

## Article

# Para-Holomorphic Statistical Structure with Cheeger Gromoll Metric

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**Abstract:** We consider the family of  $\lambda$  connections  $\nabla^{(\lambda)}$  on a statistical manifold  $M$  equipped with a pair of conjugate connections  $\nabla = \nabla^{(1)}$  and  $\nabla^* = \nabla^{(-1)}$ , where the  $\lambda$  connection is defined as  $\nabla^{(\lambda)} = \frac{1+\lambda}{2}\nabla + \frac{1-\lambda}{2}\nabla^*$ . This paper develops expressions for the vertical and horizontal distributions on the tangent bundle of the statistical manifold  $(M, g, \nabla^{(\lambda)})$  and introduces the concept of  $\lambda$ -adapted frames. We also derive the Levi-Civita connection  ${}^{CG}\hat{\nabla}^{(\lambda)}$  of the tangent bundle  $TM$ , which is equipped with the Cheeger Gromoll-type metric  ${}^{CG}g$ . We study the statistical structure  $({}^{CG}g, {}^{CG}\nabla^{(\lambda)})$  on the tangent bundle  $TM$ , which is naturally induced from the statistical manifold  $(M, g, \nabla^{(\lambda)})$ . By introducing a para-holomorphic structure on the statistical manifold  $(M, g, \nabla^{(\lambda)})$ , we construct a para-Hermitian structure on the tangent bundle  $TM$  and examine its integrability. Finally, we present the conditions under which these bundles admit a para-holomorphic structure.

**Keywords:** statistical manifolds; para-holomorphic structures; Cheeger Gromoll metric

**MSC:** 53B05; 53A15; 53C21; 53C25



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## 1. Introduction

In the 1980s, the notion of statistical structure was introduced and began to play an important role in building a very effective branch called information geometry, which is a combination of differential geometry and statistics. In fact, information geometry joins fundamental differential geometry tools like connection, metric and curvature to statistical models and makes it possible to describe statistical objects as geometric ones. Describing statistical spaces using geometry helps us better understand statistical behaviors. A detailed introduction to information geometry can be found in [1]. Its applications could be found in various fields of science such as in statistical inference, time series, linear systems, quantum systems, image processing, physics, computer science and machine learning; see [2–4].

Let  $\Theta$  be an open subset of  $\mathbb{R}^n$ , and let  $\Omega$  be a sample space with parameter  $\theta = (\theta^1, \dots, \theta^n)$ . The set of probability density functions

$$S = \{p(x; \theta) : \int_{\Omega} p(x; \theta) dx = 1, p(x; \theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n\},$$

is known as a statistical model. For a given statistical model  $S$ , the Fisher information matrix  $g(\theta) = [g_{ij}(\theta)]$  is defined as

$$g_{ij}(\theta) := \int_{\Omega} \partial_i \ell_{\theta} \partial_j \ell_{\theta} p(x; \theta) dx = E_p[\partial_i \ell_{\theta} \partial_j \ell_{\theta}], \quad (1)$$

where  $\ell_\theta = \ell(x; \theta) := \log p(x; \theta)$ ,  $\partial_i := \frac{\partial}{\partial \theta^i}$ , and  $E_p[f]$  is the expectation of  $f(x)$  with respect to  $p(x; \theta)$ . When equipped with the Fisher information matrix,  $S$  forms a statistical manifold.

Historically, Fisher introduced Equation (1) in 1920 as a mathematical measure of information (see [5]). It has been established that if the metric tensor  $g$  is positive-definite and its components converge to real values, then  $(S, g)$  defines a Riemannian manifold, where  $g$  is referred to as the Fisher metric. Later, in [6], C. R. Rao introduced a geometric framework for the space of probability distributions by using the Fisher information metric to define a Riemannian distance.

Using the Fisher metric  $g$ , Amari's  $\lambda$  connection  $\nabla^{(\lambda)}$  for any  $\lambda \in \mathbb{R}$  with respect to  $p(x; \theta)$  is expressed in terms of the Christoffel symbols as

$$\Gamma_{ij,k}^{(\lambda)} = g(\nabla_{\partial_i}^{(\lambda)} \partial_j, \partial_k) := E_p[(\partial_i \partial_j \ell_\theta + \frac{1-\lambda}{2} \partial_i \ell_\theta \partial_j \ell_\theta)(\partial_k \ell_\theta)]. \quad (2)$$

This affine connection plays a fundamental role in information geometry, providing a differential geometric framework for statistical inference. Different choices of  $\lambda$  lead to distinct structures, such as dual  $\lambda$  connections, which are central to statistical estimation theory and machine learning. In particular, the case  $\lambda = 1$  corresponds to the Amari–Chentsov connection, while  $\lambda = -1$  defines its conjugate connection [7,8].

In 1985, Amari studied the statistical manifolds from the point of view of information geometry, and he gave a new description of the statistical distributions by using the obtained geometric structures. In Amari's definition, a statistical manifold is a triple  $(M, g, \nabla)$  consisting of a smooth manifold  $M$ , a non-degenerate metric  $g$  on it and a torsion-free connection  $\nabla$  with the property that  $\nabla g$  is symmetric [9]. An alternative definition of a statistical manifold was given by H. Furuhashi and I. Hasegawa in [10]. They described it as a (pseudo-)Riemannian manifold equipped with a pair of torsion-free conjugate connections. Earlier studies on pairs of connections that are compatible with a  $g$  structure can be traced back to the work of Cruceanu and Miron in [11].

The relationship between an affine connection and different geometric structures such as a pseudo-Riemannian metric, a non-degenerate two-form, and a bundle isomorphism has been studied in various settings. A well-known example is the case of almost (para-)Hermitian manifolds  $(M, g, L)$ , where the 2-form  $\omega$  is defined by  $\omega(X, Y) = g(LX, Y)$ . In [12], Fei and Zhang showed that if the connection  $\nabla$  satisfies the Codazzi condition with respect to both the metric  $g$  and the structure  $L$ , then the manifold carries a (para-)Kähler structure. They introduced the concept of Codazzi-(para-)Kähler manifolds as a generalization of (para-)Kähler statistical manifolds. Later, in [13], this study was extended to connections with torsion. The authors explored torsion couplings between such connections and the pair  $(g, L)$  on almost (para-)Hermitian manifolds. They proved that  $(\nabla, L)$  is torsion-coupled if and only if  $\nabla$  is (para-)holomorphic and  $L$  is integrable. Statistical geometry on almost anti-Hermitian (or Norden) manifolds has also been developed. In [14,15], Salimov and Turanli introduced anti-Kähler-Codazzi manifolds. In [16], Etayo and coauthors showed that Kähler–Codazzi manifolds reduce to Kähler manifolds within all four types of  $(\alpha, \varepsilon)$ -geometry. Vilcu introduced para-Kähler-like statistical manifolds in [17] and showed that if they have constant curvature in the sense of Kurose, their statistical structure becomes Hessian. For further developments in statistical geometry, see [18–22].

The tangent bundle of a differentiable manifold plays a crucial role in various areas of differential geometry, mathematical physics, and information geometry. Understanding the geometry of tangent bundles provides deep insights into the intrinsic and extrinsic properties of the base manifold. One of the fundamental approaches in this context is the

study of natural metrics, which extend the geometric structure of the base manifold to its tangent bundle. Among these, the Cheeger Gromoll metric has attracted significant attention due to its ability to capture the underlying geometry of tangent bundles in a natural and meaningful way.

The Cheeger Gromoll metric was originally introduced in the context of Riemannian geometry by Cheeger and Gromoll [23] and later explicitly formulated on tangent bundles by Musso and Tricerri [24]. This metric, along with other natural metrics such as the Sasaki metric and Kaluza–Klein metric, has been extensively studied in relation to curvature properties, geodesics, and holonomy structures. While these studies have provided valuable insights into the geometry of tangent bundles in Riemannian settings, the extension of the Cheeger Gromoll metric to statistical manifolds remains largely unexplored.

Additionally, statistical structures on tangent bundles have been studied in recent works such as [25,26], adding new insights to the field.

This paper aims to extend the study of the Cheeger Gromoll metric to statistical manifolds. By conducting this study, we provide a new perspective on the interaction between natural metrics and statistical structures, offering deeper insights into how geometric properties evolve when a manifold is endowed with both a metric and a statistical connection. This work serves as a bridge between differential geometry and information geometry, enabling further exploration of geometric frameworks in statistical inference, machine learning, and quantum information theory.

The paper is organized as follows: In Section 2, we provide a brief overview of fundamental concepts related to statistical manifolds. In Section 3, we introduce the vertical and horizontal distributions on the tangent bundle of a statistical manifold  $(M, g, \nabla)$  and present the notion of  $\lambda$ -adapted frames. Additionally, we define the Cheeger Gromoll-type metric  ${}^{\text{CG}}g$  with respect to these adapted frames. In Section 4, we derive the Levi–Civita connection  ${}^{\text{CG}}\hat{\nabla}$  associated with the Cheeger Gromoll metric and compute its components in the adapted frame setting. In Section 5, we extend the discussion to the statistical structure of the tangent bundle  $(TM, {}^{\text{CG}}g, {}^{\text{CG}}\nabla)$ , which is naturally induced by the statistical manifold  $(M, g, \nabla^{(\lambda)})$ . In Section 6, we shift our focus to para-holomorphic structures on statistical manifolds. We construct a para-Hermitian structure on the tangent bundle of a para-holomorphic statistical manifold and examine its integrability conditions. In fact, by introducing a para-holomorphic structure on the statistical manifold  $(M, g, \nabla^{(\lambda)})$ , we construct a para-Hermitian structure on the tangent bundle  $TM$ . We then investigate its integrability and establish the conditions under which these bundles admit a para-holomorphic structure.

## 2. Statistical Manifolds

Let  $M$  be an  $n$ -dimensional manifold and  $(U, x^i)$ ,  $i = 1, \dots, n$ , be a local chart around a point  $p \in U$ . Given the coordinates  $(x^i)$  on  $M$ , the local frame on the tangent space  $T_p M$  is given by the vector fields  $\frac{\partial}{\partial x^i}|_p$ .

Consider an affine connection  $\nabla$  and a pseudo-Riemannian metric  $g$  on the manifold  $M$ .

(I) The affine connection  $\nabla^*$  on  $M$ , defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z),$$

is called the (conjugate) dual connection of  $\nabla$  with respect to  $g$ , for any  $X, Y, Z \in \mathfrak{S}_0^1(M)$ , where  $\mathfrak{S}_0^1(M)$  denotes a set of vector fields on the manifold.

(II) The affine connection  $\nabla$  is called a Codazzi connection if the cubic tensor field  $\mathcal{C} = \nabla g$  is totally symmetric. That is, the Codazzi equations hold:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z), \quad (= (\nabla_Z g)(X, Y)), \forall X, Y, Z \in \mathfrak{S}_0^1(M). \quad (3)$$

In local coordinates, the equations for the cubic tensor field  $\mathcal{C}$  take the following form:

$$(\nabla_{\partial_i} g)(\partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) - g(\nabla_{\partial_i} \partial_j, \partial_k) - g(\partial_j, \nabla_{\partial_i} \partial_k) = \mathcal{C}(\partial_i, \partial_j, \partial_k), \quad (4)$$

hence

$$\mathcal{C}_{ijk} = \partial_k g_{ij} - \Gamma_{ik}^h g_{jh} - \Gamma_{jk}^h g_{ih}, \quad \mathcal{C}_{ijk} = \mathcal{C}_{jik} = \mathcal{C}_{kij}, \quad (5)$$

where  $\partial_i = \frac{\partial}{\partial x^i}$  and  $\Gamma_{jk}^i$ 's are the Christoffel symbols of the Codazzi connection  $\nabla$ .

(III) The triplet  $(M, g, \nabla)$  is called a *statistical manifold* if  $\nabla$  is a statistical connection, i.e., a torsion-free Codazzi connection. In particular, it is known that if the cubic tensor field  $\mathcal{C}$  vanishes, a torsion-free Codazzi connection  $\nabla$  reduces to the Levi-Civita connection  $\widehat{\nabla}$ .

Consider the one-parameter family of connections given by the convex combination of the torsion-free dual connections  $\nabla$  and  $\nabla^*$ , i.e.,

$$\nabla^{(\lambda)} = \frac{1+\lambda}{2} \nabla + \frac{1-\lambda}{2} \nabla^*, \quad \lambda \in \mathbb{R}.$$

This connection, referred to as the  $\lambda$  connection, plays a central role in the study of statistical manifolds, as it provides a natural framework to generalize dual connections.

From the above equation, we can derive the following relation

$$Xg(Y, Z) = g(\nabla_X^{(\lambda)} Y, Z) + g(Y, \nabla_X^{(-\lambda)} Z), \quad (6)$$

which demonstrates that the connections  $\nabla^{(\lambda)}$  and  $\nabla^{(-\lambda)}$  are dual  $\lambda$  connections.

Several particular values of  $\lambda$  are of distinguished importance. Setting  $\lambda = -1, 1$  in (6) gives the following two fundamental connections

$$\nabla^{(-1)} = \nabla^*, \quad \nabla^{(1)} = \nabla.$$

In addition, setting  $\lambda = 0$  results in the Levi-Civita connection  $\widehat{\nabla}$  associated with the metric  $g$ , as shown by

$$\nabla^{(0)} = \frac{1}{2}(\nabla + \nabla^*) = \widehat{\nabla}.$$

Next, the covariant derivative of the metric  $g$  with respect to  $\nabla^{(\lambda)}$  is defined as

$$(\nabla_X^{(\lambda)} g)(Y, Z) = \mathcal{C}^{(\lambda)}(X, Y, Z),$$

for any  $X, Y, Z \in \mathfrak{S}_0^1(M)$ . In local coordinates, the components of  $\mathcal{C}^{(\lambda)}(\partial_i, \partial_j, \partial_k) = \mathcal{C}_{ijk}^{(\lambda)}$  are given by

$$\mathcal{C}_{ijk}^{(\lambda)} = \partial_i g_{jk} - \Gamma_{ij}^{(\lambda)r} g_{rk} - \Gamma_{ik}^{(\lambda)r} g_{jr}. \quad (7)$$

This expression reveals how the covariant derivative depends on the connection coefficients,  $\Gamma_{ij}^{(\lambda)r}$ , and the metric components. Moreover, we observe that  $\mathcal{C}^{(\lambda)}(X, Y, Z) = \lambda \mathcal{C}(X, Y, Z)$ , showing that the term  $\mathcal{C}^{(\lambda)}$  scales linearly with  $\lambda$ . Thus,  $(M, g, \nabla^{(\lambda)})$  is a statistical manifold for all  $\lambda \in \mathbb{R}$ .

**Remark 1.** For a statistical structure  $(g, \nabla^{(\lambda)})$ , we define the tensor field  $K$  by

$$K_X Y = \frac{1}{\lambda} (\nabla_X^{(\lambda)} Y - \widehat{\nabla}_X Y), \quad \forall X, Y \in \mathfrak{S}_0^1(M), \quad \lambda \neq 0. \quad (8)$$

Thus,  $K \in \mathfrak{S}_2^1(M)$  satisfies the following properties:

$$K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z). \quad (9)$$

Conversely, for a given Riemannian metric  $g$ , if a tensor field  $K \in \mathfrak{S}_2^1(M)$  satisfies the last equations, then the pair  $(\nabla^{(\lambda)} = \widehat{\nabla} + \lambda K, g)$  defines a statistical structure on  $M$ .

The skewness tensor  $\mathcal{C}$  can be used to introduce the  $\lambda$  connection as follows:

$$g(\nabla_X^{(\lambda)} Y, Z) = g(\widehat{\nabla}_X Y, Z) - \frac{1}{2} \mathcal{C}^{(\lambda)}(X, Y, Z).$$

For every statistical manifold  $(M, g, \nabla^{(\lambda)})$ , there exists a naturally associated symmetric trilinear form  $\mathcal{C}^{(\lambda)}$ .

Conversely, let  $(M, g, \mathcal{C}^{(\lambda)})$  be a semi-Riemannian manifold with symmetric trilinear form  $\mathcal{C}^{(\lambda)}$ . Then, define the tensor field  $A$  of type  $(1, 2)$  by

$$g(\lambda A(X)Y, Z) = \mathcal{C}^{(\lambda)}(X, Y, Z), \quad (10)$$

and a linear connection  $\nabla^{(\lambda)}$  by

$$\nabla^{(\lambda)} = \widehat{\nabla} - \frac{\lambda}{2} A. \quad (11)$$

Then,  $\nabla^{(\lambda)}$  is torsion-free and satisfies  $\nabla^{(\lambda)} g = \mathcal{C}^{(\lambda)}$ . Hence, the triplet  $(M, g, \nabla^{(\lambda)})$  becomes a statistical manifold.

It is noteworthy that by considering  $K = -\frac{1}{2} A$ , the two approaches described above coincide.

### 3. $\lambda$ -Adapted Frames

Let  $\nabla^{(\lambda)}$  be a  $\lambda$  connection on  $M$  and let  $TM$  be the tangent bundle of  $M$  with the natural projection  $\pi : TM \rightarrow M$ . If we consider  $\pi_* : TTM \rightarrow TM$ , then the tangent bundle  $TTM$  of  $TM$  splits into the  $\lambda$ -horizontal and  $\lambda$ -vertical subspaces  $\mathcal{H}TM$  and  $\mathcal{V}TM$  with respect to  $\nabla^{(\lambda)}$ :

$$TTM = \mathcal{H}TM \oplus \mathcal{V}TM, \quad (12)$$

where the  $\lambda$ -vertical distribution  $\mathcal{V}TM = \ker \pi_*$  and the  $\lambda$ -horizontal distribution on  $TM$  is a complementary distribution  $\mathcal{H}TM$  for  $\mathcal{V}TM$  on  $TTM$ . Namely, a system of local coordinates  $(U, x^i)$ , in the neighborhood  $U$ , induces a system of local coordinates  $(\pi^{-1}(U), x^i, y^i)$ , where  $x^i$  and  $y^i$  represent the position and direction of a point on  $TM$ , respectively. Let  $\Gamma_{ij}^{(\lambda)k}$ , with indices  $i, j, k$  running over the range  $1, \dots, n$ , be the Christoffel symbols of the affine connection  $\nabla^{(\lambda)}$ . Then

$$\left(\frac{\delta}{\delta x^i}\right)^{(\lambda)} = \frac{\partial}{\partial x^i} - y^r \Gamma_{ir}^{(\lambda)k} \frac{\partial}{\partial y^k},$$

and  $\frac{\partial}{\partial y^i}$  in  $u$ , span the spaces  $\mathcal{H}_u TM$  and  $\mathcal{V}_u TM$ , respectively.

We define  $\delta_i^{(\lambda)} = (\frac{\delta}{\delta x^i})^{(\lambda)}$  and  $\partial_{\bar{i}} = \frac{\partial}{\partial y^{\bar{i}}}$ . Then,  $\{\delta_i^{(\lambda)}, \partial_{\bar{i}}\}$  forms the  $\lambda$ -adapted local field of frames of  $TM$ . Let  $\{dx^i, (\delta y^{\bar{i}})^{(\lambda)}\}$  be the dual basis of  $\{\delta_i^{(\lambda)}, \partial_{\bar{i}}\}$ , where

$$(\delta y^{\bar{i}})^{(\lambda)} = dx^i + y^r \Gamma_{rk}^{(\lambda)i} dx^k.$$

For any  $X \in \mathfrak{X}_0^1(M)$ , we denote by  ${}^{\mathcal{H}}X^{(\lambda)}$  (and  ${}^{\mathcal{V}}X^{(\lambda)} = {}^{\mathcal{V}}X$ , respectively) the  $\lambda$ -horizontal lift (and the  $\lambda$ -vertical lift, respectively) of  $X$  to  $TM$ . According to (8) and the above expressions, considering  $K(\partial_i, \partial_j) = K_{ij}^r \partial_r$ , it can be seen that  $\delta_i^{(\lambda)}$  has the following local expression

$$\delta_i^{(\lambda)} = \widehat{\delta}_i - \lambda y^r K_{ir}^k \partial_{\bar{k}},$$

where  $\widehat{\delta}_i = \partial_i - y^r \widehat{\Gamma}_{ir}^k \partial_{\bar{k}}$  and  $\widehat{\Gamma}_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection  $\widehat{\nabla}$ . Thus, the horizontal lift  ${}^{\mathcal{H}}X$  of  $X$  to  $TM$  is obtained by

$${}^{\mathcal{H}}X^{(\lambda)} = {}^{\mathcal{H}}\widehat{X} - \lambda {}^{\mathcal{V}}K(X, y), \quad (13)$$

where  ${}^{\mathcal{H}}\widehat{X}$  is the horizontal lift of  $X$  with respect to the Levi-Civita connection  $\widehat{\nabla}$ ,  $K(\cdot, y)$  is the  $(1, 1)$  tensor field induced by  $y$  given by  $K(X, y) = \nabla_y X - \widehat{\nabla}_y X$  and  ${}^{\mathcal{V}}K(X, y)$  is the vertical lift of  $K(X, y)$ .

If  $y \in TM$ , the energy density associated with  $y$  on  $TM$  is defined as

$$\tau = y^i y^j g_{ij}.$$

**Lemma 1.** Let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold. Then, the following statements hold:

$$\partial_{\bar{k}}(\tau) = 2y_k, \quad \delta_k^{(\lambda)}(\tau) = y^i y^j C_{ijk}^{(\lambda)}, \quad (14)$$

where  $y_k = y^r g_{rk}$ .

**Proof.** Since the metric  $g$  depends only on  $(x^h)$ , we have

$$\partial_{\bar{k}}(\tau) = \partial_{\bar{k}}(y^i y^j g_{ij}) = y^j g_{kj} + y^i g_{ki} = 2y_k.$$

Additionally, we can express

$$\delta_k^{(\lambda)}(\tau) = \delta_k^{(\lambda)}(y^i y^j g_{ij}) = \delta_k^{(\lambda)}(y^i) y^j g_{ij} + y^i \delta_k^{(\lambda)}(y^j) g_{ij} + y^i y^j \partial_k(g_{ij}).$$

On the other hand, we know

$$\delta_k^{(\lambda)} y^i = -y^r \Gamma_{kr}^{(\lambda)i}.$$

From the two previous equations, it follows that

$$\delta_k^{(\lambda)}(\tau) = y^i y^j (\nabla_{\partial_k}^{(\lambda)} g)(\partial_i, \partial_j) = y^i y^j C_{ijk}^{(\lambda)}.$$

□

For a  $\lambda$  connection  $\nabla^{(\lambda)}$ , the  $\lambda$ -curvature tensor  $R^{\nabla^{(\lambda)}}$  is given by

$$R^{\nabla^{(\lambda)}}(X, Y)Z = \nabla_X^{(\lambda)} \nabla_Y^{(\lambda)} Z - \nabla_Y^{(\lambda)} \nabla_X^{(\lambda)} Z - \nabla_{[X, Y]}^{(\lambda)} Z,$$

for any  $X, Y, Z \in \mathfrak{S}_0^1(M)$ . We denote  $R^{\nabla^{(\lambda)}}$  by  $R^{(\lambda)}$  for simplicity. Locally, the components of the curvature tensor are determined by

$$R_{ijk}^{(\lambda)r} = \partial_i \Gamma_{jk}^{(\lambda)r} - \partial_j \Gamma_{ik}^{(\lambda)r} + \Gamma_{im}^{(\lambda)r} \Gamma_{jk}^{(\lambda)m} - \Gamma_{jm}^{(\lambda)r} \Gamma_{ik}^{(\lambda)m}, \quad (15)$$

where  $R^{(\lambda)}(\partial_i, \partial_j)\partial_k = R_{ijk}^{(\lambda)r}\partial_r$ . A statistical manifold is said to be  $\lambda$  flat if its  $\lambda$  curvature tensor vanishes.

According to (15), the following relations hold

$$R_{ijkl}^{(\lambda)} = -R_{jikl}^{(\lambda)}, \quad (16)$$

$$R_{ijkl}^{(\lambda)} = -R_{ijlk}^{(-\lambda)}, \quad (17)$$

$$R_{ijk}^{(\lambda)r} + R_{jki}^{(\lambda)r} + R_{kij}^{(\lambda)r} = 0, \quad (18)$$

where  $R_{ijkl}^{(\lambda)} = g_{rl}R_{ijk}^{(\lambda)r}$ . By multiplying both sides of relation (17) by  $y^l y^k$  and noting that  $k$  and  $l$  are summation indices, the following result is obtained:

$$y^l y^k R_{ijlk}^{(\lambda)} = -y^l y^k R_{ijkl}^{(-\lambda)} = -y^l y^k R_{ijlk}^{(-\lambda)}.$$

Therefore, the following identity holds

$$y^l y^k (R_{ijlk}^{(\lambda)} + R_{ijlk}^{(-\lambda)}) = 0. \quad (19)$$

In particular, for the Levi–Civita connection  $\widehat{\nabla}$ , the above equation simplifies to

$$y^l y^k \widehat{R}_{ijlk} = 0.$$

The Lie bracket of the  $\lambda$ -adapted frame of  $TM$  satisfies the following relations

$$[\delta_i^{(\lambda)}, \delta_j^{(\lambda)}] = -y^l R_{ijl}^{(\lambda)k} \partial_{\bar{k}}, \quad [\delta_i^{(\lambda)}, \partial_{\bar{j}}] = \Gamma_{ij}^{(\lambda)k} \partial_{\bar{k}}, \quad [\partial_{\bar{i}}, \partial_{\bar{j}}] = 0. \quad (20)$$

#### 4. The Levi–Civita Connection on a Statistical Manifold with the Cheeger Gromoll Metric

Let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold. The Cheeger Gromoll metric  ${}^{CG}g$  on the tangent bundle  $TM$  is a natural lift of the metric  $g$  given by

$$\begin{cases} {}^{CG}g_{(p,y)}(\mathcal{H}X^{(\lambda)}, \mathcal{H}Y^{(\lambda)}) = g_p(X, Y), \\ {}^{CG}g_{(p,y)}(\mathcal{V}X, \mathcal{H}Y^{(\lambda)}) = 0, \\ {}^{CG}g_{(p,y)}(\mathcal{V}X, \mathcal{V}Y) = \frac{1}{\alpha} \{g_p(X, Y) + g_p(X, y)g_p(Y, y)\}, \end{cases} \quad (21)$$

where  $X, Y \in \mathfrak{S}_0^1(M)$  and  $\alpha = 1 + \tau$ . Applying (13), it is straightforward to see that  ${}^{\text{CG}}g$  can be expressed as follows:

$$\begin{aligned} {}^{\text{CG}}g_{(p,y)}(\mathcal{H}\hat{X}, \mathcal{H}\hat{Y}) &= g_p(X, Y) + \frac{\lambda^2}{\alpha} \{g_p(K(X, y), K(Y, y)) + g_p(K(X, y), y)g_p(K(Y, y), y)\}, \\ {}^{\text{CG}}g_{(p,y)}(\mathcal{H}\hat{X}, \mathcal{V}\hat{Y}) &= \frac{\lambda}{\alpha} \{g_p(K(X, y), Y) + g_p(K(X, y), y)g_p(Y, y)\}, \\ {}^{\text{CG}}g_{(p,y)}(\mathcal{V}\hat{X}, \mathcal{V}\hat{Y}) &= \frac{1}{\alpha} \{g_p(X, Y) + g_p(X, y)g_p(Y, y)\}. \end{aligned}$$

The matrix representation of  ${}^{\text{CG}}g$  with respect to the adapted frame  $\{\delta_i^{(\lambda)}, \partial_i\}$  is given by

$${}^{\text{CG}}g = \begin{pmatrix} {}^{\text{CG}}g_{ij} & {}^{\text{CG}}g_{i\bar{j}} \\ {}^{\text{CG}}g_{\bar{i}j} & {}^{\text{CG}}g_{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{\alpha}(g_{ij} + y_i y_j) \end{pmatrix}.$$

This matrix has the following inverse

$$({}^{\text{CG}}g)^{-1} = \begin{pmatrix} {}^{\text{CG}}g^{ij} & {}^{\text{CG}}g^{i\bar{j}} \\ {}^{\text{CG}}g^{\bar{i}j} & {}^{\text{CG}}g^{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} g^{ij} & 0 \\ 0 & \alpha g^{ij} - y^i y^j \end{pmatrix},$$

where  $g_{ij}$  and  $g^{ij}$  are the local covariant and contravariant components of  $g$  on  $M$ .

Let  ${}^{\text{CG}}\hat{\nabla}$  denote the Levi–Civita connection of the Cheeger Gromoll metric  ${}^{\text{CG}}g$ . The explicit expression for  ${}^{\text{CG}}\hat{\nabla}$  can be obtained from the following well-known formula

$$\begin{aligned} 2{}^{\text{CG}}g({}^{\text{CG}}\hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}) &= \bar{X} {}^{\text{CG}}g(\bar{Y}, \bar{Z}) + \bar{Y} {}^{\text{CG}}g(\bar{X}, \bar{Z}) - \bar{Z} {}^{\text{CG}}g(\bar{X}, \bar{Y}) \\ &\quad + {}^{\text{CG}}g(\bar{Z}, [\bar{X}, \bar{Y}]) - {}^{\text{CG}}g(\bar{Y}, [\bar{X}, \bar{Z}]) - {}^{\text{CG}}g(\bar{X}, [\bar{Y}, \bar{Z}]), \end{aligned} \quad (22)$$

for any  $\bar{X}, \bar{Y}, \bar{Z} \in \chi(TM)$ . A direct computation using (20) and (21) leads to the following theorem:

**Theorem 1.** Let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold and let  ${}^{\text{CG}}\hat{\nabla}$  be the Levi–Civita connection of the tangent bundle  $TM$  equipped with the Cheeger Gromoll metric  ${}^{\text{CG}}g$ . Then,  ${}^{\text{CG}}\hat{\nabla}$  satisfies the following expressions:

$$\begin{aligned} {}^{\text{CG}}\hat{\nabla}_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)} &= Q_{ij}^s \delta_s^{(\lambda)} - \frac{1}{2}y^l R_{ijl}^{(\lambda)s} \partial_s, & {}^{\text{CG}}\hat{\nabla}_{\delta_i^{(\lambda)}}\partial_{\bar{j}} &= P_{i\bar{j}}^s \delta_s^{(\lambda)} + (\Gamma_{ij}^{(\lambda)s} + P_{i\bar{j}}^{\bar{s}})\partial_{\bar{s}}, \\ {}^{\text{CG}}\hat{\nabla}_{\partial_i}\delta_j^{(\lambda)} &= P_{j\bar{i}}^s \delta_s^{(\lambda)} + P_{j\bar{i}}^{\bar{s}} \partial_{\bar{s}}, & {}^{\text{CG}}\hat{\nabla}_{\partial_i}\partial_{\bar{j}} &= S_{i\bar{j}}^s \delta_s^{(\lambda)} + S_{i\bar{j}}^{\bar{s}} \partial_{\bar{s}}, \end{aligned}$$

where the coefficients are given by

$$\begin{aligned} Q_{ij}^s &= \Gamma_{ij}^{(\lambda)s} + \frac{1}{2}C_{ijk}^{(\lambda)} g^{ks}, \\ P_{i\bar{j}}^s &= \frac{1}{2}y^r R_{ikr}^{(\lambda)l} {}^{\text{CG}}g_{\bar{j}l} g^{ks}, \\ P_{i\bar{j}}^{\bar{s}} &= \frac{1}{2\alpha}(y_j y^r C_{ikr}^{(\lambda)} {}^{\text{CG}}g^{\bar{k}\bar{s}} - y^l y^r C_{irl}^{(\lambda)} \delta_j^s + \alpha C_{ijk}^{(\lambda)} g^{ks}), \\ S_{i\bar{j}}^s &= \frac{1}{2\alpha} g^{ks} (y^m y^l C_{kml}^{(\lambda)} {}^{\text{CG}}g_{\bar{i}\bar{j}} - C_{kji}^{(\lambda)} - y_i y^r C_{kjr}^{(\lambda)} - y_j y^r C_{kir}^{(\lambda)}), \\ S_{i\bar{j}}^{\bar{s}} &= \frac{1+\alpha}{\alpha^2} g_{ij} y^s + \frac{1}{\alpha^2} y_i y_j y^s - \frac{1}{\alpha} (y_i \delta_j^s + y_j \delta_i^s). \end{aligned} \quad (23)$$

**Example 1.** We consider the normal distribution manifold, which is defined as

$$M = \{p(x, \mu, \sigma) | p(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}, \mu \in \mathbb{R}, \sigma > 0\}.$$



Hence,  $M$  may be viewed as a 2-dimensional manifold with  $(\mu, \sigma)$  as a coordinate system. The Fisher metric of  $M$  is described by

$$g = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}. \quad (24)$$

The components  $\Gamma_{ij}^{(\lambda)k}$  of the  $\lambda$  connection  $\nabla^{(\lambda)}$  on  $M$  are given by

$$\Gamma_{11}^{(\lambda)2} = \frac{1-\lambda}{2\sigma}, \quad \Gamma_{12}^{(\lambda)1} = \Gamma_{21}^{(\lambda)1} = -\frac{\lambda+1}{\sigma}, \quad \Gamma_{22}^{(\lambda)2} = -\frac{2\lambda+1}{\sigma}, \quad (25)$$

while the other independent components are zero. From the above equations and (7), it follows that

$$C_{111}^{(\lambda)} = 0, \quad C_{121}^{(\lambda)} = C_{112}^{(\lambda)} = C_{211}^{(\lambda)} = \frac{2\lambda}{\sigma^3}, \quad C_{122}^{(\lambda)} = C_{212}^{(\lambda)} = C_{221}^{(\lambda)} = 0, \quad C_{222}^{(\lambda)} = \frac{8\lambda}{\sigma^3}, \quad (26)$$

which implies that  $(M, g, \nabla^{(\lambda)})$  is a statistical manifold. Applying (21), we obtain the Cheeger Gromoll metric  ${}^{CG}g$  associated with the Fisher metric (24) as follows

$$\begin{aligned} {}^{CG}g_{11} &= \frac{1}{\sigma^2}, & {}^{CG}g_{22} &= \frac{2}{\sigma^2}, & {}^{CG}g_{\bar{1}\bar{1}} &= \frac{1}{\alpha} \left( \frac{1}{\sigma^2} + y_1 y_1 \right), & {}^{CG}g_{\bar{2}\bar{2}} &= \frac{1}{\alpha} \left( \frac{2}{\sigma^2} + y_2 y_2 \right), \\ {}^{CG}g_{\bar{1}\bar{2}} &= \frac{1}{\alpha} y_1 y_2, & {}^{CG}g_{12} &= {}^{CG}g_{\bar{1}\bar{1}} = {}^{CG}g_{\bar{1}\bar{2}} = {}^{CG}g_{2\bar{1}} = {}^{CG}g_{\bar{2}\bar{2}} = 0. \end{aligned} \quad (27)$$

The  $\lambda$ -curvature tensor  $R^{(\lambda)}$  of  $\nabla^{(\lambda)}$  is determined by

$$R_{121}^{(\lambda)2} = \frac{1-\lambda^2}{2\sigma^2}, \quad R_{122}^{(\lambda)1} = \frac{\lambda^2-1}{\sigma^2}.$$

Substituting (26) and (27) into (23), we find

$$\begin{aligned}
Q_{11}^2 &= \frac{1}{2\sigma}, & Q_{12}^1 &= Q_{21}^1 = Q_{22}^2 = -\frac{1}{\sigma}, & P_{22}^1 &= -2P_{12}^2 = -\frac{y^1}{2\alpha\sigma^4} \left( (\lambda - 1) \left( 4(y^2)^2 - \lambda\sigma^2 \right) \right), \\
P_{21}^1 &= -2P_{11}^2 = -\frac{y^2}{2\alpha\sigma^4} \left( \sigma^2(\lambda^2 + \lambda - 2) + 2(\lambda - 1)(y^1)^2 \right), & P_{11}^{\bar{1}} &= \frac{\lambda}{\alpha\sigma^3} \left( (\alpha - 2) - 2y_1y^1 \right) y^1y^2, \\
P_{11}^2 &= \frac{\lambda}{\alpha\sigma^3} \left( \frac{\alpha\sigma^2}{2} (1 + y_1y^1) - 2y_1y^1y^2y^2 \right), & P_{12}^{\bar{1}} &= \frac{\lambda}{\alpha\sigma^3} \left( \alpha\sigma^2 (1 + y_2y^2) - 2y_2y^2y^1y^1 \right), \\
P_{12}^{\bar{2}} &= \frac{\lambda}{\alpha\sigma^3} \left( (\alpha - 2) - 2y_2y^2 \right) y^1y^2, & P_{21}^{\bar{1}} &= \frac{\lambda}{\alpha\sigma^3} \left( \alpha\sigma^2 - y^1y^1 - 4y^2y^2 \right) (1 + y_1y^1), \\
P_{21}^2 &= \frac{\lambda}{\alpha\sigma^3} \left( 4 \left( \frac{\alpha\sigma^2}{2} - y^2y^2 \right) - y^1y^1 \right) y_1y^2, & P_{22}^{\bar{1}} &= \frac{\lambda}{\alpha\sigma^3} \left( (\alpha\sigma^2 - y^1y^1) - 4y^2y^2 \right) y^1y_2, \\
P_{22}^2 &= \frac{\lambda}{\alpha\sigma^3} \left( 4 \left( \frac{\alpha\sigma^2}{2} - y^2y^2 \right) - y^1y^1 \right) (1 + y_2y^2), & S_{11}^1 &= \frac{2\lambda}{\alpha\sigma^3} \left( \frac{1}{\alpha} (1 + y_1y^1) - 1 \right) y^1y^2, \\
S_{11}^2 &= \frac{\lambda}{2\alpha\sigma} \left( \frac{1}{\alpha} \left( \frac{1}{\sigma^2} + y_1y^1 \right) (y^1y^1 + 4y^2y^2) - 1 - 2y_1y^1 \right), & S_{12}^1 &= \frac{\lambda}{\alpha\sigma} \left( \frac{2}{\alpha} y_1y^1y_2y^2 - \alpha \right), \\
S_{12}^2 &= \frac{\lambda}{2\alpha\sigma} \left( \frac{1}{\alpha} y_1y_2 (y^1y^1 + 4y^2y^2) - 4y_1y^2 - y_2y^1 \right), & S_{22}^1 &= \frac{2\lambda}{\alpha\sigma} \left( \left( \frac{1}{\alpha} - 1 \right) + \frac{1}{\alpha} y_2y^2 \right) y_2y^1, \\
S_{22}^2 &= \frac{\lambda}{2\alpha\sigma} \left( \frac{1}{\alpha} \left( \frac{2}{\sigma^2} + y_2y^2 \right) (y^1y^1 + 4y^2y^2) - 4 - 8y_2y^2 \right), & S_{11}^{\bar{1}} &= \frac{1}{\alpha^2\sigma^4} \left( \sigma^2(1 - \alpha) + (y^1)^2 \right) y^1, \\
S_{12}^{\bar{1}} &= S_{21}^{\bar{1}} = \frac{2}{\alpha^2\sigma^4} \left( (y^1)^2 - \alpha\sigma^2 \right) y^2, & S_{22}^{\bar{1}} &= \frac{2}{\alpha^2\sigma^4} \left( \sigma^2(1 + \alpha) + 2(y^2)^2 \right) y^1, \\
S_{11}^{\bar{2}} &= \frac{1}{\alpha^2\sigma^4} \left( \sigma^2(1 + \alpha) + (y^1)^2 \right) y^2, & S_{12}^{\bar{2}} &= S_{21}^{\bar{2}} = \frac{1}{\alpha^2\sigma^4} \left( 2(y^2)^2 - \alpha\sigma^2 \right) y^1, \\
S_{22}^{\bar{2}} &= \frac{2}{\alpha^2\sigma^4} \left( \sigma^2(1 - \alpha) + 2(y^2)^2 \right) y^2, & Q_{11}^1 &= Q_{22}^1 = Q_{12}^2 = Q_{21}^2 = P_{11}^1 = P_{12}^1 = P_{22}^2 = 0.
\end{aligned}$$

Thus, the corresponding Levi-Civita connection satisfies the following relations

$$\begin{aligned}
{}^{\text{CG}}\widehat{\nabla}_{\delta_1^{(\lambda)}}\delta_1^{(\lambda)} &= Q_{11}^2\delta_2^{(\lambda)}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_2^{(\lambda)}}\delta_2^{(\lambda)} &= Q_{22}^2\delta_2^{(\lambda)}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_1^{(\lambda)}}\delta_2^{(\lambda)} &= Q_{12}^1\delta_1^{(\lambda)} - \frac{\lambda^2 - 1}{2\sigma^2}y^2\partial_{\bar{1}} - \frac{1 - \lambda^2}{4\sigma^2}y^1\partial_{\bar{2}}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_2^{(\lambda)}}\delta_1^{(\lambda)} &= Q_{21}^1\delta_1^{(\lambda)} + \frac{\lambda^2 - 1}{2\sigma^2}y^2\partial_{\bar{1}} + \frac{1 - \lambda^2}{4\sigma^2}y^1\partial_{\bar{2}}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{1}} &= P_{11}^1\delta_1^{(\lambda)} + P_{11}^2\delta_2^{(\lambda)} + P_{11}^{\bar{1}}\partial_{\bar{1}} + (\Gamma_{11}^{(\lambda)2} + P_{11}^{\bar{2}})\partial_{\bar{2}}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{2}} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^2\delta_2^{(\lambda)} + (\Gamma_{12}^{(\lambda)1} + P_{12}^{\bar{1}})\partial_{\bar{1}} + P_{12}^{\bar{2}}\partial_{\bar{2}}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{1}} &= P_{21}^1\delta_1^{(\lambda)} + P_{21}^2\delta_2^{(\lambda)} + (\Gamma_{21}^{(\lambda)1} + P_{21}^{\bar{1}})\partial_{\bar{1}} + P_{21}^{\bar{2}}\partial_{\bar{2}}, \\
{}^{\text{CG}}\widehat{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{2}} &= P_{22}^1\delta_1^{(\lambda)} + P_{22}^2\delta_2^{(\lambda)} + P_{22}^{\bar{1}}\partial_{\bar{1}} + (\Gamma_{22}^{(\lambda)2} + P_{22}^{\bar{2}})\partial_{\bar{2}},
\end{aligned}$$

and

$$\begin{aligned}
 {}^{CG}\widehat{\nabla}_{\partial_1}\delta_1^{(\lambda)} &= P_{11}^1\delta_1^{(\lambda)} + P_{11}^2\delta_2^{(\lambda)} + P_{11}^{\bar{1}}\partial_{\bar{1}} + P_{11}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_1}\delta_2^{(\lambda)} &= P_{21}^1\delta_1^{(\lambda)} + P_{21}^2\delta_2^{(\lambda)} + P_{21}^{\bar{1}}\partial_{\bar{1}} + P_{21}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_2}\delta_1^{(\lambda)} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^2\delta_2^{(\lambda)} + P_{12}^{\bar{1}}\partial_{\bar{2}} + P_{12}^{\bar{2}}\partial_{\bar{1}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_2}\delta_2^{(\lambda)} &= P_{22}^1\delta_1^{(\lambda)} + P_{22}^2\delta_2^{(\lambda)} + P_{22}^{\bar{1}}\partial_{\bar{1}} + P_{22}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_1}\partial_{\bar{1}} &= S_{11}^1\delta_1^{(\lambda)} + S_{11}^2\delta_2^{(\lambda)} + S_{11}^{\bar{1}}\partial_{\bar{1}} + S_{11}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_1}\partial_{\bar{2}} &= {}^{CG}\widehat{\nabla}_{\partial_2}\partial_{\bar{1}} = S_{12}^1\delta_1^{(\lambda)} + S_{12}^2\delta_2^{(\lambda)} + S_{12}^{\bar{1}}\partial_{\bar{1}} + S_{12}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\widehat{\nabla}_{\partial_2}\partial_{\bar{2}} &= S_{22}^1\delta_1^{(\lambda)} + S_{22}^2\delta_2^{(\lambda)} + S_{22}^{\bar{1}}\partial_{\bar{1}} + S_{22}^{\bar{2}}\partial_{\bar{2}}.
 \end{aligned}$$

**Example 2.** We equip  $M = \mathbb{R}^2$  with the pseudo-Riemannian metric

$$g = dx^1 \otimes dx^2 + dx^2 \otimes dx^1,$$

where  $(x^1, x^2)$  denotes the standard coordinate system, and  $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$  represent the associated vector fields. For the functions  $f_i = f_i(x^1, x^2)$  with  $i = 1, \dots, 4$  on  $\mathbb{R}^2$ , define  $(1, 2)$ -tensor field  $K = K_{ij}^l \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j$ ,  $i, j, l = 1, 2$  on  $\mathbb{R}^2$  as follows:

$$K_{11}^1 = K_{12}^2 = K_{21}^2 = f_1, \quad K_{12}^1 = K_{21}^1 = K_{22}^2 = f_2, \quad K_{11}^2 = f_3, \quad K_{22}^1 = f_4.$$

Thus,  $K$  satisfies (9). Setting  $\nabla_{\partial_i}^{(\lambda)} \partial_j := \widehat{\nabla}_{\partial_i} \partial_j + \lambda K_{\partial_i} \partial_j$ , we obtain the following Christoffel symbols:

$$\Gamma_{11}^{(\lambda)1} = \Gamma_{12}^{(\lambda)2} = \Gamma_{21}^{(\lambda)2} = \lambda f_1, \quad \Gamma_{12}^{(\lambda)1} = \Gamma_{21}^{(\lambda)1} = \Gamma_{22}^{(\lambda)2} = \lambda f_2, \quad \Gamma_{11}^{(\lambda)2} = \lambda f_3, \quad \Gamma_{22}^{(\lambda)1} = \lambda f_4.$$

It follows

$$C_{111}^{(\lambda)} = -2\lambda f_3, \quad C_{112}^{(\lambda)} = C_{121}^{(\lambda)} = C_{211}^{(\lambda)} = -2\lambda f_1, \quad C_{122}^{(\lambda)} = C_{212}^{(\lambda)} = C_{221}^{(\lambda)} = -2\lambda f_2, \quad C_{222}^{(\lambda)} = -2\lambda f_4.$$

Therefore,  $(\mathbb{R}^2, g, \nabla^{(\lambda)})$  forms a statistical manifold. Now, suppose  $f_1 = f_2 = 0$ . Using (15), we obtain  $R_{ijk}^{(\lambda)r} = 0$ ,  $i, j, k, r = 1, 2$  except

$$R_{121}^{(\lambda)1} = -\lambda^2 f_3 f_4 = -R_{122}^{(\lambda)2}, \quad R_{122}^{(\lambda)1} = \lambda \partial_1 f_4, \quad R_{121}^{(\lambda)2} = -\lambda \partial_2 f_3.$$

According to (23), we obtain

$$\begin{cases}
 P_{11}^1 = -P_{21}^2 = \frac{\lambda}{2\alpha} (-\partial_2 f_3 y^1 (1 + y^1 y^2) + \partial_1 f_4 (y^2)^3 + \lambda f_3 f_4 y^2), & P_{11}^2 = -\frac{1}{\alpha} \lambda f_3 (y^1 y^2 (\alpha - y^1 y^2) + \alpha), \\
 P_{12}^1 = -P_{22}^2 = \frac{\lambda}{2\alpha} (\partial_1 f_4 y^2 (1 + y^1 y^2) - \partial_2 f_3 (y^1)^3 - \lambda f_3 f_4 y^1), & P_{22}^{\bar{1}} = -\frac{1}{\alpha} \lambda f_4 (y^1 y^2 (\alpha - y^1 y^2) + \alpha), \\
 P_{21}^2 = \frac{1}{\alpha} \lambda f_4 (y^2)^4, & P_{12}^2 = \frac{1-\alpha}{2\alpha} \lambda f_3 (y^1)^2, \quad P_{21}^{\bar{1}} = \frac{1-\alpha}{2\alpha} \lambda f_4 (y^2)^2, \quad S_{11}^2 = \lambda f_3 (1 - \frac{1}{\alpha^2} (y^1)^2 (y^2)^2), \\
 P_{12}^{\bar{1}} = \frac{1}{\alpha} \lambda f_3 (y^1)^4, & P_{11}^{\bar{1}} = \frac{1+\alpha}{2\alpha} \lambda f_3 (y^1)^2, \quad P_{22}^2 = \frac{1+\alpha}{2\alpha} \lambda f_4 (y^2)^2, \quad S_{12}^1 = \lambda f_4 (1 - \frac{1}{\alpha^2} (y^1)^2 (y^2)^2), \\
 S_{11}^2 = \frac{1}{\alpha^2} (y^2)^3, & S_{11}^{\bar{1}} = -\frac{1}{\alpha^2} \lambda f_4 (y^2)^4, \quad S_{12}^1 = \frac{\alpha-1}{2\alpha^2} \lambda f_4 (y^2)^2, \quad S_{11}^{\bar{1}} = \frac{1}{\alpha} (\frac{1}{\alpha} y^1 y^2 - 2) y^2, \quad S_{12}^{\bar{1}} = \frac{1+\alpha}{2\alpha^2} y^1, \\
 S_{22}^{\bar{1}} = \frac{1}{\alpha^2} (y^1)^3, & S_{12}^2 = \frac{\alpha-1}{2\alpha^2} \lambda f_3 (y^1)^2, \quad S_{22}^2 = -\frac{1}{\alpha^2} \lambda f_3 (y^1)^4, \quad S_{12}^2 = \frac{1+\alpha}{2\alpha^2} y^2, \quad S_{22}^2 = -\frac{1+3\alpha}{2\alpha^2} y^1.
 \end{cases}$$

From (22) and the above equations, the Levi–Civita connection on  $TM$  satisfies

$$\begin{aligned}
 {}^{CG}\hat{\nabla}_{\delta_1^{(\lambda)}}\delta_1^{(\lambda)} &= {}^{CG}\hat{\nabla}_{\delta_2^{(\lambda)}}\delta_2^{(\lambda)} = 0, \\
 {}^{CG}\hat{\nabla}_{\delta_1^{(\lambda)}}\delta_2^{(\lambda)} &= \frac{\lambda}{2}(\lambda f_3 f_4 y^1 - \partial_1 f_4 y^2)\partial_{\bar{1}} + \frac{\lambda}{2}(\partial_2 f_3 y^1 - \lambda f_3 f_4 y^2)\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\delta_2^{(\lambda)}}\delta_1^{(\lambda)} &= -\frac{\lambda}{2}(\lambda f_3 f_4 y^1 - \partial_1 f_4 y^2)\partial_{\bar{1}} - \frac{\lambda}{2}(\partial_2 f_3 y^1 - \lambda f_3 f_4 y^2)\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{1}} &= P_{1\bar{1}}^1\delta_1^{(\lambda)} + P_{1\bar{1}}^{\bar{1}}\partial_{\bar{1}} + (\Gamma_{11}^{(\lambda)2} + P_{11}^2)\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{2}} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^{\bar{1}}\partial_{\bar{1}} + P_{12}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{1}} &= P_{2\bar{1}}^2\delta_2^{(\lambda)} + P_{2\bar{1}}^{\bar{1}}\partial_{\bar{1}} + P_{2\bar{1}}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{2}} &= P_{22}^2\delta_2^{(\lambda)} + (\Gamma_{22}^{(\lambda)1} + P_{22}^{\bar{1}})\partial_{\bar{1}} + P_{22}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{1}}}\delta_1^{(\lambda)} &= P_{11}^1\delta_1^{(\lambda)} + P_{11}^{\bar{1}}\partial_{\bar{1}} + P_{11}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{1}}}\delta_2^{(\lambda)} &= P_{21}^2\delta_2^{(\lambda)} + P_{21}^{\bar{1}}\partial_{\bar{1}} + P_{21}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{2}}}\delta_1^{(\lambda)} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^{\bar{1}}\partial_{\bar{1}} + P_{12}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{2}}}\delta_2^{(\lambda)} &= P_{22}^2\delta_2^{(\lambda)} + P_{22}^{\bar{1}}\partial_{\bar{1}} + P_{22}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{1}}}\partial_{\bar{1}} &= S_{11}^1\delta_1^{(\lambda)} + S_{11}^2\delta_2^{(\lambda)} + S_{11}^{\bar{1}}\partial_{\bar{1}} + S_{11}^{\bar{2}}\partial_{\bar{2}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{1}}}\partial_{\bar{2}} &= S_{12}^1\delta_1^{(\lambda)} + S_{12}^2\delta_2^{(\lambda)} + S_{12}^{\bar{1}}\partial_{\bar{1}} + S_{12}^{\bar{2}}\partial_{\bar{2}} = {}^{CG}\hat{\nabla}_{\partial_{\bar{2}}}\partial_{\bar{1}}, \\
 {}^{CG}\hat{\nabla}_{\partial_{\bar{2}}}\partial_{\bar{2}} &= S_{22}^1\delta_1^{(\lambda)} + S_{22}^2\delta_2^{(\lambda)} + S_{22}^{\bar{1}}\partial_{\bar{1}} + S_{22}^{\bar{2}}\partial_{\bar{2}}.
 \end{aligned}$$

## 5. Statistical Structures on $(TM, {}^{CG}g)$

Assuming the Riemannian manifold  $(TM, {}^{CG}g)$  induced by the statistical manifold  $(M, g, \nabla^{(\lambda)})$ , we define a new connection on  $TM$  by  ${}^{CG}\nabla^{(\lambda)} : \chi(TM) \times \chi(TM) \rightarrow \chi(TM)$  with the coefficients  ${}^{CG}\Gamma_{BC}^{(\lambda)A}$  that are

$$\left\{ \begin{array}{ll} {}^{CG}\nabla_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)} = {}^{CG}\Gamma_{ij}^{(\lambda)k}\delta_k^{(\lambda)} + {}^{CG}\Gamma_{ij}^{(\lambda)\bar{k}}\partial_{\bar{k}}, & {}^{CG}\nabla_{\delta_i^{(\lambda)}}\partial_{\bar{j}} = {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k}\delta_k^{(\lambda)} + {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)\bar{k}}\partial_{\bar{k}}, \\ {}^{CG}\nabla_{\partial_{\bar{i}}}\delta_j^{(\lambda)} = {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k}\delta_k^{(\lambda)} + {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)\bar{k}}\partial_{\bar{k}}, & {}^{CG}\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} = {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k}\delta_k^{(\lambda)} + {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)\bar{k}}\partial_{\bar{k}}, \end{array} \right. \quad (28)$$

where  ${}^{CG}\Gamma_{BC}^{(\lambda)A}$  are smooth functions on  $TM$  and the indices  $A, B, C$  range over  $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ .

**Lemma 2.** The connection  ${}^{CG}\nabla^{(\lambda)}$  on  $TM$  is torsion-free if and only if the following relations hold:

$$\left\{ \begin{array}{ll} {}^{CG}\Gamma_{ij}^{(\lambda)k} = {}^{CG}\Gamma_{ji}^{(\lambda)k}, & {}^{CG}\Gamma_{ij}^{(\lambda)\bar{k}} - {}^{CG}\Gamma_{ji}^{(\lambda)\bar{k}} = -y^r R_{ijr}^{(\lambda)k}, \\ {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k} = {}^{CG}\Gamma_{\bar{j}i}^{(\lambda)k}, & {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)\bar{k}} - {}^{CG}\Gamma_{\bar{j}i}^{(\lambda)\bar{k}} = \Gamma_{ij}^{(\lambda)k}, \\ {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k} = {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)k}, & {}^{CG}\Gamma_{i\bar{j}}^{(\lambda)\bar{k}} = {}^{CG}\Gamma_{\bar{j}i}^{(\lambda)\bar{k}}. \end{array} \right. \quad (29)$$

**Proof.** Using (20) and the symmetry property of  ${}^{CG}\nabla^{(\lambda)}$ , we have

$${}^{CG}\nabla_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)} - {}^{CG}\nabla_{\delta_j^{(\lambda)}}\delta_i^{(\lambda)} = [\delta_i^{(\lambda)}, \delta_j^{(\lambda)}] = -y^r R_{ijr}^{(\lambda)k}\partial_{\bar{k}},$$

which gives us the first two relations in the statements. Similarly, since

$${}^{CG}\nabla_{\delta_i^{(\lambda)}}\partial_{\bar{j}} - {}^{CG}\nabla_{\partial_{\bar{j}}}\delta_i^{(\lambda)} = [\delta_i^{(\lambda)}, \partial_{\bar{j}}] = \Gamma_{ij}^{(\lambda)k}\partial_{\bar{k}},$$

we obtain the second line of relations. Finally, observing that

$${}^{CG}\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} - {}^{CG}\nabla_{\partial_{\bar{j}}}\partial_{\bar{i}} = [\partial_{\bar{i}}, \partial_{\bar{j}}] = 0,$$

we derive the third line of relations. This completes the proof.  $\square$

**Lemma 3.** Let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold. Then,  $(TM, {}^{CG}g, {}^{CG}\nabla^{(\lambda)})$  is a statistical manifold if and only if (29) holds along with the following equations:

- (i)  $\partial_i g_{jk} - \partial_j g_{ki} = {}^{CG}\Gamma_{ki}^{(\lambda)r} g_{rj} - {}^{CG}\Gamma_{kj}^{(\lambda)r} g_{ri},$
- (ii)  ${}^{CG}\Gamma_{ik}^{(\lambda)r} g_{rj} - {}^{CG}\Gamma_{jk}^{(\lambda)r} g_{ri} = y^l R_{ijl}^{(\lambda)r},$
- (iii)  $\frac{1}{\alpha}(-y^m y^l C_{iml}^{(\lambda)CG} g_{\bar{j}\bar{k}} + C_{ijk}^{(\lambda)} + y^m y_k C_{ijm}^{(\lambda)} + y^m y_j C_{ikm}^{(\lambda)}) = {}^{CG}\Gamma_{ki}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{j}} - {}^{CG}\Gamma_{jk}^{(\lambda)r} g_{ri},$
- (iv)  $\frac{2+\alpha}{\alpha^2}(g_{ki}y_j - g_{kj}y_i) = {}^{CG}\Gamma_{ik}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{j}} - {}^{CG}\Gamma_{jk}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{i}}.$

**Proof.** We begin by examining the expression for the covariant derivative of the metric

$$({}^{CG}\nabla_{\delta_i^{(\lambda)}} {}^{CG}g)(\delta_j^{(\lambda)}, \delta_k^{(\lambda)}) = \partial_i g_{jk} - {}^{CG}\Gamma_{ij}^{(\lambda)r} g_{rk} - {}^{CG}\Gamma_{ik}^{(\lambda)r} g_{rj}.$$

By interchanging the indices  $i$  and  $j$  in this equation, and applying the Codazzi equations

$$({}^{CG}\nabla_{\delta_i^{(\lambda)}} {}^{CG}g)(\delta_j^{(\lambda)}, \delta_k^{(\lambda)}) = ({}^{CG}\nabla_{\delta_j^{(\lambda)}} {}^{CG}g)(\delta_i^{(\lambda)}, \delta_k^{(\lambda)}),$$

we deduce condition (i). From

$$({}^{CG}\nabla_{\delta_j^{(\lambda)}} {}^{CG}g)(\delta_i^{(\lambda)}, \partial_{\bar{k}}) = ({}^{CG}\nabla_{\delta_j^{(\lambda)}} {}^{CG}g)(\delta_i^{(\lambda)}, \partial_{\bar{k}}),$$

it follows

$$-{}^{CG}\Gamma_{ij}^{\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{ik}^r g_{jr} = -{}^{CG}\Gamma_{ji}^{\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{jk}^r g_{ir}.$$

Furthermore, using the identity

$${}^{CG}\Gamma_{ij}^{(\lambda)\bar{r}} - {}^{CG}\Gamma_{ji}^{(\lambda)\bar{r}} = -y^l R_{ijl}^{(\lambda)r},$$

we deduce condition (ii). To derive condition (iii), apply Formula (14) to

$$({}^{CG}\nabla_{\delta_i^{(\lambda)}} {}^{CG}g)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \delta_i^{(\lambda)CG} g_{\bar{j}\bar{k}} - {}^{CG}\Gamma_{ij}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{ik}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{j}}.$$

This yields

$$\begin{aligned} ({}^{CG}\nabla_{\delta_i^{(\lambda)}} {}^{CG}g)(\partial_{\bar{j}}, \partial_{\bar{k}}) &= \frac{1}{\alpha}(-y^m y^l C_{iml}^{(\lambda)CG} g_{\bar{j}\bar{k}} + C_{ijk}^{(\lambda)} + y^m y_k C_{ijm}^{(\lambda)} + y^m y_j C_{ikm}^{(\lambda)}) \\ &\quad - {}^{CG}\Gamma_{ji}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{ki}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{j}}. \end{aligned}$$

Therefore, from

$$({}^{CG}\nabla_{\partial_{\bar{j}}} {}^{CG}g)(\delta_i^{(\lambda)}, \partial_{\bar{k}}) = -{}^{CG}\Gamma_{ji}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{jk}^{(\lambda)r} g_{ir},$$

and

$$({}^{CG}\nabla_{\delta_i^{(\lambda)}} {}^{CG}g)(\partial_{\bar{j}}, \partial_{\bar{k}}) = ({}^{CG}\nabla_{\partial_{\bar{j}}} {}^{CG}g)(\delta_i^{(\lambda)}, \partial_{\bar{k}}),$$

we conclude (iii). Finally, to establish condition (iv), we compute

$$({}^{CG}\nabla_{\partial_{\bar{i}}} {}^{CG}g)(\partial_{\bar{j}}, \partial_{\bar{k}}) = -\frac{2}{\alpha} y_i {}^{CG}g_{\bar{j}\bar{k}} + \frac{1}{\alpha} (g_{ij} y_k + g_{ik} y_j) - {}^{CG}\Gamma_{\bar{i}\bar{j}}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{k}} - {}^{CG}\Gamma_{\bar{i}\bar{k}}^{(\lambda)\bar{r}} {}^{CG}g_{\bar{r}\bar{j}}.$$

By using the Codazzi equation

$$({}^{CG}\nabla_{\partial_{\bar{i}}} {}^{CG}g)(\partial_{\bar{j}}, \partial_{\bar{k}}) = ({}^{CG}\nabla_{\partial_{\bar{j}}} {}^{CG}g)(\partial_{\bar{i}}, \partial_{\bar{k}}),$$

we have (iv).  $\square$

Now, let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold and  $\mathcal{C}^{(\lambda)}$  be the cubic tensor field of  $\nabla^{(\lambda)}$  with the coefficients

$$\mathcal{C}_{ijk}^{(\lambda)} = \partial_k g_{ij} - \Gamma_{ik}^{(\lambda)h} g_{jh} - \Gamma_{jk}^{(\lambda)h} g_{ih}.$$

We consider the horizontal lift  ${}^{\mathcal{H}}\mathcal{C}^{(\lambda)}$  of  $\mathcal{C}^{(\lambda)}$ . Then, we have

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}, \partial_{\bar{k}}) &= {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\delta_i^{(\lambda)}, \partial_{\bar{j}}, \delta_k^{(\lambda)}) = {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\partial_{\bar{i}}, \delta_j^{(\lambda)}, \delta_k^{(\lambda)}) = \nu(\mathcal{C}^{(\lambda)}(\partial_i, \partial_j, \partial_k)) = \nu \mathcal{C}_{ijk}^{(\lambda)} = \mathcal{C}_{ijk}^{(\lambda)} \circ \pi, \\ {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\delta_i^{(\lambda)}, \partial_{\bar{j}}, \partial_{\bar{k}}) &= {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\partial_{\bar{i}}, \delta_j^{(\lambda)}, \partial_{\bar{k}}) = {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\partial_{\bar{i}}, \partial_{\bar{j}}, \delta_k^{(\lambda)}) = \nu \mathcal{C}_{ijk}^{(\lambda)} = \mathcal{C}_{ijk}^{(\lambda)} \circ \pi, \\ {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}, \delta_k^{(\lambda)}) &= 0 = {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{k}}). \end{aligned}$$

We define the tensor field  $\bar{A}^{(\lambda)} := \lambda \bar{A}$  of type  $(1, 2)$  on  $TM$  by

$${}^{CG}g(\bar{A}^{(\lambda)}(\bar{X})\bar{Y}, \bar{Z}) = {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}(\bar{X}, \bar{Y}, \bar{Z}), \quad \forall \bar{X}, \bar{Y}, \bar{Z} \in \chi(TM). \quad (30)$$

Using the above equation, the coefficient of  $\bar{A}^{(\lambda)}$  can be written with respect to  $\{\delta_i^{(\lambda)}, \partial_{\bar{i}}\}$  as follows:

$$\begin{aligned} \bar{A}^{(\lambda)}(\delta_i^{(\lambda)})\delta_j^{(\lambda)} &= \bar{A}_{ij}^{(\lambda)s} \delta_s^{(\lambda)} + \bar{A}_{ij}^{(\lambda)\bar{s}} \partial_{\bar{s}}, & \bar{A}^{(\lambda)}(\delta_i^{(\lambda)})\partial_{\bar{j}} &= \bar{A}_{ij}^{(\lambda)s} \delta_s^{(\lambda)} + \bar{A}_{ij}^{(\lambda)\bar{s}} \partial_{\bar{s}}, \\ \bar{A}^{(\lambda)}(\partial_{\bar{i}})\delta_j^{(\lambda)} &= \bar{A}_{\bar{i}j}^{(\lambda)s} \delta_s^{(\lambda)} + \bar{A}_{\bar{i}j}^{(\lambda)\bar{s}} \partial_{\bar{s}}, & \bar{A}^{(\lambda)}(\partial_{\bar{i}})\partial_{\bar{j}} &= \bar{A}_{\bar{i}\bar{j}}^{(\lambda)s} \delta_s^{(\lambda)} + \bar{A}_{\bar{i}\bar{j}}^{(\lambda)\bar{s}} \partial_{\bar{s}}, \end{aligned}$$

where

$$\begin{cases} \bar{A}_{ij}^{(\lambda)\bar{s}} = (\alpha g^{rs} - y^r y^s) \mathcal{C}_{ijr}^{(\lambda)}, & \bar{A}_{ij}^{(\lambda)s} = g^{sr} \mathcal{C}_{ijr}^{(\lambda)}, & \bar{A}_{i\bar{j}}^{(\lambda)\bar{s}} = (\alpha g^{rs} - y^r y^s) \mathcal{C}_{ijr}^{(\lambda)}, & \bar{A}_{i\bar{j}}^{(\lambda)s} = 0, \\ \bar{A}_{\bar{i}j}^{(\lambda)\bar{s}} = (\alpha g^{rs} - y^r y^s) \mathcal{C}_{jir}^{(\lambda)}, & \bar{A}_{\bar{i}j}^{(\lambda)s} = g^{sr} \mathcal{C}_{jir}^{(\lambda)}, & \bar{A}_{\bar{i}\bar{j}}^{(\lambda)s} = g^{sr} \mathcal{C}_{ijr}^{(\lambda)}, & \bar{A}_{\bar{i}\bar{j}}^{(\lambda)\bar{s}} = 0. \end{cases} \quad (31)$$

As  $\mathcal{C}^{(\lambda)}$  is symmetric,  $\bar{A}^{(\lambda)}(\bar{X})$  is also symmetric with respect to  ${}^{CG}g$ .

Considering the Levi-Civita connection  ${}^{CG}\hat{\nabla}$  on  $TM$ , we define a linear connection  ${}^{CG}\bar{\nabla}$  by

$${}^{CG}\bar{\nabla}^{(\lambda)} = {}^{CG}\hat{\nabla} - \frac{1}{2} \bar{A}^{(\lambda)}. \quad (32)$$

Then,  ${}^{CG}\bar{\nabla}$  is torsion free and satisfies  ${}^{CG}\bar{\nabla}^{(\lambda)} {}^{CG}g = {}^{\mathcal{H}}\mathcal{C}^{(\lambda)}$ .

Using Theorem 1 and the last equation, we obtain

$$\begin{aligned} {}^{CG}\bar{\nabla}_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)} &= Q_{ij}^s\delta_s^{(\lambda)} - \frac{1}{2}(y^l R_{ijl}^{(\lambda)s} + \bar{A}_{ij}^{(\lambda)\bar{s}})\partial_{\bar{s}}, \\ {}^{CG}\bar{\nabla}_{\delta_i^{(\lambda)}}\partial_{\bar{j}} &= (P_{ij}^s - \frac{1}{2}\bar{A}_{ij}^{(\lambda)s})\delta_s^{(\lambda)} + (\Gamma_{ij}^{(\lambda)s} + P_{ij}^{\bar{s}} - \frac{1}{2}\bar{A}_{ij}^{(\lambda)\bar{s}})\partial_{\bar{s}}, \\ {}^{CG}\bar{\nabla}_{\partial_i}\delta_j^{(\lambda)} &= (P_{ji}^s - \frac{1}{2}\bar{A}_{ij}^{(\lambda)s})\delta_s^{(\lambda)} + (P_{ji}^{\bar{s}} - \frac{1}{2}\bar{A}_{ij}^{(\lambda)\bar{s}})\partial_{\bar{s}}, \\ {}^{CG}\bar{\nabla}_{\partial_i}\partial_{\bar{j}} &= (S_{ij}^s - \frac{1}{2}\bar{A}_{ij}^{(\lambda)s})\delta_s^{(\lambda)} + S_{ij}^{\bar{s}}\partial_{\bar{s}}. \end{aligned}$$

Next, we apply the above explanations to the following:

**Theorem 2.** Let  $(M, g, \nabla^{(\lambda)})$  be a statistical manifold with the cubic tensor field  $\mathcal{C}^{(\lambda)}$ . Then,  $(TM, {}^{CG}g, {}^{CG}\bar{\nabla}^{(\lambda)})$  is a statistical manifold where  ${}^{CG}\bar{\nabla}^{(\lambda)}$  is determined by (32).

**Example 3.** We consider the statistical manifold  $(\mathbb{R}^2, g, \nabla^{(\lambda)})$  in Example 2. According to (32), the following apply:

$$\begin{aligned} {}^{CG}\bar{\nabla}_{\delta_1^{(\lambda)}}\delta_1^{(\lambda)} &= -\frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{1}}\partial_{\bar{1}} - \frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_2^{(\lambda)}}\delta_2^{(\lambda)} &= -\frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{1}}\partial_{\bar{1}} - \frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_1^{(\lambda)}}\delta_2^{(\lambda)} &= \frac{\lambda}{2}(\lambda f_3 f_4 y^1 - \partial_1 f_4 y^2)\partial_{\bar{1}} + \frac{\lambda}{2}(\partial_2 f_3 y^1 - \lambda f_3 f_4 y^2)\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_2^{(\lambda)}}\delta_1^{(\lambda)} &= -\frac{\lambda}{2}(\lambda f_3 f_4 y^1 - \partial_1 f_4 y^2)\partial_{\bar{1}} - \frac{\lambda}{2}(\partial_2 f_3 y^1 - \lambda f_3 f_4 y^2)\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{1}} &= P_{11}^1\delta_1^{(\lambda)} - \frac{1}{2}\bar{A}_{11}^{(\lambda)2}\delta_2^{(\lambda)} + (P_{11}^{\bar{1}} - \frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{1}})\partial_{\bar{1}} + (\Gamma_{11}^{(\lambda)2} + P_{11}^{\bar{2}} - \frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{2}})\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_1^{(\lambda)}}\partial_{\bar{2}} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^{\bar{1}}\partial_{\bar{1}} + P_{12}^{\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{1}} &= P_{21}^2\delta_2^{(\lambda)} + P_{21}^{\bar{1}}\partial_{\bar{1}} + P_{21}^{\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\delta_2^{(\lambda)}}\partial_{\bar{2}} &= -\frac{1}{2}\bar{A}_{22}^{(\lambda)1}\delta_1^{(\lambda)} + P_{22}^2\delta_2^{(\lambda)} + (\Gamma_{22}^{(\lambda)1} + P_{22}^{\bar{1}} - \frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{1}})\partial_{\bar{1}} + (P_{22}^{\bar{2}} - \frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{2}})\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{1}}}\delta_1^{(\lambda)} &= P_{11}^1\delta_1^{(\lambda)} - \frac{1}{2}\bar{A}_{11}^{(\lambda)2}\delta_2^{(\lambda)} + (P_{11}^{\bar{1}} - \frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{1}})\partial_{\bar{1}} + (P_{11}^{\bar{2}} - \frac{1}{2}\bar{A}_{11}^{(\lambda)\bar{2}})\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{1}}}\delta_2^{(\lambda)} &= P_{21}^2\delta_2^{(\lambda)} + P_{21}^{\bar{1}}\partial_{\bar{1}} + P_{21}^{\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{2}}}\delta_1^{(\lambda)} &= P_{12}^1\delta_1^{(\lambda)} + P_{12}^{\bar{1}}\partial_{\bar{1}} + P_{12}^{\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{2}}}\delta_2^{(\lambda)} &= -\frac{1}{2}\bar{A}_{22}^{(\lambda)1}\delta_1^{(\lambda)} + P_{22}^2\delta_2^{(\lambda)} + (P_{22}^{\bar{1}} - \frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{1}})\partial_{\bar{1}} + (P_{22}^{\bar{2}} - \frac{1}{2}\bar{A}_{22}^{(\lambda)\bar{2}})\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{1}}}\partial_{\bar{1}} &= S_{11}^1\delta_1^{(\lambda)} + (S_{11}^2 - \frac{1}{2}\bar{A}_{11}^{(\lambda)2})\delta_2^{(\lambda)} + S_{11}^{\bar{1}}\partial_{\bar{1}} + S_{11}^{\bar{2}}\partial_{\bar{2}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{1}}}\partial_{\bar{2}} &= S_{12}^1\delta_1^{(\lambda)} + S_{12}^2\delta_2^{(\lambda)} + S_{12}^{\bar{1}}\partial_{\bar{1}} + S_{12}^{\bar{2}}\partial_{\bar{2}} = {}^{CG}\bar{\nabla}_{\partial_{\bar{2}}}\partial_{\bar{1}}, \\ {}^{CG}\bar{\nabla}_{\partial_{\bar{2}}}\partial_{\bar{2}} &= (S_{22}^1 - \frac{1}{2}\bar{A}_{22}^{(\lambda)1})\delta_1^{(\lambda)} + S_{22}^2\delta_2^{(\lambda)} + S_{22}^{\bar{1}}\partial_{\bar{1}} + S_{22}^{\bar{2}}\partial_{\bar{2}}, \end{aligned}$$

where

$$\begin{aligned}\bar{A}_{11}^{(\lambda)\bar{1}} &= 2\lambda f_3 y^1 y^1, & \bar{A}_{11}^{(\lambda)\bar{2}} &= -2\lambda f_3 (\alpha - y^1 y^2), & \bar{A}_{22}^{(\lambda)\bar{1}} &= -2\lambda f_4 (\alpha - y^1 y^2), \\ \bar{A}_{22}^{(\lambda)\bar{2}} &= 2\lambda f_4 y^2 y^2, & \bar{A}_{11}^{(\lambda)2} &= \bar{A}_{11}^{(\lambda)\bar{2}} = -2\lambda f_3, & \bar{A}_{11}^{(\lambda)\bar{1}} &= \bar{A}_{11}^{(\lambda)1} = 2\lambda f_3 y^1 y^1, \\ \bar{A}_{11}^{(\lambda)2} &= \bar{A}_{11}^{(\lambda)\bar{2}} = -2\lambda f_3 (\alpha - y^1 y^2), & \bar{A}_{22}^{(\lambda)1} &= \bar{A}_{22}^{(\lambda)\bar{1}} = -2\lambda f_4, \\ \bar{A}_{22}^{(\lambda)\bar{1}} &= \bar{A}_{22}^{(\lambda)1} = -2\lambda f_4 (\alpha - y^1 y^2), & \bar{A}_{22}^{(\lambda)2} &= \bar{A}_{22}^{(\lambda)\bar{2}} = 2\lambda f_4 y^2 y^2, \\ \bar{A}_{11}^{(\lambda)2} &= -2\lambda f_3, & \bar{A}_{22}^{(\lambda)1} &= -2\lambda f_4, \\ \bar{A}_{12}^{(\lambda)1} &= \bar{A}_{21}^{(\lambda)1} = \bar{A}_{12}^{(\lambda)2} = \bar{A}_{21}^{(\lambda)2} = \bar{A}_{12}^{(\lambda)\bar{1}} = \bar{A}_{21}^{(\lambda)\bar{1}} = \bar{A}_{12}^{(\lambda)\bar{2}} = \bar{A}_{21}^{(\lambda)\bar{2}} = 0, \\ \bar{A}_{12}^{(\lambda)\bar{1}} &= \bar{A}_{21}^{(\lambda)\bar{1}} = \bar{A}_{12}^{(\lambda)2} = \bar{A}_{21}^{(\lambda)2} = \bar{A}_{12}^{(\lambda)1} = \bar{A}_{21}^{(\lambda)1} = \bar{A}_{12}^{(\lambda)\bar{2}} = \bar{A}_{21}^{(\lambda)\bar{2}} = 0.\end{aligned}$$

The last equations satisfy (31). Also, an easy computation shows that  ${}^{CG}\bar{C}_{\alpha\beta\gamma}^{(\lambda)} = 0, \alpha, \beta, \gamma = 1, 2$ , except

$$\begin{aligned}{}^{CG}\bar{C}_{111}^{(\lambda)} &= {}^{CG}\bar{C}_{111}^{(\lambda)} = {}^{CG}\bar{C}_{111}^{(\lambda)} = -2\lambda f_3, & {}^{CG}\bar{C}_{111}^{(\lambda)} &= {}^{CG}\bar{C}_{111}^{(\lambda)} = {}^{CG}\bar{C}_{111}^{(\lambda)} = -2\lambda f_3, \\ {}^{CG}\bar{C}_{222}^{(\lambda)} &= {}^{CG}\bar{C}_{222}^{(\lambda)} = {}^{CG}\bar{C}_{222}^{(\lambda)} = -2\lambda f_4, & {}^{CG}\bar{C}_{222}^{(\lambda)} &= {}^{CG}\bar{C}_{222}^{(\lambda)} = {}^{CG}\bar{C}_{222}^{(\lambda)} = -2\lambda f_4,\end{aligned}$$

where  ${}^{CG}\bar{C}_{\alpha\beta\gamma}^{(\lambda)} = ({}^G\nabla_{e_\alpha}^{(\lambda)} {}^{CG}g)(e_\beta, e_\gamma)$  with respect to the  $\lambda$ -adapted frame  $\{e_\beta\} = \{\delta_i^{(\lambda)}, \partial_i\}$ . So,  $(T\mathbb{R}^2, {}^{CG}g, {}^{CG}\nabla^{(\lambda)})$  is a statistical manifold. Hence, Theorem 2 holds.

## 6. Para-Holomorphic Statistical Structure on $(TM, {}^{CG}g)$

An almost product structure  $F$  on a differentiable manifold  $M$  is a  $(1, 1)$ -tensor field satisfying  $F^2 = Id$ . The pair  $(M, F)$  is called an almost product manifold. An almost para-complex manifold is an almost product manifold  $(M, F)$  such that the two eigenbundles  $T^+M$  and  $T^-M$ , corresponding to the eigenvalues  $+1$  and  $-1$  of  $F$ , respectively, have the same rank. (Note that the dimension of an almost para-complex manifold must be even.) The Nijenhuis tensor  $N_F$  of an almost para-complex structure  $F$  is defined by

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + [X, Y], \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

An almost para-complex structure  $F$  is called a para-complex structure if  $N_F = 0$ , and a manifold  $M$  endowed with such a structure is called a para-complex manifold. A para-Hermitian manifold  $(M, g, F)$  is a para-complex manifold  $(M, F)$  endowed with a pseudo-Riemannian metric  $g$  such that  $g$  is compatible with the para-complex structure  $F$ , that is

$$g(FX, FY) = -g(X, Y), \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

The fundamental two-form  $\Omega$  of the para-Hermitian manifold  $(M, g, F)$  is the skew-symmetric tensor field defined by

$$\Omega(X, Y) = g(X, FY).$$

A para-Kähler manifold  $(M, g, F)$  is a para-Hermitian manifold where the fundamental 2-form  $\Omega$  is closed, meaning  $d\Omega = 0$ .

**Definition 1.** A triple  $(g, \nabla^{(\lambda)}, F)$  is called a  $\lambda$ -para-holomorphic statistical structure on  $M$  if  $(g, \nabla^{(\lambda)})$  is a statistical structure,  $(g, F)$  is a para-Kähler structure on  $M$  and

$$K_X FY + FK_X Y = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M), \quad (33)$$



where  $K$  is given by (8).

Given a vector field  $X = X^i \partial_i$ , the  $(1,1)$ -tensor field  $F$  defined by  $F \partial_i = F_i^j \partial_j$  and the tensor  $K$  as  $K_{\partial_i} \partial_j = K_{ij}^r \partial_r$  in a neighborhood  $U \subset M$ , the conditions for a  $\lambda$ -para-holomorphic structure on the manifold  $M$  can be expressed locally as follows:

$$\begin{cases} \Gamma_{ij}^{(\lambda)r} = \Gamma_{ji}^{(\lambda)r}, \quad C_{ijk}^{(\lambda)} = C_{jik}^{(\lambda)}, \quad F_i^j F_j^k = \delta_i^k, \quad F_i^k F_{jk} = -g_{ij}, \quad \Omega_{ij} = F_{ji}, \\ \partial_i F_{kj} + \partial_j F_{ik} + \partial_k F_{ji} = 0, \quad F_j^r K_{ir}^s = -K_{ij}^r F_r^s, \quad \Gamma_{ij}^{(\lambda)r} - \Gamma_{ij}^{(0)r} = \lambda K, \\ F_i^m \partial_m (F_j^s) - F_j^m \partial_m (F_i^s) - \partial_i (F_j^m) F_m^s + \partial_j (F_i^m) F_m^s = 0, \end{cases} \quad (34)$$

where  $F_{jk} = F_j^r g_{rk}$ . The above equations imply that  $F_{jk} = -F_{kj}$ , establishing the skew-symmetry of  $F$ .

**Example 4.** In Example 2, we consider the statistical manifold  $(\mathbb{R}^2, g, \nabla^{(\lambda)})$  with the para-complex structure defined by  $F \partial_1 = \partial_1$  and  $F \partial_2 = -\partial_2$ , which implies  $F_1^1 = 1, F_2^2 = -1$ . The condition (33) holds if and only if  $K_{11}^1 = K_{22}^2 = 0$ , i.e.,  $f_1 = f_2 = 0$ . It is straightforward to verify that the relations in (34) are satisfied. Therefore,  $(g, \nabla^{(\lambda)}, F)$  constitutes a  $\lambda$ -para-holomorphic statistical structure on  $\mathbb{R}^2$ .

#### 6.1. The Almost Product Structure $F_{CG}$

Let  $(M, g, \nabla^{(\lambda)}, F)$  be a  $\lambda$ -para-holomorphic statistical manifold. Define  $F_{CG}$  as a tensor field of type  $(1,1)$  on  $TM$ , which is given by the following:

$$\begin{cases} F_{CG}(\mathcal{H}X) = a \mathcal{V}(FX) - b g(X, Fy) \mathcal{V}y, \\ F_{CG}(\mathcal{V}X) = q \mathcal{H}(FX) - s g(X, y) \mathcal{H}(Fy), \end{cases} \quad (35)$$

where  $X \in \mathfrak{S}_0^1(M), y \in TM$  and  $a, b, q, s$  are differentiable real functions with respect to  $\tau$ .

It is easy to show that  $F_{CG}^2 = Id$  if and only if the coefficients satisfy the following relations:

$$aq = 1, \quad sa(a + \tau b) - b = 0.$$

Thus, locally, based on the equations above, the almost para-complex structure  $F_{CG}$  on  $TM$  can be written as

$$\begin{cases} F_{CG}(\delta_i^{(\lambda)}) = (a F_i^r - b y^l F_{li} y^r) \partial_r, \\ F_{CG}(\partial_i) = (\frac{1}{a} F_i^r - \frac{b}{a(a+\tau b)} y_i y^l F_l^r) \delta_r^{(\lambda)}. \end{cases} \quad (36)$$

#### 6.2. The Nijenhuis Tensor of $F_{CG}$

Let  $N_{F_{CG}}$  be the Nijenhuis tensor of  $F_{CG}$ , which is defined as

$$N_{F_{CG}}(\bar{X}, \bar{Y}) = [F_{CG} \bar{X}, F_{CG} \bar{Y}] - F_{CG}[F_{CG} \bar{X}, \bar{Y}] - F_{CG}[\bar{X}, F_{CG} \bar{Y}] + [\bar{X}, \bar{Y}], \quad (37)$$

for any  $\bar{X}, \bar{Y} \in \mathfrak{S}_0^1(TM)$ . The structure  $F_{CG}$  is said to be integrable if and only if  $N_{F_{CG}} = 0$ .

**Proposition 1.** The almost para-complex structure  $F_{CG}$  defined in (35) is integrable if and only if

$$N_{F_{CG}}(\mathcal{H}X, \mathcal{H}Y) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

**Proof.** Assume that  $N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{H}}Y) = 0$ , and we will show that  $N_{F_{CG}} = 0$ . First, observe that since  $F_{CG}^2 = Id$ , thus (37) implies

$$N_{F_{CG}}({}^{\mathcal{H}}X, F_{CG}{}^{\mathcal{H}}Y) = -F_{CG}N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{H}}Y) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

Applying this result and (35), we obtain the equation

$$aN_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}(FY)) - bg(Y, Fy)N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}y) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M), y \in TM. \quad (38)$$

Substituting  $Y$  by  $y$  in the above equation, we have

$$aN_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}(Fy)) - bg(y, Fy)N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}y) = 0, \quad \forall X \in \mathfrak{S}_0^1(M), y \in TM.$$

Since  $g(y, Fy) = 0$  and  $a \neq 0$ , the last equation yields

$$N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}(Fy)) = 0, \quad \forall X \in \mathfrak{S}_0^1(M), y \in TM.$$

Substituting  $y$  with  $Fy$  in (38) and using the above equation, we deduce that

$$N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}(FY)) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

Finally, replacing  $Y$  with  $FY$  in the above equation, we obtain

$$N_{F_{CG}}({}^{\mathcal{H}}X, {}^{\mathcal{V}}Y) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

By similar reasoning, we also obtain

$$N_{F_{CG}}({}^{\mathcal{V}}X, {}^{\mathcal{V}}Y) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M).$$

Therefore, we have shown that all components of the Nijenhuis tensor vanish:

$$N_{F_{CG}}(X^{\mathcal{H}}, Y^{\mathcal{H}}) = N_{F_{CG}}(X^{\mathcal{H}}, Y^{\mathcal{V}}) = N_{F_{CG}}(X^{\mathcal{V}}, Y^{\mathcal{V}}) = 0, \quad \forall X, Y \in \mathfrak{S}_0^1(M),$$

which leads to  $N_{F_{CG}} = 0$ . The converse is straightforward.  $\square$

**Proposition 2.** The Nijenhuis tensor  $N_{F_{CG}}$  can be written with respect to  $\{\delta_i^{(\lambda)}, \partial_i^{(\lambda)}\}$  as follows

$$\begin{aligned} N_{F_{CG}}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}) &= \{(F_r^s - asy_r y^l F_l^s)(\nabla_{\partial_j}^{(\lambda)} F_i^r - \nabla_{\partial_i}^{(\lambda)} F_j^r) + b(q - \tau s)(\nabla_{\partial_i}^{(\lambda)} F_{lj} - \nabla_{\partial_j}^{(\lambda)} F_{li})y^l y^r F_r^s \\ &+ (b'q - \tau b's - a's)(\mathcal{C}_{ikm}^{(\lambda)} F_{rj} - \mathcal{C}_{jkm}^{(\lambda)} F_{ri})y^k y^m y^r y^l F_l^s + a'q(\mathcal{C}_{jkm}^{(\lambda)} \delta_i^s - \mathcal{C}_{ikm}^{(\lambda)} \delta_j^s)y^m y^k\} \delta_s^{(\lambda)} \\ &+ y^l \{(2a'(a + \tau b) - ab)(F_{il} F_j^s - F_{jl} F_i^s) - R_{ijl}^{(\lambda)s}\} \partial_{\bar{s}}. \end{aligned}$$

**Proof.** By the direct computations, we obtain the following

$$[F_{CG} \delta_i^{(\lambda)}, F_{CG} \delta_j^{(\lambda)}] = (2aa' + 2\tau a'b - ab)(F_{il} F_j^s - F_{jl} F_i^s)y^l \partial_{\bar{s}},$$

and

$$\begin{aligned} F_{CG}[F_{CG} \delta_i^{(\lambda)}, \delta_j^{(\lambda)}] + F_{CG}[\delta_i^{(\lambda)}, F_{CG} \delta_j^{(\lambda)}] &= -\{(F_r^s - asy_r y^l F_l^s)(\nabla_{\partial_j}^{(\lambda)} F_i^r - \nabla_{\partial_i}^{(\lambda)} F_j^r) + b(q - \tau s)(\nabla_{\partial_i}^{(\lambda)} F_{kj} \\ &- \nabla_{\partial_j}^{(\lambda)} F_{ki})y^k y^r F_r^s + (b'q - \tau b's - a's)(\mathcal{C}_{ikm}^{(\lambda)} F_{rj} - \mathcal{C}_{jkm}^{(\lambda)} F_{ri})y^k y^m y^r y^l F_l^s + a'q(\mathcal{C}_{jkm}^{(\lambda)} \delta_i^s - \mathcal{C}_{ikm}^{(\lambda)} \delta_j^s)y^m y^k\} \delta_s^{(\lambda)}. \end{aligned}$$

Applying these results along with (20) in (37), we conclude the assertion.  $\square$

**Corollary 1.** The structure  $F_{CG}$  is integrable if and only if the following conditions hold

$$(F_r^s - asy_r y^l F_l^s)(\nabla_{\partial_j}^{(\lambda)} F_i^r - \nabla_{\partial_i}^{(\lambda)} F_j^r) + b(q - \tau s)(\nabla_{\partial_i}^{(\lambda)} F_{kj} - \nabla_{\partial_j}^{(\lambda)} F_{ki})y^k y^r F_r^s \\ + (b'q - \tau b's - a's)(C_{ikm}^{(\lambda)} F_{rj} - C_{jkm}^{(\lambda)} F_{ri})y^k y^m y^r y^l F_l^s + a'q(C_{jkm}^{(\lambda)} \delta_i^s - C_{ikm}^{(\lambda)} \delta_j^s)y^m y^k = 0,$$

and

$$y^k((2a'(a + \tau b) - ab)(F_{ik}F_j^s - F_{jk}F_i^s) - R_{ijk}^{(\lambda)s}) = 0.$$

Additionally, the para-complex structure  $({}^{CG}g, F_{CG})$  is a para-Hermitian structure on the tangent bundle  $TM$ , i.e.,

$${}^{CG}g(F_{CG}({}^{\mathcal{H}}X), F_{CG}({}^{\mathcal{H}}Y)) = -{}^{CG}g({}^{\mathcal{H}}X, {}^{\mathcal{H}}Y) = -g(X, Y), \\ {}^{CG}g(F_{CG}({}^{\mathcal{V}}X), F_{CG}({}^{\mathcal{V}}Y)) = -{}^{CG}g({}^{\mathcal{V}}X, {}^{\mathcal{V}}Y) = -\frac{1}{\alpha}(g(X, Y) + g(X, u)g(Y, u)),$$

if and only if the following conditions are satisfied

$$a^2 = \alpha, \quad (a + \tau b)^2 + b(2a + \tau b) = 0. \quad (39)$$

From (39), it follows

$$a = \sqrt{\tau + 1}, \quad b = -\frac{\sqrt{\tau + 1} \pm 1}{\tau}, \quad (40)$$

or

$$a = -\sqrt{\tau + 1}, \quad b = \frac{\sqrt{\tau + 1} \pm 1}{\tau}.$$

### 6.3. Para-Kähler Form $\Omega_{CG}$

Considering the para-Hermitian manifold  $(TM, {}^{CG}g, F_{CG})$ , we define the fundamental 2-form, para-Kähler form  $\Omega_{CG}$  of  $TM$  as follows

$$\Omega_{CG}(\tilde{X}, \tilde{Y}) = {}^{CG}g(\tilde{X}, F_{CG}\tilde{Y}), \quad \forall \tilde{X}, \tilde{Y} \in \chi(TM).$$

The para-Kähler form  $\Omega_{CG}$  with respect to  $\{\delta_i^{(\lambda)}, \partial_{\bar{i}}\}$  is given by

$$\Omega_{CG}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}) = 0 = \Omega_{CG}(\partial_{\bar{i}}, \partial_{\bar{j}}), \\ \Omega_{CG}(\delta_i^{(\lambda)}, \partial_{\bar{j}}) = -(\frac{1}{a}F_{ij} + sy_j y^l F_{li}) = -\Omega_{CG}(\partial_{\bar{j}}, \delta_i^{(\lambda)}).$$

Applying (35), we perform direct calculations and obtain the following results for the exterior derivative of  $\Omega_{CG}$ :

$$d\Omega_{CG}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{k}}) = 0,$$

$$d\Omega_{CG}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}, \delta_k^{(\lambda)}) = y^m R_{ijm}^{(\lambda)r}(\frac{1}{a}F_{kr} + sy_r y^l F_{lk}) + y^m R_{kim}^{(\lambda)r}(\frac{1}{a}F_{jr} + sy_r y^l F_{lj}) + y^m R_{jkm}^{(\lambda)r}(\frac{1}{a}F_{ir} + sy_r y^l F_{li}), \\ d\Omega_{CG}(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}, \partial_{\bar{k}}) = \frac{1}{a}(\nabla_{\partial_j}^{(\lambda)} F_{ik} - \nabla_{\partial_i}^{(\lambda)} F_{jk}) + s(\nabla_{\partial_j}^{(\lambda)} F_{li} - \nabla_{\partial_i}^{(\lambda)} F_{lj})y_k y^l + s(C_{jkr}^{(\lambda)} F_{li} - C_{ikr}^{(\lambda)} F_{lj})y^r y^l \\ + \frac{a'}{a^2}(C_{jmr}^{(\lambda)} F_{ik} - C_{imr}^{(\lambda)} F_{jk})y^m y^r + s'(C_{jmr}^{(\lambda)} F_{li} - C_{imr}^{(\lambda)} F_{lj})y_k y^l y^m y^r,$$

and

$$d\Omega_{CG}(\delta_i^{(\lambda)}, \partial_{\bar{j}}, \partial_{\bar{k}}) = \left(\frac{2a'}{a^2} - s\right)(F_{ij}y_k - F_{ik}y_j).$$

**Corollary 2.** The para-Kähler form  $\Omega_{CG}$  is closed if and only if the following conditions hold:

$$\begin{aligned} (i) & y^m R_{ijm}^{(\lambda)r} \left(\frac{1}{a} F_{kr} + s y_r y^l F_{lk}\right) + y^m R_{kim}^{(\lambda)r} \left(\frac{1}{a} F_{jr} + s y_r y^l F_{lj}\right) + y^m R_{jkm}^{(\lambda)r} \left(\frac{1}{a} F_{ir} + s y_r y^l F_{li}\right) = 0, \\ (ii) & \frac{1}{a} (\nabla_{\partial_j}^{(\lambda)} F_{ik} - \nabla_{\partial_i}^{(\lambda)} F_{jk}) + s (\nabla_{\partial_j}^{(\lambda)} F_{li} - \nabla_{\partial_i}^{(\lambda)} F_{lj}) y_k y^l + s (C_{jkr}^{(\lambda)} F_{li} - C_{ikr}^{(\lambda)} F_{lj}) y^r y^l \\ & + \frac{a'}{a^2} (C_{jmr}^{(\lambda)} F_{ik} - C_{imr}^{(\lambda)} F_{jk}) y^m y^r + s' (C_{jmr}^{(\lambda)} F_{li} - C_{imr}^{(\lambda)} F_{lj}) y_k y^l y^m y^r = 0, \\ (iii) & \left(\frac{2a'}{a^2} - s\right)(F_{ij}y_k - F_{ik}y_j) = 0. \end{aligned}$$

**Theorem 3.** Let  $(M, g, \nabla^{(\lambda)}, F)$  be a  $\lambda$ -para-holomorphic statistical manifold. Then,  $(TM, {}^{CG}g, F_{CG})$  is a para-Kähler manifold if and only if the conditions stated in Corollaries 1 and 2 are satisfied and

$$a = \pm \sqrt{\tau + 1}, \quad b = \mp \frac{\sqrt{\tau + 1} \pm 1}{\tau}.$$

**Remark 2.** Consider the statistical manifold  $(\mathbb{R}^2, g, \nabla^{(\lambda)}, F)$  introduced in Example 4. The almost para-complex structure  $F_{CG}$  on the tangent bundle of this manifold is defined as follows:

$$\begin{aligned} F_{CG}(\delta_1^{(\lambda)}) &= (a + b y^1 y^2) \partial_1 + b (y^2)^2 \partial_2, & F_{CG}(\delta_2^{(\lambda)}) &= -b (y^1)^2 \partial_1 - (a + b y^1 y^2) \partial_2, \\ F_{CG}(\partial_1) &= (q - s y_1 y^1) \delta_1^{(\lambda)} + s y_1 y^2 \delta_2^{(\lambda)}, & F_{CG}(\partial_2) &= -s y_2 y^1 \delta_1^{(\lambda)} + (-q + s y_2 y^2) \delta_2^{(\lambda)}. \end{aligned}$$

According to Corollary 1, the structure  $F_{CG}$  is integrable if and only if the following conditions are satisfied

$$\begin{cases} (b'q - \tau b's - a's) \lambda f_3 (y^1)^4 + [(b'q - \tau b's - a's) y^1 y^2 + a'q] \lambda f_4 (y^2)^2 = 0, \\ (b'q - \tau b's - a's) \lambda f_4 (y^2)^4 + [(b'q - \tau b's - a's) y^1 y^2 + a'q] \lambda f_3 (y^1)^2 = 0, \\ (2a'(a + \tau b) - ab) y^1 = -\lambda^2 f_3 f_4 y^1 + \lambda \partial_1 (f_4) y^2, \\ (2a'(a + \tau b) - ab) y^2 = -\lambda^2 f_3 f_4 y^2 + \lambda \partial_2 (f_3) y^1. \end{cases} \quad (41)$$

Additionally, from Corollary 2, the para-Kähler form  $\Omega_{CG}$  of the para-Hermitian manifold  $(T\mathbb{R}^2, {}^{CG}g, F_{CG})$  is closed if and only if the following conditions hold

$$\begin{cases} (-s + \frac{a'}{a^2} - y^1 y^2 s') \lambda f_3 (y^1)^2 - s' \lambda f_4 (y^2)^4 = 0, \\ (-s + \frac{a'}{a^2} - y^1 y^2 s') \lambda f_4 (y^2)^2 - s' \lambda f_3 (y^1)^4 = 0, \\ (s - \frac{2a'}{a^2}) y^1 = 0 = (s - \frac{2a'}{a^2}) y^2. \end{cases} \quad (42)$$

Here,  $aq = 1$  and  $s = \frac{b}{a(a+b\tau)}$ . Therefore, the statistical manifold  $(T\mathbb{R}^2, {}^{CG}g, F_{CG})$  is a para-Kähler manifold if and only if (39), (41) and (42) are satisfied.

**Example 5.** We consider the para-Hermitian manifold  $(T\mathbb{R}^2, {}^{CG}g, F_{CG})$  induced by  $\lambda$ -para-holomorphic statistical manifold  $(M, g, \nabla^{(\lambda)}, F)$  in Remark 2 with  $a = \sqrt{\tau + 1}$  and  $b = -\frac{\sqrt{\tau+1}+1}{\tau}$ . According to the third part of (42), we conclude  $s = \frac{2a'}{a^2}$ . In this case,  $(\tau + 1)\sqrt{1 + \tau} = -1$ , and this is a contradiction. Thus, this manifold does not admit a para-Kähler structure.

#### 6.4. Para-Holomorphic Statistical Structure on the Statistical Manifold $(TM, {}^{CG}g, {}^{CG}\nabla^{(\lambda)})$

Now, we study the holomorphic conditions for statistical connections on a tangent bundle with Cheeger Gromoll metric  ${}^{CG}g$ . For this aim, for the statistical structure  $({}^{CG}\nabla^{(\lambda)}, {}^{CG}g)$ , we define the difference tensor field  ${}^{CG}K$  as

$${}^{CG}K(\bar{X}, \bar{Y}) = \frac{1}{\lambda}({}^{CG}\nabla_{\bar{X}}^{(\lambda)}\bar{Y} - {}^{CG}\widehat{\nabla}_{\bar{X}}\bar{Y}), \quad \forall \bar{X}, \bar{Y} \in \chi(TM), \quad (43)$$

where  $\lambda \neq 0$ . Denoting the coefficients  ${}^{CG}K$  by  ${}^{CG}K_{BC}^A$ , we have

$$\begin{aligned} {}^{CG}K(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}) &= {}^{CG}K_{ij}^k \delta_k^{(\lambda)} + {}^{CG}K_{ij}^{\bar{k}} \partial_{\bar{k}}, \\ {}^{CG}K(\delta_i^{(\lambda)}, \partial_{\bar{j}}) &= {}^{CG}K_{ij}^k \delta_k^{(\lambda)} + {}^{CG}K_{ij}^{\bar{k}} \partial_{\bar{k}}, \\ {}^{CG}K(\partial_{\bar{i}}, \partial_{\bar{j}}) &= {}^{CG}K_{ij}^k \delta_k^{(\lambda)} + {}^{CG}K_{ij}^{\bar{k}} \partial_{\bar{k}}. \end{aligned}$$

From (43), we obtain

$$\begin{aligned} {}^{CG}K(\delta_i^{(\lambda)}, \delta_j^{(\lambda)}) &= \frac{1}{\lambda}({}^{CG}\nabla_{\delta_i^{(\lambda)}}^{(\lambda)}\delta_j^{(\lambda)} - {}^{CG}\widehat{\nabla}_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)}), \\ {}^{CG}K(\delta_i^{(\lambda)}, \partial_{\bar{j}}) &= \frac{1}{\lambda}({}^{CG}\nabla_{\delta_i^{(\lambda)}}^{(\lambda)}\partial_{\bar{j}} - {}^{CG}\widehat{\nabla}_{\delta_i^{(\lambda)}}\partial_{\bar{j}}), \\ {}^{CG}K(\partial_{\bar{i}}, \partial_{\bar{j}}) &= \frac{1}{\lambda}({}^{CG}\nabla_{\partial_{\bar{i}}}^{(\lambda)}\partial_{\bar{j}} - {}^{CG}\widehat{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}}). \end{aligned}$$

Applying Theorem 1 and (28) to the above equations, the following holds:

$$\left\{ \begin{array}{ll} {}^{CG}K_{ij}^s = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)s} - Q_{ij}^s), & {}^{CG}K_{ij}^{\bar{s}} = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)\bar{s}} + \frac{1}{2}y^l R_{ijl}^{(\lambda)s}), \\ {}^{CG}K_{ij}^{\bar{s}} = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)\bar{s}} - P_{ij}^{\bar{s}}), & {}^{CG}K_{ij}^s = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)s} - (\Gamma_{ij}^{(\lambda)s} + P_{ij}^{\bar{s}})), \\ {}^{CG}K_{ij}^s = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)s} - S_{ij}^s), & {}^{CG}K_{ij}^{\bar{s}} = \frac{1}{\lambda}({}^{CG}\Gamma_{ij}^{(\lambda)\bar{s}} - S_{ij}^{\bar{s}}). \end{array} \right. \quad (44)$$

We conduct a thorough investigation of the holomorphic conditions for the family of connections  ${}^{CG}\nabla^{(\lambda)}$ , where  $({}^{CG}\nabla^{(\lambda)}, {}^{CG}g)$  constitutes a statistical structure on the tangent bundle equipped with the Cheeger Gromoll metric  ${}^{CG}g$ . To rigorously establish these conditions, we systematically examine the fundamental equation

$${}^{CG}K_{\bar{X}}F_{CG}\bar{Y} + F_{CG}({}^{CG}K_{\bar{X}}\bar{Y}) = 0, \quad \forall \bar{X}, \bar{Y} \in \chi(TM). \quad (45)$$

Using (36) and (44) in (45), we compute the following

$$\begin{aligned} {}^{CG}K_{\delta_i^{(\lambda)}}(F_{CG}\delta_j^{(\lambda)}) + F_{CG}({}^{CG}K_{\delta_i^{(\lambda)}}\delta_j^{(\lambda)}) &= \frac{1}{\lambda} \left( (aF_j^r - by^k F_{kj}y^r)({}^{CG}\Gamma_{i\bar{r}}^{(\lambda)s} - P_{i\bar{r}}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_m y^l F_l^s \right) \right. \\ &\quad \left( {}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} + \frac{1}{2}y^l R_{ijl}^{(\lambda)m} \right) \delta_s^{(\lambda)} + \frac{1}{\lambda} \left( (aF_j^r - by^k F_{kj}y^r)({}^{CG}\Gamma_{i\bar{r}}^{(\lambda)\bar{s}} \right. \\ &\quad \left. - (\Gamma_{i\bar{r}}^{(\lambda)s} + P_{i\bar{r}}^{\bar{s}})) + (aF_m^s - by^k F_{km}y^s)({}^{CG}\Gamma_{ij}^{(\lambda)m} - Q_{ij}^m) \right) \partial_{\bar{s}}, \\ {}^{CG}K_{\partial_{\bar{i}}}(F_{CG}\partial_{\bar{j}}) + F_{CG}({}^{CG}K_{\partial_{\bar{i}}}\partial_{\bar{j}}) &= \frac{1}{\lambda} \left( \left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_j y^l F_l^r \right)({}^{CG}\Gamma_{i\bar{r}}^{(\lambda)s} - S_{i\bar{r}}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_m y^l F_l^s \right) \right. \\ &\quad \left( {}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} - S_{ij}^{\bar{m}} \right) \delta_s^{(\lambda)} + \frac{1}{\lambda} \left( \left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_j y^l F_l^r \right)({}^{CG}\Gamma_{i\bar{r}}^{(\lambda)\bar{s}} - S_{i\bar{r}}^{\bar{s}}) \right. \\ &\quad \left. + (aF_m^s - by^k F_{km}y^s)({}^{CG}\Gamma_{ij}^{(\lambda)m} - S_{ij}^m) \right) \partial_{\bar{s}}, \end{aligned}$$

$$\begin{aligned}
{}^{CG}K_{\delta_i^{(\lambda)}}(F_{CG}\partial_{\bar{j}}) + F_{CG}({}^{CG}K_{\delta_i^{(\lambda)}}\partial_{\bar{j}}) &= \frac{1}{\lambda} \left( \left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - Q_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^lF_l^s \right) \right. \\
&\quad \left. ({}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} - (\Gamma_{ij}^{(\lambda)m} + P_{ij}^{\bar{m}})) \right) \delta_s^{(\lambda)} + \frac{1}{\lambda} \left( \left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) \right. \\
&\quad \left. ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} + \frac{1}{2}y^cR_{irc}^{(\lambda)s}) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ij}^{(\lambda)m} - P_{ij}^{\bar{m}}) \right) \partial_{\bar{s}}, \\
{}^{CG}K_{\partial_i}(F_{CG}\delta_j^{(\lambda)}) + F_{CG}({}^{CG}K_{\partial_i}\delta_j^{(\lambda)}) &= \frac{1}{\lambda} \left( (aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - S_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^cF_c^s \right) \right. \\
&\quad \left. ({}^{CG}\Gamma_{ji}^{(\lambda)\bar{m}} - (\Gamma_{ji}^{(\lambda)m} + P_{ji}^{\bar{m}})) \right) \delta_s^{(\lambda)} + \frac{1}{\lambda} \left( (aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} \right. \\
&\quad \left. - S_{ir}^{\bar{s}}) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ji}^{(\lambda)m} - P_{ji}^{\bar{m}}) \right) \partial_{\bar{s}}.
\end{aligned}$$

**Theorem 4.** Let  $(M, g, \nabla, F)$  be a holomorphic statistical manifold. Then,  $(TM, {}^{CG}g, {}^{CG}\nabla, F_{CG})$  is a holomorphic statistical manifold if and only if the conditions from Theorem 3 are satisfied and moreover,

$$\begin{aligned}
(aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - P_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^cF_c^s \right) ({}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} + \frac{1}{2}y^lR_{ijl}^{(\lambda)m}) &= 0, \\
(aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} - (\Gamma_{ir}^{(\lambda)s} + P_{ir}^{\bar{s}})) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ij}^{(\lambda)m} - Q_{ij}^{\bar{m}}) &= 0, \\
\left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - S_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^lF_l^s \right) ({}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} - S_{ij}^{\bar{m}}) &= 0, \\
\left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} - S_{ir}^{\bar{s}}) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ij}^{(\lambda)m} - S_{ij}^{\bar{m}}) &= 0, \\
\left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - Q_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^lF_l^s \right) ({}^{CG}\Gamma_{ij}^{(\lambda)\bar{m}} - (\Gamma_{ij}^{(\lambda)m} + P_{ij}^{\bar{m}})) &= 0, \\
\left( \frac{1}{a}F_j^r - \frac{b}{a(a+\tau b)}y_jy^lF_l^r \right) ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} + \frac{1}{2}y^cR_{irc}^{(\lambda)s}) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ij}^{(\lambda)m} - P_{ij}^{\bar{m}}) &= 0, \\
(aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)s} - S_{ir}^s) + \left( \frac{1}{a}F_m^s - \frac{b}{a(a+\tau b)}y_my^cF_c^s \right) ({}^{CG}\Gamma_{ji}^{(\lambda)\bar{m}} - (\Gamma_{ji}^{(\lambda)m} + P_{ji}^{\bar{m}})) &= 0, \\
(aF_j^r - by^kF_{kj}y^r) ({}^{CG}\Gamma_{ir}^{(\lambda)\bar{s}} - S_{ir}^{\bar{s}}) + (aF_m^s - by^kF_{km}y^s) ({}^{CG}\Gamma_{ji}^{(\lambda)m} - P_{ji}^{\bar{m}}) &= 0.
\end{aligned}$$

## 7. Conclusions

In this paper, we explored the geometric properties of the family of  $\lambda$  connections on a statistical manifold  $(M, g, \nabla^{(\lambda)})$  and extended these structures to the tangent bundle  $TM$ . By deriving explicit expressions for the vertical and horizontal distributions, we introduced the concept of  $\lambda$ -adapted frames, providing a natural framework for studying the induced statistical structure  $({}^{CG}g, {}^{CG}\nabla^{(\lambda)})$  on  $TM$  equipped with the Cheeger Gromoll-type metric. Furthermore, by incorporating a para-holomorphic structure on  $(M, g, \nabla^{(\lambda)})$ , we constructed a para-Hermitian structure on  $TM$  and investigated its integrability.

These results have several practical applications. The  $\lambda$  connections are important tools in information geometry, where they help model learning processes and statistical inference. The Cheeger Gromoll metric extension gives new ways to describe large and complex parameter spaces, such as those that appear in deep learning and optimization problems.

The para-Hermitian and para-holomorphic structures introduced in this work can also help understand the hidden symmetries and geometric flows in machine learning models. This could lead to better training algorithms, improved generalization, and new

ideas in areas like manifold learning, representation learning, information theory, and even quantum machine learning.

In the future, the methods developed here could be used to design geometry-aware machine learning algorithms or study more complex data spaces in statistics and artificial intelligence. We believe that connecting differential geometry with modern data science will continue to create exciting new research directions.

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