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A Simultaneous Decomposition for a Quaternion Tensor Quaternity with Applications

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Abstract: Quaternion tensor decompositions have recently been the center of focus due to their wide potential applications in color data processing. In this paper, we establish a simultaneous decomposition for a quaternion tensor quaternity under Einstein product. The decomposition brings the quaternity of four quaternion tensors into a canonical form, which only has 0 and 1 entries. The structure of the canonical form is discussed in detail. Moreover, the proposed decomposition is applied to a new framework of color video encryption and decryption based on discrete wavelet transform. This new approach can realize simultaneous encryption and compression with high security.

Keywords: tensor decomposition; quaternion algebra; quaternion tensor

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1. Introduction

A tensor is a multi-dimensional array that can hold vast amounts of structured data. Tensors and decompositions of tensors are useful in data mining [1], genomic signals [2], signal processing [3], computer vision [4] and elsewhere. Numerous kinds of tensor decompositions have been discussed in the literature, including Tucker decomposition, higher-order singular value decomposition and so on (e.g., [5–9]). Kolda and Bader [10] in 2013 provided a review of existing tensor decompositions as well as their applications and related algorithms.

Quaternion algebra was introduced by Hamilton in 1943. Quaternion algebra is an associative and noncommutative division algebra over the real number field. The theory of quaternion algebra is discussed in [11,12]. Recently, quaternion algebra has attracted significant attention due to its wide applications in signal processing, control theory, computer science, quantum mechanics and others [13–19]. Particularly in the realm of color image processing, Pei and Cheng [20] proposed a quaternion model for color images. In this model, the RGB components of every pixel fit well to the three imaginary parts of a quaternion number. Therefore, the quaternion model for color images is widely used in many studies.

A tensor with quaternion entries is a quaternion tensor. Quaternion tensors can hold more information than real tensors and therefore have more potential applications. For example, Miao et al. [21] defined quaternion-based higher-order singular value decomposition and applied it in color image processing. Eigenvalues of quaternion tensors under Einstein product and applications in color video compression are investigated in [22].

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However, to our knowledge, the theory of simultaneous decomposition for multiple tensors over quaternion algebra is not so fruitful at present. Many results over real number fields cannot be directly extended to quaternion algebra due to its noncommutativity. In particular, He et al. [23,24] established simultaneous decompositions for two sets of quaternion tensor triplets under Einstein product and provided applications in color video processing.

Motivated by the wide applications of quaternion tensor decomposition and the works mentioned above, in this paper, we establish a simultaneous decomposition for a quaternion tensor quaternity under Einstein product. This decomposition brings the quaternity of four quaternion tensors into a canonical form whose entries are only 0 and 1. The structure of the canonical form is discussed in detail. These results extend the existing findings of simultaneous decomposition for multiple quaternion tensors. Moreover, we combine the proposed decomposition with discrete wavelet transform to construct a new framework of color video encryption and decryption. This new method can realize simultaneous encryption and compression with high security.

The remainder of this paper is organized as follows. In Section 2, we present some notations and necessary results about quaternion algebra, tensor and Einstein product. In Section 3, we establish a simultaneous decomposition for a quaternion tensor quaternity and discuss its structure in detail. In Section 4, we apply the proposed decomposition to color video processing.

2. Preliminaries

A tensor $\mathcal{A} = (a_{i_1,\ldots,i_N})_{1\leq i_j \leq I_j}(j = 1,\ldots,N)$ is a multi-dimensional array with $I_1 I_2 \cdots I_N$ entries. N is called the order of $\mathcal{A}$. Let $\mathbb{R}$ and $\mathbb{H}$ stand, respectively, for the real number field and the quaternion algebra:

$$\mathbb{H} = \left\{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \middle| \, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

We denote $\mathbb{H}^{I_1\times\cdots\times I_N}$ as the set of all the N-order quaternion tensors of dimensions $I_1 \times \cdots \times I_N$. A tensor $\mathcal{D} = (d_{i_1,\ldots,i_N,j_1,\ldots,j_N}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is called a diagonal tensor if all of its entries are zero except for $d_{i_1,\ldots,i_N,i_1,\ldots,i_N}$. If all the $d_{i_1,\ldots,i_N,i_1,\ldots,i_N} = 1$, then $\mathcal{D}$ is called a unit tensor and denoted by $\mathcal{I}$. A tensor is called a zero tensor if all of its entries are zero. A zero tensor with an appropriate order size is denoted by 0.

In particular, a matrix is a second-order tensor. We use normal uppercase letters to represent a matrix, for example, A. An identity matrix of appropriate size is denoted by I. Let $A \in \mathbb{H}^{m\times n}$; the symbols $r(A)$ and A^{-1} stand for the rank of A and the inverse of A if A is invertible, respectively. For a more detailed review of the quaternion matrix, readers can refer to [12].

Next, we give the definition of Einstein product.

Definition 1 (Einstein product, [25])**.** *For two tensors with compatible sizes* $\mathcal{A} = (a_{i_1,\ldots,i_N,j_1,\ldots,j_N}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ *and* $\mathcal{B} = (b_{j_1,\ldots,j_N,k_1,\ldots,k_M}) \in \mathbb{H}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_M}$*, the Einstein product of* $\mathcal{A}$ *and* $\mathcal{B}$ *is defined by the operation* $*_N$ *via*

$$(\mathcal{A} *_N \mathcal{B})_{i_1,\ldots,i_N,k_1,\ldots,j_M} = \sum_{j_1,\ldots,j_N} a_{i_1,\ldots,i_N,j_1,\ldots,j_N} b_{j_1,\ldots,j_N,k_1,\ldots,k_M}.$$

Thus, $(\mathcal{A} *_N \mathcal{B}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_M}$.

Navasca et al. in [6] defined a transformation from tensor to matrix over a real number field. We now give a similar and more precise definition of transformation from tensor to matrix over quaternion algebra.

Definition 2 (Transformation)**.** *Define the transformation*

$$f_{I_1,\cdots,I_N,J_1,\cdots,J_N} : \mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N} \to A \in \mathbb{H}^{I_1\cdot I_2\cdots I_{N-1}\cdot I_N\times J_1\cdot J_2\cdots J_{N-1}\cdot J_N}$$

with $f_{I_1,\cdots,I_N,J_1,\cdots,J_N}(\mathcal{A}) = A$ *defined entry-wise as*

$$(\mathcal{A})_{i_1,\ldots,i_N,j_1,\ldots,j_N} \xrightarrow{f_{I_1,\cdots,I_N,J_1,\cdots,J_N}} (A)_{[i_1+\sum_{k=2}^{N}(i_k-1)\prod_{s=1}^{k-1} I_s],[j_1+\sum_{k=2}^{N}(j_k-1)\prod_{s=1}^{k-1} J_s]}. \tag{1}$$

Remark 1. *Compared with the transformation f defined in [6], here, we add certain subscripts to f. This would help distinguish different transformations that deal with tensors of different sizes. In particular, we can drop the subscripts of f if they are clear from the context.*

Navasca et al. in [6] discussed some properties of the map f over a real number field. By a similar approach, these results can be generalized to quaternion algebra.

Lemma 1. *Let f be the map defined in (1). Then, the following properties hold:*

1. *For any $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, the map $f_{I_1,\cdots,I_N,J_1,\cdots,J_N}$ is a bijection and its inverse map $f^{-1}_{I_1,\cdots,I_N,J_1,\cdots,J_N}$, or simply, f^{-1}, if its subscripts are clear from the context, is given by*

$$f^{-1} : A \in \mathbb{H}^{I_1\cdot I_2\cdots I_{N-1}\cdot I_N\times J_1\cdot J_2\cdots J_{N-1}\cdot J_N} \to \mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$$

with $f^{-1}(A) = \mathcal{A}$ defined entry-wise as

$$(A)_{i,j} \xrightarrow{f^{-1}} (\mathcal{A})_{i_1,\ldots,i_N,j_1,\ldots,j_N} \tag{2}$$

where

$$i_t = \begin{cases} \left[\!\left[\dfrac{i-1}{\prod_{s=1}^{N-1} I_s}\right]\!\right] + 1 & \text{if } t = N; \\ \left[\!\left[\dfrac{i-1-\sum_{k=t+1}^{N}(i_t-1)\prod_{s=1}^{k-1} I_s}{\prod_{s=1}^{t-1} I_s}\right]\!\right] + 1 & \text{if } t = 2,\cdots,N-1; \\ i - \sum_{k=2}^{N}(i_k-1)\prod_{s=1}^{k-1} I_s & \text{if } t = 1, \end{cases} \tag{3}$$

$$j_t = \begin{cases} \left[\!\left[\dfrac{j-1}{\prod_{s=1}^{N-1} J_s}\right]\!\right] + 1 & \text{if } t = N; \\ \left[\!\left[\dfrac{j-1-\sum_{k=t+1}^{N}(j_t-1)\prod_{s=1}^{k-1} J_s}{\prod_{s=1}^{t-1} J_s}\right]\!\right] + 1 & \text{if } t = 2,\cdots,N-1; \\ j - \sum_{k=2}^{N}(j_k-1)\prod_{s=1}^{k-1} J_s & \text{if } t = 1, \end{cases} \tag{4}$$

and $[\![x]\!]$ is the greatest integer less than or equal to the real number x.

2. *For any $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{H}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$, the map f satisfies $f(\mathcal{A} *_N \mathcal{B}) = f(\mathcal{A}) \cdot f(\mathcal{B})$, where "$\cdot$" is the usual matrix multiplication.*

Proof. We only prove that, under the foundation of f being a bijection, the expression of f^{-1} is given as (2)–(4). Readers can refer to [6,26] for the other parts of the proof.

If $N = 1$, then f and f^{-1} are simply identity maps.

Now, we consider the case of $N \geq 2$. Suppose

$$(A)_{i,j} \xrightarrow{f^{-1}} (\mathcal{A})_{i_1,\ldots,i_N,j_1,\ldots,j_N}.$$

Then, there must be

$$(\mathcal{A})_{i_1,\ldots,i_N,j_1,\ldots,j_N} \xrightarrow{f} (A)_{i,j}$$

since f is a bijection. It follows from the definition of f in (1) that

$$i_1 + \sum_{k=2}^{N} (i_k - 1) \prod_{s=1}^{k-1} I_s = i, \tag{5}$$

and

$$j_1 + \sum_{k=2}^{N} (j_k - 1) \prod_{s=1}^{k-1} J_s = j. \tag{6}$$

We can first obtain from (5) that

$$(i_N - 1) \prod_{s=1}^{N-1} I_s = i - i_1 - \sum_{k=2}^{N-1} (i_k - 1) \prod_{s=1}^{k-1} I_s \leq i - 1,$$

i.e.,

$$i_N \leq \frac{i-1}{\prod_{s=1}^{N-1} I_s} + 1.$$

Since i_N is a positive integer, we see that

$$i_N \leq \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] + 1.$$

Now, we want to prove $i_N = \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] + 1$. Suppose $i_N < \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] + 1$, that is, $i_N \leq \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right]$. Then, we have

$$\begin{aligned} i &= i_1 + \sum_{k=2}^{N} (i_k - 1) \prod_{s=1}^{k-1} I_s \\ &\leq I_1 + (I_2 - 1) I_1 + (I_3 - 1) I_1 I_2 + \cdots + (I_{N-1} - 1) \prod_{s=1}^{N-2} I_s + \left(\left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] - 1 \right) \prod_{s=1}^{N-1} I_s \\ &= \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] \prod_{s=1}^{N-1} I_s \leq i - 1, \end{aligned}$$

which is a contradiction. Hence, we have $i_N = \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s} \right]\!\right] + 1$. Then, we can rewrite (5) as

$$i_1 + \sum_{k=2}^{N-1} (i_k - 1) \prod_{s=1}^{k-1} I_s = i - (i_N - 1) \prod_{s=1}^{N-1} I_s.$$

By the same approach, we can obtain the expressions of $i_{N-1}, i_{N-2}, \ldots, i_1$. Similarly, we can give the expressions of $j_N, j_{N-1}, \ldots, j_1$. $\square$

According to Lemma 1, we can immediately obtain the following property of f^{-1}.

Lemma 2. *For any two matrices* $A \in \mathbb{H}^{I_1 \cdot I_2 \cdots I_{N-1} \cdot I_N \times J_1 \cdot J_2 \cdots J_{N-1} \cdot J_N}$ *and* $B \in \mathbb{H}^{J_1 \cdot J_2 \cdots J_{N-1} \cdot J_N \times K_1 \cdot K_2 \cdots K_{N-1} \cdot K_N}$*, the map* f^{-1} *satisfies* $f^{-1}_{I_1,\cdots,I_N,K_1,\cdots,K_N}(A \cdot B) = f^{-1}_{I_1,\cdots,I_N,J_1,\cdots,J_N}(A) *_N f^{-1}_{J_1,\cdots,J_N,K_1,\cdots,K_N}(A \cdot B)$.

Proof.

$$\begin{aligned} f^{-1}(A \cdot B) &= f^{-1}(f(f^{-1}(A)) \cdot f(f^{-1}(B))) \\ &= f^{-1}(f(f^{-1}(A) *_N f^{-1}(B))) = f^{-1}(A) *_N f^{-1}(B) \end{aligned}$$

□

We finally give the definition of the inverse of an even-order tensor.

Definition 3 (Inverse of an even-order tensor). *A tensor* $\mathcal{A} \in \mathbb{H}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ *is invertible if there exists* $\mathcal{X} \in \mathbb{H}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ *such that*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} = \mathcal{I}.$$

In this case, $\mathcal{X}$ *is called the inverse of* $\mathcal{A}$ *and is denoted by* $\mathcal{A}^{-1}$.

Since $f(\mathcal{I}) = I$, where I is an identity matrix with appropriate size, together with the multiplicative properties of f and f^{-1}, we have the following.

Lemma 3. *A tensor* $\mathcal{A} \in \mathbb{H}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ *is invertible if and only if matrix* $f(\mathcal{A})$ *is invertible. In this case,* $\mathcal{A}^{-1} = f^{-1}(f(\mathcal{A})^{-1})$.

Remark 2. *The above several properties of* f *and* f^{-1} *admit a group structure on* $\mathbb{H}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. *The transformation of* f *and* f^{-1} *between the quaternion tensor and quaternion matrix is the main proof idea of the results in the next section.*

3. A Simultaneous Decomposition for a Quaternion Tensor Quaternity

In this section, we give a simultaneous decomposition for a quaternion tensor quaternity via Einstein product. We first present the lemma of an equivalence canonical form of a quaternion matrix quaternity.

Lemma 4 ([27,28]). *Given four matrices of compatible sizes,* $A \in \mathbb{H}^{p_1 \times q_1}$, $B \in \mathbb{H}^{p_2 \times q_1}$, $C \in \mathbb{H}^{p_2 \times q_2}$ *and* $D \in \mathbb{H}^{p_2 \times q_3}$*, there exist nonsingular matrices* $P_1 \in \mathbb{H}^{p_1 \times p_1}$, $P_2 \in \mathbb{H}^{p_2 \times p_2}$, $Q_1 \in \mathbb{H}^{q_1 \times q_1}$, $Q_2 \in \mathbb{H}^{q_2 \times q_2}$ *and* $Q_3 \in \mathbb{H}^{q_3 \times q_3}$*, such that*

$$P_1 A Q_1 = S_A, \quad P_2 B Q_1 = S_B, \quad P_2 C Q_2 = S_C, \quad P_2 D Q_3 = S_D,$$

where

$$S_A = \begin{array}{c c} & \begin{array}{cc} r_1 & \end{array} \\ \begin{array}{c} r_1 \\ \end{array} & \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad S_B = \begin{array}{c c} & \begin{array}{cccc} r_2 & r_1 - r_2 & r_3 & \end{array} \\ \begin{array}{c} r_3 \\ r_2 \\ \end{array} & \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad S_C = \begin{array}{c c} & \begin{array}{cccc} r_6 & r_4 & r_5 & \end{array} \\ \begin{array}{c} r_5 \\ r_3 - r_5 \\ r_4 \\ r_2 - r_4 \\ r_6 \\ \end{array} & \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and

$$S_D = \begin{array}{l c}
 & \begin{array}{cccccccccc} r_{15} & r_{13} & r_{10} & r_{14} & r_{12} & r_9 & r_{11} & r_8 & r_7 & \end{array} \\
\begin{array}{l} r_7 \\ r_5 - r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_3 - r_5 - r_8 - r_9 - r_{10} \\ r_9 \\ r_{11} \\ r_4 - r_9 - r_{11} \\ r_{12} \\ r_{13} \\ r_2 - r_4 - r_{12} - r_{13} \\ r_{13} \\ r_{10} \\ r_{14} \\ r_6 - r_{13} - r_{10} - r_{14} \\ r_{15} \\ \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}.$$

The expressions for the block dimensions r_1–r_{15} are given by

$$r_1 = r(A),\ r_2 = r(A) + r(B) - r\begin{pmatrix} A \\ B \end{pmatrix},\ r_3 = r\begin{pmatrix} A \\ B \end{pmatrix} - r(A),$$

$$r_4 = r\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} - r\begin{pmatrix} A \\ B \end{pmatrix} - r\begin{pmatrix} B & C \end{pmatrix} + r(B),$$

$$r_5 = r\begin{pmatrix} A \\ B \end{pmatrix} + r(C) - r\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},\ r_6 = r\begin{pmatrix} B & C \end{pmatrix} - r(B),$$

$$r_7 = r\begin{pmatrix} A \\ B \end{pmatrix} + r(C) + r(D) - r\begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & C & D \end{pmatrix},\ r_8 = r\begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & C & D \end{pmatrix} - r\begin{pmatrix} A & 0 \\ B & D \end{pmatrix} - r(C),$$

$$r_9 = r\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} + r\begin{pmatrix} A & 0 \\ B & D \end{pmatrix} + r\begin{pmatrix} B & C & 0 \\ 0 & C & D \end{pmatrix} - r\begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & C & D \end{pmatrix} - r\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & C & 0 \\ B & 0 & C & D \end{pmatrix},$$

$$r_{10} = r\begin{pmatrix} C & D \end{pmatrix} + r\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & C & 0 \\ B & 0 & C & D \end{pmatrix} - r\begin{pmatrix} A & 0 & 0 \\ B & C & D \end{pmatrix} - r\begin{pmatrix} B & C & 0 \\ 0 & C & D \end{pmatrix},$$

$$r_{11} = r(B) + r\begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & C & D \end{pmatrix} - r\begin{pmatrix} A \\ B \end{pmatrix} - r\begin{pmatrix} B & C & 0 \\ 0 & C & D \end{pmatrix},$$

$$r_{12} = r\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & C & 0 \\ B & 0 & C & D \end{pmatrix} - r\begin{pmatrix} B & D \end{pmatrix} - r\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

$$r_{13} = r\begin{pmatrix} B & C \end{pmatrix} + r\begin{pmatrix} B & D \end{pmatrix} - r\begin{pmatrix} B & C & D \end{pmatrix} + r\begin{pmatrix} A & 0 & 0 \\ B & C & D \end{pmatrix} - r\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & C & 0 \\ B & 0 & C & D \end{pmatrix},$$

$$r_{14} = r\begin{pmatrix} B & C & 0 \\ 0 & C & D \end{pmatrix} - r(B) - r\begin{pmatrix} C & D \end{pmatrix},\ r_{15} = r\begin{pmatrix} B & C & D \end{pmatrix} - r\begin{pmatrix} B & C \end{pmatrix}.$$

Now, we give the main theorem of this paper.

Theorem 1. *Given four tensors of compatible sizes,* $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, $\mathcal{B} \in \mathbb{H}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$, $\mathcal{C} \in \mathbb{H}^{J_1\times\cdots\times J_N\times L_1\times\cdots\times L_N}$ *and* $\mathcal{D} \in \mathbb{H}^{J_1\times\cdots\times J_N\times R_1\times\cdots\times R_N}$, *there exist invertible tensors* $\mathcal{P}_1 \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, $\mathcal{P}_2 \in \mathbb{H}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$, $\mathcal{Q}_1 \in \mathbb{H}^{K_1\times\cdots\times K_N\times K_1\times\cdots\times K_N}$, $\mathcal{Q}_2 \in \mathbb{H}^{L_1\times\cdots\times L_N\times L_1\times\cdots\times L_N}$ *and* $\mathcal{Q}_3 \in \mathbb{H}^{R_1\times\cdots\times R_N\times R_1\times\cdots\times R_N}$, *such that*

$$\begin{aligned} &\mathcal{P}_1 *_N \mathcal{A} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{A}}, \quad \mathcal{P}_2 *_N \mathcal{B} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{B}}, \\ &\mathcal{P}_2 *_N \mathcal{C} *_N \mathcal{Q}_2 = \mathcal{S}_{\mathcal{C}}, \quad \mathcal{P}_2 *_N \mathcal{D} *_N \mathcal{Q}_3 = \mathcal{S}_{\mathcal{D}}, \end{aligned} \tag{7}$$

where

$$\mathcal{S}_{\mathcal{A}} = f^{-1}_{I_1,\cdots,I_N,K_1,\cdots,K_N}(S_A), \quad \mathcal{S}_{\mathcal{B}} = f^{-1}_{J_1,\cdots,J_N,K_1,\cdots,K_N}(S_B),$$

$$\mathcal{S}_{\mathcal{C}} = f^{-1}_{J_1,\cdots,J_N,L_1,\cdots,L_N}(S_C), \quad \mathcal{S}_{\mathcal{D}} = f^{-1}_{J_1,\cdots,J_N,R_1,\cdots,R_N}(S_D),$$

and

$$S_A = \begin{array}{c} \\ v_1 \end{array}\!\!\begin{array}{c} \quad v_1 \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array},\ S_B = \begin{array}{c} \\ v_3 \\ v_2 \\ \\ \end{array}\!\!\begin{array}{c} v_2 \quad v_1 - v_2 \quad v_3 \\ \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},\ S_C = \begin{array}{c} \\ v_5 \\ v_3 - v_5 \\ v_4 \\ v_2 - v_4 \\ v_6 \\ \\ \end{array}\!\!\begin{array}{c} v_6 \quad v_4 \quad v_5 \\ \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{8}$$

$$
S_D = \begin{array}{l c}
 & \begin{array}{cccccccccc} v_{15} & v_{13} & v_{10} & v_{14} & v_{12} & v_9 & v_{11} & v_8 & v_7 & \end{array} \\
\begin{array}{l}
v_7 \\
v_5 - v_7 \\
v_8 \\
v_9 \\
v_{10} \\
v_3 - v_5 - v_8 - v_9 - v_{10} \\
v_9 \\
v_{11} \\
v_4 - v_9 - v_{11} \\
v_{12} \\
v_{13} \\
v_2 - v_4 - v_{12} - v_{13} \\
v_{13} \\
v_{10} \\
v_{14} \\
v_6 - v_{13} - v_{10} - v_{14} \\
v_{15} \\

\end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}. \quad (9)
$$

The expressions for the block dimensions v_1–v_{15} are given by

$$v_1 = r\Big(f(\mathcal{A})\Big),\ v_2 = r\Big(f(\mathcal{A})\Big) + r\Big(f(\mathcal{B})\Big) - r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix},\ v_3 = r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix} - r\Big(f(\mathcal{A})\Big),$$

$$v_4 = r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix} - r\Big(f(\mathcal{B})\ f(\mathcal{C})\Big) + r\Big(f(\mathcal{B})\Big),$$

$$v_5 = r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix} + r\Big(f(\mathcal{C})\Big) - r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix},\ v_6 = r\Big(f(\mathcal{B})\ f(\mathcal{C})\Big) - r\Big(f(\mathcal{B})\Big),$$

$$v_7 = r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix} + r\Big(f(\mathcal{C})\Big) + r\Big(f(\mathcal{D})\Big) - r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix},$$

$$v_8 = r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{D}) \end{pmatrix} - r\Big(f(\mathcal{C})\Big),$$

$$v_9 = r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix} + r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{D}) \end{pmatrix} + r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 & 0 \\ 0 & f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ f(\mathcal{B}) & 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix},$$

$$v_{10} = r\Big(f(\mathcal{C})\ f(\mathcal{D})\Big) + r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 & 0 \\ 0 & f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ f(\mathcal{B}) & 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix},$$

$$v_{11} = r\Big(f(\mathcal{B}))\Big) + r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) \\ f(\mathcal{B}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix},$$

$$v_{12} = r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 & 0 \\ 0 & f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ f(\mathcal{B}) & 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix},$$

$$v_{13} = r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix} + r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} + r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{A}) & 0 & 0 & 0 \\ 0 & f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ f(\mathcal{B}) & 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix},$$

$$v_{14} = r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & 0 \\ 0 & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix}, \; v_{15} = r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} - r\begin{pmatrix} f(\mathcal{B}) & f(\mathcal{C}) \end{pmatrix}.$$

The exact structures of $\mathcal{S}_{\mathcal{A}}$, $\mathcal{S}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{S}_{\mathcal{D}}$ are given in Theorem 2.

Proof. Note that the matrices $f(\mathcal{A}), f(\mathcal{B}), f(\mathcal{C})$, and $f(\mathcal{D})$ can be arranged in the following matrix array:

$$\begin{array}{c} \\ \prod_{i=1}^{N} I_i \\ \prod_{i=1}^{N} J_i \end{array} \begin{array}{c} \begin{array}{ccc} \prod_{i=1}^{N} K_i & \prod_{i=1}^{N} L_i & \prod_{i=1}^{N} R_i \end{array} \\ \begin{pmatrix} f(\mathcal{A}) & 0 & 0 \\ f(\mathcal{B}) & f(\mathcal{C}) & f(\mathcal{D}) \end{pmatrix} \end{array}. \tag{10}$$

Applying Lemma 4 to (10), we have nonsingular matrices P_1, P_2, Q_1, Q_2, and Q_3 with appropriate sizes such that

$$P_1 f(\mathcal{A}) Q_1 = S_A, \quad P_2 f(\mathcal{B}) Q_1 = S_B, \quad P_2 f(\mathcal{C}) Q_2 = S_C, \quad P_2 f(\mathcal{D}) Q_3 = S_D$$

where S_A, S_B, S_C, and S_D are exactly in the forms of (8) and (9). Moreover, it follows from the properties of f and f^{-1} in Lemmas 1 and 2 that

$$f^{-1}(S_A) = f^{-1}(P_1 f(\mathcal{A}) Q_1) = f^{-1}(P_1) *_N f^{-1}(f(\mathcal{A})) *_N f^{-1}(Q_1) \Rightarrow \mathcal{P}_1 *_N \mathcal{A} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{A}},$$

$$f^{-1}(S_B) = f^{-1}(P_2 f(\mathcal{B}) Q_1) = f^{-1}(P_2) *_N f^{-1}(f(\mathcal{B})) *_N f^{-1}(Q_1) \Rightarrow \mathcal{P}_2 *_N \mathcal{B} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{B}},$$

$$f^{-1}(S_C) = f^{-1}(P_2 f(\mathcal{C}) Q_2) = f^{-1}(P_2) *_N f^{-1}(f(\mathcal{C})) *_N f^{-1}(Q_2) \Rightarrow \mathcal{P}_2 *_N \mathcal{C} *_N \mathcal{Q}_2 = \mathcal{S}_{\mathcal{C}},$$

$$f^{-1}(S_D) = f^{-1}(P_2 f(\mathcal{D}) Q_3) = f^{-1}(P_2) *_N f^{-1}(f(\mathcal{D})) *_N f^{-1}(Q_3) \Rightarrow \mathcal{P}_2 *_N \mathcal{D} *_N \mathcal{Q}_3 = \mathcal{S}_{\mathcal{D}},$$

where $\mathcal{P}_1 := f^{-1}(P_1), \mathcal{P}_2 := f^{-1}(P_2), \mathcal{Q}_1 := f^{-1}(Q_1), \mathcal{Q}_2 := f^{-1}(Q_2)$, and $\mathcal{Q}_3 := f^{-1}(Q_3)$ are invertible tensors by Lemma 3; $\mathcal{S}_{\mathcal{A}} := f^{-1}(S_A), \mathcal{S}_{\mathcal{B}} := f^{-1}(S_B), \mathcal{S}_{\mathcal{C}} := f^{-1}(S_C)$, and $\mathcal{S}_{\mathcal{D}} := f^{-1}(S_D)$ are real tensors whose nonzero entries are all 1. □

To determine the position of 1 in $\mathcal{S}_{\mathcal{A}}$, $\mathcal{S}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{S}_{\mathcal{D}}$, we first give the following definition.

Definition 4. *We denote $\langle N \rangle = \{1, 2, \cdots, N\}$ and define the map*

$$h_{I_1, \cdots, I_N, J_1, \cdots, J_N} : \langle I_1 \cdot I_2 \cdots I_{N-1} \cdot I_N \rangle \times \langle J_1 \cdot J_2 \cdots J_{N-1} \cdot J_N \rangle \to \langle I_1 \rangle \times \cdots \times \langle I_N \rangle \times \langle J_1 \rangle \times \cdots \times \langle J_N \rangle$$

as $h_{I_1,\cdots,I_N,J_1,\cdots,J_N}(i,j) = (i_1,\ldots,i_N,j_1,\ldots j_N)$ *with*

$$i_t = \begin{cases} \left[\!\left[\frac{i-1}{\prod_{s=1}^{N-1} I_s}\right]\!\right] + 1 & \textit{if } t = N; \\ \left[\!\left[\frac{i-1-\sum_{k=t+1}^{N}(i_t-1)\prod_{s=1}^{k-1} I_s}{\prod_{s=1}^{t-1} I_s}\right]\!\right] + 1 & \textit{if } t = 2,\cdots,N-1; \\ i - \sum_{k=2}^{N}(i_k-1)\prod_{s=1}^{k-1} I_s & \textit{if } t = 1, \end{cases}$$

$$j_t = \begin{cases} \left[\!\left[\frac{j-1}{\prod_{s=1}^{N-1} J_s}\right]\!\right] + 1 & \textit{if } t = N; \\ \left[\!\left[\frac{j-1-\sum_{k=t+1}^{N}(j_t-1)\prod_{s=1}^{k-1} J_s}{\prod_{s=1}^{t-1} J_s}\right]\!\right] + 1 & \textit{if } t = 2,\cdots,N-1; \\ j - \sum_{k=2}^{N}(j_k-1)\prod_{s=1}^{k-1} J_s & \textit{if } t = 1. \end{cases}$$

We also define

$$h_{I_1,\cdots,I_N,J_1,\cdots,J_N}(G) := \{h_{I_1,\cdots,I_N,J_1,\cdots,J_N}(i,j)\,|(i,j) \in G\}$$

where $G \subseteq \langle I_1 \cdot I_2 \cdots I_{N-1} \cdot I_N\rangle \times \langle J_1 \cdot J_2 \cdots J_{N-1} \cdot J_N\rangle$.

Now, we can easily use the map h to translate the position of 1 in S_A, S_B, S_C, and S_D to the position of 1 in $\mathcal{S}_{\mathcal{A}}$, $\mathcal{S}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{S}_{\mathcal{D}}$.

Theorem 2. *The structures of* $\mathcal{S}_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}}, \mathcal{S}_{\mathcal{C}}$ *and* $\mathcal{S}_{\mathcal{D}}$ *in Theorem 1 are as follows:*

$$(\mathcal{S}_{\mathcal{A}})_{i_1,\ldots,i_N,k_1,\ldots,k_N} = \begin{cases} 1, & \textit{if } (i_1,\ldots,i_N,k_1,\ldots,k_N) \in h_{I_1,\cdots,I_N,K_1,\cdots,K_N}(H_A), \\ 0, & \textit{otherwise}, \end{cases}$$

$$(\mathcal{S}_{\mathcal{B}})_{j_1,\ldots,j_N,k_1,\ldots,k_N} = \begin{cases} 1, & \textit{if } (j_1,\ldots,j_N,k_1,\ldots,k_N) \in h_{J_1,\cdots,J_N,K_1,\cdots,K_N}(H_B), \\ 0, & \textit{otherwise}, \end{cases}$$

$$(\mathcal{S}_{\mathcal{C}})_{j_1,\ldots,j_N,l_1,\ldots,l_N} = \begin{cases} 1, & \textit{if } (j_1,\ldots,j_N,l_1,\ldots,l_N) \in h_{J_1,\cdots,J_N,L_1,\cdots,L_N}(H_C), \\ 0, & \textit{otherwise}, \end{cases}$$

$$(\mathcal{S}_{\mathcal{D}})_{j_1,\ldots,j_N,r_1,\ldots,r_N} = \begin{cases} 1, & \textit{if } (j_1,\ldots,j_N,r_1,\ldots,r_N) \in h_{J_1,\cdots,J_N,R_1,\cdots,R_N}(H_D), \\ 0, & \textit{otherwise}. \end{cases}$$

where

$$H_A = \{(a_1,a_1)|a_1 \in \langle v_1\rangle\},$$

$$\begin{aligned} H_B = &\{(v_3+b_1,b_1)|b_1 \in \langle v_2\rangle\} \\ &\cup \{(b_2,v_1+b_2)|b_2 \in \langle v_3\rangle\}, \end{aligned}$$

$$\begin{aligned} H_C = &\{(v_2+v_3+c_1,c_1)|c_1 \in \langle v_6\rangle\} \\ &\cup \{(v_3+c_2,v_6+c_2)|c_2 \in \langle v_4\rangle\} \\ &\cup \{(c_3,v_6+v_4+c_3)|c_3 \in \langle v_5\rangle\}, \end{aligned}$$

$$
\begin{aligned}
H_D = &\{(v_2+v_3+v_6+d_1, d_1) | d_1 \in \langle v_{15} \rangle\} \\
&\cup \{(v_2+v_3+d_2, v_{15}+d_2) | d_2 \in \langle v_{13}+v_{10}+v_{14} \rangle\} \\
&\cup \{(v_3+v_4+v_{12}+d_3, v_{15}+d_3) | d_3 \in \langle v_{13} \rangle\} \\
&\cup \{(v_3+v_4+d_4, v_{15}+v_{13}+v_{10}+v_{14}+d_4) | d_4 \in \langle v_{12} \rangle\} \\
&\cup \{(v_3+d_5, v_{15}+v_{13}+v_{10}+v_{14}+v_{12}+d_5) | d_5 \in \langle v_9+v_{11} \rangle\} \\
&\cup \{(v_5+v_8+v_9+d_6, v_{15}+v_{13}+d_6) | d_6 \in \langle v_{10} \rangle\} \\
&\cup \{(v_5+v_8+d_7, v_{15}+v_{13}+v_{10}+v_{14}+v_{12}+d_7) | d_7 \in \langle v_9 \rangle\} \\
&\cup \{(v_5+d_8, v_{15}+v_{13}+v_{10}+v_{14}+v_{12}+v_9+v_{11}+d_8) | d_8 \in \langle v_8 \rangle\} \\
&\cup \{(d_9, v_{15}+v_{13}+v_{10}+v_{14}+v_{12}+v_9+v_{11}+v_8+d_9) | d_9 \in \langle v_7 \rangle\}.
\end{aligned}
$$

4. An Application of the Proposed Decomposition in Color Video Processing

In this section, an new framework of color video encryption using the proposed simultaneous decomposition (7) is presented. In the quaternion model for color images [20], a color image can be represented by a quaternion matrix. Analogously, a color video can be represented by a third-order quaternion tensor $\mathcal{A} \in \mathbb{H}^{m\times n\times p}$, where p is the number of frames of the video, and m and n are the height and the width of each frame, respectively. When the video has even frames, it can be further represented by a fourth-order quaternion tensor $\mathcal{A} \in \mathbb{H}^{m\times n\times \frac{p}{2}\times 2}$. $\mathcal{A}(:,:,:,1)$ and $\mathcal{A}(:,:,:,2)$ represent the first half and the second half of the video, respectively.

Now, we want to apply the simultaneous decomposition for four quaternion tensors to encrypt a color video. In the process, we might perform the f transformation to transform a color video $\mathcal{A} \in \mathbb{H}^{m\times n\times \frac{p}{2}\times 2}$ into a matrix $A \in \mathbb{H}^{mn\times p}$. However, in the cases where the video is short, we have $mn \gg p$, which would result in the matrix A being ill-conditioned for further processing. Hence, in the following, we equivalently use $\mathcal{A} \in \mathbb{H}^{m\times \frac{p}{2}\times n\times 2}$ to represent a color video, where the meanings of m, n and p are the same as above. We also assume that m and n are even numbers.

The steps of the new method of color video encryption using the simultaneous decomposition for four quaternion tensors are as follows:

Step 1. Perform the discrete wavelet transform [29] to each frame of the original video. The LL, LH, HL and HH sub-bands form four sub-videos of the same sizes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{H}^{\frac{m}{2}\times \frac{p}{2}\times \frac{n}{2}\times 2}$.

Step 2. Note that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ satisfy the conditions of Theorem 1. We can conduct the simultaneous decomposition for these four tensors:

$$
\begin{aligned}
\mathcal{P}_1 *_N \mathcal{A} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{A}}, \quad \mathcal{P}_2 *_N \mathcal{B} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{B}}, \\
\mathcal{P}_2 *_N \mathcal{C} *_N \mathcal{Q}_2 = \mathcal{S}_{\mathcal{C}}, \quad \mathcal{P}_2 *_N \mathcal{D} *_N \mathcal{Q}_3 = \mathcal{S}_{\mathcal{D}},
\end{aligned}
$$

where $\mathcal{S}_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}}, \mathcal{S}_{\mathcal{C}}, \mathcal{S}_{\mathcal{D}} \in \mathbb{H}^{\frac{m}{2}\times \frac{p}{2}\times \frac{n}{2}\times 2}$ only have entries 0 and 1.

Step 3. Put $\mathcal{S}_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}}, \mathcal{S}_{\mathcal{C}}$, and $\mathcal{S}_{\mathcal{D}}$ together to form the encrypted video $\mathcal{V}_e \in \mathbb{H}^{m\times \frac{p}{2}\times n\times 2}$. Save $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_3$ as keys.

The encryption and the corresponding decryption processes are summarized in Algorithm 1 and Algorithm 2, respectively.

In the experiment, we used the first 20 frames of the color video *rhinos.avi* from MATLAB R2022a as the original video. Each frame is of size 240×320. This color video can be represented as $\mathcal{V} \in \mathbb{H}^{240\times 10\times 320\times 2}$. We performed the Haar discrete wavelet transform and obtained four sub-bands of the same sizes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{H}^{120\times 10\times 160\times 2}$. Then, we applied the simultaneous decomposition for four tensors to obtain the encrypted video and keys. Finally, we used the keys to decrypt the encrypted video.

The results of the encryption and the decryption are shown in Figures 1 and 2. It can be seen that each frame of the encrypted video is a binary image, which only has white and black pixels. Therefore, the information of the original video is highly concealed through the encryption process. Furthermore, the decrypted video is almost identical to the original one, which shows that the decryption effect is also great.

It is worth noting that DWT is also useful in color video compression. Hence, our framework can also realize simultaneously encryption and compression. Moreover, we can perform DWT more times in the first step of encryption to shrink the size of the original video and find a balance between speed and effect.

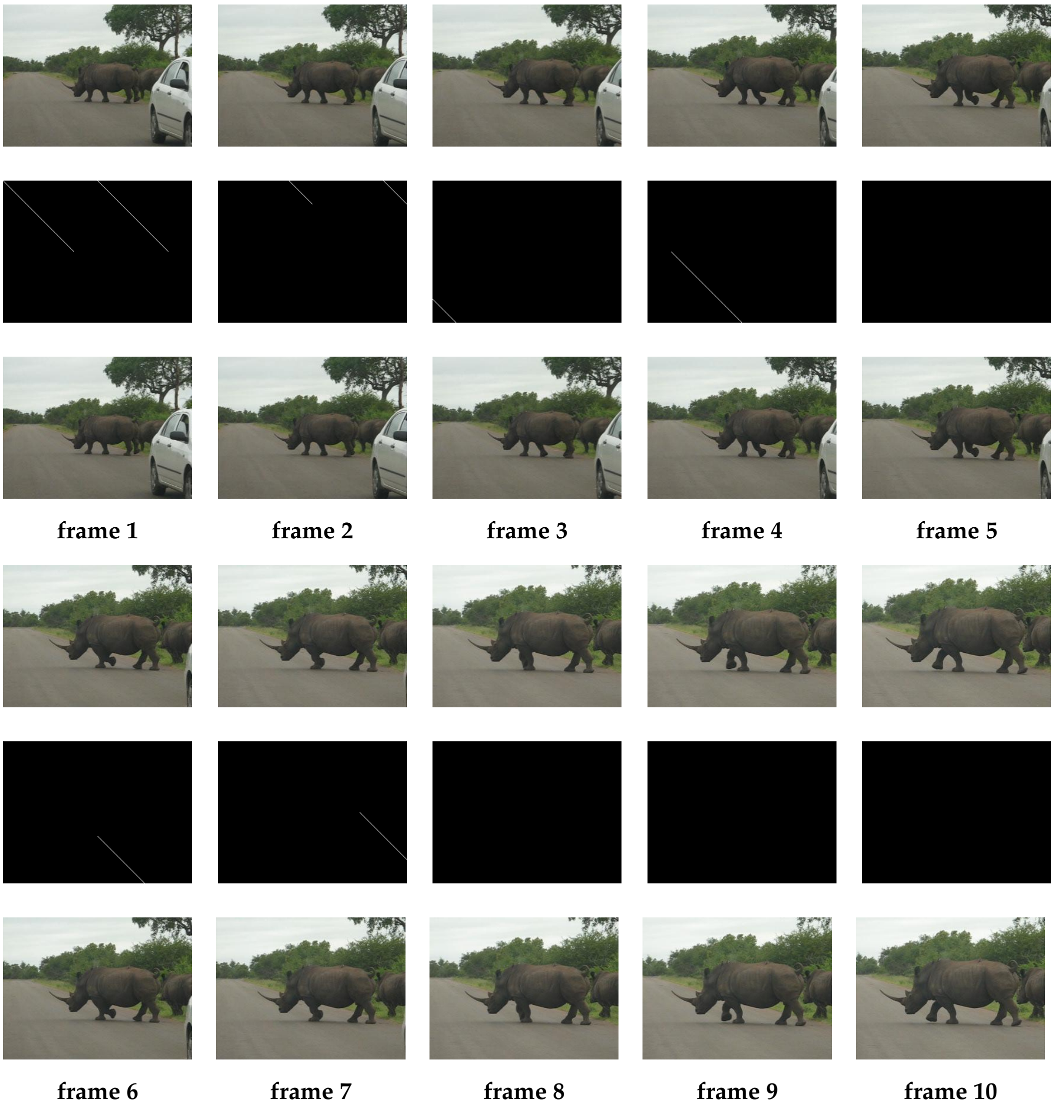

Figure 1. Encryption and decryption results of the video *rhinos.avi* from frame 1 to frame 10. The original frames from the video are listed in the first row and the fourth row. The encryption frames are listed in the second row and the fifth row. The decryption frames are listed in the third row and the sixth row.

Algorithm 1: Encryption process

Input :Original video $\mathcal{V} \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$.
Output:Encrypted video $\mathcal{V}_e \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$ and keys $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$.

1 For $i = 1, 2, \ldots, \frac{p}{2}, j = 1, 2$, perform DWT to each frame $\mathcal{V}_{i,j} \in \mathbb{H}^{m\times n}$ of the original video. The obtained LL, LH, HL and HH sub-bands $\mathcal{A}_{i,j}, \mathcal{B}_{i,j}, \mathcal{C}_{i,j}, \mathcal{D}_{i,j} \in \mathbb{H}^{\frac{m}{2}\times\frac{n}{2}}$ of each frame form four sub-videos $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{H}^{\frac{m}{2}\times\frac{p}{2}\times\frac{n}{2}\times 2}$.

2 By Theorem 1, compute the simultaneous decomposition

$$\mathcal{P}_1 *_N \mathcal{A} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{A}}, \quad \mathcal{P}_2 *_N \mathcal{B} *_N \mathcal{Q}_1 = \mathcal{S}_{\mathcal{B}},$$
$$\mathcal{P}_2 *_N \mathcal{C} *_N \mathcal{Q}_2 = \mathcal{S}_{\mathcal{C}}, \quad \mathcal{P}_2 *_N \mathcal{D} *_N \mathcal{Q}_3 = \mathcal{S}_{\mathcal{D}}.$$

3 Save $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_3$ as keys and the encrypted video $\mathcal{V}_e \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$ is the combination of the equivalence canonical form $\mathcal{S}_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}}, \mathcal{S}_{\mathcal{C}}, \mathcal{S}_{\mathcal{D}} \in \mathbb{H}^{\frac{m}{2}\times\frac{p}{2}\times\frac{n}{2}\times 2}$.

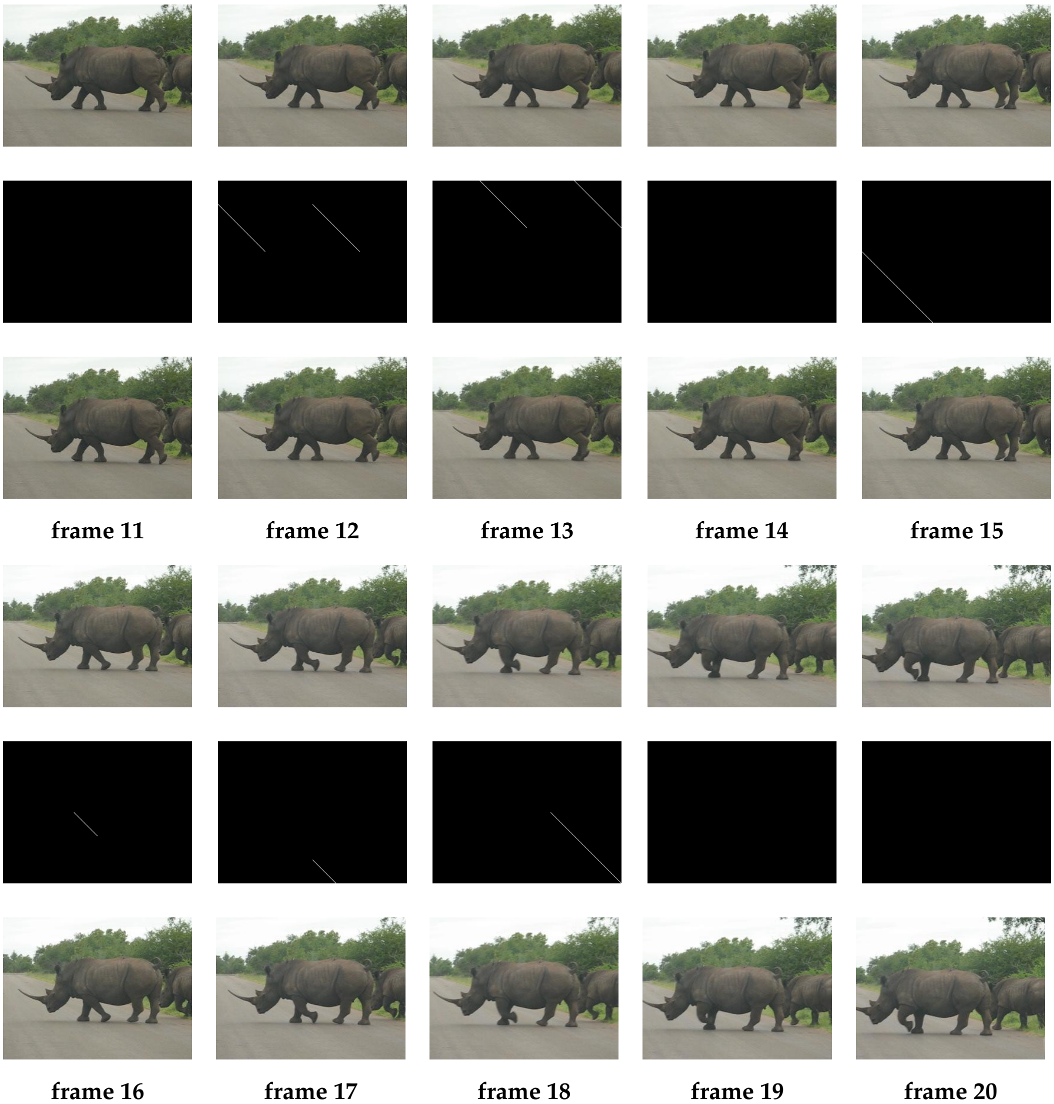

Figure 2. Encryption and decryption results of the video *rhinos.avi* from frame 11 to frame 20. The original frames from the video are listed in the first row and the fourth row. The encryption frames are listed in the second row and the fifth row. The decryption frames are listed in the third row and the sixth row.

Algorithm 2: Decryption process

Input : Encrypted video $\mathcal{V}_e \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$ and keys $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$.
Output: Decrypted video $\mathcal{V}_d \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$.

1 Seperate the encrypted video $\mathcal{V}_e$ into four quarters $\mathcal{S}_\mathcal{A}, \mathcal{S}_\mathcal{B}, \mathcal{S}_\mathcal{C}, \mathcal{S}_\mathcal{D} \in \mathbb{H}^{\frac{m}{2}\times\frac{p}{2}\times\frac{n}{2}\times 2}$.
2 Use the keys to decrypt the four quarters:

$$\mathcal{A} = \mathcal{P}_1^{-1} *_N \mathcal{S}_\mathcal{A} *_N \mathcal{Q}_1^{-1},\ \mathcal{B} = \mathcal{P}_2^{-1} *_N \mathcal{S}_\mathcal{B} *_N \mathcal{Q}_1^{-1},$$
$$\mathcal{C} = \mathcal{P}_2^{-1} *_N \mathcal{S}_\mathcal{C} *_N \mathcal{Q}_2^{-1},\ \mathcal{D} = \mathcal{P}_2^{-1} *_N \mathcal{S}_\mathcal{D} *_N \mathcal{Q}_3^{-1}.$$

3 Apply inverse DWT on $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ to reconstruct the video $\mathcal{V}_d \in \mathbb{H}^{m\times\frac{p}{2}\times n\times 2}$.

5. Conclusions

We have reviewed results in relation to the Einstein product of tensors and provided a more precise definition of transformation (1). We have derived a simultaneous decomposition for a quaternion tensor quaternity in Theorem 1 that brings the given tensors into a canonical form with only 0 and 1 entries. The structure of the canonical form has been discussed in detail in Theorem 2 as well. Furthermore, we have applied the proposed simultaneous decomposition combined with DWT to construct a new framework of color video encryption and decryption. The framework can realize simultaneous encryption and compression.

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