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On the Solution of the Navier Problem for the 3-Harmonic Equation in the Unit Ball

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Abstract: In this paper, based on a new representation of the Green's function of the Navier problem for the 3-harmonic equation in the unit ball, an integral representation of the solution of the corresponding Navier problem is found. Then, for the Navier problem for a homogeneous 3-harmonic equation, the obtained representation of the solution is reduced to a form that does not explicitly contain the Green's function.

Keywords: 3-harmonic equation; biharmonic equation; Navier problem; Green's function

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1. Introduction

The explicit form of Green's functions for elliptic boundary value problems is presented in many papers. For example, in the two-dimensional case, in [1], based on the known harmonic Green's function, Green's functions of various biharmonic problems are presented. The explicit form of the Green's function in a sector for the biharmonic and 3-harmonic equations is found in [2,3], and in [4,5], the explicit form of the Green's functions for the same equations in the unit ball is given. In [6,7], for the m-harmonic equation in a ball, the Green's function of the Dirichlet problem is constructed. In [8], based on the integral representation of functions from the class $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$, where $D \subset \mathbb{R}^n$ is a bounded domain with piecewise-smooth boundary, an integral representation of the solution of the Navier problem [9–11] is also given for the m-harmonic equation in the unit ball, and the Green's function $G^r_{2m}(x, \xi)$ of this problem is presented. In [12], the solvability of the Dirichlet–Riquier problem for the biharmonic equation, which includes the Dirichlet, Neumann, Navier and Riquier–Neumann problems, is investigated, and in [13], the construction of a solution to the Neumann problem can be found. The articles [14–18] study the Dirichlet, Neumann, and Robin problems in the upper half-plane and in unbounded domains. Let us present some works on the construction of the Green's function for various boundary value problems [19–24] and works on the application of Green's functions in problems of mechanics and physics [25–28].

Let Δ be the Laplace operator, and Δ^3 and Δ^2 be its third and second powers, respectively. In this paper, we find a new representation of the Green's function $G^n_6(x, \xi)$ [8] of the Navier problem for the 3-harmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$:

$$\Delta^3 u(x) = f(x), \quad x \in S, \tag{1}$$

$$u|_{\partial S} = \varphi_0, \quad \Delta u\big|_{\partial S} = \varphi_1, \quad \Delta^2 u|_{\partial S} = \varphi_2, \tag{2}$$

and, based on this representation, a solution to the Navier problem for a homogeneous equation is obtained without explicit use of the Green's function, as is performed in [29] for the Dirichlet problem.

The 3-harmonic equation $\Delta^3 u = f(x)$ is an elliptic partial differential equation of the sixth order, which is encountered in areas of continuum mechanics, linear elasticity theory, and viscous flow problems. These are the areas where the obtained results can be applied. For example, a plane slowly rotating the flow of a highly viscous fluid in small cavities is modeled by the 3-harmonic equation for the stream function [30]. An iterative method for solving a boundary value problem for the 3-harmonic equation is proposed in [31]. Note that the accuracy of such an approximate method can be tested using the exact solutions we obtained in this study.

This paper is organized as follows. The introduction presents the problem statement and provides information on known results and methods related to the area under study. Theorems 1 and 2 from Section 2.2 provide the necessary auxiliary results. Section 2 also presents preliminary results on elementary solutions $E_4(x,\xi)$ and $E_6(x,\xi)$ [29]. In Lemmas 1 and 2 from Section 2.3, one property of the Gegenbauer polynomials $C_k^{\lambda}[t]$ [32] is described. Based on this property, in Section 2.4, expansions of the functions $E_4(x,\xi)$ and $G_4^n(x,\xi)$ in the complete system of homogeneous harmonic polynomials $\{H_k^{(i)}(x)\}$ [33] orthogonal on ∂S are found. In Theorem 4 from Section 2.5, a similar expansion is obtained for $E_6(x,\xi)$ [5]. In Section 3, the Green's function $G_6^n(x,\xi)$ is studied. For this purpose, in Theorem 5 from Section 3.1, based on Lemmas 4–9, an expansion of the Green's function $G_6^n(x,\xi)$ in the system $\{H_k^{(i)}(x)\}$ is obtained. Then, in Section 3.2, a function $\mathcal{G}_6^n(x,\xi)$ is defined that has an explicit singularity $E_6(x,\xi)$, and in Theorem 6, it is proven that $G_6^n(x,\xi) = \mathcal{G}_6^n(x,\xi)$ for $n > 6$. In Section 4, the biharmonic Navier problem is studied. In Lemma 13, the Green's function $\mathcal{G}_4^n(x,\xi)$ is introduced, and in Theorem 7, an integral representation of the solution to the Navier problem using $\mathcal{G}_4^n(x,\xi)$ is established. In Theorem 8 from Section 4.2, a representation of the solution of the Navier problem for a homogeneous equation is found. Based on these results, in Section 5, the 3-harmonic Navier problem is investigated. In Theorem 9, for $n > 4$, an integral representation of the solution of the Navier problem through the Green's function $\mathcal{G}_6^n(x,\xi)$ is found. Then, in Theorem 10, the obtained solution for $f = 0$ is rewritten in a different form, without explicitly using the Green's function $\mathcal{G}_6^n(x,\xi)$. Finally, in Theorem 11, it is proved that the function $u(x)$ obtained in Theorem 10 is a solution of the Navier problem for all $n \geq 2$. The directions for further development of this research are outlined in the Conclusion section.

2. Preliminary Results

2.1. Notation and Terminology

For further presentation, some preliminary information is necessary. In [4], an elementary solution $E_4(x,\xi)$ of the biharmonic equation is introduced as follows:

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n > 4,\ n = 3, \\ -\dfrac{1}{4}\ln|x-\xi|, & n = 4, \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi| - 1\big), & n = 2, \end{cases} \tag{3}$$

and in [5], an elementary solution $E_6(x,\xi)$ for the 3-harmonic equation is defined as

$$E_6(x,\xi)=\begin{cases}\dfrac{|x-\xi|^{6-n}}{2\cdot 4(n-2)(n-4)(n-6)}, & n\geq 3, n\neq 4,6,\\ -\dfrac{1}{64}\ln|x-\xi|, & n=6,\\ \dfrac{|x-\xi|^2}{32}\Big(\ln|x-\xi|-\dfrac{3}{4}\Big), & n=4,\\ -\dfrac{|x-\xi|^4}{64}\Big(\ln|x-\xi|-\dfrac{3}{2}\Big), & n=2.\end{cases} \tag{4}$$

In addition, in these papers, the Green's functions $G_4(x,\xi)$ and $G_6(x,\xi)$ corresponding to the Dirichlet problems in S are found. If we denote $E^*_{2k}(x,\xi)=E_{2k}\Big(\frac{x}{|x|},|x|\xi\Big)$, then the Green's function $G_6(x,\xi)$ has the form

$$G_6(x,\xi)=E_6(x,\xi)-E^*_6(x,\xi)-\frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E^*_4(x,\xi)\\ -\frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}E^*_2(x,\xi), \tag{5}$$

where $E_2(x,\xi)=|x-\xi|^{2-n}/(n-2)$ is an elementary solution of the Laplace equation defined by A.V. Bitsadze [34]. In [8] (Lemma 2.1), the elementary functions $E_{2k}(x,\xi)$ are defined for a natural $k\geq 2$ (for $k=2$ and $k=3$, see above), and it is proved that for $x\neq\xi$, the equalities are true:

$$\Delta_\xi E_{2k}(x,\xi)=-E_{2k-2}(x,\xi),\quad \Delta_\xi E_2(x,\xi)=0. \tag{6}$$

Note that the elementary solutions $E_4(x,\xi)$ and $E_6(x,\xi)$ are very similar to the fundamental solutions used by S.L. Sobolev [35], but are not equal to them. In the paper [36] (Theorem 2), it is established (the Green's function is designated as $G^r_4(x,\xi)$, since in some papers the Navier problem is called the Riquier problem [37,38]) that the Green's function of the Navier problem for the biharmonic equation can be found by the equality

$$G^n_4(x,\xi)=\frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy,$$

where $G_2(x,\xi)=E_2(x,\xi)-E^*_2(x,\xi)$ is the well-known Green's function of the Dirichlet problem for the Laplace equation [34], and ω_n is the area of the unit sphere. In [8], the following definition of the Green's function $G^n_{2m}(x,\xi)$ is given.

Definition 1. *A symmetric function of the form $G^n_{2m}(x,\xi)=E_{2m}(x,\xi)+g^n_{2m}(x,\xi)$, where $g^n_{2m}(x,\xi)$ is an m-harmonic function in the variables $x,\xi\in S$, such that the following equalities are true:*

$$G^n_{2m}(x,\xi)|_{\xi\in\partial S}=\Delta_\xi G^n_{2m}(x,\xi)|_{\xi\in\partial S}=\ldots=\Delta^{m-1}_\xi G^n_{2m}(x,\xi)|_{\xi\in\partial S}=0,\quad x\in S,$$

is called the Green's function of the Navier problem for the m-harmonic equation.

2.2. Auxiliary Results

The following statement has been proven ([8], Theorem 4.1).

Theorem 1. *The function $G^n_{2m}(x,\xi)$ for $\xi\neq x\in S$, defined recursively by the equality*

$$G^n_{2m}(x,\xi)=\frac{1}{\omega_n}\int_S G^n_{2m-2}(x,y)G_2(y,\xi)\,dy,\quad m\geq 2,$$

where $G_2^n(x,y) = G_2(x,y)$ is the Green's function of the Dirichlet problem ($m = 1$), is the Green's function of the Navier problem for the m-harmonic equation in S. The Green's function $G_{2m}^r(x,\xi)$ has continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$, $x \neq \xi$ and is symmetric.

Based on Theorem 1, the function $G_6^n(x,\xi)$ has the form

$$G_6^n(x,\xi) = \frac{1}{\omega_n}\int_S G_4^n(x,y)G_2(y,\xi)\,dy. \tag{7}$$

The singularity nature of the function $G_6^n(x,\xi)$ is not clearly visible here; therefore, in accordance with Definition 1, we give another representation of the Green's function.

In [8] (Theorem 4.2), an integral representation of the solution of the Navier problem for the m-harmonic equation is obtained. For $m = 3$, this result can be written as follows:

Theorem 2. *The function $u \in C^6(S) \cap C^5(\bar{S})$, which is a solution to Navier roblem (1)-(2), can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S} -\frac{\partial G_2^n(x,\xi)}{\partial \nu}\varphi_0(\xi) + \frac{\partial G_4^n(x,\xi)}{\partial \nu}\varphi_1(\xi) - \frac{\partial G_6^n(x,\xi)}{\partial \nu}\varphi_2(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_S G_6^n(x,\xi)f(\xi)\,d\xi,\ x \in S. \tag{8}$$

If $\varphi_0 \in C(\partial S)$, $\varphi_1, \varphi_2 \in C^{1+\varepsilon}(\partial S)$ $(\varepsilon > 0)$ and $f \in C^1(\bar{S})$, then the function $u(x)$ determined from (8) is a solution of Navier roblem (1)-(2).

2.3. Elementary Solutions $E_4(x,\xi)$ and $E_6(x,\xi)$

We investigate the elementary solution $E_6(x,\xi)$ for $n > 6$. Denote by $C_k^\lambda[t]$ the Gegenbauer polynomial of degree k. It can be represented in the form ([32], p. 177)

$$C_k^\lambda[t] = \sum_{m=0}^{[k/2]} \frac{(-1)^m(\lambda)_{k-m}}{m!(k-2m)!}(2t)^{k-2m}, \tag{9}$$

where $(\lambda)_k = \lambda(\lambda+1)\dots(\lambda+k-1)$ aside from $(\lambda)_0 = 1$. It is not difficult to calculate that $C_0^\lambda[t] = 1$ and

$$C_1^\lambda[t] = \sum_{m=0}^{0} \frac{(-1)^0(\lambda)_1}{0!1!}(2t)^1 = 2\lambda t.$$

In what follows, we assume that $C_k^\lambda[t] = 0$ if $k < 0$. For $|\xi| = 1$, $|x| < 1$, and $m > 2$, the following representation is true [4]:

$$|x-\xi|^{2-m} = \sum_{k=0}^{\infty} C_k^{m/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k,$$

where (x,ξ) denotes the scalar product in $\mathbb{R}^n$. If we take here $m = n-4$, then for $n > 6$, we obtain

$$|x-\xi|^{6-n} = \sum_{k=0}^{\infty} C_k^{n/2-3}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k. \tag{10}$$

Lemma 1. *If $n > 6$ and $|\xi| = 1$, the following function $C_k^{n/2-3}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k$ is a 3-harmonic polynomial of degree $k \in \mathbb{N}_0$ on x.*

Proof. Using the representation (9) of the Gegenbauer polynomials, we can write

$$C_k^{n/2-3}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=\sum_{m=0}^{[k/2]}\frac{(-1)^m(n/2-3)_{k-m}}{m!(k-2m)!}|x|^{2m}(2(x,\xi))^{k-2m}=$$
$$=\sum_{m=0}^{[k/2]}\frac{(-1)^m(n-6,2)_{k-m}}{(2m)!!(k-2m)!}|x|^{2m}(x,\xi)^{k-2m},$$

where $(n,2)_k=n(n+2)\dots(n+2k-2)$ is the generalized Pochhammer symbol with the convention $(n,2)_0=1$. From here, we can see that it is a polynomial of degree k with respect to x. Using the equalities

$$\Delta(x,\xi)^m=m(m-1)|\xi|^2(x,\xi)^{m-2},\quad \Delta|x|^{2l}=2l(2l+n-2)|x|^{2l-2},$$

it is not difficult to see that

$$\Delta|x|^{2l}(x,\xi)^m=m(m-1)|\xi|^2|x|^{2l}(x,\xi)^{m-2}+2l(2l+n-2+2m)|x|^{2l-2}(x,\xi)^m.$$

It follows that

$$\Delta|x|^{2m}(x,\xi)^{k-2m}=(k-2m)(k-2m-1)|\xi|^2|x|^{2m}(x,\xi)^{k-2m-2}$$
$$+2m(2k-2m+n-2)|x|^{2m-2}(x,\xi)^{k-2m}.$$

Taking into account the obtained equality, we have

$$\begin{aligned}\Delta C_k^{n/2-3}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=&\Delta\sum_{m=0}^{[k/2]}\frac{(-1)^m(n-6,2)_{k-m}}{(2m)!!(k-2m)!}|x|^{2m}(x,\xi)^{k-2m}\\
=&\sum_{m=0}^{[k/2-1]}\frac{(-1)^m(n-6,2)_{k-m}}{(2m)!!(k-2m-2)!}|\xi|^2|x|^{2m}(x,\xi)^{k-2m-2}\\
&+\sum_{m=1}^{[k/2]}\frac{(-1)^m(n-6,2)_{k-m+1}}{(2m-2)!!(k-2m)!}|x|^{2m-2}(x,\xi)^{k-2m}\\
&+4\sum_{m=1}^{[k/2]}\frac{(-1)^m(n-6,2)_{k-m}}{(2m-2)!!(k-2m)!}|x|^{2m-2}(x,\xi)^{k-2m}\\
=&(|\xi|^2-1)\sum_{m=0}^{[k/2-1]}\frac{(-1)^m(n-6,2)_{k-m}}{(2m)!!(k-2m-2)!}|x|^{2m}(x,\xi)^{k-2m-2}\\
&-4\sum_{m=0}^{[k/2-1]}\frac{(-1)^m(n-6,2)_{k-m-1}}{(2m)!!(k-2m-2)!}|x|^{2m}(x,\xi)^{k-2m-2}\\
=&-4(n-6)\sum_{m=0}^{[k/2-1]}\frac{(-1)^m(n-4,2)_{k-m-2}}{(2m)!!(k-2m-2)!}|x|^{2m}(x,\xi)^{k-2m-2}\\
=&-4(n-6)C_{k-2}^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^{k-2}.\end{aligned}$$

To obtain the last line of these equalities, the following equality is used:

$$C_k^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=\sum_{m=0}^{[k/2]}\frac{(-1)^m(n-4,2)_{k-m}}{(2m)!!(k-2m)!}|x|^{2m}(x,\xi)^{k-2m},$$

which follows from [4] (Lemma 3.3). Since for $|\xi| = 1$, the polynomial $C_k^{n/2-2}\big[(x/|x|, \xi)\big]|x|^k$ is biharmonic ([4], Lemma 3.3), then the polynomial $C_k^{n/2-3}\big[(x/|x|, \xi)\big]|x|^k$ is 3-harmonic. Thus, the lemma is proven. □

Lemma 2. *Let $n > 6$ and $k \in \mathbb{N}_0$. The following equality holds:*

$$C_k^{n/2-3}[t] = \frac{n-6}{2k+n-6}\big(C_k^{n/2-2}[t] - C_{k-2}^{n/2-2}[t]\big). \tag{11}$$

Proof. First, suppose that $k \geq 2$. The following equalities are valid:

$$\begin{aligned}
&C_k^{n/2-2}[t] - C_{k-2}^{n/2-2}[t] \\
=&\sum_{m=0}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m} - \sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-4,2)_{k-m-2}}{(2m)!!(k-2m-2)!} t^{k-2m-2} \\
=&\sum_{m=0}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m} + \sum_{m=1}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m-1}}{(2m-2)!!(k-2m)!} t^{k-2m} \\
=&\frac{2k+n-6}{n-6} \sum_{m=1}^{[k/2]} \frac{(-1)^m (n-6,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m} + \frac{(n-4,2)_k}{k!} t^k.
\end{aligned}$$

The last term can be transformed to the form

$$\frac{(n-4,2)_k}{k!} t^k = \frac{2k+n-6}{n-6} \frac{(n-6,2)_k}{k!} t^k,$$

which means

$$\begin{aligned}
C_k^{n/2-2}[t] - C_{k-2}^{n/2-2}[t] =& \frac{2k+n-6}{n-6} \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-6,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m} \\
=& \frac{2k+n-6}{n-6} C_k^{n/2-3}[t].
\end{aligned}$$

This implies equality (11). If we assume, as noted above, that $C_{-2}^{n/2-1}[t] = 0$ and $C_{-1}^{n/2-1}[t] = 0$, then the equality (11) is also true for $k = 0$ (the values $C_0^\lambda[t]$ and $C_1^\lambda[t]$ have been calculated above): $C_0^{n/2-3}[t] = 1 = \frac{n-6}{n-6} C_0^{n/2-2}[t]$. For $k = 1$, (11) is also true:

$$\begin{aligned}
C_1^{n/2-3}[t] =& 2(n/2-3)t = (n-6)t = \frac{n-6}{2+n-6}(n-4)t \\
=& \frac{n-6}{2+n-6} 2(n/2-2)t = \frac{n-6}{2+n-6} C_1^{n/2-2}[t].
\end{aligned}$$

The lemma is proven. □

2.4. Expansions of Functions $E_4(x, \xi)$ and $G_4^n(x, \xi)$

Let $\big\{H_k^{(i)}(x) : i = 1, \ldots, h_k,\ k \in \mathbb{N}_0\big\}$ be a complete system of homogeneous harmonic polynomials of degree $k \in \mathbb{N}_0$ that are orthogonal on ∂S (see, for example, [33]), and normalized so that $\int_{\partial S} \big(H_k^{(i)}(\xi)\big)^2 ds_\xi = \omega_n$, where $h_k = \frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}$ for $n > 2$ ($h_k = 2$ for $n = 2$) is the dimension of the basis of homogeneous harmonic polynomials of degree k. Consider the following symmetric harmonic polynomials:

$$\mathcal{H}_k(x, \xi) = \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad k \in \mathbb{N}_0.$$

Note that for $k = 0$, we have $h_0 = 1$ and $H_0^{(1)}(x) = 1$, and also $\mathcal{H}_k(x, \xi) = \mathcal{H}_k(\xi, x)$.

1. Expansion of $E_4(x,\xi)$. In [4] (Theorem 4.1), for $|\xi| < |x| < 1$, the following representations of elementary solutions are obtained:

$$E_2(x,\xi) = \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)}}{2k+n-2} \mathcal{H}_k(x,\xi), \quad (n>2), \tag{12}$$

$$E_4(x,\xi) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)}}{2k+n-2} \left(\frac{|x|^2}{2k+n-4} - \frac{|\xi|^2}{2k+n} \right) \mathcal{H}_k(x,\xi), \quad (n>4). \tag{13}$$

Moreover, in the case of $|x| < |\xi| < 1$, the variables x and ξ must be swapped under the summation sign. The series written above converge uniformly in x and ξ when $|\xi| < a < |x|$. Let us denote

$$\Phi_{k,i}^{(2)}(x,\xi) = \frac{|x|^{-(2k+n-2)} H_k^{(i)}(x)}{2k+n-2} \cdot H_k^{(i)}(\xi).$$

Remark 1. *In the notation $\Phi_{k,i}^{(2)}(x,\xi)$, the harmonic polynomials $H_k^{(i)}(\xi)$ from the complete system of orthogonal harmonic polynomials are multiplied by the Kelvin transform of the same polynomials $K[H_k^{(i)}](x) = |x|^{-(2k+n-2)} H_k^{(i)}(x)$, but with respect to the variable x. In the representation of the function $E_2(x,\xi)$ for $|\xi| < |x|$, the functions $\Phi_{k,i}^{(2)}(x,\xi)$ are summed over all values of $k \in \mathbb{N}_0$ and $i = 1, \ldots, h_k$, i.e.,*

$$E_2(x,\xi) = \sum_{k,i} \Phi_{k,i}^{(2)}(x,\xi).$$

It is clear that $\Phi_{0,1}^{(2)}(x,\xi) = E_2(x,0) \cdot 1$. In the case of $n = 2$, the value of $\Phi_{0,1}^{(2)}(x,\xi)$ is undefined. However, as shown in [4] (Lemma 3.1), instead of this term, we should take the function $-\ln|x| = E_2(x,0)$ and set $\Phi_{0,1}^{(2)}(x,\xi) \equiv -\ln|x| \cdot 1$.

Next, we consider the structure of the representation of the function $E_4(x,\xi)$ from (13). If we denote

$$\Phi_{k,i}^{(4,1)}(x,\xi) = \frac{|x|^{-(2k+n-4)} H_k^{(i)}(x) \cdot H_k^{(i)}(\xi)}{2(2k+n-2)(2k+n-4)},$$

$$\Phi_{k,i}^{(4,2)}(x,\xi) = \frac{|x|^{-(2k+n-2)} H_k^{(i)}(x) \cdot |\xi|^2 H_k^{(i)}(\xi)}{2(2k+n-2)(2k+n)},$$

then for $|\xi| < |x|$ and $n > 4$, we obtain the equality

$$E_4(x,\xi) = \sum_{k,i} \left(\Phi_{k,i}^{(4,1)}(x,\xi) - \Phi_{k,i}^{(4,2)}(x,\xi) \right).$$

Remark 2. *It can be noted that all the functions $\Phi_{k,i}^{(4,j)}(x,\xi)$ have singularities at $x = 0$ and they are biharmonic in $x \neq 0$, and with respect to the variable ξ, they are biharmonic polynomials and*

$$\begin{aligned} &\Delta_x \Phi_{k,i}^{(4,1)}(x,\xi) = -\Phi_{k,i}^{(2)}(x,\xi), \quad \Delta_x \Phi_{k,i}^{(4,2)}(x,\xi) = 0, \\ &\Delta_\xi \Phi_{k,i}^{(4,1)}(x,\xi) = 0, \qquad\qquad \Delta_\xi \Phi_{k,i}^{(4,2)}(x,\xi) = \Phi_{k,i}^{(2)}(x,\xi). \end{aligned} \tag{14}$$

It is clear that the function $\Phi_{k,i}^{(4,1)}(x,\xi)$ is biharmonic in x and harmonic in ξ, and the function $\Phi_{k,i}^{(4,2)}(x,\xi)$ is harmonic in x and biharmonic in ξ; thus, the equality (14) is satisfied. In addition, $\Phi_{0,1}^{(4,1)}(x,0) = E_4(x,0) \cdot 1$, $\Phi_{0,1}^{(4,2)}(x,\xi) = -\Delta_x E_4(x,0)|\xi|^2 = E_2(x,0)|\xi|^2$.

If $n = 4$, then the function $\Phi_{0,1}^{(4,1)}(x,0)$ is not defined and it is necessary, according to the second line of (3), to set $\Phi_{0,1}^{(4,1)}(x,\xi) \equiv E_4(x,0) = -\frac{1}{4}\ln|x| \cdot 1$. In this case, all equalities (14) are preserved. It then remains only to check the case when $k = 0$:

$$\Delta_x \Phi_{0,1}^{(4,1)}(x,\xi) = -\Delta_x \frac{1}{4}\ln|x| = \frac{|x|^{-2}}{2} = \Phi_{0,1}^{(2)}(x,\xi),$$

where it is taken into account that $H_0^{(1)} = 1$. The function $\Phi_{0,1}^{(4,2)}(x,\xi)$ does not change.

Hypothesis 1. *For $n = 4$, the following equality holds:*

$$E_4(x,\xi) = -\frac{1}{4}\ln|x| \cdot 1 - \frac{1}{2}|x|^{-2}\frac{1}{8}|\xi|^2 + \frac{1}{2}\sum_{k=1}^{\infty}\frac{|x|^{-(2k+2)}}{2k+2}\left(\frac{|x|^2}{2k} - \frac{|\xi|^2}{2k+4}\right)\mathcal{H}_k(x,\xi).$$

Note that the functions $\Phi_{k,i}^{(4,j)}(x,\xi)$ and $\Phi_{k,i}^{(2)}(x,\xi)$ also depend on the dimension n, but in order not to complicate the calculations, this dependence is not explicitly indicated, but it must be kept in mind. Similarly, it is also easy to assume a formula for representing $E_4(x,\xi)$ for $n = 2$. In this case, it is necessary to define the functions $\Phi_{0,1}^{(4,1)}(x,\xi)$, $\Phi_{0,1}^{(4,2)}(x,\xi)$ and $\Phi_{1,i}^{(4,1)}(x,\xi)$. From the third line of (3), we find $\Phi_{0,1}^{(4,1)}(x,\xi) = E_4(x,0) = |x|^2\frac{\ln|x|-1}{4}$, since by (6), we have $\Delta_x\Phi_{0,1}^{(4,1)}(x,\xi) = -\ln|x| = \Phi_{0,1}^{(2)}(x,\xi)$, where $\Phi_{0,1}^{(2)}(x,\xi)$ was already defined above. Obviously, $\Phi_{0,1}^{(4,2)}(x,\xi) = -\ln|x|$ as in the case of $E_2(x,\xi)$. To find $\Phi_{1,i}^{(4,1)}(x,\xi)$, we note that $\Phi_{1,i}^{(2)}(x,\xi) = \frac{1}{2}|x|^{-2}H_1^{(i)}(x)H_1^{(i)}(\xi)$. Further, taking into account the homogeneity of the harmonic polynomials $H_1^{(i)}(x)$, we write $\Delta_x \ln|x| H_1^{(i)}(x) = 2|x|^{-2}H_1^{(i)}(x)$, and therefore,

$$\Delta_x \frac{1}{4}\ln|x|H_1^{(i)}(x)H_1^{(i)}(\xi) = \frac{1}{2}|x|^{-2}H_1^{(i)}(x)H_1^{(i)}(\xi) = \Phi_{1,i}^{(2)}(x,\xi).$$

Therefore, to satisfy the equality (14) for $k = 1$, we should set

$$\Phi_{1,i}^{(4,1)}(x,\xi) = -\frac{1}{4}\ln|x|H_1^{(i)}(x)H_1^{(i)}(\xi).$$

Hypothesis 2. *For $n = 2$, the following equality holds:*

$$\begin{aligned} E_4(x,\xi) =& |x|^2\frac{\ln|x|-1}{4}\cdot 1 + \ln|x|\cdot\frac{1}{4}|\xi|^2 - \left(\frac{1}{4}\ln|x| + \frac{|x|^{-2}|\xi|^2}{16}\right)\mathcal{H}_1(x,\xi) \\ &+\frac{1}{2}\sum_{k=2}^{\infty}\frac{|x|^{-2k}}{2k}\left(\frac{|x|^2}{2k-2} - \frac{|\xi|^2}{2k+2}\right)\mathcal{H}_k(x,\xi). \end{aligned}$$

If $|x| < |\xi| < 1$, then the variables ξ and x on the right must be swapped.

2. Expansion of $G_4^n(x,\xi)$. From (12) for $n > 2$ and $|\xi| < |x| < 1$, the following equalities hold:

$$E_2(x,\xi) = \sum_{k=0}^{\infty}\frac{|x|^{-(\mathbf{k}-2)}}{\mathbf{k}-2}\mathcal{H}_k(x,\xi), \quad E_2^*(x,\xi) = \sum_{k=0}^{\infty}\frac{1}{\mathbf{k}-2}\mathcal{H}_k(x,\xi), \tag{15}$$

and therefore, it is not difficult to obtain equality:

$$G_2(x,\xi) = E_2(x,\xi) - E_2^*(x,\xi) = \sum_{k=0}^{\infty}\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}-2}\mathcal{H}_k(x,\xi).$$

Further, from Theorem 1, it follows that the Green's function $G_4^n(x,\xi)$ is defined as

$$G_4^n(x,\xi) = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy.$$

Based on the decomposition (12), taking into account the notation $\mathbf{k} = 2k+n$, which allows us to write formulas in a more concise form, it is easy to prove the following statement.

Theorem 3. *For $n > 4$ and $|\xi| < |x| < 1$, the expansion below holds*

$$G_4^n(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)} - \frac{|x|^2-1}{(\mathbf{k}-2)\mathbf{k}} - \frac{(|x|^{-(\mathbf{k}-2)}-1)}{(\mathbf{k}-2)\mathbf{k}}|\xi|^2\right)\mathcal{H}_k(x,\xi), \tag{16}$$

where $\mathbf{k} = 2k+n$ is applied to shorten the notation. If $|x| < |\xi| < 1$, then the variables ξ and x under the sum sign must be swapped.

The representation (16) is useful for finding integrals of the form $\int_S G_4^n(x,\xi)|\xi|^m H(\xi)\,d\xi$ with the function $G_4^n(x,\xi)$ as the kernel, where $H(\xi)$ is some harmonic polynomial [36].

2.5. Expansion of the Elementary Solution $E_6(x,\xi)$

Now, we are ready to find the expansion of the function $E_6(x,\xi)$ in the complete system of harmonic polynomials: $\{H_k^{(i)}(x) : i = 1,\dots,h_k,\ k \in \mathbb{N}_0\}$.

Theorem 4. *For the function $E_6(x,\xi)$ with $n > 6$ and $|x| < |\xi|$, the following expansion holds:*

$$E_6(x,\xi) = \frac{1}{8}\sum_{k=0}^{\infty}\frac{|\xi|^{-(\mathbf{k}-2)}}{\mathbf{k}-2}\left(\frac{|\xi|^4}{(\mathbf{k}-6)(\mathbf{k}-4)} - \frac{2|\xi|^2|x|^2}{(\mathbf{k}-4)\mathbf{k}} + \frac{|x|^4}{\mathbf{k}(\mathbf{k}+2)}\right)\mathcal{H}_k(x,\xi). \tag{17}$$

If $|\xi| < |x|$, then the corresponding formula can be obtained from (17) by interchanging the variables x and ξ under the summation sign on the right.

Proof. Consider the 3-harmonic polynomial $C_k^{n/2-3}\left[(x/|x|,\xi)\right]|x|^k$ from Lemma 1. We transform it using (11) from Lemma 2:

$$C_k^{n/2-3}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \frac{n-6}{2k+n-6}\left(C_k^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k - |x|^2 C_{k-2}^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^{k-2}\right). \tag{18}$$

To perform this, first, we transform $C_k^{n/2-2}\left[(x/|x|,\xi)\right]|x|^k$. We use the equality from [4] (Lemma 3.4)

$$C_k^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \frac{n-4}{2k+n-4}\left(C_k^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k - |x|^2 C_{k-2}^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^{k-2}\right)$$

and [4] (Theorem 4.1)

$$C_k^{n/2-1}\left[(x/|x|,\xi)\right]|x|^k = \frac{n-2}{2k+n-2}\mathcal{H}_k(x,\xi),$$

which are true for $|\xi| = 1$ and $|x| < 1$. We then have

$$C_k^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \frac{(n-4,2)_2}{2k+n-4}\left(\frac{\mathcal{H}_k(x,\xi)}{2k+n-2} - |x|^2\frac{\mathcal{H}_{k-2}(x,\xi)}{2k+n-6}\right).$$

Now, we can transform the equality (18) to the form

$$C_k^{n/2-3}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=\frac{n-6}{2k+n-6}\frac{(n-4,2)_2}{2k+n-4}\Big(\frac{\mathcal{H}_k(x,\xi)}{2k+n-2}-|x|^2\frac{\mathcal{H}_{k-2}(x,\xi)}{2k+n-6}\Big)$$
$$-\frac{n-6}{2k+n-6}\frac{(n-4,2)_2}{2k+n-8}\Big(|x|^2\frac{\mathcal{H}_{k-2}(x,\xi)}{2k+n-6}-|x|^4\frac{\mathcal{H}_{k-4}(x,\xi)}{2k+n-10}\Big).$$

For brevity, we denote $\mathbf{k}=2k+n$. Let us calculate the coefficient at $|x|^2\mathcal{H}_{k-2}(x,\xi)$:

$$-\frac{(n-6,2)_3}{(\mathbf{k}-6)^2}\Big(\frac{1}{\mathbf{k}-4}+\frac{1}{\mathbf{k}-8}\Big)=-\frac{2(n-6,2)_3}{(\mathbf{k}-8,2)_3}$$

and so, in the end, we have

$$C_k^{n/2-3}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=\frac{(n-6,2)_3}{(\mathbf{k}-6,2)_3}\mathcal{H}_k-\frac{2(n-6,2)_3}{(\mathbf{k}-8,2)_3}|x|^2\mathcal{H}_{k-2}+\frac{(n-6,2)_3}{(\mathbf{k}-10,2)_3}|x|^4\mathcal{H}_{k-4},$$

where it is necessary to take into account that $\mathcal{H}_k(x,\xi)=0$ when $k<0$. Using the obtained equality, from (10), when $|\xi|=1$ and $|x|<1$, we obtain

$$|x-\xi|^{6-n}=\sum_{k=0}^{\infty}C_k^{n/2-3}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k=\sum_{k=0}^{\infty}\frac{(n-6,2)_3}{(\mathbf{k}-6,2)_3}\mathcal{H}_k(x,\xi)$$
$$-|x|^2\sum_{k=2}^{\infty}\frac{2(n-6,2)_3}{(\mathbf{k}-8,2)_3}\mathcal{H}_{k-2}(x,\xi)+|x|^4\sum_{k=4}^{\infty}\frac{(n-6,2)_3}{(\mathbf{k}-10,2)_3}\mathcal{H}_{k-4}(x,\xi)$$
$$=(n-6,2)_3\sum_{k=0}^{\infty}\Big(\frac{1}{(\mathbf{k}-6,2)_3}-\frac{2|x|^2}{(\mathbf{k}-4,2)_3}+\frac{|x|^4}{(\mathbf{k}-2,2)_3}\Big)\mathcal{H}_k(x,\xi).$$

Let us remove the constraint $|\xi|=1$, but assume that $|x|<|\xi|$. In this case, $|x/|\xi||<1$ and $|\xi/|\xi||=1$, and therefore,

$$|x-\xi|^{6-n}=|\xi|^{6-n}\Big|\frac{x}{|\xi|}-\frac{\xi}{|\xi|}\Big|^{6-n}$$
$$=(n-6,2)_3\sum_{k=0}^{\infty}|\xi|^{-(2k+n-6)}\Big(\frac{1}{(\mathbf{k}-6,2)_3}-\frac{2|x|^2/|\xi|^2}{(\mathbf{k}-4,2)_3}+\frac{|x|^4/|\xi|^4}{(\mathbf{k}-2,2)_3}\Big)\mathcal{H}_k(x,\xi)$$
$$=2\cdot4(n-6,2)_3\frac{1}{8}\sum_{k=0}^{\infty}|\xi|^{-(2k+n-2)}\Big(\frac{|\xi|^4}{(\mathbf{k}-6,2)_3}-\frac{2|x|^2|\xi|^2}{(\mathbf{k}-4,2)_3}+\frac{|x|^4}{(\mathbf{k}-2,2)_3}\Big)\mathcal{H}_k(x,\xi).$$

Hence, taking into account the definition (4) of the function $E_6(x,\xi)$ for $n>6$ and the notation $(\mathbf{k}-6,2)_3=(\mathbf{k}-6)(\mathbf{k}-4)(\mathbf{k}-2)$, $\mathbf{k}=2k+n$, we obtain the representation (17).

For the final proof of this theorem, we note that in the following equality (see [4], Theorem 4.1):

$$C_k^{n/2-1}\big[(x,\xi)\big]=\frac{n-2}{2k+n-2}\mathcal{H}_k(x,\xi),\quad |\xi|=|x|=1,$$

on the basis of which the proof is carried out, variables x and ξ are equal in rights. Therefore, if we swap them, then in deriving the formula for $E_6(x,\xi)$ on condition $|\xi|<|x|$, we obtain (17), in which the variables x and ξ are interchanged. The theorem is thus proven. □

3. Green's Function $G_6^n(x,\xi)$

Green's function $G_6^n(x,\xi)$ was defined earlier in (7). To transform it, we prove the following statement.

Lemma 3. *The polynomials $\mathcal{H}_k(x,\xi)$ have the property*

$$\frac{1}{\omega_n}\int_{\partial S}\mathcal{H}_k(x,y)\mathcal{H}_m(y,\xi)\,ds_y=\begin{cases}0, & k\neq m,\\ \mathcal{H}_k(x,\xi), & k=m.\end{cases}$$

Proof. Indeed, the equalities below are true

$$\begin{aligned}\frac{1}{\omega_n}\int_{\partial S}\mathcal{H}_k(x,y)\mathcal{H}_m(y,\xi)\,ds_y&=\frac{1}{\omega_n}\int_{\partial S}\sum_{i=1}^{h_k}H_k^{(i)}(x)H_k^{(i)}(y)\sum_{j=1}^{h_m}H_m^{(j)}(y)H_m^{(j)}(\xi)\,ds_y\\&=\sum_{i=1}^{h_k}\sum_{j=1}^{h_m}H_k^{(i)}(x)H_m^{(j)}(\xi)\frac{1}{\omega_n}\int_{\partial S}H_k^{(i)}(y)H_m^{(j)}(y)ds_y\\&=\begin{cases}0, & k\neq m,\\ \sum_{i=1}^{h_k}H_k^{(i)}(x)H_k^{(i)}(\xi)=\mathcal{H}_k(x,\xi), & k=m,\end{cases}\end{aligned}$$

since, by orthogonality on ∂S of the system of polynomials $\left\{H_k^{(i)}(x):i=1,\dots,h_k,\,k\in\mathbb{N}_0\right\}$, we have

$$\frac{1}{\omega_n}\int_{\partial S}H_k^{(i)}(y)H_m^{(j)}(y)ds_y=\begin{cases}1, & k=m,i=j,\\ 0, & \text{otherwise.}\end{cases}$$

The lemma is proved. □

3.1. Expansion of $G_6^n(x,\xi)$

Let us formulate a theorem on the expansion of $G_6^n(x,\xi)$ into a series.

Theorem 5. *For $n>6$ and $|\xi|\neq|x|<1$, the representation is valid:*

$$\begin{aligned}G_6^n(x,\xi)&=E_6(x,\xi)-E_6^*(x,\xi)\\&+\frac{(|x|^4-1)(|\xi|^4-1)}{8}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2,2)_3}-\frac{(|x|^2-1)(|\xi|^2-1)}{2}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2},\end{aligned}\tag{19}$$

where $\mathbf{k}=2k+n$.

Proof. Let $|\xi|<|x|<1$; then, since the integral from (7) has integrable singularities at $y=\xi$ and $y=x$, we write

$$\begin{aligned}G_6^n(x,\xi)=\frac{1}{\omega_n}\int_S G_4^n(x,y)G_2(y,\xi)\,dy&=\left(\int_0^{|\xi|}+\int_{|\xi|}^{|x|}+\int_{|x|}^1\right)\rho^{n-1}\,d\rho\\&\times\frac{1}{\omega_n}\int_{\partial S}G_4^n(x,\rho y)G_2(\rho y,\xi)\,ds_y\equiv I_1+I_2+I_3.\end{aligned}$$

Taking into account the equality $G_2(y,\xi)=E_2(y,\xi)-E_2^*(y,\xi)$, we divide each of the integrals I_j into two more parts. We denote by $I_{j,1}$ and $I_{j,2}$ the parts of the integral I_j in which instead of $G_2(y,\xi)$, we first take the function $E_2(y,\xi)$, and then the function $-E_2^*(y,\xi)$, respectively. We interrupt the proof of the theorem and present auxiliary lemmas. □

To prove the following lemmas, we use (15), (16), and Lemma 3. To perform this, we note that the power series in these formulas converge uniformly in x and ξ for $|\xi|<a<|x|$ ([4], Theorem 4.1), and therefore, they can be integrated term by term. In all the lemmas in the section below, we assume that $n>6$ and $|\xi|<|x|<1$.

Lemma 4. *For the function $I_{1,1}(x,\xi)$, the following representation is valid:*

$$I_{1,1}(x,\xi)=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)\mathbf{k}}|\xi|^2+\frac{|x|^2-1}{\mathbf{k}^2}|\xi|^2-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}(\mathbf{k}+2)}|\xi|^4\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.$$

Proof. First, by Lemma 3, we calculate the inner integral in $I_{1,1}(x,\xi)$. Since $|y|<|\xi|<|x|$, then

$$\begin{aligned}\frac{1}{\omega_n}\int_{\partial S}G_4^n(x,\rho y)E_2(\rho y,\xi)ds_y&=\frac{1}{\omega_n}\int_{\partial S}\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)}\right.\\&\left.+\frac{|x|^2-1}{(\mathbf{k}-2)\mathbf{k}}-\frac{(|x|^{-(\mathbf{k}-2)}-1)}{(\mathbf{k}-2)\mathbf{k}}\rho^2\right)\rho^k\,\mathcal{H}_k(x,y)\sum_{m=0}^{\infty}\frac{|\xi|^{-(\mathbf{m}-2)}}{\mathbf{m}-2}\rho^m\,\mathcal{H}_m(y,\xi)\\&=\frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{\mathbf{k}-4}+\frac{|x|^2-1}{\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}}\rho^2\right)\frac{|\xi|^{-(\mathbf{k}-2)}}{(\mathbf{k}-2)^2}\mathcal{H}_k(x,\xi).\end{aligned}$$

Then, for the entire integral $I_{1,1}(x,\xi)$, we have

$$\begin{aligned}I_{1,1}&=\sum_{k=0}^{\infty}\int_0^{|\xi|}\frac{\rho^{\mathbf{k}-1}}{2}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}}\rho^2\right)d\rho\frac{|\xi|^{-(\mathbf{k}-2)}}{(\mathbf{k}-2)^2}\times\\\mathcal{H}_k(x,\xi)&=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}|\xi|^2+\frac{|x|^2-1}{\mathbf{k}}|\xi|^2-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}+2)}|\xi|^4\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2\mathbf{k}},\end{aligned}$$

from which the assertion of the lemma follows. The lemma is proved. □

Lemma 5. *For the function $I_{1,2}(x,\xi)$, the following representation is valid:*

$$\begin{aligned}&I_{1,2}(x,\xi)\\&\quad=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)\mathbf{k}}|\xi|^{\mathbf{k}}+\frac{|x|^2-1}{\mathbf{k}^2}|\xi|^{\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}(\mathbf{k}+2)}|\xi|^{\mathbf{k}+2}\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.\end{aligned}$$

Proof. Similar to the proof of Lemma 4 for the inner integral in $I_{1,2}$, we write

$$\begin{aligned}&-\frac{1}{\omega_n}\int_{\partial S}G_4^n(x,\rho y)E_2^*(\rho y,\xi)ds_y\\&\quad=-\frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)}+\frac{|x|^2-1}{(\mathbf{k}-2)\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)\mathbf{k}}\rho^2\right)\frac{1}{\mathbf{k}-2}\mathcal{H}_k(x,\xi).\end{aligned}$$

Then, for the entire integral $I_{1,2}(x,\xi)$, we have

$$\begin{aligned}&I_{1,2}(x,\xi)\\&\quad=-\sum_{k=0}^{\infty}\int_0^{|\xi|}\frac{\rho^{\mathbf{k}-1}}{2}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}}\rho^2\right)d\rho\,\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}\\&\quad=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)\mathbf{k}}|\xi|^{\mathbf{k}}+\frac{|x|^2-1}{\mathbf{k}^2}|\xi|^{\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}(\mathbf{k}+2)}|\xi|^{\mathbf{k}+2}\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2},\end{aligned}$$

from which the assertion of the lemma follows. The lemma is proved. □

Lemma 6. *For the function $I_{2,1}(x,\xi)$, the following representation is valid:*

$$I_{2,1}(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}\right)\frac{|x|^2-|\xi|^2}{2}-\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}}\frac{|x|^4-|\xi|^4}{4}\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.$$

Proof. Note that in the integral $I_{2,1}(x,\xi)$, the inequalities $|\xi|<|y|<|x|$ are true. For the inner integral $I_{2,1}(x,\xi)$, taking into account Lemma 3, we write

$$\begin{aligned}\frac{1}{\omega_n}\int_{\partial S} G_4^n(x,\rho y)E_2(\rho y,\xi)ds_y &= \frac{1}{\omega_n}\int_{\partial S}\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)}\right.\\ &\left.+\frac{|x|^2-1}{(\mathbf{k}-2)\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)\mathbf{k}}\rho^2\right)\rho^k\mathcal{H}_k(x,y)\sum_{m=0}^{\infty}\frac{\rho^{-(\mathbf{m}-2)}}{\mathbf{m}-2}\rho^m\mathcal{H}_m(y,\xi)\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)}+\frac{|x|^2-1}{(\mathbf{k}-2)\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)\mathbf{k}}\rho^2\right)\frac{\rho^{-(\mathbf{k}-2)}}{\mathbf{k}-2}\mathcal{H}_k(x,\xi).\end{aligned}$$

Then, for the entire integral $I_{2,1}(x,\xi)$, we have

$$\begin{aligned}I_{2,1} &= \frac{1}{2}\sum_{k=0}^{\infty}\int_{|\xi|}^{|x|}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)^2}\rho+\frac{|x|^2-1}{(\mathbf{k}-2)^2\mathbf{k}}\rho-\frac{(|x|^{-(\mathbf{k}-2)}-1)}{(\mathbf{k}-2)^2\mathbf{k}}\rho^3\right)d\rho\,\mathcal{H}_k(x,\xi)\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left(\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}\right)\frac{|x|^2-|\xi|^2}{2(\mathbf{k}-2)^2}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)^2\mathbf{k}}\frac{|x|^4-|\xi|^4}{4}\right)\mathcal{H}_k(x,\xi),\end{aligned}$$

from which the statement of the lemma follows. The lemma is proved. □

Lemma 7. *For the function $I_{2,2}(x,\xi)$, the following representation is valid:*

$$\begin{aligned}I_{2,2}(x,\xi) = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)\mathbf{k}}+\frac{|x|^2-1}{\mathbf{k}^2}\right)(|x|^{\mathbf{k}}-|\xi|^{\mathbf{k}})\right.\\ \left.+\frac{|x|^{(\mathbf{k}+2)}-|x|^4}{\mathbf{k}(\mathbf{k}+2)}+\frac{|x|^{-(\mathbf{k}-2)}-1}{\mathbf{k}(\mathbf{k}+2)}|\xi|^{\mathbf{k}+2}\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.\end{aligned}$$

Proof. Similar to the proof of Lemma 6 for the inner integral in $I_{2,2}$, we write

$$\begin{aligned}-\frac{1}{\omega_n}\int_{\partial S} G_4^n(x,\rho y)E_2^*(\rho y,\xi)ds_y\\ = -\frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)^2}+\frac{|x|^2-1}{(\mathbf{k}-2)^2\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)^2\mathbf{k}}\rho^2\right)\mathcal{H}_k(x,\xi).\end{aligned}$$

Then, for the entire integral $I_{2,2}$, we have

$$\begin{aligned}I_{2,2}(x,\xi) &= -\frac{1}{2}\sum_{k=0}^{\infty}\int_{|\xi|}^{|x|}\rho^{\mathbf{k}-1}\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)^2}+\frac{|x|^2-1}{(\mathbf{k}-2)^2\mathbf{k}}-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)^2\mathbf{k}}\rho^2\right)d\rho\,\mathcal{H}_k(x,\xi)\\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\left(\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}\right)\frac{|x|^{\mathbf{k}}-|\xi|^{\mathbf{k}}}{(\mathbf{k}-2)^2\mathbf{k}}\right.\\ &\left.-\frac{|x|^{-(\mathbf{k}-2)}-1}{(\mathbf{k}-2)^2\mathbf{k}}\frac{|x|^{\mathbf{k}+2}-|\xi|^{\mathbf{k}+2}}{(\mathbf{k}+2)}\right)\mathcal{H}_k(x,\xi),\end{aligned}$$

from which the assertion of the lemma follows if we multiply the last two fractions in the outer bracket and take out the common factor. The lemma is proved. □

Lemma 8. *For the function $I_{3,1}(x,\xi)$, the following representation is valid:*

$$I_{3,1}(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-6)}-1}{(\mathbf{k}-6)(\mathbf{k}-4)} + \frac{1-|x|^4}{4\mathbf{k}} + \frac{|x|^2-|x|^{-(\mathbf{k}-6)}}{(\mathbf{k}-4)\mathbf{k}} + \frac{1-|x|^2}{2}\Big(\frac{|x|^2}{\mathbf{k}} - \frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}}\Big)\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.$$

Proof. Note that in the integral $I_{3,1}(x,\xi)$, the inequalities $|\xi| < |x| < |y| < 1$ are true. For the inner integral $I_{3,1}(x,\xi)$, taking into account Lemma 3, we write

$$\begin{aligned}\frac{1}{\omega_n}\int_{\partial S} G_4^n(x,\rho y)E_2(\rho y,\xi)ds_y &= \frac{1}{\omega_n}\int_{\partial S}\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\rho^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)} + \frac{\rho^2-1}{(\mathbf{k}-2)\mathbf{k}}\right.\\ &\left. - \frac{(\rho^{-(\mathbf{k}-2)}-1)}{(\mathbf{k}-2)\mathbf{k}}|x|^2\right)\rho^k\mathcal{H}_k(x,y)\sum_{m=0}^{\infty}\frac{\rho^{-(\mathbf{m}-2)}}{\mathbf{m}-2}\rho^m\mathcal{H}_m(y,\xi)\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{\rho^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)(\mathbf{k}-2)} + \frac{\rho^2-1}{(\mathbf{k}-2)\mathbf{k}} - \frac{(\rho^{-(\mathbf{k}-2)}-1)}{(\mathbf{k}-2)\mathbf{k}}|x|^2\right)\frac{\rho^{-(\mathbf{k}-2)}}{\mathbf{k}-2}\mathcal{H}_k(x,\xi).\end{aligned}$$

Then, for the entire integral $I_{3,1}(x,\xi)$, we have

$$\begin{aligned}I_{3,1}(x,\xi) &= \frac{1}{2}\sum_{k=0}^{\infty}\int_{|x|}^{1}\left(\frac{\rho^{-(\mathbf{k}-5)}-\rho}{(\mathbf{k}-4)} + \frac{\rho^3-\rho}{\mathbf{k}} - \frac{\rho^{-(\mathbf{k}-3)}-\rho}{\mathbf{k}}|x|^2\right)d\rho\,\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\int_{|x|}^{1}\left(\frac{\rho^{-(\mathbf{k}-5)}}{(\mathbf{k}-4)} + \frac{\rho^3}{\mathbf{k}} - \frac{\rho^{-(\mathbf{k}-3)}}{\mathbf{k}}|x|^2 + \rho\Big(\frac{-1}{(\mathbf{k}-4)} - \frac{1}{\mathbf{k}} + \frac{|x|^2}{\mathbf{k}}\Big)\right)d\rho\,\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^{-(\mathbf{k}-6)}-1}{(\mathbf{k}-6)(\mathbf{k}-4)} + \frac{1-|x|^4}{4\mathbf{k}} + \frac{|x|^2-|x|^{-(\mathbf{k}-6)}}{(\mathbf{k}-4)\mathbf{k}}\right.\\ &\left. + \frac{1-|x|^2}{2}\Big(\frac{|x|^2}{\mathbf{k}} - \frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}}\Big)\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.\end{aligned}$$

The lemma is proved. □

Lemma 9. *For the function $I_{3,2}(x,\xi)$, the following representation is valid:*

$$I_{3,2}(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^4-1}{4(\mathbf{k}-4)} + \frac{|x|^{\mathbf{k}+2}-1}{\mathbf{k}(\mathbf{k}+2)} + \frac{|x|^2-|x|^4}{2\mathbf{k}} + \frac{|x|^{\mathbf{k}}-1}{\mathbf{k}}\Big(\frac{|x|^2}{\mathbf{k}} - \frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}}\Big)\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.$$

Proof. Similar to Lemma 8 for the inner integral in $I_{3,2}(x,\xi)$, we write

$$\begin{aligned}&-\frac{1}{\omega_n}\int_{\partial S} G_4^n(x,\rho y)E_2^*(\rho y,\xi)ds_y\\ &\quad= -\frac{1}{2}\sum_{k=0}^{\infty}\rho^{2k}\left(\frac{\rho^{-(\mathbf{k}-4)}}{(\mathbf{k}-4)} + \frac{\rho^2}{\mathbf{k}} - \frac{\rho^{-(\mathbf{k}-2)}}{\mathbf{k}}|x|^2 + \Big(\frac{-1}{(\mathbf{k}-4)} - \frac{1}{\mathbf{k}} + \frac{|x|^2}{\mathbf{k}}\Big)\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2},\end{aligned}$$

and therefore, for the entire integral $I_{3,2}$, we have

$$I_{3,2}(x,\xi) = -\frac{1}{2}\sum_{k=0}^{\infty}\int_{|x|}^{1}\left(\frac{\rho^3}{(\mathbf{k}-4)}+\frac{\rho^{(\mathbf{k}+1)}}{\mathbf{k}}-\frac{\rho}{\mathbf{k}}|x|^2+\rho^{\mathbf{k}-1}\Big(\frac{|x|^2}{\mathbf{k}}-\frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}}\Big)\right)d\rho\,\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{|x|^4-1}{4(\mathbf{k}-4)}+\frac{|x|^{\mathbf{k}+2}-1}{\mathbf{k}(\mathbf{k}+2)}+\frac{|x|^2-|x|^4}{2\mathbf{k}}+\frac{|x|^{\mathbf{k}}-1}{\mathbf{k}}\Big(\frac{|x|^2}{\mathbf{k}}-\frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}}\Big)\right)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)^2}.$$

The lemma is proved. □

Continuation of the Proof of Theorem 5. To find the expansion of the function $G_6^n(x,\xi)$ in a series, it is necessary to add the found values of the integrals $I_{i,1}$ and $I_{i,2}$, $i=1,2,3$. We obtain a rather cumbersome expression, and therefore, we first exclude some terms.

Let us calculate the sum of the coefficients of the terms containing $|\xi|^{\mathbf{k}}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2\mathbf{k}}$, which are in $I_{1,2}$ and $I_{2,2}$:

$$-\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}-\frac{|x|^2-1}{\mathbf{k}}+\frac{|x|^{-(\mathbf{k}-4)}-1}{(\mathbf{k}-4)}+\frac{|x|^2-1}{\mathbf{k}}=0.$$

Similar calculations show that the sums of the coefficients of the terms containing the expressions $|\xi|^{\mathbf{k}+2}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2\mathbf{k}}$, $|x|^{\mathbf{k}}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2\mathbf{k}}$ and $|x|^{\mathbf{k}+2}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2\mathbf{k}}$ are also equal to zero. Next, we calculate the sum of the coefficients of the terms containing $|x|^{-(\mathbf{k}-6)}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$, which are in $I_{2,1}$ and $I_{3,1}$:

$$\frac{1}{2(\mathbf{k}-4)}-\frac{1}{4\mathbf{k}}+\frac{1}{(\mathbf{k}-6)(\mathbf{k}-4)}-\frac{1}{(\mathbf{k}-4)\mathbf{k}}=\frac{\mathbf{k}-2}{4(\mathbf{k}-6)(\mathbf{k}-4)}.$$

The sum of the coefficients of $|x|^{-(\mathbf{k}-4)}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ that are in $I_{1,1}$ and $I_{2,1}$ is equal to $-\frac{(\mathbf{k}-2)|\xi|^2}{2(\mathbf{k}-4)\mathbf{k}}$, and the sum of the coefficients of $|x|^{-(\mathbf{k}-2)}\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ that are in $I_{1,1}$ and $I_{2,1}$ is equal to $\frac{(\mathbf{k}-2)}{4\mathbf{k}(\mathbf{k}+2)}|\xi|^4$. Adding up all the integrals $I_{i,j}(x,\xi)$, we find

$$G_6^n(x,\xi)=\sum_{i=1}^{3}\sum_{j-1}^{2}I_{i,j}(x,\xi)$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{|x|^{-(\mathbf{k}-6)}}{4(\mathbf{k}-6,2)_3}-\frac{|x|^{-(\mathbf{k}-4)}|\xi|^2}{2(\mathbf{k}-4,2)_3}+\frac{|x|^{-(\mathbf{k}-2)}|\xi|^4}{4(\mathbf{k}-2,2)_3}\Big)\mathcal{H}_k(x,\xi)+\sum_{k=0}^{\infty}F_k(x,\xi)\mathcal{H}_k(x,\xi),$$

where the function $F_k(x,\xi)$ denotes the sum of the terms of the obtained series containing $\mathcal{H}_k(x,\xi)$, but not containing the powers of $|x|$ and $|\xi|$ depending on k—they still need to be calculated. Note that in accordance with (17) and the notation $\mathbf{k}=2k+n$, the sum of the fractions in the expression above gives the function $E_6(x,\xi)$ from Theorem 4 for $n>6$ and $|\xi|<|x|$, and therefore, we obtain

$$G_6^n(x,\xi)=E_6(x,\xi)+\sum_{k=0}^{\infty}F_k(x,\xi)\mathcal{H}_k(x,\xi). \tag{20}$$

Next, we denote the coefficients of $|\xi|^4\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ and $|\xi|^2\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ as b_4 and b_2, respectively. From Lemmas 4 and 6, we find

$$b_4=-\frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)},\quad b_2=\frac{\mathbf{k}-2}{2(\mathbf{k}-4)\mathbf{k}}-(|x|^2-1)\frac{\mathbf{k}-2}{2\mathbf{k}^2}=\frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2}-\frac{\mathbf{k}-2}{2\mathbf{k}^2}|x|^2.$$

Let us denote the coefficients of $|x|^4 \frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ and $|x|^2 \frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$, independent of $|\xi|$, as a_4 and a_2, accordingly. From Lemmas 6–9, we find

$$a_4 = \frac{1}{2\mathbf{k}} + \frac{1}{4\mathbf{k}} - \frac{1}{\mathbf{k}(\mathbf{k}-4)} + \frac{1}{\mathbf{k}(\mathbf{k}+2)} - \frac{1}{4\mathbf{k}} - \frac{1}{2\mathbf{k}} + \frac{1}{4(\mathbf{k}-4)} - \frac{1}{2\mathbf{k}} = -\frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)},$$
$$a_2 = -\frac{1}{2(\mathbf{k}-4)} - \frac{1}{2\mathbf{k}} + \frac{1}{\mathbf{k}(\mathbf{k}-4)} + \frac{1}{2\mathbf{k}} + \frac{\mathbf{k}-2}{(\mathbf{k}-4)\mathbf{k}} + \frac{1}{2\mathbf{k}} - \frac{1}{\mathbf{k}^2} = \frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2}.$$

From Lemmas 8 and 9, we obtain coefficients of $\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$ that do not depend on $|x|$ and $|\xi|$

$$a_0 = -\frac{1}{(\mathbf{k}-6)(\mathbf{k}-4)} + \frac{1}{4\mathbf{k}} - \frac{\mathbf{k}-2}{(\mathbf{k}-4)\mathbf{k}} - \frac{1}{4(\mathbf{k}-4)} - \frac{1}{\mathbf{k}(\mathbf{k}+2)} + \frac{2(\mathbf{k}-2)}{(\mathbf{k}-4)\mathbf{k}^2}$$
$$= -\frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2} + \frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)} - \frac{(\mathbf{k}-2)}{4(\mathbf{k}-6)(\mathbf{k}-4)}.$$

Thus, taking into account the calculated coefficients, we obtain

$$F_k(x,\xi) = \Big(-\frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)}|\xi|^4 + \frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2}|\xi|^2 - \frac{\mathbf{k}-2}{2\mathbf{k}^2}|\xi|^2|x|^2 - \frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)}|x|^4$$
$$+ \frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2}|x|^2 - \frac{(\mathbf{k}-2)^2}{(\mathbf{k}-4)\mathbf{k}^2} + \frac{\mathbf{k}-2}{4\mathbf{k}(\mathbf{k}+2)} - \frac{(\mathbf{k}-2)}{4(\mathbf{k}-6)(\mathbf{k}-4)}\Big)\frac{\mathcal{H}_k(x,\xi)}{2(\mathbf{k}-2)^2}$$
$$= \frac{1}{2}\Big(-\frac{|\xi|^4+|x|^4-1}{4\mathbf{k}(\mathbf{k}+2)} - \frac{|\xi|^2|x|^2}{2\mathbf{k}^2} + \frac{|\xi|^2+|x|^2-1}{(\mathbf{k}-4)\mathbf{k}^2}(\mathbf{k}-2)$$
$$- \frac{1}{4(\mathbf{k}-6)(\mathbf{k}-4)}\Big)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)}.$$

If we now take into account that, according to (17), the following equality is true:

$$E_6^*(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{1}{4(\mathbf{k}-6)(\mathbf{k}-4)} - \frac{|\xi|^2|x|^2}{2(\mathbf{k}-4)\mathbf{k}} + \frac{|\xi|^4|x|^4}{4\mathbf{k}(\mathbf{k}+2)}\Big)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)},$$

and also $\frac{1}{2(\mathbf{k}-4)\mathbf{k}} + \frac{1}{2\mathbf{k}^2} = \frac{\mathbf{k}-2}{(\mathbf{k}-4)\mathbf{k}^2}$, then we have

$$F_k(x,\xi) = -[E_6^*(x,\xi)]_k$$
$$+ \frac{1}{8}\frac{|\xi|^4|x|^4-|\xi|^4-|x|^4+1}{(\mathbf{k}-2)\mathbf{k}(\mathbf{k}+2)}\mathcal{H}_k(x,\xi) - \frac{1}{2}\frac{|\xi|^2|x|^2-|\xi|^2-|x|^2+1}{(\mathbf{k}-4)\mathbf{k}^2}\mathcal{H}_k(x,\xi)$$
$$= -[E_6^*(x,\xi)]_k + \frac{(|x|^4-1)(|\xi|^4-1)}{8}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2,2)_3} - \frac{(|x|^2-1)(|\xi|^2-1)}{2}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2},$$

where

$$[E_6^*(x,\xi)]_k = \frac{1}{2}\Big(\frac{1}{4(\mathbf{k}-6)(\mathbf{k}-4)} - \frac{|\xi|^2|x|^2}{2(\mathbf{k}-4)\mathbf{k}} + \frac{|\xi|^4|x|^4}{4\mathbf{k}(\mathbf{k}+2)}\Big)\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)}$$

denotes the k-th coefficient of $\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2}$ in the series expansion of $E_6^*(x,\xi)$. Substituting the found value of the function $F_k(x,\xi)$ into (20), we obtain (19).

If the inequality $|x| < |\xi| < 1$ is true for the variables x, ξ, then repeating the calculations performed for the function $G_6(x,\xi)$, we arrive at Formula (19), in the right-hand side of which the variables x and ξ must be swapped due to the symmetry of the representations of the elementary solutions $E_6(x,\xi)$ and $E_4(x,\xi)$ used in the proof. However, the right-hand

side of Formula (19) is symmetric with respect to x and ξ, and therefore, the result will be the same expression. The theorem is proved. □

3.2. Green's Function $\mathcal{G}_6^n(x,\xi)$

Let us transform the result of Theorem 5, namely, express the harmonic functions included in (19) and presented in the form of series through the elementary solutions. We will now obtain a formula similar to (5) for the Green's function $\mathcal{G}_6^n(x,\xi)$. For this, we need the following operators, which we will be applied to the harmonic in S functions:

$$I_\alpha v(\xi) = \int_0^1 v(t\xi)t^{\alpha-1}\,dt, \quad \Lambda v(\xi) = \sum_{k=1}^n \xi_i v_{\xi_i}(\xi),$$

where $\alpha > 0$, and the functions $v(\xi)$ are differentiable in S. It is easy to see that for $\alpha < 0$, the operator $I_\alpha v(\xi)$ can be applied to the functions $v(\xi)$ if $v(\xi) = O(|x|^k)$, $x \to 0$, and $\alpha + k > 0$, since in this case, the improper integral $I_\alpha v(\xi)$ converges.

Lemma 10. *The equalities below are satisfied:*

$$(\Lambda_\xi + \alpha)I_\alpha v(\xi) = v(\xi), \quad I_\alpha(\Lambda_\xi + \alpha)v(\xi) = v(\xi), \quad I_\alpha I_\beta v(\xi) = I_\beta I_\alpha v(\xi).$$

Proof. Using integration by parts and taking into account that $\alpha > 0$, we obtain

$$\begin{aligned}\Lambda_\xi I_\alpha v(\xi) =& \Lambda_\xi \int_0^1 v(t\xi)t^{\alpha-1}\,dt = \int_0^1 \sum_{k=1}^n t\xi_i v_{\xi_i}(t\xi)t^{\alpha-1}\,dt = \int_0^1 (v(t\xi))'_t t^\alpha\,dt \\ =& v(t\xi)t^\alpha\big|_0^1 - \alpha\int_0^1 v(t\xi)t^{\alpha-1}\,dt = v(\xi) - \alpha I_\alpha v(\xi).\end{aligned}$$

If we now move $\alpha I_\alpha v(\xi)$ to the left side of the equality, we obtain the first equality of the lemma. Similarly, we obtain the second equality of the lemma from the equality

$$I_\alpha \Lambda_\xi v(\xi) = \int_0^1 \sum_{k=1}^n t\xi_i v_{\xi_i}(t\xi)t^{\alpha-1}\,dt = \int_0^1 (v(t\xi))'_t t^\alpha\,dt = v(\xi) - \alpha I_\alpha v(\xi).$$

It is easy to see that for $\alpha \neq \beta$,

$$\begin{aligned}I_\alpha I_\beta v(\xi) = I_\alpha \int_0^1 v(t\xi)t^{\beta-1}\,dt &= \int_0^1 \tau^{\alpha-1}\int_0^1 v(t\tau\xi)t^{\beta-1}\,dt\,d\tau \\ = \int_0^1 \tau^{\alpha-\beta-1}\int_0^s v(s\xi)s^{\beta-1}\,ds\,d\tau &= \int_0^1 v(s\xi)s^{\beta-1}\int_s^1 \tau^{\alpha-\beta-1}\,d\tau\,ds \\ = \int_0^1 v(s\xi)s^{\beta-1}\frac{1-s^{\alpha-\beta}}{\alpha-\beta}\,ds &= \frac{1}{\alpha-\beta}\int_0^1 v(s\xi)(s^{\beta-1}-s^{\alpha-1})\,ds \\ &= \frac{1}{\alpha-\beta}(I_\beta - I_\alpha)v(\xi), \quad (21)\end{aligned}$$

where the substitution $t\tau = s$ is made under the integral. From here, we obtain the third equality of the lemma

$$I_\alpha I_\beta v(\xi) = \frac{1}{\alpha-\beta}(I_\beta - I_\alpha)v(\xi) = \frac{1}{\beta-\alpha}(I_\alpha - I_\beta)v(\xi) = I_\beta I_\alpha v(\xi),$$

which is obviously true for $\alpha = \beta$. The lemma is proved. □

Remark 3. *If we denote by $\hat{I}_\alpha$ the operator I_α, but applied to the function $E_2^*(x,\xi)$ by the variable x, then due to the fact that this function has symmetric arguments $E_2^*(x,\xi) = (1 - 2x\cdot\xi +$*

$|x|^2|\xi|^2)^{2-n}/(n-2)$, *it does not matter by which variable the operator* I_α *is applied, i.e., the equalities are true:*

$$I_\alpha E_2^*(x,\xi) = \int_0^1 E_2^*(x,t\xi)t^{\alpha-1}\,dt = \int_0^1 E_2^*(tx,\xi)t^{\alpha-1}\,dt = \hat{I}_\alpha E_2^*(x,\xi).$$

Definition 2. *Let us define the function* $\mathcal{G}_6^n(x,\xi)$ *for* $n>4$ *and* $x\neq\xi$ *by the equality*

$$\mathcal{G}_6^n(x,\xi) = E_6(x,\xi) - E_6^*(x,\xi) + \frac{(|x|^4-1)(|\xi|^4-1)}{32} I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi) - \frac{(|x|^2-1)(|\xi|^2-1)}{16}(I_{\frac{n}{2}-2}+I_{\frac{n}{2}}) I_{\frac{n}{2}} E_2^*(x,\xi). \quad (22)$$

The form of the function $\mathcal{G}_6^n(x,\xi)$ is somewhat similar to the function from (5).

Theorem 6. *For* $n>6$ *and* $x\neq\xi\in\bar{S}$, *the equality* $\mathcal{G}_6^n(x,\xi) = G_6^n(x,\xi)$ *holds.*

Proof. Let $n>6$ and $|\xi|<|x|<1$, then the equality (19) is true. Let us transform the series from the second line of (19). Consider the harmonic function $\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2,2)_3}$. It is easy to check that

$$I_{\frac{n}{2}}\mathcal{H}_k(x,\xi) = \int_0^1 \mathcal{H}_k(x,\xi)t^{k+n/2-1}dt = \mathcal{H}_k(x,\xi)\frac{1}{k+n/2} = 2\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}}, \quad (23)$$

and therefore,

$$\frac{1}{4} I_{\frac{n}{2}+1} I_{\frac{n}{2}} \mathcal{H}_k(x,\xi) = \frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}(\mathbf{k}+2)}.$$

Consequently, taking into account (15) and the uniform convergence of the series for $|\xi|<a<|x|$, we obtain

$$\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2,2)_3} = \sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)\mathbf{k}(\mathbf{k}+2)} = \frac{1}{4}\sum_{k=0}^{\infty} I_{\frac{n}{2}+1} I_{\frac{n}{2}}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)} = \frac{1}{4} I_{\frac{n}{2}+1} I_{\frac{n}{2}}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)} = \frac{1}{4} I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi).$$

To transform the harmonic function $\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2}$, we use the equality

$$\frac{1}{(\mathbf{k}-4)\mathbf{k}^2} = \frac{1}{2}\Big(\frac{1}{(\mathbf{k}-2)\mathbf{k}^2} + \frac{1}{(\mathbf{k}-4)(\mathbf{k}-2)\mathbf{k}}\Big),$$

from which the following is obtained:

$$\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2} = \frac{1}{2}\Big(\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)\mathbf{k}^2} + \sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)(\mathbf{k}-2)\mathbf{k}}\Big).$$

Considering that $n>6$ and similarly to what was executed above, we find

$$\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)\mathbf{k}^2} = \frac{1}{4} I_{\frac{n}{2}}^2\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2}, \quad \frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)(\mathbf{k}-2)\mathbf{k}} = \frac{1}{4} I_{\frac{n}{2}-2} I_{\frac{n}{2}}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2},$$

and therefore, using the uniform convergence of the series for $|\xi|<a<|x|$, we obtain

$$\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{(\mathbf{k}-4)\mathbf{k}^2} = \frac{1}{8} I_{\frac{n}{2}}^2\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2} + \frac{1}{8} I_{\frac{n}{2}-2} I_{\frac{n}{2}}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2} = \frac{1}{8}(I_{I_{\frac{n}{2}-2}+\frac{n}{2}}) I_{\frac{n}{2}} E_2^*(x,\xi).$$

If we substitute the expressions found for the series into (19), we obtain the right-hand side of (22), i.e., $G_6^n(x,\xi) = \mathcal{G}_6^n(x,\xi)$. If $|x| < |\xi| < 1$, then, due to the symmetry of (19) acting in the same way as above, we obtain the equality (22), but the operators I_α need to be applied to the variable x. However, according to Remark 3, by changing x to ξ in the integral, we obtain the same result. Finally, for $|x| = |\xi|$, but $x \neq \xi$, the equality $G_6^n(x,\xi) = \mathcal{G}_6^n(x,\xi)$ is preserved due to the continuity of the functions included in it for $x \neq \xi \in \bar{S}$. The theorem is proved. □

Remark 4. *Note that the function $G_6^n(x,\xi)$ is defined for $n \geq 2$, but $\mathcal{G}_6^n(x,\xi)$ for $n > 4$.*

4. Biharmonic Navier Problem

The biharmonic Navier problem has the form

$$\Delta^3 u(x) = f(x),\ x \in S, \quad u|_{\partial S} = \varphi_0,\ \Delta u\big|_{\partial S} = \varphi_1. \tag{24}$$

Let us study the properties of the Green's function $G_4^n(x,\xi)$ of this problem.

4.1. Green's Function $\mathcal{G}_4^n(x,\xi)$

Lemma 11. *Let $\varphi(t)$ and $v(x)$ be some twice differentiable functions for $t \in [0,1)$ and $x \in S$, respectively. Then, the following equality holds:*

$$\Delta\big(\varphi(|x|^2)v(x)\big) = 4\big(|x|^2\varphi''(|x|^2) + \varphi'(|x|^2)(\Lambda + n/2)\big)v(x) + \varphi(|x|^2)\Delta v(x). \tag{25}$$

Proof. It is easy to verify the validity of the following equalities:

$$\begin{aligned}\Delta\big(\varphi(|x|^2)v(x)\big) =& v(x)\Delta\varphi(|x|^2) + 2\sum_{i=1}^{n}\frac{\partial\varphi(|x|^2)}{\partial x_i}\frac{\partial v(x)}{\partial x_i} + \varphi(|x|^2)\Delta v(x)\\ =& \big(\Delta\varphi(|x|^2) + 4\varphi'(|x|^2)\Lambda + \varphi(|x|^2)\Delta\big)v(x).\end{aligned}$$

If we now use another equality, i.e.,

$$\begin{aligned}\Delta\varphi(|x|^2) = 2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(x_i\varphi'(|x|^2)\big) &= 2\sum_{i=1}^{n}\big(\varphi'(|x|^2) + 2x_i^2\varphi''(|x|^2)\big)\\ &= 4\Big(\frac{n}{2}\varphi'(|x|^2) + |x|^2\varphi''(|x|^2)\Big),\end{aligned}$$

then we immediately obtain (25). The lemma is proved. □

Corollary 1. *For $x, \xi \in S$, the following equality is true:*

$$\begin{aligned}&\Delta_\xi\big(\varphi(|\xi|^2)E_{2k}^*(x,\xi)\big)\\ &\quad= \big(4|\xi|^2\varphi''(|\xi|^2) + \varphi'(|\xi|^2)(2n + 4\Lambda_\xi)\big)E_{2k}^*(x,\xi) - |x|^2\varphi(|\xi|^2)E_{2k-2}^*(x,\xi).\end{aligned} \tag{26}$$

Proof. Let us use (25) for $v(\xi) = E_{2k}^*(x,\xi)$ and take into account the equality $\Delta_\xi E_{2k}^*(x,\xi) = -|x|^2E_{2k-2}^*(x,\xi)$, which can easily be obtained from (6). Then, we obtain (26). □

Lemma 12. *The biharmonic function $E_4^*(x,\xi)$ for $n > 4$ satisfies the equality*

$$E_4^*(x,\xi) = \frac{1}{4}\Big(I_{\frac{n}{2}-2} - |x|^2|\xi|^2 I_{\frac{n}{2}}\Big)E_2^*(x,\xi). \tag{27}$$

Proof. Let $|\xi| < |x| < 1$. From (23), it follows that

$$I_{\frac{n}{2}}\mathcal{H}_k(x,\xi) = \frac{2}{\mathbf{k}}\mathcal{H}_k(x,\xi), \quad I_{\frac{n}{2}-2}\mathcal{H}_k(x,\xi) = \frac{2}{\mathbf{k}-4}\mathcal{H}_k(x,\xi),$$

and therefore, since $E_2^*(x,\xi) = \sum_{k=0}^{\infty}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2}$, then from (13), we find

$$E_4^*(x,\xi) = \frac{1}{4}\sum_{k=0}^{\infty}\Big(\frac{2\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)(\mathbf{k}-4)} - |x|^2|\xi|^2\frac{2\mathcal{H}_k(x,\xi)}{(\mathbf{k}-2)\mathbf{k}}\Big) = \frac{1}{4}\sum_{k=0}^{\infty} I_{\frac{n}{2}-2}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2}$$
$$- \frac{1}{4}|x|^2|\xi|^2\sum_{k=0}^{\infty} I_{\frac{n}{2}}\frac{\mathcal{H}_k(x,\xi)}{\mathbf{k}-2} = \frac{1}{4}\Big(I_{\frac{n}{2}-2}E_2^*(x,\xi) - |x|^2|\xi|^2 I_{\frac{n}{2}}E_2^*(x,\xi)\Big),$$

from which (27) follows. For $|\xi| < |x| < 1$, due to the symmetry of the variables x and ξ in the functions $E_4^*(x,\xi)$ and $E_2^*(x,\xi)$, we obtain the same formula, but in which the operators I_α are applied to the variable x. Due to Remark 3, we can again pass to the variable ξ, i.e., we have (27). If $|\xi| = |x|$, then due to the continuity of the functions included in (27), both on the left-hand and right-hand sides of this equality, it is preserved in this case as well. The lemma is proved. □

Similarly to the function $\mathcal{G}_6^n(x,\xi)$ from Definition 2, we introduce the function $\mathcal{G}_4^n(x,\xi)$.

Lemma 13. *The biharmonic function for $\xi \neq x \in S$,*

$$\mathcal{G}_4^n(x,\xi) = E_4(x,\xi) - E_4^*(x,\xi) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}(I_{\frac{n}{2}}E_2^*)(x,\xi), \tag{28}$$

has the property $\Delta_\xi\mathcal{G}_4^n(x,\xi) = -G_2(x,\xi)$, where $G_2(x,\xi)$ is the Green's function of the harmonic Dirichlet problem in S. The function $\mathcal{G}_4^n(x,\xi)$ is symmetric and $\mathcal{G}_4^n(x,\xi)|_{\xi\in\partial S} = 0$ for $x \in S$.

Proof. From (6), it is easy to obtain the equalities

$$\Delta_\xi(E_4(x,\xi) - E_4^*(x,\xi)) = -E_2(x,\xi) + |x|^2E_2^*(x,\xi)$$
$$= -E_2(x,\xi) + E_2^*(x,\xi) + (|x|^2-1)E_2^*(x,\xi).$$

Using Lemma 11 with $v(\xi) = I_{\frac{n}{2}}E_2^*(x,\xi)$ and $\varphi(t) = (t-1)/2$, and then applying Lemma 10, we obtain

$$\Delta_\xi\frac{|\xi|^2-1}{2}(I_{\frac{n}{2}}E_2^*)(x,\xi) = 2\Big(\Lambda_\xi + \frac{n}{2}\Big)(I_{\frac{n}{2}}E_2^*)(x,\xi) = 2E_2^*(x,\xi).$$

Thus, in the end, we find

$$\Delta_\xi\mathcal{G}_4^n(x,\xi) = -E_2(x,\xi) + E_2^*(x,\xi) + (|x|^2-1)E_2^*(x,\xi) - (|x|^2-1)E_2^*(x,\xi) = -G_2(x,\xi).$$

Since the function $E_2^*(x,\xi)$ is symmetric, it follows from the form of the function $\mathcal{G}_4^n(x,\xi)$ that for its symmetry, it is sufficient to verify that only the function $(I_{\frac{n}{2}}E_2^*)(x,\xi)$ is symmetric. To perform this, note that the function $E_2^*(x,t\xi)$ is a function of the argument $\big|x/|x| - |x|t\xi\big|$, and for $t > 0$, the equality $\big|x/|x| - |x|t\xi\big| = \big|tx/|tx| - |tx|\xi\big|$ is true, which means $E_{2k}^*(x,t\xi) = E_{2k}^*(tx,\xi)$. Further, since

$$\big|x/|x| - |x|t\xi\big| = (1 - 2x\cdot(t\xi) + |x|^2|t\xi|^2)^{1/2} = \big|\xi/|\xi| - |\xi|tx\big|,$$

we then obtain $E_2^*(x,t\xi) = E_2^*(\xi,tx)$, from which $(I_{\frac{n}{2}}E_2^*)(x,\xi) = (I_{\frac{n}{2}}E_2^*)(\xi,x)$ follows. Since $E_4(x,\xi) = E_4^*(x,\xi)$, $\xi \in \partial S$, then $\mathcal{G}_4^n(x,\xi)|_{\xi\in\partial S} = 0$. The lemma is proved. □

Theorem 7. *The function $u \in C^4(S) \cap C^3(\bar{S})$, which is a solution to the Navier problem for the biharmonic Equation (24), can be represented as*

$$u(x) = \frac{1}{\omega_n} \int_{\partial S} -\frac{\partial G_2(x,\xi)}{\partial \nu} \varphi_0(\xi) + \frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu} \varphi_1(\xi)\, ds_\xi + \frac{1}{\omega_n} \int_S \mathcal{G}_4^n(x,\xi) f(\xi)\, d\xi. \quad (29)$$

If $\varphi_0 \in C(\partial S)$, $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ $(\varepsilon > 0)$ and $f \in C^1(\bar{S})$, then the function $u(x)$ found from (29) is a solution to the Navier problem for the biharmonic equation.

Proof. From Lemma 13, taking into account that $E_4(x,\xi)|_{\xi\in\partial S} = E_4^*(x,\xi)|_{\xi\in\partial S}$ and the properties of the Green's function $G_2(x,\xi)$, it follows that

$$\mathcal{G}_4^n(x,\xi)|_{\xi\in\partial S} = 0, \quad \Delta_\xi \mathcal{G}_4^n(x,\xi)|_{\xi\in\partial S} = -G_2(x,\xi)|_{\xi\in\partial S} = 0.$$

Therefore, in accordance with Definition 1, the biharmonic function $\mathcal{G}_4^n(x,\xi)$ taken from (28) for $\xi \neq x \in S$ has a singularity of the elementary solution type $E_4(x,\xi)$ for $\xi = x$, which is the Green's function of the Navier problem for the biharmonic equation in S. Therefore, similarly to the proof of [8], Theorem 4.2, we can obtain

$$u(x) = \frac{1}{\omega_n} \int_{\partial S} \frac{\partial \Delta_\xi \mathcal{G}_4^n(x,\xi)}{\partial \nu} \varphi_0(\xi)\, ds_\xi + \frac{1}{\omega_n} \int_{\partial S} \frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu} \varphi_1(\xi)\, ds_\xi + \frac{1}{\omega_n} \int_S \mathcal{G}_4^n(x,\xi) f(\xi)\, d\xi.$$

Since by Lemma 13, we have $\Delta_\xi \mathcal{G}_4^n(x,\xi) = -G_2(x,\xi)$, then (29) follows. The theorem is proved. □

Remark 5. *Using Theorem 3 and the representations (12) and (13), we can prove that for $n > 4$, we have $\mathcal{G}_4^n(x,\xi) = G_4^n(x,\xi)$.*

4.2. Another Representation of the Solution

The solution (29) of the Navier problem for a homogeneous biharmonic equation obtained from Theorem 7 can be written in another, simpler form, without using the function $\mathcal{G}_4^n(x,\xi)$. Since (29) contains normal derivatives, the following auxiliary statement is necessary.

Lemma 14. *Let $\varphi(t)$ be some function differentiable at $t > 0$, then*

$$\Lambda_\xi(\varphi(|x-\xi|) - \varphi(|x/|x| - |x|\xi|))|_{\xi\in\partial S} = -(|x|^2 - 1)\frac{\varphi'(|x-\xi|)}{|x-\xi|}\Big|_{\xi\in\partial S}. \quad (30)$$

Proof. Taking into account the homogeneity of the operator Λ and the equality $\Lambda(uv) = v\Lambda u + u\Lambda v$, it is easy to obtain that

$$\Lambda_\xi \varphi(|x-\xi|) = \frac{\varphi'(|x-\xi|)}{2|x-\xi|}\Lambda_\xi(|x|^2 - 2x\cdot\xi + |\xi|^2) = \varphi'(|x-\xi|)\frac{|\xi|^2 - x\cdot\xi}{|x-\xi|}, \quad (31)$$

$$\Lambda_\xi \varphi(|x/|x| - \xi|x||) = \frac{\varphi'(|x/|x| - \xi|x||)}{2|x/|x| - |x|\xi|}\Lambda_\xi(1 - 2x\cdot\xi + |x|^2|\xi|^2) = \varphi'(|x/|x| - \xi|x||)\frac{|\xi|^2|x|^2 - x\cdot\xi}{|x/|x| - \xi|x||}. \quad (32)$$

Therefore, since $|\frac{x}{|x|} - \xi|x|| = |x-\xi|$ for $\xi \in \partial S$, and subtracting (32) from (31), we obtain (30). The lemma is proved. □

Theorem 8. *The solution* (29) *of the biharmonic Navier problem for* $f = 0$, $n > 2$, *and* φ_0 $\varphi_1 \in C(\partial S)$ *can be represented in the form*

$$u(x) = u_0(x) + \frac{|x|^2 - 1}{4} I_{\frac{n}{2}} u_1(x), \tag{33}$$

where the harmonic in S *functions* $u_k(x)$ *satisfy the conditions* $u_k|_{\partial S} = \varphi_k$ *for* $k = 0, 1$.

Proof. The equality (29) for $f = 0$ contains two terms. We transform each of them separately. Since $\mathcal{G}_2^n(x,\xi) \equiv G_2^n(x,\xi)$, then from the properties of the Green's function of the Dirichlet problem $G_2(x,\xi)$, it follows that

$$-\frac{1}{\omega_n} \int_{\partial S} \frac{\partial \mathcal{G}_2^n(x,\xi)}{\partial \nu} \varphi_0(\xi)\, ds_\xi = u_0(x).$$

Since $\frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu} = \Lambda_\xi \mathcal{G}_4^n(x,\xi)$ on ∂S, let us transform the function $\Lambda_\xi \mathcal{G}_4^n(x,\xi)|_{\partial S}$. From (28), we find

$$\Lambda_\xi \mathcal{G}_4^n(x,\xi) = \Lambda_\xi (E_4(x,\xi) - E_4^*(x,\xi)) - \frac{|x|^2 - 1}{4} \Lambda_\xi \big((|\xi|^2 - 1)(I_{\frac{n}{2}} E_2^*)(x,\xi)\big). \tag{34}$$

Let us calculate the first term from the right side of the resulting equality.

(1) Let $n \neq 2, 4$. Then, taking into account (3), and by Lemma 14 for $\varphi(t) = \frac{t^{4-n}}{2(n-2)(n-4)}$, we get

$$\Lambda_\xi (E_4(x,\xi) - E_4^*(x,\xi))|_{\xi \in \partial S} = \frac{|x-\xi|^{2-n}}{2(n-2)}\Big|_{\xi \in \partial S} (|x|^2 - 1) = \frac{|x|^2 - 1}{2} E_2^*(x,\xi)|_{\xi \in \partial S}.$$

(2) If $n = 4$, then, again taking into account (3), by Lemma 14 with $\varphi(t) = -\frac{1}{4} \ln t$, we obtain the same equality

$$\Lambda_\xi (E_4(x,\xi) - E_4^*(x,\xi))|_{\xi \in \partial S} = \frac{1}{4} |x-\xi|^{2-4}\big|_{\xi \in \partial S} (|x|^2 - 1) = \frac{|x|^2 - 1}{2} E_2^*(x,\xi)|_{\xi \in \partial S}.$$

Now, we calculate the second term from (34) without the factor depending on $|x|$

$$\Lambda_\xi \big((|\xi|^2 - 1)(I_{\frac{n}{2}} E_2^*)(x,\xi)\big)|_{\xi \in \partial S} = 2|\xi|^2 (I_{\frac{n}{2}} E_2^*)(x,\xi)|_{\xi \in \partial S} = 2(I_{\frac{n}{2}} E_2^*)(x,\xi)$$

and in total, we obtain

$$\Lambda_\xi \mathcal{G}_4^n(x,\xi)|_{\xi \in \partial S} = \frac{|x|^2 - 1}{2} \big(E_2^*(x,\xi) - (I_{\frac{n}{2}} E_2^*)(x,\xi)\big)|_{\xi \in \partial S}.$$

Let us transform the function $w(x,\xi) = E_2^*(x,\xi) - (I_{\frac{n}{2}} E_2^*)(x,\xi)$. By Lemma 10, we have

$$w(x,\xi) = \big(\Lambda_\xi + \frac{n}{2}\big)(I_{n/2} E_2^*)(x,\xi) - (I_{\frac{n}{2}} E_2^*)(x,\xi) = \big(\Lambda_\xi + \frac{n}{2} - 1\big) I_{\frac{n}{2}} E_2^*(x,\xi),$$

and using Remark 3, we write

$$w(x,\xi) = \int_0^1 (\Lambda_\xi + n/2 - 1) E_2^*(tx,\xi) t^{n/2-1} dt = \hat{I}_{\frac{n}{2}} (\Lambda_\xi + n/2 - 1) E_2^*(x,\xi).$$

In accordance with (32) for $\varphi(t) = \frac{t^{2-n}}{n-2}$, we find

$$(2\Lambda_\xi + n - 2)E_2^*(x,\xi)|_{\xi\in\partial S} = -\frac{2|x|^2 - 2x\cdot\xi}{|x/|x| - \xi|x||^n} + \frac{1 - 2x\cdot\xi + |x|^2}{|x/|x| - \xi|x||^n}\Big|_{\xi\in\partial S} = \frac{1-|x|^2}{|x/|x| - \xi|x||^n}\Big|_{\xi\in\partial S}.$$

At the same time, according to (30), and also for $\varphi(t) = \frac{t^{2-n}}{n-2}$, we obtain

$$\Lambda_\xi G_2(x,\xi)|_{\xi\in\partial S} = \Lambda_\xi(E_2(x,\xi) - E_2^*(x,\xi))|_{\xi\in\partial S} = -\frac{1-|x|^2}{|x-\xi|^n}\Big|_{\xi\in\partial S}.$$

Comparing the equalities obtained above, we are convinced that the following equality is true:

$$(2\Lambda_\xi + n - 2)E_2^*(x,\xi)|_{\xi\in\partial S} = -\Lambda_\xi G_2(x,\xi)|_{\xi\in\partial S}, \tag{35}$$

and therefore, when $\xi \in \partial S$,

$$w(x,\xi) = -\frac{1}{2}\int_0^1 \Lambda_\xi G_2(tx,\xi)t^{n/2-1}dt = -\frac{1}{2}\hat{I}_{\frac{n}{2}}G_2(x,\xi).$$

From here, recalling the notation for $w(x,\xi)$, we obtain

$$\Lambda_\xi \mathcal{G}_4^n(x,\xi)|_{\xi\in\partial S} = \frac{|x|^2-1}{2}w(x,\xi)|_{\xi\in\partial S} = -\frac{|x|^2-1}{4}\int_0^1 \Lambda_\xi G_2(tx,\xi)t^{n/2-1}dt|_{\xi\in\partial S}.$$

Thus, the second integral from (29) for $f = 0$ and $n > 2$ has the form

$$\frac{1}{\omega_n}\int_{\partial S}\frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu}\varphi_1(\xi)\,ds_\xi = -\frac{|x|^2-1}{4}\frac{1}{\omega_n}\int_{\partial S}\int_0^1 \Lambda_\xi G_2(tx,\xi)t^{n/2-1}dt\varphi_1(\xi)\,ds_\xi$$
$$= \frac{|x|^2-1}{4}\int_0^1\Big(-\frac{1}{\omega_n}\int_{\partial S}\Lambda_\xi G_2(tx,\xi)\varphi_1(\xi)\,ds_\xi\Big)t^{n/2-1}dt.$$

From here, using the properties of the Green function $G_2(x,\xi)$ of the Dirichlet problem, we obtain

$$\frac{1}{\omega_n}\int_{\partial S}\frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu}\varphi_1(\xi)\,ds_\xi = \frac{|x|^2-1}{4}\int_0^1 u_1(tx)t^{n/2-1}\,dt = \frac{|x|^2-1}{4}I_{\frac{n}{2}}u_1(x).$$

Thus, when $f = 0$, the equality (29) is transformed to (33). The theorem is proved. □

Remark 6. *It is easy to verify directly that in the case of $n = 2$, the equality (33) also defines a solution to the Navier problem for a homogeneous equation.*

5. Triharmonic Navier Problem

In Theorem 6, in the case of $n > 6$, the equality $\mathcal{G}_6^n(x,\xi) = G_6^n(x,\xi)$ is established, but the function $\mathcal{G}_6^n(x,\xi)$ from (22) is defined for $n > 4$. Therefore, it is necessary to verify that all the properties of the Green's function of the Navier problem are fulfilled for the function $\mathcal{G}_6^n(x,\xi)$.

Theorem 9. *The function $\mathcal{G}_6^n(x,\xi)$, defined for $x \neq \xi \in S$ and $n > 4$ from the equality (22) is 3-harmonic in $\xi \neq x \in S$, satisfying the equalities $\Delta_\xi \mathcal{G}_6^n(x,\xi) = -\mathcal{G}_4^n(x,\xi)$, $\Delta_\xi^2 \mathcal{G}_6^n(x,\xi) = G_2(x,\xi)$; and, for $x \in S$, the following conditions are set on the boundary:*

$$\mathcal{G}_6^n(x,\xi)|_{\xi\in\partial S} = \Delta_\xi \mathcal{G}_6^n(x,\xi)|_{\xi\in\partial S} = \Delta_\xi^2 \mathcal{G}_6^n(x,\xi)|_{\xi\in\partial S} = 0. \tag{36}$$

To represent the solution of Navier roblem (1)-(2) from the class $u \in C^6(S) \cap C^5(\bar{S})$, the following equality is valid:

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\Big(-\frac{\partial \mathcal{G}_2^n(x,\xi)}{\partial \nu}\varphi_0(\xi) + \frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu}\varphi_1(\xi) - \frac{\partial \mathcal{G}_6^n(x,\xi)}{\partial \nu}\varphi_2(\xi)\Big)ds_\xi - \frac{1}{\omega_n}\int_S \mathcal{G}_6^n(x,\xi)f(\xi)\,d\xi,\ x \in S. \quad (37)$$

Proof. To prove the equality (36), we establish that the equality $\Delta_\xi \mathcal{G}_6^n(x,\xi) = -\mathcal{G}_4^n(x,\xi)$ holds for $x \neq \xi \in S$. In this case, by virtue of Lemma 13, the remaining equalities follow from it. Therefore, in accordance with Definition 1, the function $\mathcal{G}_6^n(x,\xi)$ from (22), which has a singularity of the elementary solution type $E_6(x,\xi)$ for $\xi = x$, is the Green's function of the Navier problem for the 3-harmonic equation in S. Therefore, by [8] (Theorem 4.2), the integral representation (37) similar to (8) is true for the solution $u(x)$.

In the equality (22), we introduce the notation

$$v_1(x,\xi) = E_6(x,\xi) - E_6^*(x,\xi), \quad v_2(x,\xi) = \frac{(|x|^4-1)(|\xi|^4-1)}{32} I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi),$$
$$v_3(x,\xi) = -\frac{(|x|^2-1)(|\xi|^2-1)}{16}(I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) I_{\frac{n}{2}} E_2^*(x,\xi). \quad (38)$$

Then, we have $\mathcal{G}_6^n(x,\xi) = v_1(x,\xi) + v_2(x,\xi) + v_3(x,\xi)$.

(a) From (6), it immediately follows that

$$\Delta_\xi v_1(x,\xi) = -E_4(x,\xi) + |x|^2 E_4^*(x,\xi) = -E_4(x,\xi) + E_4^*(x,\xi) + (|x|^2-1)E_4^*(x,\xi).$$

(b) Let us use Lemma 11 for $\varphi(t) = t^2 - 1$ and $v(\xi) = I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi)$. With the notations made according to (25), taking into account Lemma 10 and the harmonicity of $v(\xi)$, we obtain

$$\Delta_\xi v_2(x,\xi) = \frac{|x|^4-1}{32}\Delta_\xi(|\xi|^4-1) I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi)$$
$$= \frac{(|x|^4-1)|\xi|^2}{4}(\Lambda_\xi + n/2 + 1) I_{\frac{n}{2}+1} I_{\frac{n}{2}} E_2^*(x,\xi) = \frac{(|x|^4-1)|\xi|^2}{4} I_{\frac{n}{2}} E_2^*(x,\xi).$$

(c) Again, we use Lemma 11, but for $\varphi(t) = t - 1$ and $v(\xi) = (I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) I_{\frac{n}{2}} E_2^*(x,\xi)$. Taking into account Lemma 10 and the harmonicity of $v(\xi)$, we have

$$\Delta_\xi v_3(x,\xi) = -\frac{(|x|^2-1)}{16}\Delta_\xi(|\xi|^2-1)(I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) I_{\frac{n}{2}} E_2^*(x,\xi)$$
$$= -\frac{(|x|^2-1)}{4}(\Lambda_\xi + n/2) I_{\frac{n}{2}}(I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) E_2^*(x,\xi) = -\frac{(|x|^2-1)}{4}(I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) E_2^*(x,\xi).$$

Summing up the obtained equalities, we find

$$\Delta_\xi \mathcal{G}_6^n(x,\xi) = -E_4(x,\xi) + E_4^*(x,\xi) + \frac{|x|^2-1}{4}$$
$$\times \Big(4E_4^*(x,\xi) + (|x|^2+1)|\xi|^2 I_{\frac{n}{2}} E_2^*(x,\xi) - (I_{\frac{n}{2}-2} + I_{\frac{n}{2}}) E_2^*(x,\xi)\Big) = -E_4(x,\xi) + E_4^*(x,\xi)$$
$$+ \frac{|x|^2-1}{4}\Big(4E_4^*(x,\xi) + (|x|^2|\xi|^2 I_{\frac{n}{2}} - I_{\frac{n}{2}-2}) E_2^*(x,\xi)\Big) + \frac{(|x|^2-1)(|\xi|^2-1)}{4} I_{\frac{n}{2}} E_2^*(x,\xi).$$

Now, if we use (27), then the equality obtained above takes the form

$$\Delta_\xi \mathcal{G}_6^n(x,\xi) = -E_4(x,\xi) + E_4^*(x,\xi) + \frac{(|x|^2-1)(|\xi|^2-1)}{4} I_{\frac{n}{2}} E_2^*(x,\xi).$$

Recalling (13), we have $\Delta_\xi \mathcal{G}_6^n(x,\xi) = -\mathcal{G}_4^n(x,\xi)$. The theorem is proved. □

For another representation of the solution to the Navier problem, one equality is needed.

Lemma 15. *For $n > 4$ and $n = 3$, the following equality holds:*

$$\Lambda_\xi v_1(x,\xi)|_{\xi\in\partial S} \equiv \Lambda_\xi (E_6(x,\xi) - E_6^*(x,\xi))|_{\xi\in\partial S} = \frac{|x|^2-1}{4} E_4^*(x,\xi)|_{\xi\in\partial S}. \tag{39}$$

Proof. (a) Let $n > 6$, or $n = 3$, or $n = 5$; then, $E_6(x,\xi)$ is found from the first line of (4). According to Lemma 14, for $\varphi(t) = \frac{t^{6-n}}{8(n-2)(n-4)(n-6)}$, we obtain

$$\Lambda_\xi (E_6(x,\xi) - E_6^*(x,\xi))|_{\xi\in\partial S} = \frac{|x-\xi|^{4-n}}{8(n-2)(n-4)} (|x|^2-1)|_{\xi\in\partial S},$$

and since by (3),

$$\frac{|x|^2-1}{4} E_4^*(x,\xi) = \frac{|\xi/|\xi| - x|\xi||^{4-n}}{8(n-2)(n-4)} (|x|^2-1),$$

then (39) is fulfilled.

(b) Let $n = 6$. Then, from (4), we find $E_6(x,\xi) = -\frac{1}{64}\ln|\xi - x|$, and by Lemma 14 with $\varphi(t) = -\frac{\ln t}{64}$, taking into account (3) and equalities $8(n-2)(n-4) = 64$ and $4 - n = -2$, we obtain (39):

$$\Lambda_\xi (E_6(x,\xi) - E_6^*(x,\xi))|_{\xi\in\partial S} = -(|x|^2-1)\frac{-|\xi-x|^{-2}}{64}\Big|_{\xi\in\partial S} = \frac{|x-\xi|^{4-n}}{8(n-2)(n-4)} (|x|^2-1)\big|_{\xi\in\partial S}.$$

For $n = 4$, the equality (39) is not satisfied. The lemma is proved. □

Let us formulate a statement generalizing Theorem 8 to the 3-harmonic problem.

Theorem 10. *The solution of the Navier roblem (1)-(2), written according to (37) for $f = 0$ and $n > 4$, can be represented in the form*

$$u(x) = u_0(x) + \frac{|x|^2-1}{4} I_{\frac{n}{2}} u_1(x) - \Big(\frac{|x|^2-1}{16} I_{\frac{n}{2}} - \frac{|x|^4-1}{32} I_{\frac{n}{2}+1}\Big) I_{\frac{n}{2}} u_2(x), \tag{40}$$

where the harmonic functions $u_k(x)$ in S satisfy the conditions $u_k|_{\partial S} = \varphi_k$ for $k = 0,1,2$, and the operators I_α are applied by the variable x.

The equality (40) is similar to the equality for representing the solution of the Dirichlet problem from [29], but instead of the operators $\Lambda + \alpha$, their inverse operators I_α are used here.

Proof. Let $n > 4$. By Theorem 8, the sum of the first two terms in the equality (37) for $f = 0$ and $n > 2$ has the form

$$\frac{1}{\omega_n}\int_{\partial S}\Big(-\frac{\partial \mathcal{G}_2^n(x,\xi)}{\partial \nu}\varphi_0(\xi) + \frac{\partial \mathcal{G}_4^n(x,\xi)}{\partial \nu}\varphi_1(\xi)\Big)\, ds_\xi = u_0(x) + \frac{|x|^2-1}{4} I_{\frac{n}{2}} u_1(x), \tag{41}$$

which coincides with the first two terms in (40). Therefore, it is sufficient to transform the third integral included in (37):

$$J(x) = -\frac{1}{\omega_n}\int_{\partial S}\frac{\partial \mathcal{G}_6^n(x,\xi)}{\partial \nu}\varphi_2(\xi)\,ds_\xi,$$

where the function $\mathcal{G}_6^n(x,\xi)$ is found from (22). Similar to the proof of Theorem 8, we study the function $\Lambda_\xi \mathcal{G}_6^n(x,\xi)$ on ∂S. To perform this, we recall the notation (38), under which we have

$$\Lambda_\xi \mathcal{G}_6^n(x,\xi) = \Lambda_\xi v_1(x,\xi) + \Lambda_\xi v_2(x,\xi) + \Lambda_\xi v_3(x,\xi).$$

Let us examine each term in this equality.

(1) Using Lemma 15 we can obtain

$$\Lambda_\xi v_1(x,\xi)|_{\xi\in\partial S} \equiv \Lambda_\xi(E_6(x,\xi) - E_6^*(x,\xi))|_{\xi\in\partial S} = \frac{|x|^2-1}{4}E_4^*(x,\xi)|_{\xi\in\partial S}.$$

By means of the equality (27) from Lemma 12, the resulting equality can be written as

$$\begin{aligned}\Lambda_\xi v_1(x,\xi)|_{\xi\in\partial S} =& \Big(\frac{|x|^2-1}{16}I_{\frac{n}{2}-2} - |x|^2\frac{|x|^2-1}{16}I_{\frac{n}{2}}\Big)E_2^*(x,\xi)|_{\xi\in\partial S}\\ =& \Big(\frac{|x|^2-1}{16}(I_{\frac{n}{2}-2}+I_{\frac{n}{2}}) - \frac{|x|^4-1}{16}I_{\frac{n}{2}}\Big)E_2^*(x,\xi)|_{\xi\in\partial S}.\end{aligned}$$

(2) For the second term, we have

$$\begin{aligned}\Lambda_\xi v_2(x,\xi)|_{\xi\in\partial S} =& \frac{|x|^4-1}{32}\Lambda_\xi\big((|\xi|^4-1)I_{\frac{n}{2}+1}I_{\frac{n}{2}}E_2^*(x,\xi)\big)|_{\xi\in\partial S}\\ =& \frac{|x|^4-1}{8}I_{\frac{n}{2}+1}I_{\frac{n}{2}}E_2^*(x,\xi)|_{\xi\in\partial S}.\end{aligned}$$

(3) Similarly, for the third term, we obtain

$$\begin{aligned}\Lambda_\xi v_3(x,\xi)|_{\xi\in\partial S} =& -\frac{|x|^2-1}{16}\Lambda_\xi\big((|\xi|^2-1)(I_{\frac{n}{2}-2}+I_{\frac{n}{2}})I_{\frac{n}{2}}E_2^*(x,\xi)\big)|_{\xi\in\partial S}\\ =& -\frac{|x|^2-1}{8}(I_{\frac{n}{2}-2}+I_{\frac{n}{2}})I_{\frac{n}{2}}E_2^*(x,\xi)|_{\xi\in\partial S}.\end{aligned}$$

If we sum up the found values of $\Lambda_\xi v_i(x,\xi)|_{\partial S}$, then we obtain

$$\begin{aligned}\Lambda_\xi \mathcal{G}_6^n(x,\xi)|_{\xi\in\partial S} = \Big(&\frac{|x|^2-1}{16}\big(I_{\frac{n}{2}-2}+I_{\frac{n}{2}} - 2I_{\frac{n}{2}-2}I_{\frac{n}{2}} - 2I_{\frac{n}{2}}^2\big)\\ &- \frac{|x|^4-1}{16}\big(I_{\frac{n}{2}} - 2I_{\frac{n}{2}+1}I_{\frac{n}{2}}\big)\Big)E_2^*(x,\xi)|_{\xi\in\partial S}.\end{aligned}$$

If we denote $J_1 = I_{\frac{n}{2}-2} + I_{\frac{n}{2}} - 2I_{\frac{n}{2}-2}I_{\frac{n}{2}} - 2I_{\frac{n}{2}}^2$ and $J_2 = I_{\frac{n}{2}} - 2I_{\frac{n}{2}+1}I_{\frac{n}{2}}$, then

$$\Lambda_\xi \mathcal{G}_6^n(x,\xi)|_{\partial S} = \frac{|x|^2-1}{16}J_1E_2^*(x,\xi)|_{\partial S} - \frac{|x|^4-1}{16}J_2E_2^*(x,\xi)|_{\partial S}.$$

(a) Let us transform the operator J_1. Using (21), we find $I_{\frac{n}{2}-2}I_{\frac{n}{2}} = \frac{1}{2}(I_{\frac{n}{2}-2} - I_{\frac{n}{2}})$, and therefore,

$$J_1 = I_{\frac{n}{2}-2} + I_{\frac{n}{2}} - I_{\frac{n}{2}-2} + I_{\frac{n}{2}} - 2I_{\frac{n}{2}}^2 = 2(1-I_{\frac{n}{2}})I_{\frac{n}{2}}.$$

Using Lemma 10. we write $1 - I_{\frac{n}{2}} = (\Lambda_\xi + n/2)I_{\frac{n}{2}} - I_{\frac{n}{2}} = (\Lambda_\xi + n/2 - 1)I_{\frac{n}{2}}$, and therefore, we have $J_1 = (2\Lambda_\xi + n - 2)I_{\frac{n}{2}}$. Now, using Remark 3 and equality (35), we find

$$J_1 E_2^*(x,\xi)|_{\xi\in\partial S} = (2\Lambda_\xi + n - 2)\hat{I}_{\frac{n}{2}} E_2^*(x,\xi)|_{\xi\in\partial S} = -\hat{I}_{\frac{n}{2}}\Lambda_\xi G_2(x,\xi)|_{\xi\in\partial S}.$$

(b) Let us transform the operator J_2. Taking into account the properties of the operators I_α and Λ_ξ from Lemma 10, we can write

$$J_2 = I_{\frac{n}{2}} - 2I_{\frac{n}{2}+1}I_{\frac{n}{2}} = (\Lambda_\xi + n/2 + 1)I_{\frac{n}{2}}I_{\frac{n}{2}+1} - 2I_{\frac{n}{2}}I_{\frac{n}{2}+1} = \frac{1}{2}(2\Lambda_\xi + n - 2)I_{\frac{n}{2}}I_{\frac{n}{2}+1}.$$

If we now use Remark 3 and equality (35), we find

$$J_2 E_2^*(x,\xi)|_{\xi\in\partial S} = \frac{1}{2}(2\Lambda_\xi + n - 2)\hat{I}_{\frac{n}{2}}\hat{I}_{\frac{n}{2}+1}E_2^*(x,\xi)|_{\xi\in\partial S} = -\frac{1}{2}\hat{I}_{\frac{n}{2}}\hat{I}_{\frac{n}{2}+1}\Lambda_\xi G_2(x,\xi)|_{\xi\in\partial S}.$$

Given the found values of $J_1 E_2^*(x,\xi)|_{\xi\in\partial S}$ and $J_2 E_2^*(x,\xi)|_{\xi\in\partial S}$, we can write

$$\Lambda_\xi \mathcal{G}_6^n(x,\xi)|_{\partial S} = -\Big(\frac{|x|^2-1}{16}\hat{I}_{\frac{n}{2}} - \frac{|x|^4-1}{32}\hat{I}_{\frac{n}{2}}\hat{I}_{\frac{n}{2}+1}\Big)\Lambda_\xi G_2(x,\xi)|_{\xi\in\partial S}.$$

Let us calculate the integral $J(x)$. Multiplying the resulting equality by $-\varphi_2(\xi)/\omega_n$, then integrating it over $\xi \in \partial S$ and using the following property of Green's function $G_2(x,\xi)$:

$$-\frac{1}{\omega_n}\int_{\partial S}\Lambda_\xi G_2(x,\xi)\varphi_2(\xi)\,ds_\xi = u_2(x),$$

we obtain

$$\begin{aligned} J(x) &= -\frac{1}{\omega_n}\int_{\partial S}\frac{\partial \mathcal{G}_6^n(x,\xi)}{\partial \nu}\varphi_2(\xi)\,ds_\xi = -\frac{1}{\omega_n}\int_{\partial S}\Lambda_\xi \mathcal{G}_6^n(x,\xi)\varphi_2(\xi)\,ds_\xi \\ &= \Big(\frac{|x|^2-1}{16}\hat{I}_{\frac{n}{2}} - \frac{|x|^4-1}{32}\hat{I}_{\frac{n}{2}}\hat{I}_{\frac{n}{2}+1}\Big)\frac{1}{\omega_n}\int_{\partial S}\Lambda_\xi G_2(x,\xi)\varphi_2(\xi)\,ds_\xi \\ &= -\Big(\frac{|x|^2-1}{16}\hat{I}_{\frac{n}{2}} - \frac{|x|^4-1}{32}\hat{I}_{\frac{n}{2}}\hat{I}_{\frac{n}{2}+1}\Big)u_2(x). \end{aligned}$$

Since the operators $\hat{I}_\alpha$ in this equality act on the function $u_2(x)$ of one variable, they can be replaced by the operators I_α. Recalling (41), the right-hand side of (37) at $f = 0$ is transformed to the right-hand side of (40). The theorem is proved. □

The condition $n > 4$ from Theorem 10, as it turns out, is not essential.

Theorem 11. *The solution of the Navier roblem (1)-(2) for the homogeneous Equation (1) for $n \geq 2$ can be written as (40). If $\varphi_k \in C(\partial S)$, $k = 0,1,2$, then u, Δu, $\Delta^2 u \in C(\bar{S})$, and the boundary conditions (2) of the Navier problem are satisfied.*

Proof. (1) Since, according to the conditions of the theorem, $\varphi_k \in C(\partial S)$ and $k = 0,1,2$, then $u_k \in C(\bar{S})$, and hence, from (40), we obtain $u \in C(\bar{S})$. Obviously, $u|_{\partial S} = u_0|_{\partial S} = \varphi_0$.

(2) Let us find Δu. Using the equality (25) from Lemma 11, and taking into account the preservation of the harmonicity of the function by the operators I_α and the properties of the operators I_α from Lemma 10, we write

$$\begin{aligned} \Delta u(x) = 0 + \Big(\Lambda + \frac{n}{2}\Big)I_{\frac{n}{2}}u_1(x) - \Big(\frac{1}{4}\big(\Lambda + \frac{n}{2}\big)I_{\frac{n}{2}} \\ - \frac{|x|^2}{4}\big(1 + \Lambda + \frac{n}{2}\big)I_{\frac{n}{2}+1}\Big)I_{\frac{n}{2}}u_2(x) = u_1(x) - \frac{1-|x|^2}{4}I_{\frac{n}{2}}u_2(x). \end{aligned}$$

From this, it follows that $\Delta u \in C(\bar{S})$ and $\Delta u|_{\partial S} = u_1|_{\partial S} = \varphi_1$.

(3) Let us find $\Delta^2 u$. Using the found value of $\Delta u(x)$, (25), and the properties of the operators I_α from Lemma 10, we write

$$\Delta^2 u(x) = \Delta\Big(u_1(x) + \frac{|x|^2-1}{4} I_{\frac{n}{2}} u_2(x)\Big) = 0 + \Big(\Lambda + \frac{n}{2}\Big) I_{\frac{n}{2}} u_2(x) = u_2(x).$$

It follows that $\Delta^2 u \in C(\bar{S})$ and $\Delta^2 u|_{\partial S} = u_2|_{\partial S} = \varphi_2$. The theorem is proved. □

Remark 7. *A slightly different form of writing the solution* (40) *to the Navier problem is possible in the form*

$$u(x) = u_0(x) + \frac{|x|^2-1}{4} I_{\frac{n}{2}} u_1(x) - \frac{|x|^2-1}{16}\Big(I_{\frac{n}{2}} - \frac{|x|^2-1}{2}\Big) I_{\frac{n}{2}+1} I_{\frac{n}{2}} u_2(x).$$

This is so because $|x|^4 - 1 = (|x|^2-1)^2 + 2(|x|^2-1)$ *and* $I_{\frac{n}{2}} I_{\frac{n}{2}+1} + I_{\frac{n}{2}+1} = I_{\frac{n}{2}}$.

Example 1. *Consider the Navier roblem* (1)-(2) *with simple data:*

$$\Delta^3 u(x) = 1,\ x \in S; \quad u|_{\partial S} = 0,\ \Delta u|_{\partial S} = -1,\ \Delta^2 u|_{\partial S} = 1.$$

Using (40)*, we first find* $\hat{u}$*—the solution to the Navier problem for a homogeneous equation. Since* $u_0 = 0$, $u_1 = -1$, $u_2 = 1$, *then*

$$\hat{u}(x) = \frac{|x|^2-1}{4} I_{\frac{n}{2}}[-1] - \Big(\frac{|x|^2-1}{16} I_{\frac{n}{2}} - \frac{|x|^4-1}{32} I_{\frac{n}{2}+1}\Big) I_{\frac{n}{2}}[1].$$

It is not difficult to see that

$$I_{\frac{n}{2}}[-1] = \int_0^1 (-1) t^{n/2-1}\, dt = -\frac{2}{n}, \quad I_{\frac{n}{2}+1}[1] = \frac{2}{n+2},$$

and that means

$$\hat{u}(x) = -\frac{|x|^2-1}{2n} - \frac{2}{n}\Big(\frac{|x|^2-1}{8n} - \frac{|x|^4-1}{16(n+2)}\Big) = -\frac{2n+1}{4n^2}(|x|^2-1) + \frac{|x|^4-1}{8n(n+2)}.$$

Let us find the solution $u_h(x)$ *of the homogeneous Navier problem, i.e., at* $\varphi_k = 0$, $k = 0, 1, 2$, *using* (8)*. In [39], it is established that*

$$\frac{1}{\omega_n} \int_{\partial S} G_2(x,\xi) |x|^{2l} H_k^{(i)}(\xi)\, ds_\xi = -\frac{|x|^{2l+2}-1}{c_{l,k}} H_k^{(i)}(x),$$

where $c_{l,k} = (2l+2)(2l+2k+n)$*. Making calculations using* (7)*, we obtain*

$$\frac{1}{\omega_n} \int_{|\xi|<1} G_6^n(x,\xi) |x|^{2l} H_k^{(i)}(\xi)\, d\xi = \Big(\frac{-|x|^{2l+6}+1}{c_{l,k}c_{l+1,k}c_{l+2,k}} + \frac{|x|^4-1}{c_{l,k}c_{0,k}c_{1,k}} + \Big(\frac{1}{c_{l+1,k}} - \frac{1}{c_{0,k}}\Big)\frac{|x|^2-1}{c_{0,k}c_{l,k}}\Big) H_k^{(i)}(x).$$

Therefore, since in our problem $H_0^{(1)}(x) = 1$, *then taking into account that* $c_{0,0} = 2n$, $c_{1,0} = 4(n+2)$ $c_{2,0} = 6(n+4)$, *we obtain*

$$\begin{aligned} u_h(x) =& -\frac{1}{\omega_n}\int_{|\xi|<1} G_6^n(x,\xi)\,d\xi \\ =& \frac{|x|^6-1}{48n(n+2)(n+4)} - \frac{|x|^4-1}{16n^2(n+2)} + \frac{(n+4)(|x|^2-1)}{16n^3(n+2)}. \end{aligned}$$

Thus, we find the desired solution to the Navier problem, which is written in the form

$$\begin{aligned} u(x) =& \hat{u}(x) + u_h(x) = -\frac{2n+1}{4n^2}(|x|^2-1) \\ &+ \frac{|x|^4-1}{8n(n+2)} + \frac{|x|^6-1}{48n(n+2)(n+4)} - \frac{|x|^4-1}{16n^2(n+2)} + \frac{(n+4)(|x|^2-1)}{16n^3(n+2)}. \end{aligned}$$

It is easy to check that $u|_{\partial S} = 0$, $\Delta u|_{\partial S} = -1$, $\Delta^2 u|_{\partial S} = 1$ *and* $\Delta^3 u = 1$. *For* $n = 2$, *we obtain*

$$u(x,y) = \frac{665}{2304} - \frac{77x^2}{256} - \frac{77y^2}{256} + \frac{3x^4}{256} + \frac{3x^2y^2}{128} + \frac{3y^4}{256} + \frac{x^6}{2304} + \frac{x^2y^4}{768} + \frac{x^4y^2}{768} + \frac{y^6}{2304}.$$

6. Conclusions

In this study, based on the properties of elementary solutions $E_6(x,\xi)$ and $E_4(x,\xi)$, a new representation of the Green's function $\mathcal{G}_6^n(x,\xi)$ of the Navier problem for the 3-harmonic equation in a ball is obtained. Then, a representation of the solution of the Navier problem for a homogeneous equation without explicitly using the Green's function $\mathcal{G}_6^n(x,\xi)$ is found. The initiated research can be continued in two directions: constructing Green's functions of other problems for the 3-harmonic equation, for example, the Dirichlet-2 problem [31], and finding a representation of the Green's function of the Navier problem for the polyharmonic equation.

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