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Second-Order Gradient Estimates for the Porous Medium Equation on Riemannian Manifolds

Jingjing Yang † and Guangwen Zhao *,†

School of Mathematics and Statistics, Wuhan University of Technology, Wuhan 430070, China; jjyang2025@163.com

* Correspondence: gwzhao@whut.edu.cn

† These authors contributed equally to this work.

Abstract: In this paper, we derive second-order gradient estimates for positive solutions of the porous medium equation $\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t)^p, \quad p \in \left(1, 1+\frac{1}{\sqrt{n-1}}\right)$ on an n-dimensional Riemannian manifold under certain curvature conditions.

Keywords: gradient estimate; porous medium equation; Riemannian manifold

MSC: 58J35; 35K55; 35B45

1. Introduction

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold. Consider the parabolic equation of the form

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t)^p, \tag{1}$$

where $p > 0$ and Δ is the Laplacian with respect to g. This class of equations plays a significant role in mathematical physics. When $p = 1$, Equation (1) reduces to the classical heat equation, which is used to study processes involving conduction. For $p > 1$, Equation (1) becomes the so-called porous medium equation (see [1,2]) used to model fluid flow through porous media. Equation (1) with $p \in (0,1)$ is the fast diffusion equation, which is applied in the study of fast diffusions, see [3–5].

The study of gradient estimates for these equations has been a central topic in geometric analysis. The seminal work of Li–Yau [6] established a fundamental gradient estimate for positive solutions of the heat equation under Ricci curvature constraints. Subsequent developments on this subject can be found in [7–17] and the reference therein.

Higher-order gradient estimates are also particularly significant for reasons including but not limited to the following:

- Regularity theory: they provide crucial information about solution smoothness;
- Geometric interpretation: second-order estimates directly relate to curvature through the second derivatives of the metric;
- Functional analysis: second-order estimates connect to important results like Riesz transforms (see [18]), Sobolev inequalities (see [19]) and Perelman's entropy (see [20]).

In particular, Li [18] derived parabolic-type second-order gradient estimates for heat kernels on a noncompact complete manifold and used them to show that the Riesz transform on complete manifolds with non-negative Ricci curvature is of weak type (1, 1). In 1994, Li [19] used the boundedness of the Riesz transform to prove the Sobolev inequality

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on manifolds with some constraints. Later, Hamilton [21] revealed that the Li–Yau Harnack estimate of the heat equation is a trace of a full matrix inequality. In 2009, Liu [12] generalized the work of Li [18] and obtained the second-order gradient estimates for positive solutions of the heat equation under Ricci flow. Later, Sun [16] extended Liu's work to encompass general geometric flows. In 2016, Han–Zhang [22] proved second-order gradient estimates for the logarithmic solution of the heat equation on Riemannian manifolds. Then, Li [11] studied the second-order gradient estimates for the V-heat equation

$$u_t = \Delta u + V \cdot \nabla u$$

on Riemannian manifolds with a lower bound on the Bakry–Émery Ricci curvature, which generalized and improved the results of Han–Zhang [22]. Using the method of Han–Zhang, Zhang [23] established upper bounds for the second-order gradient estimates for solutions of the Allen–Cahn equation

$$\Delta u + u(1 - u^2) = 0$$

on Riemannian manifolds. Subsequently, Wang [24] obtained second-order gradient estimates for any positive solution of the nonlinear parabolic equation

$$u_t = (\Delta - q(x,t))u + au \log u$$

on Riemannian manifolds along the Ricci flow. Recently, Wang–Zhou–Xie–Yang [25] obtained second-order gradient estimates for any positive solution of the nonlinear parabolic equation

$$u_t = \Delta_{g(t)} u + \lambda u^\alpha$$

on Riemannian manifolds with a fixed metric, as well as along the Ricci flow.

This paper focuses on the porous medium equation on Riemannian manifolds, that is, Equation (1) with $p > 1$. For works on gradient estimates of this equation, see [26–30] and the references therein. In particular, for any positive solution of Equation (1) on $\mathbb{R}^n$ with $p > 1 - \frac{2}{n}$, Aronson–Bénilan [31] obtained the famous second-order differential inequality

$$\sum_i \frac{\partial}{\partial x^i}\left(pu^{p-2}\frac{\partial u}{\partial x^i}\right) \geq -\frac{n}{t(n(p-1)+2)}.$$

Vázquez [2] generalized the Aronson–Bénilan estimate to positive solutions of the porous medium equation on complete Riemannian manifolds with non-negative Ricci curvature. Lu–Ni–Vázquez–Villani [28] obtained the more general Aronson–Bénilan-type estimate under the weaker assumption that the Ricci curvature is bounded below by a negative constant. Later, Huang–Huang–Li [30] derived Li–Yau-type estimates and Harnack inequalities for positive solutions of the porous medium equation and improved on the results of [28]. In 2020, Qiu [32] studied a general porous medium equation for the V-Laplacian. He derived Li–Yau-type estimates with a Bakry–Émery Ricci curvature bounded from below and improved on the results of [28].

In this paper, inspired by [25], we derive second-order gradient estimates for positive solutions of the porous medium equation on noncompact Riemannian manifolds. Let $v = \frac{p}{p-1}u^{p-1}$. By adding a first-order differential operator $\frac{v'\nabla|\nabla^2 v|}{|\nabla^2 v|} \cdot \nabla$ to the modified heat operator $v'\Delta - \partial_t$, where v' is the derivative of v with respect to u, and using the maximum principle, we can prove the main theorem of this paper as follows.

Theorem 1. *Let* $(\mathbf{M}, g)$ *be an n-dimensional Riemannian manifold with* $n \geq 2$. *Suppose that* $(\mathbf{M}, g)$ *satisfies*

$$\mathrm{Ric} \geq -K_1 \quad \text{and} \quad |\mathrm{Rm}| \leq K_1$$

for some non-negative constant K_1, *where* Ric *is the Ricci curvature and* Rm *is the Riemannian curvature tensor. Suppose that* $u(x,t)$ *is a positive smooth solution of* (1) *with* $p \in \left(1, 1+\frac{1}{\sqrt{n-1}}\right)$ *on* $Q_{R,T} = B_R(x_0) \times [t_0 - T, t_0] \subset \mathbf{M} \times (-\infty, +\infty)$. *Then for any* $\delta \in (0,1)$ *and* $\beta \geq \max\left\{\sqrt{\frac{1}{1-\delta}}, \frac{2\sqrt{n}}{\delta} + \frac{4}{\delta(p-1)}\right\}$, *we have*

$$\begin{aligned}\left|\nabla^2 v\right| v^{\frac{3}{p-1}+\frac{7}{2}} \leq & CM^{\frac{3}{p-1}+\frac{15}{4}} + CM^{\frac{3}{p-1}+\frac{9}{2}}\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{T} + \frac{1}{TM}\right) \\ & + CM^{\frac{3}{p-1}+6}\left(\frac{1}{R^4} + \frac{1}{T^2} + K_1^2\right)\left(\frac{1}{R^2} + \frac{1}{T} + K_1\right)\end{aligned}$$

on $Q_{R/2,T/2}$, *where* $M = \max_{Q_{R,T}} u(x,t)$, $C = C(n, \beta, \delta, p)$, *and* K_2 *is a constant depending on n and* K_1.

The proof of this theorem will be presented in the next section. Here, we briefly summarize the steps of the proof.

Set $P = \frac{|\nabla^2 v|}{v} + \beta\frac{|\nabla v|^2}{v^2}$ → Calculate $\left(v'\Delta - \partial_t + \frac{v'\nabla|\nabla^2 v|}{|\nabla^2 v|}\cdot\nabla\right)(\varphi P)$, φ: cut-off function

→ With the maximum principle⇒a quadratic inequality of φP

→ Solve this quadratic inequality to obtain the desired estimate.

By letting $R, T \to \infty$ in Theorem 1, we obtain the following global estimate for a positive *ancient solution* (a solution defined in $\mathbf{M} \times (-\infty, t_1)$ for some finite number t_1).

Corollary 1. *Let* $(\mathbf{M}, g)$ *be an n-dimensional noncompact Riemannian manifold with* $n \geq 2$. *Suppose that* $(\mathbf{M}, g)$ *satisfies*

$$\mathrm{Ric} \geq -K_1 \quad \text{and} \quad |\mathrm{Rm}| \leq K_1$$

for some non-negative constant K_1. *Suppose also that* $u(x,t)$ *is a positive smooth ancient solution of* (1) *with* $p \in \left(1, 1+\frac{1}{\sqrt{n-1}}\right)$ *and* $u(x,t) \leq M$. *Then for any* $\delta \in (0,1)$ *and* $\beta \geq \max\left\{\sqrt{\frac{1}{1-\delta}}, \frac{2\sqrt{n}}{\delta} + \frac{4}{\delta(p-1)}\right\}$, *we have*

$$\left|\nabla^2 v\right| v^{\frac{3}{p-1}+\frac{7}{2}} \leq CM^{\frac{3}{p-1}+\frac{15}{4}} + C(K_1 + K_2)M^{\frac{3}{p-1}+\frac{9}{2}} + CK_1^3 M^{\frac{3}{p-1}+6},$$

where $C = C(n, \beta, \delta, p)$ *and* K_2 *is a constant depending on n and* K_1.

The rest of this paper is organized as follows. In Section 2, we provide a lemma needed to prove the main theorem. In Section 3, we provide the proof of Theorem 1.

2. A Lemma

To prove Theorem 1, we first provide a lemma. Let $v = \frac{p}{p-1}u^{p-1}$. Then $v' = pu^{p-1} = (p-1)v$; then,

$$v_t = v'\Delta v + |\nabla v|^2. \tag{2}$$

Denote

$$P = \frac{|\nabla^2 v|}{v} + \beta \frac{|\nabla v|^2}{v^2}, \tag{3}$$

and we have the following lemma.

Lemma 1. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}(v'\Delta - \partial_t)P \geq & v'\left(-2\nabla P \cdot \nabla \log v - C_0K_1P - C_0K_2P - \frac{C_0K_2}{\beta} + 2\beta\delta P^2 - 4\delta\beta^2 P\frac{|\nabla v|^2}{v^2}\right) \\ & - \frac{C_1|\nabla v||\nabla^3 v|}{v} - \frac{2|\nabla^2 v|^2}{v} - (p-1)P\Delta v - C_2\frac{|\nabla v|^2|\nabla^2 v|}{v^2} \\ & + v'\frac{|\nabla^3 v|^2 - |\nabla|\nabla^2 v||^2}{v|\nabla^2 v|},\end{aligned} \tag{4}$$

where $C_1 = 2 + 2(p-1)\sqrt{n}$, $C_2 = 4\beta + (p-1)\sqrt{n}\beta$ *and* C *is a constant depending only on* n.

Proof.

Step 1. Calculate $(v'\Delta - \partial_t)P$ using the Bochner formula and the curvature conditions. Firstly, calculate $(v'\Delta - \partial_t)\left(\frac{|\nabla^2 v|}{v}\right)$. Simple computation yields the following:

$$\Delta|\nabla^2 v|^2 = 2|\nabla^2 v|\Delta|\nabla^2 v| + 2|\nabla|\nabla^2 v||^2$$

and

$$\Delta|\nabla^2 v|^2 = \sum_{i,j,k}\left(v_{ij}^2\right)_{kk} = 2\sum_{i,j,k} v_{ijk}^2 + 2\sum_{i,j,k} v_{ij}v_{ijkk} = 2|\nabla^3 v|^2 + 2\sum_{i,j,k} v_{ij}v_{ijkk}.$$

From the above equations, we have

$$\Delta|\nabla^2 v| = \frac{|\nabla^3 v|^2 + \sum_{i,j,k} v_{ij}v_{ijkk} - |\nabla|\nabla^2 v||^2}{|\nabla^2 v|}.$$

Using the Ricci identity, we deduce that

$$v_{ijkk} = v_{kkij} + \sum_l R_{kjkl,i}v_l + \sum_l R_{kijl,k}v_l + \sum_l R_{kikl}v_{lj} + \sum_l R_{kjkl}v_{li} + 2\sum_l R_{kijl}v_{kl}.$$

With $|\mathrm{Rm}| \leq K_1$, we have $|\nabla \mathrm{Rm}| \leq K_2$ (see [33]), where K_2 is a constant depending on n and K_1. The curvature conditions imply that

$$\Delta|\nabla^2 v| \geq \frac{|\nabla^3 v|^2 - |\nabla|\nabla^2 v||^2 + \nabla^2 v \cdot \nabla^2(\Delta v)}{|\nabla^2 v|} - C_0K_1|\nabla^2 v| - C_0K_2|\nabla v|,$$

where C_0 is a constant depending only on n. Direct computation gives us

$$\partial_t\left(|\nabla^2 v|\right) = \frac{\partial_t\left(|\nabla^2 v|^2\right)}{2|\nabla^2 v|} = \frac{\sum_{ij} v_{ij}v_{ijt}}{|\nabla^2 v|} = \frac{\nabla^2 v \cdot \nabla^2(\partial_t v)}{|\nabla^2 v|}.$$

It follows that

$$\begin{aligned}(v'\Delta - \partial_t)|\nabla^2 v| \geq & \frac{\nabla^2 v \cdot \left(v'\nabla^2(\Delta v) - \nabla^2(\partial_t v)\right)}{|\nabla^2 v|} \\ & + v'\frac{|\nabla^3 v|^2 - |\nabla|\nabla^2 v||^2}{|\nabla^2 v|} - C_0K_1v'|\nabla^2 v| - C_0K_2v'|\nabla v|.\end{aligned} \tag{5}$$

For two functions f and h, we have

$$(v'\Delta - \partial_t)\frac{f}{h} = \frac{1}{h}(v'\Delta - \partial_t)f - \frac{f}{h^2}(v'\Delta - \partial_t)h - v'\frac{2}{h}\nabla\frac{f}{h}\cdot\nabla h.$$

First, we calculate $(v'\Delta - \partial_t)\left(\frac{|\nabla^2 v|}{v}\right)$. It follows that

$$(v'\Delta - \partial_t)\left(\frac{|\nabla^2 v|}{v}\right) = -v'\frac{2}{v}\nabla\frac{|\nabla^2 v|}{v}\cdot\nabla v + \frac{1}{v}(v'\Delta - \partial_t)\left|\nabla^2 v\right| - \frac{|\nabla^2 v|}{v^2}(v'\Delta - \partial_t)v.$$

Combining this equation with (2) and (5), we obtain

$$\begin{aligned}(v'\Delta - \partial_t)\left(\frac{|\nabla^2 v|}{v}\right) \geq & -v'\frac{2}{v}\nabla\frac{|\nabla^2 v|}{v}\cdot\nabla v - v'C_0K_1\frac{|\nabla^2 v|}{v} \\ & - v'C_0K_2\frac{|\nabla v|}{v} + \frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v) - \nabla^2(\partial_t v)\right)}{|\nabla^2 v|v} \\ & + v'\frac{\left|\nabla^3 v\right|^2 - |\nabla|\nabla^2 v||^2}{v|\nabla^2 v|} + \frac{|\nabla^2 v|}{v^2}|\nabla v|^2.\end{aligned} \tag{6}$$

Secondly, calculate $(v'\Delta - \partial_t)\left(\beta\frac{|\nabla v|^2}{v^2}\right)$. By the Bochner formula, we have

$$\begin{aligned}(v'\Delta - \partial_t)\left(\frac{|\nabla v|^2}{v^2}\right) =& v'\left(\frac{2\sum v_{ij}^2}{v^2} + \frac{2\sum v_i v_{jji}}{v^2} + 2\frac{\mathrm{Ric}(\nabla v, \nabla v)}{v^2}\right. \\ & \left. -8\frac{\sum v_{ij}v_i v_j}{v^3} - 2\frac{\sum v_i^2 v_{jj}}{v^3} + 6\frac{\sum v_i^2 v_j^2}{v^4}\right) - 2\frac{\sum v_i v_{it}}{v^2} + 2\frac{\sum v_i^2 v_t}{v^3} \\ \geq & v'\left(\frac{2\sum v_{ij}^2}{v^2} - 8\frac{\sum v_{ij}v_i v_j}{v^3} + 6\frac{\sum v_i^2 v_j^2}{v^4} - 2K_1\frac{|\nabla v|^2}{v^2}\right) \\ & + \frac{2|\nabla v|^4}{v^3} + \frac{2\sum v_i[v'(\Delta v)_i - v_{ti}]}{v^2} \\ = & \frac{2|\nabla v|^4}{v^3} + v'\left(-2\nabla\frac{|\nabla v|^2}{v^2}\cdot\nabla\log v + \frac{2\sum v_{ij}^2}{v^2} - 4\frac{\sum v_{ij}v_i v_j}{v^3}\right. \\ & \left. +2\frac{\sum v_i^2 v_j^2}{v^4} - 2K_1\frac{|\nabla v|^2}{v^2}\right) + \frac{2\sum v_i[v'(\Delta v)_i - v_{ti}]}{v^2}.\end{aligned} \tag{7}$$

Using Cauchy's and Young's inequalities, for any $0 < \delta < 1$, we can infer that

$$4\frac{\sum v_{ij}v_i v_j}{v^3} \leq 2(1-\delta)\frac{\sum v_{ij}^2}{v^2} + \frac{2}{1-\delta}\left(\frac{\sum v_i^2}{v^2}\right)^2.$$

On substituting the above inequality into (7), we obtain

$$\begin{aligned}(v'\Delta - \partial_t)\left(\beta\frac{|\nabla v|^2}{v^2}\right) \geq & v'\left[-2\nabla\left(\beta\frac{|\nabla v|^2}{v^2}\right)\cdot\nabla\log v + 2\beta\delta\frac{\left|\nabla^2 v\right|^2}{v^2} - \frac{2\delta\beta}{1-\delta}\left(\frac{\sum v_i^2}{v^2}\right)^2\right. \\ & \left. - 2\beta K_1\frac{|\nabla v|^2}{v^2}\right] + \beta\left[\frac{2|\nabla v|^4}{v^3} + \frac{2\sum v_i[v'(\Delta v)_i - v_{ti}]}{v^2}\right].\end{aligned} \tag{8}$$

From (6) and (8), we obtain

$$\begin{aligned}(v'\Delta-\partial_t)P=&(v'\Delta-\partial_t)\left(\frac{|\nabla^2 v|}{v}+\beta\frac{|\nabla v|^2}{v^2}\right)\\ \geq& v'\left(-2\nabla P\cdot\nabla\log v-C_0K_1P-C_0K_2\frac{|\nabla v|}{v}+2\beta\delta\frac{|\nabla^2 v|^2}{v^2}-\frac{2\beta}{1-\delta}\frac{|\nabla v|^4}{v^4}\right)\\ &+\beta\left[\frac{2|\nabla v|^4}{v^3}+\frac{2\sum v_i[v'(\Delta v)_i-v_{ti}]}{v^2}\right]+\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|v}\\ &+\frac{|\nabla^2 v||\nabla v|^2}{v^2}+v'\frac{|\nabla^3 v|^2-|\nabla|\nabla^2 v||^2}{v|\nabla^2 v|}.\end{aligned}$$

According to (3), we have

$$\begin{aligned}(v'\Delta-\partial_t)P\geq& v'\left[-2\nabla P\cdot\nabla\log v-C_0K_1P-C_0K_2\frac{|\nabla v|}{v}\right.\\ &\left.+2\beta\delta P^2-4\delta\beta^2P\frac{|\nabla v|^2}{v^2}+2\delta\beta\left(\beta^2-\frac{1}{1-\delta}\right)\frac{|\nabla v|^4}{v^4}\right]\\ &+\beta\left[\frac{2|\nabla v|^4}{v^3}+\frac{2\sum v_i[v'(\Delta v)_i-v_{ti}]}{v^2}\right]+\frac{|\nabla^2 v||\nabla v|^2}{v^2}\\ &+\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|v}+v'\frac{|\nabla^3 v|^2-|\nabla|\nabla^2 v||^2}{v|\nabla^2 v|}.\end{aligned}\tag{9}$$

Step 2. Further estimates of some terms on the right side of (9).
Using (2) and Cauchy's inequality, we obtain

$$\begin{aligned}2\beta\frac{\sum v_i[v'(\Delta v)_i-v_{ti}]}{v^2}&=2\beta\frac{\sum v_i\left[v'(\Delta v)_i-\left(v'\Delta v+|\nabla v|^2\right)_i\right]}{v^2}\\ &=2\beta\frac{-\sum v_i\left(|\nabla v|^2\right)_i-\sum v_i(v')_i\Delta v}{v^2}\\ &\geq\beta\frac{-4|\nabla v|^2|\nabla^2 v|-2(p-1)|\nabla v|^2\Delta v}{v^2}.\end{aligned}$$

Using (2) again, a direct computation gives

$$\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|v}=\frac{\sum v_{ij}\left[\left(-|\nabla v|^2\right)_{ij}-2(\Delta v)_i(v')_j-(v')_{ij}\Delta v\right]}{|\nabla^2 v|v}.\tag{10}$$

It follows from Cauchy's inequality that

$$\begin{aligned}\sum_{i,j}v_{ij}\left(|\nabla v|^2\right)_{ij}&=\sum_{i,j,k}(2v_kv_{kij}v_{ij}+2v_{ki}v_{kj}v_{ij})\\ &\leq 2\left|\nabla^2 v\right||\nabla v|\left|\nabla^3 v\right|+2\left|\nabla^2 v\right|^3\end{aligned}\tag{11}$$

and

$$\sum_{i,j}2(p-1)v_{ij}(\Delta v)_iv_j\leq 2(p-1)\left|\nabla^2 v\right||\nabla v|\sqrt{\sum_i[(\Delta v)_i]^2}.\tag{12}$$

Again, using Cauchy's inequality, we have

$$\sqrt{\sum_i[(\Delta v)_i]^2}=\sqrt{\sum_i\left(\sum_k v_{kki}\right)^2}\leq\sqrt{n}\sqrt{\sum_{i,j,k}v_{ijk}^2}=\sqrt{n}\left|\nabla^3 v\right|.$$

Substitution of the above inequality into (12) yields

$$\sum_{i,j} 2(p-1)v_{ij}(\Delta v)_i v_j \le 2(p-1)\left|\nabla^2 v\right||\nabla v|\sqrt{n}\left|\nabla^3 v\right|. \tag{13}$$

On the other hand, we have

$$\begin{aligned}\sum_{i.j} v_{ij}(v')_{ij}\Delta v &= \sum_{i,j}(p-1)v_{ij}v_{ij}\Delta v\\ &=(p-1)\Delta v\left|\nabla^2 v\right|^2.\end{aligned} \tag{14}$$

On substituting (11), (13) and (14) into (10), we obtain

$$\begin{aligned}\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|v} \ge& \frac{-2|\nabla v||\nabla^3 v|}{v}-\frac{2(p-1)\sqrt{n}|\nabla v||\nabla^3 v\,|}{v}\\ &-\frac{2\left|\nabla^2 v\right|^2}{v}-\frac{(p-1)\Delta v|\nabla^2 v|}{v}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|v}+2\beta\frac{\sum v_i[v'(\Delta v)_i-v_{ti}]}{v^2}\\ \ge&\beta\frac{-4|\nabla v|^2|\nabla^2 v|-2(p-1)|\nabla v|^2\Delta v}{v^2}+\frac{-2|\nabla v||\nabla^3 v|}{v}\\ &-\frac{2\left|\nabla^2 v\right|^2}{v}-\frac{2(p-1)\sqrt{n}|\nabla v||\nabla^3 v\,|}{v}-\frac{(p-1)\Delta v|\nabla^2 v|}{v}\\ =&\beta\frac{-4|\nabla v|^2|\nabla^2 v|-(p-1)|\nabla v|^2\Delta v}{v^2}+\frac{-2|\nabla v||\nabla^3 v|}{v}\\ &-\frac{2\left|\nabla^2 v\right|^2}{v}-\frac{2(p-1)\sqrt{n}|\nabla v||\nabla^3 v\,|}{v}-(p-1)P\Delta v.\end{aligned}$$

Using $\Delta v \le \sqrt{n}\left|\nabla^2 v\right|$, we can obtain

$$\begin{aligned}&\frac{\nabla^2 v\cdot\left(v'\nabla^2(\Delta v)-\nabla^2(\partial_t v)\right)}{|\nabla^2 v|}+2\beta\frac{\sum v_i[v'(\Delta v)_i-v_{ti}]}{v^2}\\ \ge&-\frac{C_1|\nabla v||\nabla^3 v|}{v}-\frac{2\left|\nabla^2 v\right|^2}{v}-(p-1)P\Delta v-C_2\frac{|\nabla v|^2|\nabla^2 v|}{v^2},\end{aligned} \tag{15}$$

where $C_1 = 2+2(p-1)\sqrt{n}$ and $C_2 = 4\beta+(p-1)\sqrt{n}\beta$. Since $\beta^2-\frac{1}{1-\delta}$, $2\beta\frac{|\nabla v|^4}{v^3}$ and $\frac{|\nabla^2 v||\nabla v|^2}{v^2}$ are both non-negative, we can omit these three terms. We can substitute (15) into (9) to obtain

$$\begin{aligned}\left(v'\Delta-\partial_t\right)P \ge& v'\left(-2\nabla P\cdot\nabla\log v-C_0K_1P-C_0K_2\frac{|\nabla v|}{v}+2\beta\delta P^2-4\delta\beta^2P\frac{|\nabla v|^2}{v^2}\right)\\ &-\frac{C_1|\nabla v||\nabla^3 v|}{v}-\frac{2\left|\nabla^2 v\right|^2}{v}-(p-1)P\Delta v-C_2\frac{|\nabla v|^2|\nabla^2 v|}{v^2}\\ &+v'\frac{\left|\nabla^3 v\right|^2-|\nabla|\nabla^2 v||^2}{v|\nabla^2 v|}.\end{aligned}$$

Once again applying Cauchy's inequality and $\frac{|\nabla v|}{v} \leq \sqrt{\frac{P}{\beta}}$, we have

$$-C_0K_2\frac{|\nabla v|}{v} \geq -C_0K_2\sqrt{\frac{P}{\beta}} \geq -C_0K_2P - \frac{C_0K_2}{\beta}.$$

Inequality (4) follows from the above two inequalities. □

3. Proof of the Main Theorem

In this section, with the help of Lemma 1, we prove the main theorem using the maximum principle. For ease of writing, during the proof of Theorem 1, we use the same letter, C, to represent different constants that depend at most on n, β, δ and p.

Proof of Theorem 1.

Step 1. A quadratic inequality is derived by using the well-known cut-off function and the maximum principle.

We introduce the cut-off function from Souplet–Zhang [15] and Li–Yau [6]. Let $\varphi = \varphi(x,t)$ be a smooth cut-off function supported in $Q_{R,T}$ that satisfies the following properties:

1. $\varphi = \varphi(d(x,x_0),t) \equiv \varphi(r,t); \varphi(r,t) = 1$ in $Q_{R/2,T/2}, 0 \leq \varphi \leq 1$.
2. φ is decreasing as a radial function in the spatial variables.
3. $|\partial_r\varphi|/\varphi^a \leq C_a/R, \left|\partial_r^2\varphi\right|/\varphi^a \leq C_a/R^2$ when $0 < a < 1$.
4. $|\partial_t\varphi|/\varphi^{1/2} \leq C/T$.

According to Lemma 1, we can infer that

$$\begin{aligned}
&\left(v'\Delta - \partial_t + \frac{v'\nabla\left|\nabla^2 v\right|}{|\nabla^2 v|}\cdot\nabla\right)(\varphi P)\\
=&\varphi\left(v'\Delta - \partial_t\right)P + v'P\Delta\varphi + 2v'\nabla\varphi\nabla P - P\varphi_t + \frac{v'}{|\nabla^2 v|}\nabla\left|\nabla^2 v\right|\cdot\nabla(\varphi P)\\
\geq&\varphi\left[v'\left(-2\nabla P\cdot\nabla\log v - CK_1P - CK_2P - \frac{CK_2}{\beta} + 2\beta\delta P^2 - 4\delta\beta^2P\frac{|\nabla v|^2}{v^2}\right)\right.\\
&\left. - \frac{C_1|\nabla v||\nabla^3 v|}{v} - \frac{2\left|\nabla^2 v\right|^2}{v} - (p-1)P\Delta v - C_2\frac{|\nabla v|^2|\nabla^2 v|}{v^2}\right]\\
&+\varphi v'\frac{\left|\nabla^3 v\right|^2 - |\nabla|\nabla^2 v||^2}{v|\nabla^2 v|} + \frac{\varphi v'|\nabla|\nabla^2 v||^2}{v|\nabla^2 v|} - \frac{v'\varphi\nabla\left|\nabla^2 v\right|\cdot\nabla v}{v^2}\\
&+\frac{v'\varphi\beta}{|\nabla^2 v|}\nabla\left|\nabla^2 v\right|\cdot\nabla\left(\frac{|\nabla v|^2}{v^2}\right) + \frac{v'P}{|\nabla^2 v|}\nabla\left|\nabla^2 v\right|\cdot\nabla\varphi\\
&+v'P\Delta\varphi + 2v'\nabla\varphi\cdot\nabla P - P\varphi_t.
\end{aligned} \tag{16}$$

By using

$$\left|\nabla\left(\frac{|\nabla v|^2}{v^2}\right)\right|^2 = \frac{|\nabla|\nabla v|^2|^2}{v^4} - \frac{4|\nabla v|^2\nabla v\cdot\nabla|\nabla v|^2}{v^5} + \frac{4|\nabla v|^6}{v^6}$$

and $\nabla v\cdot\nabla|\nabla v|^2 = 2\nabla^2 v(\nabla v,\nabla v)$, we arrive at

$$\left|\nabla\left(\frac{|\nabla v|^2}{v^2}\right)\right| \leq \frac{2|\nabla v|\left|\nabla^2 v\right|}{v^2} + \frac{2\sqrt{2}|\nabla v|^2\left|\nabla^2 v\right|^{\frac{1}{2}}}{v^{\frac{5}{2}}} + \frac{2|\nabla v|^3}{v^3}.$$

From the above inequalities, Young's inequality and $\left|\nabla^3 v\right| \geq |\nabla|\nabla^2 v||$, we have

$$\frac{C_1|\nabla v|\left|\nabla^3 v\right|\varphi}{v} \leq \frac{1}{7}\frac{v'\left|\nabla^3 v\right|^2\varphi}{|\nabla^2 v|v} + C\frac{|\nabla v|^2\varphi|\nabla^2 v|}{v^2}, \tag{17}$$

$$\begin{aligned}\frac{v'\varphi\nabla\left|\nabla^2 v\right|\cdot\nabla v}{v^2} \leq & \frac{v'\varphi|\nabla|\nabla^2 v|||\nabla v|}{v^2} \\ \leq & \frac{1}{7}\frac{v'\left|\nabla^3 v\right|^2\varphi}{|\nabla^2 v|v} + C\frac{v'|\nabla v|^2\varphi|\nabla^2 v|}{v^3},\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{v'\varphi\beta}{|\nabla^2 v|}\nabla\left|\nabla^2 v\right|\cdot\nabla\left(\frac{|\nabla v|^2}{v^2}\right) \leq & \frac{v'\varphi\beta}{|\nabla^2 v|}\left|\nabla\left|\nabla^2 v\right|\right|\left|\nabla\frac{|\nabla v|^2}{v^2}\right| \\ \leq & \frac{2v'\beta\varphi|\nabla v||\nabla|\nabla^2 v||}{v^2} \\ & + \frac{2\sqrt{2}|\nabla v|^2 v'\beta\varphi|\nabla|\nabla^2 v||}{v^{\frac{5}{2}}|\nabla^2 v|^{\frac{1}{2}}} + \frac{2|\nabla v|^3 v'\beta\varphi|\nabla|\nabla^2 v||}{v^3|\nabla^2 v|} \\ \leq & \frac{3}{7}\frac{v'\left|\nabla^3 v\right|^2\varphi}{|\nabla^2 v|v} + C\frac{v'\varphi\beta^2|\nabla v|^2|\nabla^2 v|}{v^3} \\ & + C\frac{v'\beta^2\varphi|\nabla v|^4}{v^4} + C\frac{v'\beta^2\varphi|\nabla v|^6}{v^5|\nabla^2 v|}\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{v'P}{|\nabla^2 v|}\nabla\left|\nabla^2 v\right|\cdot\nabla\varphi \leq & \frac{v'P|\nabla|\nabla^2 v|||\nabla\varphi|}{|\nabla^2 v|} \\ \leq & \frac{2}{7}\frac{v'\left|\nabla^3 v\right|^2\varphi}{|\nabla^2 v|v} + C\frac{v'|\nabla\varphi|^2|\nabla^2 v|}{\varphi v} + C\frac{v'\beta^2|\nabla\varphi|^2|\nabla v|^4}{|\nabla^2 v|v^3\varphi} \\ \leq & \frac{2}{7}\frac{v'\left|\nabla^3 v\right|^2\varphi}{|\nabla^2 v|v} + C\frac{v'\left|\nabla^2 v\right|}{vR^2} + C\frac{v'\beta^2|\nabla v|^4}{|\nabla^2 v|v^3R^2}.\end{aligned} \tag{20}$$

On substituting (17)–(20) into (16), we obtain

$$\begin{aligned}\left(v'\Delta - \partial_t + \frac{v'\nabla\left|\nabla^2 v\right|}{|\nabla^2 v|}\cdot\nabla\right)(\varphi P) \geq & \varphi\left[v'\left(-2\nabla P\cdot\nabla\log v - CK_1P - CK_2P - \frac{CK_2}{\beta}\right.\right. \\ & \left. + 2\beta\delta P^2 - 4\delta\beta^2 P\frac{|\nabla v|^2}{v^2}\right) - \frac{2\left|\nabla^2 v\right|^2}{v} \\ & \left. - (p-1)P\Delta v - \frac{|\nabla v|^2|\nabla^2 v|}{v^2}C\right] - C\frac{v'\varphi|\nabla v|^4}{v^4} \\ & - C\frac{v'\varphi|\nabla v|^6}{v^5|\nabla^2 v|} - C\frac{v'\left|\nabla^2 v\right|}{R^2 v} - C\frac{v'|\nabla v|^4}{R^2 v^3|\nabla^2 v|} \\ & + v'P\Delta\varphi + 2v'\nabla\varphi\nabla P - P\varphi_t.\end{aligned} \tag{21}$$

By using Young's inequality, we can obtain

$$C\frac{v'\varphi|\nabla v|^6}{v^5|\nabla^2 v|} \leq C\left(\frac{|\nabla v|^{12}\varphi^2}{v^8} + \frac{1}{|\nabla^2 v|^2}\right), \tag{22}$$

$$C\frac{v'|\nabla v|^4}{R^2 v^3|\nabla^2 v|} \leq C\left(\frac{|\nabla v|^8}{R^4 v^4} + \frac{1}{|\nabla^2 v|^2}\right). \tag{23}$$

Suppose that φP reaches its maximum at the point (x_1, t_1). According to [6], without loss of generality, we can assume that x_1 is not on the cut-locus of M^n. Therefore, at the point (x_1, t_1), we have

$$\nabla(\varphi P) = 0, \quad \Delta(\varphi P) \leq 0, \quad \partial_t(\varphi P) \geq 0.$$

Hence, we have

$$\left(v'\Delta - \partial_t + \frac{v'\nabla|\nabla^2 v|}{|\nabla^2 v|} \cdot \nabla\right)(\varphi P) \leq 0, \quad \varphi \nabla P = -P\nabla\varphi. \tag{24}$$

By substituting (22) and (23) into (21), with (24), we obtain

$$\begin{aligned} 2\varphi P^2 \leq & \frac{\varphi}{\delta\beta}\left(-2\frac{P\nabla\varphi\cdot\nabla\log v}{\varphi} + CK_1P + CK_2P + \frac{CK_2}{\beta} + 4\delta\beta^2 P\frac{|\nabla v|^2}{v^2}\right) \\ & + \frac{\varphi}{v'\beta\delta}\left(\frac{2|\nabla^2 v|^2}{v} + (p-1)P\Delta v + C\frac{|\nabla v|^2|\nabla^2 v|}{v^2}\right) + \frac{CP}{R^2\beta\delta} \\ & + \frac{C\varphi|\nabla v|^4}{\beta\delta v^4} + \frac{C|\nabla v|^{12}\varphi^2}{v^9\beta\delta} + \frac{C}{|\nabla^2 v|^2 v'\beta\delta} + \frac{C|\nabla v|^8}{R^4\beta\delta v^5} \\ & - \frac{P\Delta\varphi}{\delta\beta} + \frac{2P}{\delta\beta}\frac{|\nabla\varphi|^2}{\varphi} + \frac{P\varphi_t}{v'\delta\beta}. \end{aligned} \tag{25}$$

Now, we use Young's inequality to estimate some terms on the right-hand side of this inequality. For the first term, from $\beta\frac{|\nabla v|^2}{v^2} \leq P$, we obtain

$$\begin{aligned} -\frac{2}{\delta\beta}P\nabla\varphi\cdot\nabla\log v \leq & \frac{2}{\delta\beta}P|\nabla\varphi||\nabla\log v| \\ \leq & \frac{2}{\delta\beta^{\frac{3}{2}}}P^{\frac{3}{2}}|\nabla\varphi| \\ \leq & \frac{1}{6}\varphi P^2 + \frac{C}{\delta^4\beta^6}\frac{|\nabla\varphi|^4}{\varphi^3} \\ \leq & \frac{1}{6}\varphi P^2 + \frac{C}{\delta^4\beta^6}\frac{1}{R^4}. \end{aligned} \tag{26}$$

For the sixth, seventh and eighth terms, we have

$$\begin{aligned} \frac{2\varphi|\nabla^2 v|^2}{v'\beta\delta v} = & \frac{2\varphi}{\beta\delta(p-1)}\left(P^2 - 2P\beta\frac{|\nabla v|^2}{v^2} + \beta^2\frac{|\nabla v|^4}{v^4}\right) \\ \leq & \frac{2\varphi}{\beta\delta(p-1)}\left(P^2 + \beta^2\frac{|\nabla v|^4}{v^4}\right), \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{(p-1)\varphi}{v'\beta\delta}P\Delta v \leq & \frac{\varphi\sqrt{n}|\nabla^2 v|}{\beta\delta v}P \\ = & \frac{\varphi\sqrt{n}}{\beta\delta}\left(P^2 - \beta\frac{|\nabla v|^2}{v^2}P\right) \\ \leq & \frac{\varphi\sqrt{n}}{\beta\delta}\left(P^2 + \beta\frac{|\nabla v|^2}{v^2}P\right) \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{C\varphi}{v'\beta\delta}\frac{|\nabla v|^2|\nabla^2 v|}{v^2} = & \frac{C\varphi}{(p-1)\beta\delta}\frac{|\nabla v|^2}{v^2}P + \frac{-C\varphi|\nabla v|^2}{(p-1)\delta v^2} \\ \leq & \frac{C\varphi}{(p-1)\beta\delta}\frac{|\nabla v|^2}{v^2}P. \end{aligned} \tag{29}$$

For the third-to-last term, combined with the Laplacian comparison theorem, we have

$$-\frac{1}{\delta\beta}P\Delta\varphi \le \frac{1}{\delta\beta}|P||\Delta\varphi| \le \frac{1}{6}\varphi P^2 + \frac{C}{\delta^2\beta^2}\left(\frac{1}{R^4}+\frac{K_1}{R^2}\right). \tag{30}$$

For the penultimate term, we have

$$\frac{2P}{\delta\beta}\frac{|\nabla\varphi|^2}{\varphi} \le \frac{C}{\delta\beta R^2}P \tag{31}$$

by the third property of φ. For the last term, we have

$$\frac{1}{v'\delta\beta}P\varphi_t \le \frac{C}{\delta}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\frac{\varphi^{\frac{1}{2}}P}{v'\beta} \le \frac{1}{6}\varphi P^2 + \frac{C}{(v')^2\delta^2\beta^2}\frac{1}{T^2}. \tag{32}$$

With the use of (26)–(32), at the point (x_1, t_1), inequality (25) reduces to

$$\begin{aligned}\left(2-\frac{1}{2}-\frac{\sqrt{n}}{\beta\delta}-\frac{2}{\beta\delta(p-1)}\right)\varphi P^2 \le& \left(\frac{C}{\delta\beta}(K_1+K_2)+C\frac{|\nabla v|^2}{v^2}\right)\varphi P+\frac{C}{\delta\beta R^2}P+\frac{CK_2\varphi}{\delta\beta^2}\\ &+\frac{C}{\delta^4\beta^6}\frac{1}{R^4}+\frac{2\varphi\beta}{\delta(p-1)}\frac{|\nabla v|^4}{v^4}+\frac{C\varphi|\nabla v|^4}{\beta\delta v^4}\\ &+\frac{C|\nabla v|^{12}\varphi}{v^9\beta\delta}+\frac{C}{|\nabla^2 v|^2 v'\beta\delta}+\frac{C|\nabla v|^8}{R^4\beta\delta v^5}\\ &+\frac{C}{\delta^2\beta^2}\left(\frac{1}{R^4}+\frac{K_1}{R^2}\right)+\frac{C}{\delta^2\beta^2}\frac{1}{T^2}\frac{1}{(v')^2}.\end{aligned}$$

Multiplying through by φ and note that $0 \le \varphi \le 1$, we obtain

$$\begin{aligned}\left(2-\frac{1}{2}-\frac{\sqrt{n}}{\beta\delta}-\frac{2}{\beta\delta(p-1)}\right)\varphi^2 P^2 \le& \left(\frac{C}{\delta\beta}(K_1+K_2)+C\frac{|\nabla v|^2}{v^2}+\frac{C}{\delta\beta R^2}\right)\varphi P+\frac{CK_2}{\delta\beta^2}\\ &+\frac{C}{\delta^4\beta^6}\frac{1}{R^4}+\frac{2\beta}{\delta(p-1)}\frac{|\nabla v|^4}{v^4}+\frac{C|\nabla v|^4}{\beta\delta v^4}\\ &+\frac{C|\nabla v|^{12}}{v^9\beta\delta}+\frac{C}{|\nabla^2 v|^2 v'\beta\delta}+\frac{C|\nabla v|^8}{R^4\beta\delta v^5}\\ &+\frac{C}{\delta^2\beta^2}\left(\frac{1}{R^4}+\frac{K_1}{R^2}\right)+\frac{C}{\delta^2\beta^2}\frac{1}{T^2}\frac{1}{(v')^2}.\end{aligned}$$

Step 2. Solve this quadratic inequality to obtain the estimate of φP and then obtain the desired second-order gradient estimate.

From the inequality $ax^2 - bx - c \le 0$, we have $x \le \frac{b}{a} + \sqrt{\frac{c}{a}}$, where a is a positive number and b and c are two non-negative numbers. Noting that $2-\frac{1}{2}-\frac{\sqrt{n}}{\beta\delta}-\frac{2}{\beta\delta(p-1)} \ge 1$ and that $\sqrt{x+y} \le \sqrt{x}+\sqrt{y}$ for any $x, y \ge 0$, we have

$$\begin{aligned}\varphi P \le& \frac{C}{\delta\beta}(K_1+K_2)+C\frac{|\nabla v|^2}{v^2}+\frac{C}{\delta\beta}\frac{1}{R^2}+\sqrt{\frac{CK_2}{\delta\beta^2}}\\ &+\frac{C}{\delta^2\beta^3}\frac{1}{R^2}+\sqrt{\frac{2\beta}{\delta(p-1)}}\frac{|\nabla v|^2}{v^2}+\frac{C}{\sqrt{\delta\beta}}\frac{|\nabla v|^2}{v^2}\\ &+\frac{C}{\sqrt{\delta\beta}}\frac{|\nabla v|^6}{v^{\frac{9}{2}}}+\frac{C}{\sqrt{\delta\beta v}|\nabla^2 v|}+\frac{C|\nabla v|^4}{\sqrt{\delta\beta}R^2 v^{\frac{5}{2}}}+\frac{C}{\delta\beta}\left(\frac{1}{R^2}+\frac{\sqrt{K_1}}{R}\right)+\frac{C}{\delta\beta}\frac{1}{Tv'}.\end{aligned}$$

Since $\varphi(x,t) = 1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $P(x,t) \leq (\varphi P)(x_1,t_1)$, we can conclude that

$$\begin{aligned}\frac{|\nabla^2 v|}{v} \leq P \leq & C\frac{|\nabla v|^2}{v^2} + \frac{C}{\sqrt{\delta\beta}}\frac{|\nabla v|^6}{v^{\frac{9}{2}}} + \frac{C}{\sqrt{\delta\beta v}|\nabla^2 v|} + \frac{C|\nabla v|^4}{\sqrt{\delta\beta}R^2 v^{\frac{5}{2}}} \\ & + \frac{C}{\delta\beta}\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{Tv'}\right),\end{aligned}$$

which implies

$$\left|\nabla^2 v\right|^2 \leq Cv\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{Tv'} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla v|^4}{R^2 v^{\frac{5}{2}}} + \frac{|\nabla v|^6}{v^{\frac{9}{2}}}\right)\left|\nabla^2 v\right| + C\sqrt{v}.$$

Thus, we have

$$\left|\nabla^2 v\right| \leq Cv\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{Tv'}\right) + C\frac{|\nabla v|^2}{v} + \frac{C|\nabla v|^4}{R^2 v^{\frac{3}{2}}} + \frac{C|\nabla v|^6}{v^{\frac{7}{2}}} + Cv^{\frac{1}{4}}.$$

By using the results of Huang–Xu–Zeng [29] , for $1 < p < 1 + \frac{1}{\sqrt{n-1}}$ and $v \leq M$, we have

$$v^{\frac{1}{2(p-1)}}|\nabla v| \leq CM^{1+\frac{1}{2(p-1)}}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K_1}\right).$$

We conclude that

$$\begin{aligned}\left|\nabla^2 v\right| v^{\frac{3}{p-1}+\frac{7}{2}} \leq & C\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{Tv}\right)v^{\frac{3}{p-1}+\frac{9}{2}} + CM^{\frac{1}{p-1}+2}v^{\frac{2}{p-1}+\frac{5}{2}}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K_1}\right)^2 \\ & + \frac{C}{R^2}M^{\frac{2}{p-1}+4}v^{\frac{1}{p-1}+2}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K_1}\right)^4 + CM^{\frac{3}{p-1}+6}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K_1}\right)^6 \\ & + Cv^{\frac{3}{p-1}+\frac{15}{4}}.\end{aligned}$$

Again, if $v \leq M$, using the power mean inequality, for non-negative real numbers $a_1, \cdots, a_n$ and any real number $p \geq q$, there are

$$\left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q + a_2^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}}.$$

From the above inequality, we have

$$\begin{aligned}\left|\nabla^2 v\right| v^{\frac{3}{p-1}+\frac{7}{2}} \leq & CM^{\frac{3}{p-1}+\frac{15}{4}} + CM^{\frac{3}{p-1}+\frac{9}{2}}\left(K_1 + K_2 + \frac{1}{R^2} + \frac{1}{T} + \frac{1}{TM}\right) \\ & + CM^{\frac{3}{p-1}+6}\left(\frac{1}{R^4} + \frac{1}{T^2} + K_1^2\right)\left(\frac{1}{R^2} + \frac{1}{T} + K_1\right).\end{aligned}$$

This completes the proof. □

4. Conclusions

In this paper, we investigated second-order gradient estimates for a positive solution u of the porous medium equation

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t)^p, \quad p \in \left(1, 1 + \frac{1}{\sqrt{n-1}}\right)$$

on Riemannian manifolds. We first constructed an auxiliary function

$$P = \frac{|\nabla^2 v|}{v} + \beta \frac{|\nabla v|^2}{v^2} \quad \text{with } v = \frac{p}{p-1} u^{p-1}$$

and applied the modified heat operator $v'\Delta - \partial_t$ to P, obtaining a crucial inequality. Subsequently, we introduced a first-order operator $\frac{v'\nabla|\nabla^2 v|}{|\nabla^2 v|} \cdot \nabla$ into the modified heat operator and applied the resulting operator to φP, where φ is a smooth cut-off function. This approach aimed to eliminate the third-order derivative term v through a series of inequality estimations. Finally, by employing the maximum principle, combining the Laplacian comparison theorem and Young's inequality and leveraging the first-order estimates established by Huang–Xu–Zeng [29], we proved the desired second-order gradient estimates.

As mentioned in the introduction, the second-order gradient estimate is important and attracts considerable attention. Furthermore, here we would like to outline some possible directions for future research. In fact, when the metric of a manifold evolves under a geometric flow (such as Ricci flow), obtaining second-order gradient estimates for positive solutions of the porous medium equation becomes a problem worthy of study. In addition, it is also possible to consider the case where **M** is a submanifold of a Riemannian manifold. When the second fundamental form and its covariant derivative are bounded, the estimate of the positive solution of the porous medium equation on the submanifold can be considered.

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