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# Pinching Results on Totally Real Submanifolds of a Locally Conformal Kähler Manifolds

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**Abstract:** This paper investigates the relationship between pseudo-umbilical and minimal totally real submanifolds in locally conformal Kähler space forms. Some rigidity theorems and an integral inequality are obtained using the moving-frame method and the DDVV inequality. Our results extend this line of previous research.

**Keywords:** locally conformal Kähler space form; totally real submanifolds; second fundamental form; pinching theorems

**MSC:** 53C21; 53C25; 53C50

## 1. Introduction and Motivations

Minimal submanifolds are a key topic in differential geometry, with significant applications in general relativity, as seen in references [1–4], among others. The gap phenomenon in minimal submanifolds and their generalizations has been studied in great detail. For example, let $\sigma$ denote the length of the second fundamental form of a compact minimal submanifold $N^n$ in $\mathbb{S}^{n+m}$ with codimension $m$, satisfying the inequality $0 \leq \sigma \leq \left(\frac{n}{2-\frac{1}{m}}\right)$, then either $\sigma = 0$, or $\sigma = \left(\frac{n}{2-\frac{1}{m}}\right)$. Also, N is either a Clifford hypersurface or the Veronese surface in $\mathbb{S}^4$. Later, Li [5] and Chen-Xu [6] improved the pinching constant from $\frac{n}{(2-\frac{1}{m})}$ to $\frac{2n}{3}$. They showed that if the inequality $0 \leq \sigma \leq \frac{2n}{3}$ holds, then either $\sigma = 0$ or $\sigma = \frac{2n}{3}$. Similarly, N is the Veronese surface in $\mathbb{S}^4$. Following the initial motivation by Simons [2] and preliminary developments (for example, [3,5–7]), this topic has received considerable attention.

It is well established that pseudo-umbilical submanifolds generalize minimal submanifolds. In [8], Du explored the relationship between totally real and minimal submanifolds, demonstrating that pseudo-umbilical totally real submanifolds with parallel mean curvature vectors must be minimal.Moreover, when such submanifolds are compact, the condition of having parallel mean curvature reduces to having constant mean curvature. This research has since been extended to rigidity results concerning both minimal submanifolds and those with parallel mean curvature in space forms; (see [9–17]). Moreover, space forms are instrumental in the study of geometric analysis. Numerous authors have examined the first eigenvalues of submanifolds situated within various space forms; see, for example, [18–23].

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It should be noted that relatively little work has been done on rigidity theorems for totally real submanifolds within space form geometry. Therefore, motivated by previous work, we established rigidity results for a totally real submanifolds in locally conformal Kähler manifold space forms. In this paper, we extend these results in two ways: first, for pseudo-umbilical totally real manifolds, and second, for those with parallel mean curvature vector.We present two theorems that assertain certain types of totally real submanifolds must be pseudo-umbilical submanifolds under specific conditions. We present an example of a totally real submanifold in a locally conformal Kähler manifold.

**Example 1** ([24]). *Let us consider a Hopf manifold, a classical example of a locally conformal Kähler manifold. A Hopf manifold can be constructed as follows:*

$$\mathrm{M} = \left(\mathbb{C}^n \setminus \{0\}\right) / < X >$$

*where* $X(z) = q(z)$ *for some* $0 < |q| < 1$ *and* $< X >$ *is the group generated by this contraction. Let* $\mathbb{R}^n \subset \mathbb{C}^n$ *be the real subspace (i.e., the standard inclusion of real coordinates into complex coordinates). Then, the Hopf manifold is a totally real submanifold because the complex structure* J *on* $\mathbb{C}^n$ *maps real tangent vectors into purely imaginary directions, which lie outside the tangent space* $T(\mathbb{R}^n)$*. Also, the Hopf manifold inherits its complex structure and LCK metric from* $\mathbb{C}^n \setminus \{0\}$*, and the image of* $\mathbb{R}^n \setminus \{0\}$ *descends to a totally real submanifold of the LCK Hopf manifold.*

The study of totally real submanifolds in locally conformal Kähler (l.c.K.) manifolds may have important implications in theoretical physics, especially within the frameworks of complex and almost complex geometry used in string theory, where l.c.K. manifolds sometimes arise in supersymmetric sigma models and flux compactifications.The rigidity results and pinching conditions proven in this paper may be related to stability criteria for certain brane configurations or compactification geometries.

## 2. Preliminaries

If an open cover $\{\mathrm{U}_i\}_{i\in I}$ of $\mathrm{M}^{2n}$ and a family $\{\rho_i\}_{i\in I}$ of $\mathrm{C}^\infty$ functions $\rho_i : \mathrm{U}_i \longrightarrow \mathbb{R}$ on the $n$-dimensional Hermitian manifold $(\mathrm{M}^{2n}, \mathrm{J}, \tilde{g})$ satisfy the Kählerian metric [24–26] such that

$$\tilde{g}_i = exp(-\rho_i)\tilde{g}_{|\mathrm{U}_i}$$

where J denotes the complex structure and $g$ is the Hermitian metric, then $(\mathrm{M}^{2n}, \mathrm{J}, \tilde{g})$ is a locally conformal Kähler (l.c.K.) manifold. Moreover, an inclusion is defined as $g_{|\mathrm{U}_i} = i_i^* \tilde{g}$, where $i_i : \mathrm{U}_i \longrightarrow \mathrm{M}^{2n}$. The fundamental 2-form (Kähler form) corresponding to the almost complex structure J is defined by

$$\Omega(\mathrm{X}_2, \mathrm{Y}_2) = \tilde{g}(\mathrm{JX}_2, \mathrm{Y}_2)$$

for any $\mathrm{X}_2, \mathrm{Y}_2 \in \Gamma(T\mathrm{M})$. There exists a closed 1-form $\rho_1$ on a locally conformal Kähler manifold $(\mathrm{M}, \tilde{g})$, which is smoothly defined everywhere on the manifold M, which is

$$\begin{aligned}(\bar{\nabla}_{\mathrm{Z}_2}\Omega)(\mathrm{X}_2, \mathrm{Y}_2) =& \rho_1(\mathrm{Y}_2)g(\mathrm{X}_2, \mathrm{Z}_2) - \rho_1(\mathrm{X}_2)g(\mathrm{Z}_2, \mathrm{Y}_2) \\ &+ \rho_2(\mathrm{Y}_2)\Omega(\mathrm{X}_2, \mathrm{Z}_2) - \rho_2(\mathrm{Y}_2)\Omega(\mathrm{X}_2, \mathrm{Z}_2)\end{aligned}$$

provided that $\rho_1(\mathrm{X}_2) = -\rho_2(\mathrm{JX}_2)$, where $\rho_1$ is a 1-form and $\bar{\nabla}$ is a Levi–Cita connection. Moreover, 1-form $\rho_2$ is called the Lee form and its dual vector field is called the Lee

vector field. Now, we define a symmetric $(0,2)$ tensor P on a locally conformal Kähler manifold as:

$$\mathrm{P}(\mathrm{X}_2,\mathrm{Y}_2) = -(\bar{\nabla}_{\mathrm{X}_2}\rho_2)\mathrm{Y}_2 - \rho_2(\mathrm{X}_2)\rho_2(\mathrm{Y}_2) + \frac{1}{2}\|\rho_2\|^2 g(\mathrm{X}_2,\mathrm{Y}_2)$$

where $\|\rho_2\|$ is the norm of $\rho_2$ in the direction of $g$. Further, a skew-symmetric $(0,2)$ tensor $\tilde{\mathrm{P}}$ is derived as:

$$\tilde{\mathrm{P}}(\mathrm{X}_2,\mathrm{Y}_2) = \mathrm{P}(\mathrm{J}\mathrm{X}_2,\mathrm{Y}_2). \tag{1}$$

If $\epsilon$ is the holomorphic constant sectional curvature of a locally conformal Kähler manifold $(\mathrm{M},\tilde{g})$, then it is called a locally conformal Kähler space form, and denoted by $\mathrm{M}(\epsilon)$ [27–29]. Therefore, the curvature tensor R of $\mathrm{M}(\epsilon)$ is defined as:

$$\begin{aligned}\mathrm{R}(\mathrm{X}_2,\mathrm{Y}_2)\mathrm{Z}_2 =& \frac{\epsilon}{4}\Big\{g(\mathrm{Z}_2,\mathrm{Y}_2)\mathrm{X}_2 - g(\mathrm{X}_2,\mathrm{Z}_2)\mathrm{Y}_2 + \Omega(\mathrm{Z}_2,\mathrm{Y}_2)\mathrm{J}\mathrm{X}_2 - \Omega(\mathrm{Y}_2,\mathrm{Z}_2)\mathrm{J}\mathrm{Y}_2 \\ &- 2\Omega(\mathrm{X}_2,\mathrm{Y}_2)\mathrm{J}\mathrm{Z}_2\Big\} \\ &+ \frac{3}{4}\Big\{g(\mathrm{Y}_2,\mathrm{Z}_2)\mathrm{P}_1\mathrm{X}_2 - g(\mathrm{X}_2,\mathrm{Z}_2)\mathrm{P}_1\mathrm{Y}_2 + \mathrm{P}(\mathrm{Y}_2,\mathrm{Z}_2)\mathrm{X}_2 - \mathrm{P}(\mathrm{X}_2,\mathrm{Z}_2)\mathrm{Y}_2\Big\} \\ &- \frac{1}{4}\Big\{\Omega(\mathrm{Z}_2,\mathrm{Y}_2)\tilde{\mathrm{P}}_1\mathrm{X}_2 - \Omega(\mathrm{Y}_2,\mathrm{Z}_2)\tilde{\mathrm{P}}_1\mathrm{Y}_2 + \tilde{\mathrm{P}}(\mathrm{Y}_2,\mathrm{Z}_2)\mathrm{J}\mathrm{X}_2 - \tilde{\mathrm{P}}(\mathrm{X}_2,\mathrm{Z}_2)\mathrm{J}\mathrm{Y}_2 \\ &- 2\tilde{\mathrm{P}}(\mathrm{X}_2,\mathrm{Y}_2)\mathrm{J}\mathrm{Z}_2 - 2\Omega(\mathrm{X}_2,\mathrm{Y}_2)\tilde{\mathrm{P}}_1\mathrm{Z}_2\Big\}\end{aligned} \tag{2}$$

provided that $g(\mathrm{P}_1\mathrm{X}_2,\mathrm{Y}_2) = \mathrm{P}(\mathrm{X}_2,\mathrm{Y}_2)$ *and* $g(\tilde{\mathrm{P}}_1\mathrm{X}_2,\mathrm{Y}_2) = \tilde{\mathrm{P}}(\mathrm{X}_2,\mathrm{Y}_2)$ for any $\mathrm{X}_2,\mathrm{Y}_2 \in \Gamma(T\mathrm{M}(\epsilon))$.

A submanifold N of a locally conformal Kähler manifold M is termed a totally real submanifold if, for every point, the almost complex structure J maps the tangent space of N into the normal space [30]. In this case, $\Omega(\mathrm{X}_2,\mathrm{Y}_2) = 0$ for each $\mathrm{X}_2,\mathrm{Y}_2$ tangent to N, (see [28,29]). Then, (2) is reduced to the following

$$\begin{aligned}g(\mathrm{R}(\mathrm{X}_2,\mathrm{Y}_2)\mathrm{Z}_2,\mathrm{W}_2) =& \frac{\epsilon}{4}\Big\{g(\mathrm{Z}_2,\mathrm{Y}_2)g(\mathrm{X}_2,\mathrm{W}_2) - g(\mathrm{X}_2,\mathrm{Z}_2)g(\mathrm{Y}_2,\mathrm{W}_2)\Big\} \\ &+ \frac{3}{4}\Big\{g(\mathrm{X}_2,\mathrm{W}_2)\mathrm{P}(\mathrm{Y}_2,\mathrm{Z}_2) - g(\mathrm{Y}_2,\mathrm{W}_2)\mathrm{P}(\mathrm{X}_2,\mathrm{Z}_2) \\ &+ \mathrm{P}(\mathrm{X}_2,\mathrm{W}_2)g(\mathrm{Y}_2,\mathrm{Z}_2) - \mathrm{P}(\mathrm{Y}_2,\mathrm{W}_2)g(\mathrm{X}_2,\mathrm{Z}_2)\Big\}\end{aligned} \tag{3}$$

for any $\mathrm{X}_2,\mathrm{Y}_2,\mathrm{Z}_2,\mathrm{W}_2 \in \Gamma(T\mathrm{N})$. The curvature tensors $\tilde{\mathrm{R}}$ and R of $\mathrm{N}^m$ and M are connected by the Gaus equation, that is

$$\begin{aligned}\tilde{\mathrm{R}}(\mathrm{X}_2,\mathrm{Y}_2,\mathrm{Z}_2,\mathrm{W}_2) =& \mathrm{R}(\mathrm{X}_2,\mathrm{Y}_2,\mathrm{Z}_2,\mathrm{W}_2) + \tilde{g}(\mathrm{Y}(\mathrm{X}_2,\mathrm{W}_2),\mathrm{Y}(\mathrm{Y}_2,\mathrm{Z}_2)) \\ &- \tilde{g}(\mathrm{Y}(\mathrm{X}_2,\mathrm{Z}_2),\mathrm{Y}(\mathrm{Y}_2,\mathrm{W}_2)).\end{aligned} \tag{4}$$

We considered orthonormal frames $\{e_1\cdots e_m, e_{m+1}\cdots e_{m+h}, e_1^* = \mathrm{J}e_1\cdots e_m^* = \mathrm{J}e_m, e_{(m+1)^*} = \mathrm{J}e_{m+1}\cdots e_{(m+h)^*} = \mathrm{J}e_{m+h}\}$ in $\mathrm{M}^{m+h}(\epsilon)$ are restricted to $\mathrm{N}^m$ with codimension $h$; that is, $e_1\cdots e_m$ are tangent to $\mathrm{N}^m$. We provide the indices as follows:

$$\begin{aligned}\mathcal{A},\mathcal{B},\mathcal{C}\cdots =& 1,\cdots,m+h,1^*\cdots,m+h^* \\ a,b,c\cdots =& 1,\cdots,m; \alpha,\beta,\gamma = m+1,\cdots,m+h,1^*,\cdots,m+h^*.\end{aligned}$$

Equation (3) is expressed in terms of local coordinates,

$$\begin{aligned}\widetilde{\mathrm{K}}_{\mathcal{ABCD}} = &\frac{\epsilon}{4}\left(\delta_{\mathcal{AC}}\delta_{\mathcal{BD}} - \delta_{\mathcal{AD}}\delta_{\mathcal{BC}}\right)\\ &+\frac{3}{4}\left(\delta_{\mathcal{AD}}\mathrm{P}_{\mathcal{BC}} - \delta_{\mathcal{BD}}\mathrm{P}_{\mathcal{AC}} + \mathrm{P}_{\mathcal{AD}}\delta_{\mathcal{BC}} - \mathrm{P}_{\mathcal{BD}}\delta_{\mathcal{AC}}\right)\end{aligned} \tag{5}$$

where $\widetilde{\mathrm{K}}$ is the sectional curvature of $\mathrm{M}^{m+h}(\epsilon)$ and Kronecker delta is defined as

$$\delta_{\mathcal{AB}} = \begin{cases} 1, & \text{if } \mathcal{A} = \mathcal{B}, \\ 0, & \text{if } \mathcal{A} \neq \mathcal{B}. \end{cases}$$

Let $\Pi$ denote the squared length of the second fundamental form Y of $\mathrm{N}^m$, which is defined by

$$\Pi = \sum_{abk} (\mathrm{Y}^k_{ab})^2.$$

Similarly, the mean curvature of $\mathrm{N}^m$ is calculated as follows:

$$\mathrm{H} = \frac{1}{m}\sum_{ak} \mathrm{Y}^k_{aa} e_k.$$

The curvature tensor of the indices for a submanifold is as follows:

$$\mathrm{R}_{abcl} = \widetilde{\mathrm{K}}_{abcl} + \sum_{\alpha}\left(\mathrm{Y}^{\alpha}_{ac}\mathrm{Y}^{\alpha}_{bl} - \mathrm{Y}^{\alpha}_{al}\mathrm{Y}^{\alpha}_{bc}\right). \tag{6}$$

The Ricci curvature for a totally real submanifold, from (3), is defined as follows:

$$\begin{aligned}\mathrm{R}_{ab} =&\frac{\epsilon}{4}(m-1)\delta_{ab} + \frac{3}{4}(m-2)\mathrm{P}_{ab} + \frac{3}{4}trace(\mathrm{P}|\mathrm{N})\\ &+\sum_{a}\left(\mathrm{Y}^k_{ab}\sum_{c}\mathrm{Y}^k_{cc} - \sum_{c}\mathrm{Y}^k_{ac}\mathrm{Y}^k_{cb}\right).\end{aligned} \tag{7}$$

From the above, we correct certain notations and the mean curvature vector $\xi$ as:

$$\Pi = \|\mathrm{Y}\|^2, \quad \mathrm{H} = |\xi|, \quad \mathrm{H}_{\alpha} = (\mathrm{Y}^{\alpha}_{ab})_{m\times m}. \tag{8}$$

If $e_{m+1} \parallel \mathrm{H}$, then

$$tr\mathrm{H}_{m+1} = m\mathrm{H}, \quad \mathrm{H}_{\alpha} = 0, \quad \alpha \neq m+1 \tag{9}$$

such that $tr$ is the trace of the matrix $\mathrm{H}_{\alpha} = (\mathrm{Y}^{\alpha}_{ab})$. Taking into account (5), (6), and (9), we have the scalar curvature as

$$\mathrm{R} = \frac{1}{4}m(m-1)(\epsilon + 6\mu) + m^2\mathrm{H}^2 - \Pi \tag{10}$$

where H is the mean curvature vector of $\mathrm{N}^m$, considering notation

$$\mu = \frac{1}{m}\sum_{a=1}^{m} \mathrm{P}(e_a, e_a)$$

such that P is a symmetric $(0,2)$-tensor. From (10), we find that the scalar curvature R is constant, if and only if, $\Pi$ and H are constant due to $\mu$ and $\epsilon$ being constant. Let $\mathrm{Y}^k_{abc}$ denote the second covariant derivative of $\mathrm{Y}^k_{ab}$, we have

$$\sum_c \mathrm{Y}^{\alpha}_{abc}\omega_c = \mathrm{dY}^{\alpha}_{ab} - \sum_c \mathrm{Y}^{\alpha}_{cb}\omega_{ca} - \sum_c \mathrm{Y}^{\alpha}_{ac}\omega_{cb} + \sum_t \mathrm{Y}^{\alpha}_{ab}\omega_{tk}$$

where $\{\omega_t k\}$ is the dual orthonormal frame of $\mathrm{N}^m$. Taking the exterior derivative of the above equation, we obtain

$$\sum_l \mathrm{Y}^{\alpha}_{abcl}\omega_l = \mathrm{dY}^{\alpha}_{abc} - \sum_l \mathrm{Y}^{\alpha}_{abc}\omega_{la} + \sum_t \mathrm{Y}^{t}_{abc}\omega_{tk} - \sum_l \mathrm{Y}^{\alpha}_{alc}\omega_{ab} - \sum_l \mathrm{Y}^{\alpha}_{abl}\omega_{lc}.$$

Moreover, the Laplacian of $\mathrm{Y}^{\alpha}_{ab}$ is

$$\Delta \mathrm{Y}^{\alpha}_{ab} = \sum_c \mathrm{Y}^{\alpha}_{abcc} = \sum_c \mathrm{Y}^{\alpha}_{ccab} + \sum_{cd}\left(\mathrm{Y}^{\alpha}_{cd}\mathrm{R}_{dabc} + \mathrm{Y}^{\alpha}_{da}\mathrm{R}_{dcbc}\right) - \sum_{\beta c}\mathrm{Y}^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc}. \tag{11}$$

We have some useful Lemmas for further use:

**Lemma 1** ([31]). *Let* $\mathrm{T}_1, \cdots, \mathrm{T}_n$ *be symmetric* $(m \times m)$*-matrices, then*

$$\sum_{r,s=1}^{n} \|[\mathrm{T}_r, \mathrm{T}_s]\|^2 \leq \left(\sum_{r=1}^{n} \|\mathrm{T}_r\|^2\right)^2$$

*such that the equality holds if and only if, under some rotation, all* $\mathrm{T}_r$*'s are zero except for two, which can be written in the following matrix form,*

$$\mathrm{T}_r = P\begin{pmatrix} 0 & \mu & 0\cdots 0 \\ \mu & 0 & 0\cdots 0 \\ 0 & 0 & 0\cdots 0 \\ \vdots & \vdots & \vdots\ddots\vdots \\ 0 & 0 & 0\cdots 0 \end{pmatrix}P^t, \quad \mathrm{T}_s = P\begin{pmatrix} \mu & 0 & 0\cdots 0 \\ 0 & -\mu & 0\cdots 0 \\ 0 & 0 & 0\cdots 0 \\ \vdots & \vdots & \vdots\ddots\vdots \\ 0 & 0 & 0\cdots 0 \end{pmatrix}P^t$$

*where* $P$ *is an orthogonal* $(m \times m)$*-matrix,* $[\mathrm{T}_r, \mathrm{T}_s] = \mathrm{T}_r\mathrm{T}_s - \mathrm{T}_s\mathrm{T}_r$ *is the commutator of the matrices* $\mathrm{T}_r, \mathrm{T}_s$.

**Lemma 2** ([32]). *Let* $\mathcal{T}_1, \mathcal{T}_2, \cdots \mathcal{T}_m (n \geq 2)$ *be symmetric* $(m \times m)$*-matrices.Then,*

$$-2\sum_{\alpha,\beta=1}^{m}\left(tr(\mathcal{T}_\alpha^2\mathcal{T}_\beta^2 - tr(\mathcal{T}_\alpha\mathcal{T}_\beta)^2\right) \geq \sum_{\alpha,\beta=1}^{m}[tr(\mathcal{T}_\alpha\mathcal{T}_\beta)]^2 - \frac{3}{2}\Big(\sum_{\alpha=1}^{m} tr(\mathcal{T}_\alpha^2)\Big)^2. \tag{12}$$

We can now estimate our first main result, which implies the following.

**Theorem 1.** *Let* $\mathrm{N}^m$ *be an m-dimensional compact totally real submanifold in locally conformal Kähler space form* $\mathrm{M}^{m+h}(\epsilon)$ *admitting parallel mean curvature and satisfying the following inequality.*

$$\mathrm{R}_\mathrm{N} \geq \left(\frac{m+2h-1}{2(m+2h)}\right)\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right) \tag{13}$$

*then,* $\mathrm{N}^m$ *is a totally umbilical sphere* $\mathbb{S}^m\left(\frac{1}{\sqrt{\frac{\epsilon+6\mu}{4}+\mathrm{H}^2}}\right)$.

**Proof.** Assume that $\mathrm{N}^m$ is a totally real submanifold of locally conformal Kähler space form $\mathrm{M}^{m+h}(\epsilon)$ with a parallel mean curvature vector H. Consider $e_{m+1}$, such that it is parallel to the mean curvature vector $\xi$ and

$$tr\mathrm{H}_{m+1} = m\mathrm{H}, \quad tr\mathrm{H}_{\alpha} = 0, \quad \alpha = m+1. \tag{14}$$

Due to the mean curvature vector $\xi$ being parallel, we have

$$\mathrm{D}^{\perp}\xi = \mathrm{d}\mathrm{H}e_{m+1} + \mathrm{H}\mathrm{D}^{\perp}e_{m+1} = \mathrm{d}\mathrm{H}e_{m+1} + \mathrm{H}\sum_{\beta}\omega_{m+1\beta}e_{\beta} = 0. \tag{15}$$

From the structure equation and (15), we derive

$$\begin{aligned}\mathrm{d}\omega_{m+1\beta} &= -\sum_{\gamma}\omega_{m+1\gamma}\wedge\omega_{\gamma\beta} + \frac{1}{2}\sum_{cl}\mathrm{R}_{m+1\beta cl}\omega_c\wedge\omega_l \\ &= \frac{1}{2}\sum_{cl}\mathrm{R}_{m+1\beta cl}\omega_c\wedge\omega_l = 0\end{aligned} \tag{16}$$

where $\{\omega_a^{\alpha}\}$ are the dual orthonormal 1-forms on $\mathrm{N}^m$. Since the mean curvature vector $\xi$ is parallel and $\sum_c \mathrm{H}^{\alpha}_{ccab} = 0$, one derives the following

$$\begin{aligned}\frac{1}{2}\Delta\Pi_H &= \sum_{abc}(\mathrm{Y}^{m+1}_{abc})^2 + \sum_{ij}\mathrm{Y}^{m+1}_{ab}\Delta\mathrm{Y}^{m+1}_{ab} \\ &= \sum_{abc}(\mathrm{Y}^{m+1}_{abc})^2 + \sum_{abcl}\mathrm{Y}^{m+1}_{ab}\Big(\mathrm{Y}^{m+1}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{m+1}_{la}\mathrm{R}_{lcbc}\Big).\end{aligned} \tag{17}$$

If a 2-plane $\pi \subset T_p\mathrm{N}$ at the point $p \in \mathrm{N}^m$, the sectional curvature is denoted by $\mathrm{R}_{\mathrm{N}}(p,\pi)$. Then, we define

$$\mathrm{R}_{\min}(p) = \min_{\pi\subset T_p\mathrm{N}}\mathrm{R}_{\mathrm{N}}(p,\pi).$$

Now, consider orthonormal fields $\{e_i\}$ and eigenvalues $\lambda_i$, such that $\mathrm{Y}^{m+1}_{ab} = \lambda_i\delta_{ab}$. Hence, we obtain

$$\begin{aligned}\sum_{abcl}\mathrm{Y}^{m+1}_{ab}\Big(\mathrm{Y}^{m+1}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{m+1}_{la}\mathrm{R}_{mlcbc}\Big) &= \frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{abab} \\ &\geq \frac{1}{2}\sum_{ab}\Big(\lambda_a-\lambda_b\Big)^2\mathrm{R}_{\min}.\end{aligned} \tag{18}$$

Considering (17) and (18), we have

$$\frac{1}{2}\Delta\Pi_{\mathrm{H}} \geq \sum_{abc}(\mathrm{Y}^{m+1}_{abc})^2 + \frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{\min}. \tag{19}$$

If we assume that the inequality $\mathrm{R}_{\mathrm{N}} \geq \frac{m+2h-1}{2(m+2h)}(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2)$ holds, and that $\Pi_{\mathrm{H}}$ is constant, we derive

$$\frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{\min} = 0. \tag{20}$$

It is implied that $\lambda_a = \lambda_b$. Then, $\mathrm{N}^m$ is pseudo-umbilical.

Again, using (11), we have that $\sum_c \mathrm{H}^{\alpha}_{ccab} = 0$ and the mean curvature vector of $\mathrm{N}^m$ is parallel, one constructs

$$\frac{1}{2}\Delta\tau = \sum_{\alpha\neq m+1}\sum_{abc}(\mathrm{Y}^{\alpha}_{abc})^2 + \sum_{\alpha\neq m+1}\sum_{abcl}\mathrm{Y}^{\alpha}_{ab}\Big(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) - \sum_{\alpha\neq m+1}\sum_{\beta abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc}. \tag{21}$$

From (6) and (2), we obtain

$$\begin{aligned}\sum_{\alpha\neq m+1}\sum_{abcl}\mathrm{Y}^{\alpha}_{ab}\Big(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) =& m\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right)\tau \\ &+ \sum_{\alpha,\beta\neq m+1}\Big(\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})^2 - \mathrm{tr}(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta})\Big) \\ &- \sum_{\alpha,\beta\neq m+1}\Big(\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})\Big)^2.\end{aligned} \tag{22}$$

Again (6), we derive

$$\sum_{\alpha\neq m+1}\sum_{\beta abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc} = \sum_{i} tr\mathrm{H}^2_{i^*} - \sum_{\alpha,\beta\neq m+1}\Big(tr(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})^2 - tr(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta})\Big). \tag{23}$$

Inserting (23) and (22) into (21), we obtain

$$\begin{aligned}\frac{1}{2}\Delta\tau =& \sum_{\alpha\neq m+1}\sum_{abc}(\mathrm{Y}^{\alpha}_{abc})^2 + \sum_{i} tr\mathrm{H}^2_{i^*} - a'm\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right)\tau \\ &+ (1+a')\sum_{\alpha\neq m+1}\sum_{abcl}\mathrm{Y}^{\alpha}_{ab}(\mathrm{Y}^{\alpha}_{cl}R_{labc} + \mathrm{Y}^{\alpha}_{la}R_{lcbc}) + a'\sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})\Big)^2 \\ &+ (1-a')\sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})^2 - \mathrm{tr}(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta})\Big)\end{aligned} \tag{24}$$

where $\tau$ is a scalar curvature. Setting $\mathrm{Y}^{\alpha}_{ab} = \lambda^{\alpha}_{a}\delta_{ab}$ with respect to the orthonormal frame fields $\{e_a\}$ and for a fixed $\alpha$, we obrain from Lemma 2

$$\begin{aligned}\sum_{abcl}\mathrm{Y}^{\alpha}_{ab}\Big(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) =& \frac{1}{2}\sum_{ab}(\lambda^{\alpha}_{a}-\lambda^{\alpha}_{b})^2\mathrm{R}_{abab} \\ \geq& \frac{1}{2}\sum_{ab}(\lambda^{\alpha}_{i}-\lambda^{\alpha}_{j})^2\mathrm{R}_{\min} \\ =& m\mathrm{tr}\mathrm{H}^2_{\alpha}\mathrm{R}_{\min}\end{aligned}$$

which implies that

$$\sum_{\alpha\neq m+1}\sum_{abcl}\mathrm{Y}^{\alpha}_{ab}\Big(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) \geq m\tau\mathrm{R}_{min}. \tag{25}$$

In the implementation of the DDVV inequality from Lemma 1, we construct the following

$$\begin{aligned}\sum_{\alpha\beta\neq m+1}\Big\{\mathrm{tr}(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta}) - \mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})^2\Big\} =& \frac{1}{2}\sum_{\alpha\beta\neq m+1}\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta} - \mathrm{H}_{\beta}\mathrm{H}_{\alpha})^2 \\ \leq& \frac{1}{2}\Big(\sum_{\alpha\neq m+1}\mathrm{tr}\mathrm{H}^2_{\alpha}\Big)^2 \\ =& \frac{1}{2}\tau^2.\end{aligned} \tag{26}$$

On the other hand, we have

$$\sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathrm{H}_\alpha\mathrm{H}_\beta)\Big)^2 \geq \frac{\tau^2}{m+2h-1}. \tag{27}$$

Setting $a' = \frac{m+2h-1}{m+2h+1}$ in (24), combining (25), (26) and (27), we derive

$$\frac{1}{2}\Delta\tau \geq \left\{-\left(\frac{m+2h-1}{m+2h+1}\right)\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right)+\left(\frac{2m+4h}{m+2h+1}\right)\mathrm{R}_{min}\right\}m\tau. \tag{28}$$

If our assumption in (13) is satisfied, then we have

$$\frac{1}{2}\Delta\tau \geq 0$$

using the Hopf lemma. This concludes that $\Delta\tau = 0$. Hence, we obtain the following

$$\tau = 0 \quad or \quad \mathrm{R_N} = \left(\frac{m+2h-1}{2(m+2h)}\right)\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right).$$

For the first case $\tau = 0$, then $\mathrm{N}^m$ is totally umbilical. For the second case, using (6), we derive

$$\mathrm{R}_{abab} = \frac{\epsilon+6\mu}{4}+\mathrm{H}^2,$$

which proves that $\mathrm{N}^m$ is a totally umbilical sphere $\mathbb{S}^m\left(\frac{1}{\sqrt{\frac{\epsilon+6\mu}{4}+\mathrm{H}^2}}\right)$. Moreover, the inequalities (25), (26), (27), and (28) are changed to equalities if $\mathrm{R_N} = \left(\frac{m+2h-1}{2(m+2h)}\right)\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right)$.

The equalities of (26) imply that either all $\mathrm{H}'_\alpha s$ are zero or two of the $\mathrm{H}'_\alpha s$ are nonzero for $\alpha \neq m+1$. We estimate that if the inequalities in (27) and (28) are converted into equalities, that is

$$tr\mathrm{H}^2_\alpha = tr\mathrm{H}^2_\beta \quad (\alpha,\beta \neq m+1), \quad \text{and} \quad \sum_t tr\mathrm{H}^2_{t^*} = 0.$$

Thus, $\mathrm{N}^m$ is totally umbilical $\mathrm{R}_{abab} = \frac{\epsilon+6\mu}{4}+\mathrm{H}^2$, as $\mathrm{H}'^s_\alpha$ are zero ($\alpha \neq m+1$). This presents a contradiction. This completes the proof of the theorem. □

In the following results, we establish the following:

**Theorem 2.** *Let $\mathrm{N}^m$ be an $(m \geq 2)$-dimensional totally real submanifold in a locally conformal Kähler space form $\mathrm{M}^{m+h}(\epsilon)$. If $\mathrm{J}\xi$ is normal to $\mathrm{N}^m$, then either $\mathrm{N}^m$ is totally umbilical or it satisfies the inequality*

$$\inf\tau \leq m\left(\frac{\epsilon+6\mu}{4}+\mathrm{H}^2\right)\left(m-\frac{5}{3}\right) \tag{29}$$

*where the scalar curvature, the mean curvature, and the mean curvature vector are represented by $\tau, \mathrm{H}$, and $\xi$, respectively.*

**Proof.** Let $\mathrm{J}\xi$ be normal to $\mathrm{N}^m$, and $e_{m+1}$ be parallel to $\xi$, we have

$$tr\mathrm{H}_{m+1} = m\mathrm{H}, \quad tr\mathrm{H}_\alpha = 0, \quad \alpha \neq m+1. \tag{30}$$

From (11), we have

$$\frac{1}{2}\Delta\Pi = \sum_{\alpha ijk}(\mathrm{Y}^{\alpha}_{abc})^2 + \sum_{\alpha abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\alpha}_{ccab} + \sum_{\alpha abcl}\mathrm{Y}^{\alpha}_{ab}\left(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\right) - \sum_{\alpha\beta abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc}. \tag{31}$$

From (6) and (31), $\mathrm{N}^m$ is totally umbilical, and we obtain

$$\sum_{\alpha abcl}\mathrm{Y}^{\alpha}_{ab}\left(\mathrm{Y}^{\alpha}_{cl}\mathrm{R}_{labc} + \mathrm{Y}^{\alpha}_{la}\mathrm{R}_{lcbc}\right) = m\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right)\Pi - m^2\mathrm{H}^2 + \sum_{\alpha\beta}\left\{\mathrm{tr}\left(\mathrm{H}_{\alpha}\mathrm{H}_{\beta}\right)^2 - \mathrm{tr}\left(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta}\right)\right\} - \sum_{\alpha\beta}\left(\mathrm{tr}\left(\mathrm{H}_{\alpha}\mathrm{H}_{\beta}\right)\right)^2, \tag{32}$$

$$\sum_{\alpha\beta abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc} = -\sum_{\alpha\beta}\left\{\mathrm{tr}(\mathrm{H}_{\alpha}\mathrm{H}_{\beta})^2 - \mathrm{tr}(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta})\right\} - \sum_{a}\mathrm{tr}\mathrm{H}^2_{a^*}. \tag{33}$$

Considering (30) and $\mathrm{Y}^{m+1}_{ab} = \mathrm{H}\delta_{ab}$ for pseudo-umbilical $\mathrm{N}^m$, one derives

$$\sum_{\alpha abc}\mathrm{Y}^{\alpha}_{ab}\mathrm{Y}^{\alpha}_{ccab} = m\mathrm{H}\Delta\mathrm{H}, \tag{34}$$

$$\sum_{\alpha abc}(\mathrm{Y}^{\alpha}_{abc})^2 \geq \sum_{ac}(\mathrm{Y}^{m+1}_{aac})^2 = m\sum_{a}(\nabla_a\mathrm{H})^2, \tag{35}$$

$$\frac{1}{2}\Delta\mathrm{H}^2 = \mathrm{H}\Delta\mathrm{H} + \sum_{a}(\nabla_a\mathrm{H})^2. \tag{36}$$

Using Lemma 2 and $\mathrm{Y}^{m+1}_{ab} = \mathrm{H}\delta_{ab}$ for pseudo-umbilical $\mathrm{N}^m$, we obtain

$$\begin{aligned} 2\sum_{\alpha\beta}\left\{\mathrm{tr}(\mathrm{H}_{\mathrm{ff}}\mathrm{H}_{\mathrm{fi}})^2 - \mathrm{tr}\left(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta}\right)\right\} - \sum_{\alpha\beta}\mathrm{tr}\left(\mathrm{H}_{\alpha}\mathrm{H}_{\beta}\right)^2 &= 2\sum_{\alpha\beta\neq m+1}\left\{\mathrm{tr}\left(\mathrm{H}_{\alpha}\mathrm{H}_{\beta}\right)^2 - \mathrm{tr}\left(\mathrm{H}^2_{\alpha}\mathrm{H}^2_{\beta}\right)\right\} \\ &\quad - \sum_{\alpha\beta\neq m+1} tr\left(\mathrm{H}_{\alpha}\mathrm{H}_{\beta}\right)^2 - \left(\mathrm{tr}\mathrm{H}^2_{m+1}\right)^2 \\ &\geq -\frac{3}{2}\tau^2 - m^2\mathrm{H}^4 \\ &= -\frac{3}{2}(\Pi - m\mathrm{H}^2)^2 - m^2\mathrm{H}^4. \end{aligned} \tag{37}$$

Substituting (32)–(37) into (31), we have

$$\begin{aligned} \frac{1}{2}\Delta\Pi &\geq \frac{1}{2}m\Delta\mathrm{H}^2 + m\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right)\Pi - \frac{3}{2}\left(\Pi - m\mathrm{H}^2\right)^2 - m^2\mathrm{H}^4 - m^2\mathrm{H}^2 \\ &= \frac{1}{2}m\Delta\mathrm{H}^2 + \left(\Pi - m\mathrm{H}^2\right)\left\{m\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right) - \frac{3}{2}\left(\Pi - m\mathrm{H}^2\right)\right\} \\ &= \frac{1}{2}m\Delta\mathrm{H}^2 + \tau\left\{m\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right) - \frac{3}{2}\tau\right\}. \end{aligned} \tag{38}$$

Using the same argument as in [12], we conclude that either $\mathrm{N}^m$ is totally umbilical or

$$\inf\rho \leq m\left(\frac{\epsilon+6\mu}{4} + \mathrm{H}^2\right)\left(m - \frac{5}{3}\right).$$

This completes the proof of the theorem. □

**Theorem 3.** *Let* $\mathrm{N}^m$ *be a* $(m \geq 2)$*-dimensional totally real submanifold in a locally conformal Kähler space form* $\mathrm{M}^{m+h}(\epsilon)$*. If* $\mathrm{J}\xi$ *is normal to* $\mathrm{N}^m$*, then the following inequality holds,*

$$\int \left\{ 2\left(\frac{\epsilon + 6\mu}{4} + 4\mathrm{H}^2\right) m\Pi - 3\Pi^2 - 5m^2\mathrm{H}^4 - 4m^2\mathrm{H}^2 + 2m\mathrm{H}^2 \right\} \mathrm{dV} \leq 0 \tag{39}$$

*where* H *and* $\Pi$ *denote the mean curvature of* $\mathrm{N}^m$ *and the squared norm of the second fundamental form of* $\mathrm{N}^m$*.*

**Proof.** Without a loss of generality, we consider $e_{1^*}$, such that it is parallel to $\xi$ and $tr\mathrm{H}_{1^*} = m\mathrm{H}$. Next, $tr\mathrm{H}_\alpha = 0$, for $\alpha \neq 1^*$ and $\mathrm{J}\xi$ is normal to $\mathrm{N}^m$. These conditions, together with (6), yield

$$\begin{aligned}
\sum_{\alpha\beta abc} \mathrm{Y}^\alpha_{ab}\mathrm{Y}^\beta_{ca}\mathrm{R}_{\alpha\beta bc} =& m^2\mathrm{H}^2 - \sum_a \mathrm{tr}\mathrm{H}^2_{a^*} - \sum_{\alpha\beta}\left\{ \mathrm{tr}(\mathrm{H}_\alpha\mathrm{H}_\beta)^2 - \mathrm{tr}(\mathrm{H}^2_\alpha\mathrm{H}^2_\beta) \right\} \\
\leq& m^2\mathrm{H}^2 - \mathrm{tr}\mathrm{H}^2_{1^*} - \sum_{\alpha\beta}\left\{ \mathrm{tr}(\mathrm{H}_\alpha\mathrm{H}_\beta)^2 - \mathrm{tr}(\mathrm{H}^2_\alpha\mathrm{H}^2_\beta) \right\} \\
=& m^2\mathrm{H}^2 - m\mathrm{H}^2 - \sum_{\alpha\beta}\left\{ \mathrm{tr}(\mathrm{H}_\alpha\mathrm{H}_\beta)^2 - \mathrm{tr}(\mathrm{H}^2_\alpha\mathrm{H}^2_\beta) \right\}.
\end{aligned} \tag{40}$$

Using the same argument as in Theorem 2, we conclude that

$$\begin{aligned}
\frac{1}{2}\Delta\Pi \geq& \frac{1}{2}m\Delta\mathrm{H}^2 + m\left(\frac{\epsilon + 6\mu}{4} + \mathrm{H}^2\right)\Pi \\
&- \frac{3}{2}(\Pi - m\mathrm{H}^2)^2 - m^2\mathrm{H}^4 - 2m^2\mathrm{H}^2 + m\mathrm{H}^2.
\end{aligned}$$

As the boundary of $\mathrm{N}^m$ is compact, using the Stokes' theorem, we obtain

$$\int \left\{ m\left(\frac{\epsilon + 6\mu}{4} + \mathrm{H}^2\right)\Pi - \frac{3}{2}(\Pi - m\mathrm{H}^2)^2 - m^2\mathrm{H}^4 - 2m^2\mathrm{H}^2 + m\mathrm{H}^2 \right\} \leq 0$$

which implies (39). This completes the proof of the theorem. □

For a minimal submanifold, we have the following Corollary from Theorem 3

**Corollary 1.** *Let* $\mathrm{N}^m$ *be an* $(m \geq 2)$*-dimensional minimal totally real submanifold of a locally conformal Kähler space form* $\mathrm{M}^{m+h}(\epsilon)$*. If* $\mathrm{J}\xi$ *is normal to* $\mathrm{N}^m$*, then the following inequality holds,*

$$\int \left\{ \left(\frac{\epsilon + 6\mu}{2}\right) m\Pi - 3\Pi^2 \right\} \mathrm{dV} \leq 0$$

*where* $\Pi$ *denotes the squared length of the second fundamental form of* $\mathrm{N}^m$*.*

**Remark 1.** *It can be seen that Theorems 1, 2, and 3 are extended versions of Theorems 1 and 2 in [33]. Also, see [34,35] for similar results.*

**Example 2.** *The generalized Hopf manifold is a locally conformal manifold with parallel Lee form. One such example of a locally conformal Kähler manifold that is not Kähler is the Hopf manifold, which is diffeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^{2m-1}$*.*

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