mathcal{J}(\tilde{\mathbb{G}}_F) = \frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_z \mathbf{Y}_s(t) \mathbb{C}_z^\top] dt \quad (\text{A19})$$

From (A16) and (A19), we deduce that

$$\mathcal{J}(\mathbb{G}_F) \geq \mathcal{J}(\tilde{\mathbb{G}}_F) = \frac{1}{h} \int_0^h \text{Tr}[\mathbb{C}_z \mathbf{Y}_s(t) \mathbb{C}_z^\top] dt \quad (\text{A20})$$

This shows that the filter $\mathbb{G}_F$ achieves the minimal value of the cost function (5), which means that its output $z_F(t)$ provides the best estimation of the signal $z(t)$. Finally, employing (60), we deduce that (A19) coincides with (78). Thus, the proof ends.

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