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Quasilinear Fractional Neumann Problems

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Abstract: We study an elliptic quasilinear fractional problem with fractional Neumann boundary conditions, proving an existence and multiplicity result without assuming the classical Ambrosetti–Rabinowitz condition. Improving previous results, we also provide the weak formulation of solutions without regularity assumptions and we provide an example, even in the linear case, for which no regularity can indeed be assumed.

Keywords: mixed local and fractional p-Laplacians; Neumann boundary conditions; regularity; superlinear problems

MSC: 35R11; 35A15; 47J30; 45G05

1. Introduction

The aim of this paper is to study quasilinear problems driven by mixed operators with fractional Neumann boundary conditions. More precisely, we consider the operator

$$\mathcal{L} := -\alpha\Delta_p + \beta(-\Delta)_p^s. \tag{1}$$

Here, $\alpha, \beta > 0$, $s \in (0,1)$, $p > 1$, Δ_p is the classical $p-$Laplacian, acting as $\Delta_p u := \mathrm{div}(|Du|^{p-2}Du)$, and $(-\Delta)_p^s$ is the fractional p-Laplacian, namely

$$(-\Delta)_p^s u(x) = C_{N,s,p}\, PV \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy. \tag{2}$$

Here, PV stands for the Cauchy principal value and the constant $C_{N,s,p}$, defined as

$$C_{N,s,p} := \frac{s2^{2s-1}\Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{N/2}\Gamma(1-s)},$$

is the usual normalization constant for $(-\Delta)_p^s$ (see [1] for more details), but its value will not play a role in our analysis.

The operator in (1) acts on a bounded open subset $\Omega \subset \mathbb{R}^N$ with a smooth boundary of class C^1.

Mixed operators of the form (1) have raised increasing interest in recent years, mainly in the Hilbert setting (namely, when $p = 2$), but the quasilinear case has also been the object of several results about existence, multiplicity, and qualitative properties of solutions, both in the elliptic and in the parabolic case. The list of references being huge, we just quote

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some very recent ones and their related bibliography, [2–16]. We also refer to [17], where a more general superposition of local and nonlocal operators is considered.

As for the boundary conditions, we combine the operator in (1) with the so-called (α, β)-Neumann conditions, introduced in [18] for $p = 2$ and in [19] for general p. These conditions are made of two contributions. The first one corresponds to the local part and is defined on $\partial\Omega$, while the second corresponds to the nonlocal part and is defined on $\mathbb{R}^N \setminus \overline{\Omega}$, namely

$$(\alpha, \beta)\text{-Neumann conditions:} \quad \begin{cases} \mathcal{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here,

$$\mathcal{N}_{s,p} u(x) := C_{N,s,p} \int_\Omega |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x - y|^{N+ps}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{4}$$

is the *nonlocal normal $p-$derivative*, or *fractional $p-$Neumann boundary condition* and describes the natural Neumann boundary condition in the presence of the fractional $p-$Laplacian. It was introduced in [20,21] as an extension of the notion of nonlocal normal derivative introduced in [22] for the fractional Laplacian, i.e., for $p = 2$. Nonlocal Neumann conditions have been treated recently, for instance in [23–31], as they imply a number of mathematical advantages that different conditions do not permit, see [22].

From (3) it is clear that the name (α, β)-Neumann conditions is related to the fact that they consider both the local part and the nonlocal part of the operator in (1).

From now on, for the sake of simplicity, we will set $C_{N,s,p} = 1$.

In this framework, we are interested in dealing with problems of type

$$\begin{cases} -\alpha \Delta_p u + \beta (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ \text{with } (\alpha, \beta)\text{-homogeneous Neumann conditions.} \end{cases} \tag{5}$$

More precisely, we start in Section 2, giving the definition of the suitable functional space to study problems such as (5). We also recall the nonlocal counterpart of the divergence theorem and the integration by parts formula stated in [20,21]. However, we give these results in a more general setting, and we also give an example to better explain why such a generalization is needed. In particular, we consider the *linear case in dimension 1*, showing that, in some cases, solutions cannot be differentiable in the whole space, see Example 1. Moreover, we give the definition of weak solutions and some properties that they satisfy.

We give our main results in Sections 3 and 4. In the former, under a suitable hypothesis, we give an L^∞ estimate for weak solutions; namely, we prove that they are bounded in the whole of $\mathbb{R}^N$. In this case, the proof mainly relies on a suitable choice of test functions and an iteration argument.

In Section 4, we deal with a superlinear problem in the presence of a source term, which does not satisfy the so-called Ambrosetti–Rabinowitz condition. In particular, we prove the existence of two nontrivial weak solutions, which do not change signs. Here, the strategy is to apply a Mountain Pass argument to suitable truncated functionals. Moreover, the absence of the Ambrosetti–Rabinowitz condition makes it harder to prove a compactness property for such functionals, as they do not satisfy the Palais–Smale condition. To overcome this difficulty, we prove that these functionals satisfy the Cerami condition. After that, by using the Mountain Pass Theorem with the Cerami condition, we prove the existence of one nontrivial solution for problem (5).

As far as we know, these are the first results in the quasilinear case for operators of this type.

Finally, in Appendix A we give the details on the results stated in Example 1 of Section 2.

2. Functional Setting

In this section, we give the correct framework in order to study the operator in (1) with (α, β)-Neumann conditions. First, we introduce the norm

$$\|u\|^p := \alpha \int_\Omega |Du|^p \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dxdy + \int_\Omega |u(x)|^p \, dx,$$

where $\mathcal{Q} = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$. Thus, we can define the space

$$X_{\alpha,\beta} := \{u : \mathbb{R}^N \to \mathbb{R} : \|u\| < +\infty\}.$$

We observe that, setting

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \, : \, u \in L^p(\Omega) \text{ and } \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dxdy < +\infty \right\},$$

we can write $X_{\alpha,\beta} = W^{1,p}(\Omega) \cap X$. Then, it is not hard to see that $X_{\alpha,\beta}$ is an uniformly convex Banach space. Moreover, if

$$p^* = \begin{cases} \frac{pN}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases}$$

we have that the embedding of $X_{\alpha,\beta}$ in $L^r(\Omega)$ is compact for every $r \in [1, p^*)$.

Now, we recall the analogous of the divergence theorem and of the integration by parts formula for the nonlocal case:

Proposition 1. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be such that $u \in C^2(\Omega)$ and*

$$(-\Delta)_p^s u \in L^1(\Omega).$$

Assume that the function

$$\Omega \times (\mathbb{R}^N \setminus \Omega) \ni (x,y) \mapsto \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{N+ps}} \quad \textit{belongs to } L^1(\Omega \times (\mathbb{R}^N \setminus \Omega)).$$

Then,

$$\int_\Omega (-\Delta)_p^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_{s,p} u \, dx. \tag{6}$$

Proposition 2. *Let $u, v : \mathbb{R}^N \to \mathbb{R}$ be such that $u, v \in C^2(\Omega)$ and*

$$\mathcal{Q} \ni (x,y) \mapsto \frac{|u(x) - u(y)|^{p-1}(v(x) - v(y))}{|x-y|^{N+ps}} \quad \textit{belongs to } L^1(\mathcal{Q}).$$

Assume, also, that

$$(-\Delta)_p^s u \, v \in L^1(\Omega)$$

and that the function

$$(\mathbb{R}^N \setminus \Omega) \times \Omega \ni (x,y) \mapsto \frac{|u(x) - u(y)|^{p-1} v(x)}{|x-y|^{N+ps}} \quad \textit{belongs to } L^1((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

Then,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy\\ =\int_\Omega v(-\Delta)_p^s u\,dx+\int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_{s,p}u\,dx.\end{aligned}\tag{7}$$

A few comments on Propositions 1 and 2 are mandatory. These results were first given in [20] (Theorem 6.3), where u and v are assumed to be in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. They are also stated in [21] (Propositions 2.5 and 2.6) for any u and v bounded and of class $C^2(\mathbb{R}^N)$.

However, these regularity assumptions on u and v may be too much as solutions of problem (5) may not even be of class $C^1(\mathbb{R}^N)$ as we will show in the forthcoming Example 1. For this reason, in a similar fashion to [32] (Lemmas 5.1 and 5.2), which cover the case $p=2$, we give Propositions 1 and 2 in a more general setting. We stress that our assumptions are enough to guarantee that the integrals in (6) and (7) are finite. Moreover, the proofs remain the same.

Next, we give an example of a solution for problem (5), which is not of class $C^1(\mathbb{R}^N)$ when $s\in(0,1/2]$. For more details, see Appendix A.

Example 1. *Let $N=1$, $p=2$ and $\Omega=(-1,1)$. We start defining $u:(-1,1)\to\mathbb{R}$ as*

$$u(x)=x(x-1)^2(x+1)^2.$$

Clearly,

$$\lim_{x\to-1^+}u'(x)=\lim_{x\to1^-}u'(x)=0.$$

Then, defining for every $x>1$

$$g_s(x):=\frac{\displaystyle\int_{-1}^1\frac{y(y-1)^2(y+1)^2}{(x-y)^{1+2s}}\,dy}{\displaystyle\int_{-1}^1\frac{dy}{(x-y)^{1+2s}}},$$

we can extend u in $\mathbb{R}$ as

$$u(x)=\begin{cases}-g_s(-x) & x<-1\\ x(x-1)^2(x+1)^2 & -1\le x\le1\\ g_s(x) & x>1.\end{cases}$$

We note that by extending u in this way, we have $u\in C(\mathbb{R})$ and $\mathcal{N}_s u(x)=0$ for any $x\in\mathbb{R}\setminus[-1,1]$. In addition,

$$\lim_{x\to1^+}u'(x)=\lim_{x\to-1^-}u'(x)=\begin{cases}+\infty & s\in(0,\frac12),\\ \frac43 & s=\frac12,\\ 0 & s\in(\frac12,1),\end{cases}$$

so $u\notin C^1(\mathbb{R})$ if $s\in(0,1/2]$.

On the other hand, computing $u''(x)$ and $(-\Delta)^s u(x)$ in $(-1,1)$ (see also [33] for more details on the computation of $(-\Delta)^s u(x)$), we can define

$$f_s(x):=-u''(x)+(-\Delta)^s u(x).$$

In this way, we have that u is a solution of

$$\begin{cases} -u''(x) + (-\Delta)^s u(x) = f_s(x) & x \in (-1,1), \\ u'(x) = 0 & x \in \{-1,1\}, \\ \mathcal{N}_s u = 0 & x \in \mathbb{R} \setminus [-1,1], \end{cases}$$

where $f_s \in L^q$ for any $q \geq 1$ (see Appendix A).

We also stress that, even if $s \in (0,1/2]$, u satisfies the hypotheses of Proposition 1. This is in agreement with the computation

$$\int_{-1}^{1} (-\Delta)^s u(x)\, dx = 0.$$

The integration by parts formula in Proposition 2 leads to the definition of weak solutions, which shows that $X_{\alpha,\beta}$ is the natural space for problems ruled by the operator in (1). We remark that we give the definition of weak solutions and the next two results in the case of non homogeneous boundary conditions. In this case, the functional space that we consider is

$$X_{\alpha,\beta}^{g,h} := \left\{ u : \mathbb{R}^N \to \mathbb{R} \, : \|u\| + \int_{\partial\Omega} hu\, d\sigma + \int_{\mathbb{R}^N \setminus \overline{\Omega}} gu\, dx \in \mathbb{R} \right\}$$

Definition 1. *Let $f \in L^{p'}(\Omega)$, $g \in L^1(\mathbb{R}^N \setminus \overline{\Omega})$ and $h \in L^1(\partial\Omega)$. We say that $u \in X_{\alpha,\beta}^{g,h}$ is a weak solution of*

$$\begin{cases} -\alpha\Delta_p u + \beta(-\Delta)_p^s u = f(x) & \text{in } \Omega, \\ \beta \mathcal{N}_{s,p} u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega, \end{cases} \tag{8}$$

whenever

$$\begin{aligned} &\alpha \int_\Omega Du \cdot Dv\, dx + \frac{\beta}{2} \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy \\ &= \int_\Omega fv\, dx + \int_{\partial\Omega} hv\, d\sigma + \int_{\mathbb{R}^N \setminus \overline{\Omega}} gv\, dx \end{aligned} \tag{9}$$

for every $v \in X_{\alpha,\beta}^{g,h}$, where

$$J_p(u(x)-u(y)) := |u(x)-u(y)|^{p-2}(u(x)-u(y)).$$

Now, we give a sort of maximum principle for weak solutions of (8).

Proposition 3. *Let $f \in L^{p'}(\Omega)$, $g \in L^1(\mathbb{R}^N \setminus \Omega)$, and $h \in L^1(\partial\Omega)$. Let $u \in X_{\alpha,\beta}^{g,h}$ be a weak solution of (8) with $f \geq 0$, $g \geq 0$ and $h \geq 0$. Then, u is constant.*

Proof. First, we notice that $v \equiv 1$ belongs to $X_{\alpha,\beta}^{g,h}$. So, using it as a test function in (9), we obtain

$$0 \leq \int_\Omega f\, dx = -\int_{\partial\Omega} h d\sigma - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g\, dx \leq 0.$$

Hence, $f = 0$ a.e. in Ω and $g = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Now, taking $v = u$ as a test function in (9), we obtain

$$\int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy = 0,$$

so u must be constant. □

The next result states that, if u is a weak solution of (8), then the nonlocal boundary condition in $\mathbb{R}^N \setminus \overline{\Omega}$ is satisfied almost everywhere, as in related classical results for local operators.

Theorem 1. *Let $u \in X^{g,h}_{\alpha,\beta}$ be a weak solution of (8). Then, $\beta \mathscr{N}_{s,p} u = g$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$.*

Proof. First, we take $v \in X^{g,h}_{\alpha,\beta}$ such that $v \equiv 0$ in $\overline{\Omega}$ as a test function in (9), obtaining

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus \overline{\Omega}} g v \, dx &= -\frac{\beta}{2} \int_{\Omega} \int_{\mathbb{R}^N \setminus \overline{\Omega}} \frac{J_p(u(x) - u(y))v(y)}{|x-y|^{N+ps}} \, dydx \\
&\quad + \frac{\beta}{2} \int_{\mathbb{R}^N \setminus \overline{\Omega}} \int_{\Omega} \frac{J_p(u(x) - u(y))v(x)}{|x-y|^{N+ps}} \, dydx \\
&= -\beta \int_{\Omega} \int_{\mathbb{R}^N \setminus \overline{\Omega}} \frac{J_p(u(x) - u(y))v(y)}{|x-y|^{N+ps}} \, dydx \\
&= -\beta \int_{\mathbb{R}^N \setminus \overline{\Omega}} v(y) \int_{\Omega} \frac{J_p(u(x) - u(y))}{|x-y|^{N+ps}} \, dxdy \\
&= \beta \int_{\mathbb{R}^N \setminus \overline{\Omega}} v(y) \mathscr{N}_{s,p} u(y) \, dy.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} (\beta \mathscr{N}_{s,p} u(x) - g(x)) v(x) \, dx = 0$$

for every $v \in X^{g,h}_{\alpha,\beta}$, which is 0 in Ω. In particular, this is true for every $v \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$, and, so, $\beta \mathscr{N}_{s,p} u(x) = g(x)$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$. □

3. L^∞ Estimate

In this section, we give a boundedness result for weak solutions. First, it is useful to introduce the following notation for the norm in $W^{1,p}(\Omega)$, namely

$$\|u\|_{W^{1,p}(\Omega)} := \left(\alpha \int_{\Omega} |Du|^p dx + \int_{\Omega} |u|^p dx \right)^{1/p}.$$

Clearly, this norm is equivalent to the usual norm in $W^{1,p}(\Omega)$.

The main result of this section is stated as follows.

Theorem 2. *Let $N \geq 3$, $f \in L^q(\Omega)$ with $q > N/p$ and $\alpha\beta \neq 0$. If u is a weak solution of*

$$\begin{cases} -\alpha \Delta_p u + \beta (-\Delta)^s_p u + |u|^{p-2} u = f(x) & \text{in } \Omega, \\ \text{with } (\alpha, \beta) - \text{Neumann conditions}, \end{cases} \tag{10}$$

then, $u \in L^\infty(\mathbb{R}^N)$ and

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C(\|f\|_{L^q(\Omega)} + \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}) \tag{11}$$

for some constant $C > 0$.

Proof. If the weak solution u is identically zero, we have nothing to prove. Otherwise, we take $\delta \in (0,1)$ and set

$$\tilde{u} = \frac{\delta u}{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}} \quad \text{and} \quad \tilde{f} = \left(\frac{\delta}{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}} \right)^{p-1} f, \tag{12}$$

(if $N \leq p$, we replace p^* with any number larger than p by the usual Sobolev embedding theorems, see e.g., [34]). With this definition, we find that $\tilde{u}$ is a weak solution of

$$\begin{cases} -\alpha\Delta_p\tilde{u} + \beta(-\Delta)_p^s\tilde{u} + |\tilde{u}|^{p-2}\tilde{u} = \tilde{f} & \text{in } \Omega, \\ \text{with } (\alpha,\beta) - \text{Neumann conditions.} \end{cases} \tag{13}$$

For every $k \in \mathbb{N}$, we define $c_k := 1 - 2^{-k}$,

$$v_k := \tilde{u} - c_k, \quad w_k := (v_k)_+ := \max\{v_k, 0\}, \quad U_k := \|w_k\|^p_{L^{p^*}(\Omega)}.$$

From the Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} U_k = \lim_{k\to\infty} \|w_k\|^p_{L^{p^*}(\Omega)} = \|(\tilde{u}-1)_+\|^p_{L^{p^*}(\Omega)}. \tag{14}$$

Moreover, for $k = 0$, we have $w_0 = (v_0)_+ = \tilde{u_+}$, and, so,

$$U_0 = \left(\int_\Omega w_0^{p^*}(x)\,dx\right)^{p/p^*} = \left(\int_\Omega \tilde{u}_+^{p^*}(x)\,dx\right)^{p/p^*} \leq \|\tilde{u}\|^p_{L^{p^*}(\Omega)} \leq \delta^p. \tag{15}$$

Now, we can use w_{k+1} as a test function in (13), obtaining

$$\begin{aligned} &\alpha\int_\Omega |D\tilde{u}|^{p-2}D\tilde{u}\cdot Dw_{k+1}\,dx \\ &+\frac{\beta}{2}\iint_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(\tilde{u}(x)-\tilde{u}(y))(w_{k+1}(x)-w_{k+1}(y))}{|x-y|^{N+ps}}\,dxdy \\ &+\int_\Omega |\tilde{u}|^{p-2}\tilde{u}w_{k+1}dx = \int_\Omega \tilde{f}w_{k+1}\,dx. \end{aligned} \tag{16}$$

We note that, for a.e. $x, y \in \mathbb{R}^N$, simple algebraic reasoning shows that

$$\begin{aligned} &|w_{k+1}(x) - w_{k+1}(y)|^p = |(v_{k+1})_+(x) - (v_{k+1})_+(y)|^p \\ \leq &|(v_{k+1})_+(x) - (v_{k+1})_+(y)|^{p-2}((v_{k+1})_+(x) - (v_{k+1})_+(y))(v_{k+1}(x) - v_{k+1}(y)) \\ = &|w_{k+1}(x) - w_{k+1}(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(\tilde{u}(x) - \tilde{u}(y)), \end{aligned} \tag{17}$$

so that, being that $w_{k+1} = \tilde{u} - c_{k+1}$ where $\tilde{u} > c_{k+1}$, we have

$$\begin{aligned} &\iint_{\mathcal{Q}} \frac{J_p(\tilde{u}(x)-\tilde{u}(y))(w_{k+1}(x)-w_{k+1}(y))}{|x-y|^{N+ps}}\,dxdy \\ &\geq \iint_{\mathcal{Q}} \frac{|w_{k+1}(x)-w_{k+1}(y)|^p}{|x-y|^{N+ps}}\,dxdy \geq 0. \end{aligned} \tag{18}$$

Moreover,

$$\int_\Omega |D\tilde{u}|^{p-2}D\tilde{u}\cdot Dw_{k+1}\,dx = \int_{\Omega\cap\{\tilde{u}>c_{k+1}\}} |D\tilde{u}|^{p-2}D\tilde{u}\cdot Dv_{k+1}\,dx = \int_\Omega |Dw_{k+1}|^p\,dx \tag{19}$$

and

$$\int_\Omega |\tilde{u}|^{p-2}\tilde{u}w_{k+1}dx = \int_{\{\tilde{u}>c_{k+1}\}} |\tilde{u}|^{p-2}\tilde{u}(\tilde{u}-c_{k+1})dx \geq \int_\Omega w_{k+1}^p dx. \tag{20}$$

By using (18)–(20) in (16), we obtain

$$\alpha\int_\Omega |Dw_{k+1}|^p\,dx + \int_\Omega w_{k+1}^p dx \leq \int_\Omega \tilde{f}w_{k+1}\,dx.$$

Then, by the Sobolev inequality,

$$U_{k+1} = \left(\int_\Omega w_{k+1}^{p^*} \, dx \right)^{p/p^*} \leq C \| w_{k+1} \|^p_{W^{1,p}(\Omega)} \leq \int_\Omega w_{k+1} \tilde{f} \, dx, \tag{21}$$

for some $C > 0$. Moreover, by definition, $v_{k+1} \leq v_k$, and, so,

$$w_{k+1} \leq w_k. \tag{22}$$

We note that

$$w_{k+1} = (\tilde{u} - c_{k+1})_+ = \left(\tilde{u} - 1 + \frac{1}{2^{k+1}} + \frac{1}{2^k} - \frac{1}{2^k} \right)_+ = \left(v_k + \frac{1}{2^{k+1}} - \frac{1}{2^k} \right)_+,$$

and, consequently,

$$\{ w_{k+1} > 0 \} = \{ v_{k+1} > 0 \} = \left\{ w_k > \frac{1}{2^{k+1}} \right\}. \tag{23}$$

We also note that

$$p^* - \frac{p^*}{q} - 1 > p^* - \frac{p^*}{N/p} - 1 = p - 1.$$

As a consequence,

$$\tau := p^* \left(p^* - \frac{p^*}{q} - 1 \right)^{-1} < \frac{p^*}{p-1}, \tag{24}$$

and we also observe that

$$\tau > \frac{p^*}{p^* - 1} > 1.$$

A simple computation shows that

$$\frac{1}{p^*} + \frac{1}{q} + \frac{1}{\tau} = 1. \tag{25}$$

With (25) in mind, we can use the generalized Hölder Inequality with exponents p^*, q, and τ, and, together with (23) and the definition of U_{k+1}, we have

$$\begin{aligned} \int_\Omega w_{k+1} |\tilde{f}| \, dx &= \int_{\Omega \cap \{ w_{k+1} > 0 \}} w_{k+1} |\tilde{f}| \, dx \\ &\leq \| \tilde{f} \|_{L^q(\Omega)} \| w_{k+1} \|_{L^{p^*}(\Omega)} |\Omega \cap \{ w_{k+1} > 0 \}|^{1/\tau} \\ &\leq \| f \|^{2-p}_{L^q(\Omega)} \| w_k \|_{L^{p^*}(\Omega)} \left| \Omega \cap \left\{ w_k > \frac{1}{2^{k+1}} \right\} \right|^{1/\tau} \\ &\leq \| f \|^{2-p}_{L^q(\Omega)} U_k^{1/p} \left(2^{p^*(k+1)} \int_{\Omega \cap \left\{ w_k > \frac{1}{2^{k+1}} \right\}} w_k^{p^*} \right)^{1/\tau} \\ &\leq C_1^k U_k^{1/p} U_k^{p^*/p\tau}, \end{aligned} \tag{26}$$

for some $C_1 > 0$. We define the exponent

$$\gamma := \frac{1}{p} + \frac{p^*}{p\tau}$$

and observe that, from (24),

$$\gamma > 1. \tag{27}$$

Now, from (21) and (26), we have

$$U_{k+1} \leq C_1^k U_k^\gamma.$$

By iterating, from (27), we find

$$U_{k+1} \leq C_1^{\frac{(k+1)k}{2}\gamma^k} U_0^{\gamma^{k+1}}.$$

Then, if we take δ sufficiently small in (15), we can conclude that

$$\lim_{k\to\infty} U_k = 0,$$

and, recalling (14), we obtain

$$\|(\tilde{u}-1)_+\|^p_{L^{p^*}(\Omega)} = 0,$$

hence, $\tilde{u} \leq 1$. So, from (12),

$$u(x) \leq \frac{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}}{\delta} \tag{28}$$

for every $x \in \Omega$.

A similar argument can be made for $-u$. Clearly, setting

$$\bar{u} = \frac{-\delta u}{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}} \quad \text{and} \quad \bar{f} = -\left(\frac{\delta}{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}}\right)^{p-1} f,$$

it follows that $\bar{u}$ is a weak solution of

$$\begin{cases} -\alpha\Delta_p \bar{u} + \beta(-\Delta)^s_p \bar{u} + |\bar{u}|^{p-2}\bar{u} = \bar{f} & \text{in } \Omega, \\ \text{with } (\alpha,\beta) - \text{Neumann conditions.} \end{cases}$$

Reasoning as above, we can conclude that $\bar{u} \leq 1$, hence

$$-u(x) \leq \frac{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}}{\delta}$$

for every $x \in \Omega$. This, together with (28), implies that

$$|u(x)| \leq \frac{\|u\|_{L^{p^*}(\Omega)} + \|f\|_{L^q(\Omega)}}{\delta} \tag{29}$$

for every $x \in \Omega$.

On the other hand, using u as a test function in (10) and recalling (25), we obtain

$$\begin{aligned} \|u\|^p_{W^{1,p}(\Omega)} &= \alpha \int_\Omega |Du|^p \, dx + \int_\Omega |u|^p \, dx \\ &\leq \alpha \int_\Omega |Du|^p \, dx + \frac{\beta}{2} \iint_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dxdy + \int_\Omega |u|^p \, dx \\ &= \int_\Omega uf \, dx \\ &\leq \|u\|_{L^{p^*}(\Omega)} \|f\|_{L^q(\Omega)} |\Omega|^{1/\tau}. \end{aligned}$$

From this last inequality and the Sobolev inequality, we can deduce

$$\|u\|^{p-1}_{L^{p^*}(\Omega)} \leq C_2 \|f\|_{L^q(\Omega)},$$

for some $C_2 > 0$. Using this in (29), we obtain

$$\|u\|_{L^\infty(\Omega)} \leq C_3(\|f\|_{L^q(\Omega)} + \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)})$$

for some $C_3 > 0$. From Theorem 1, we easily obtain $\mathcal{N}_{s,p}u = 0$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$, which implies

$$u(x) = \frac{\displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}\,dy}{\displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}\,dy} \tag{30}$$

so that

$$|u(x)| \leq \|u\|_{L^\infty(\Omega)},\ x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{31}$$

see also [21]. This concludes the proof. □

4. A Superlinear Problem Without the Ambrosetti–Rabinowitz Condition

In this section, we consider the problem

$$\begin{cases} -\alpha\Delta_p u + \beta(-\Delta)^s_p u + |u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann conditions}, \end{cases} \tag{32}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for almost every $x \in \Omega$. In addition, we assume the following hypotheses taken from [21] as improvements of those in [35,36]:

(f_1) there exist $a \in L^q(\Omega)$, $a \geq 0$, with $q \in ((p^*)', p)$, $c > 0$ and $r \in (p, p^*)$, such that

$$|f(x,t)| \leq a(x) + c|t|^{r-1}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_2) denoting $F(x,t) = \int_0^t f(x,\sigma)\,d\sigma$, we have

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^p} = +\infty$$

uniformly for a.e. $x \in \Omega$;

(f_3) if $\sigma(x,t) = f(x,t)t - pF(x,t)$, then there exist $\theta > 1$ and $\beta^* \in L^1(\Omega)$, $\beta^* \geq 0$, such that

$$\sigma(x,t_1) \leq \theta\sigma(x,t_2) + \beta^*$$

for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$;

(f_4)

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$$

uniformly for a.e. $x \in \Omega$.

Definition 2. *Let $u \in X_{\alpha,\beta}$. With the same assumptions on f as above, we say that u is a weak solution of* (32) *if*

$$\alpha \int_\Omega |Du|^{p-2}Du \cdot Dv\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy + \int_\Omega |u|^{p-2}uv\,dx = \int_\Omega f(x,u)v\,dx$$

for every $v \in X_{\alpha,\beta}$.

Following this definition, we have that any critical point of the C^1 functional $I : X_{\alpha,\beta} \to \mathbb{R}$, defined as

$$I(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)\,dx$$

is a weak solution of (32).

Proposition 4. *Setting $A(u) = \|u\|^p$, the functional $A' : X_{\alpha,\beta} \to X'_{\alpha,\beta}$ satisfies the (S) property, that is, for every sequence $(u_n)_n$ such that $u_n \rightharpoonup u$ in $X_{\alpha,\beta}$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \langle A'(u_n), u_n - u\rangle = 0,$$

there holds

$$\lim_{n\to\infty} \|u_n\| = \|u\|.$$

Proof. Clearly, A is weakly lower semicontinuous in $X_{\alpha,\beta}$, so that

$$A(u) \le \liminf_{n\to\infty} A(u_n). \tag{33}$$

On the other hand, A is convex and, so,

$$A(u) \ge A(u_n) + \langle A'(u_n), u_n - u\rangle,$$

which, passing to the limit, gives

$$A(u) \ge \limsup_{n\to\infty} A(u_n). \tag{34}$$

Thus, combining (33) and (34), we obtain

$$A(u) = \lim_{n\to\infty} A(u_n)$$

as desired. □

We have the following result.

Theorem 3. *If hypotheses (f_1)-(f_4) hold, then problem* (32) *has two nontrivial constant sign solutions.*

In order to prove Theorem 3, we introduce the functionals

$$I_\pm(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u_\pm)\,dx,$$

where $u_+ := \max\{u,0\}$ and $u_- := \max\{-u,0\}$ denote the classical positive and negative parts of u, respectively.

Now, we want to prove that both $I_\pm$ satisfy the Cerami condition, (C) for short, which states that any sequence $(u_n)_n$ in $X_{\alpha,\beta}$ such that $(I_\pm(u_n))_n$ is bounded and $(1+\|u_n\|)I'_\pm(u_n) \to 0$ as $n \to \infty$, admits a convergent subsequence.

Proposition 5. *Under the assumptions of Theorem 3, the functionals $I_\pm$ satisfy the (C) condition.*

Proof. We give the proof for I_+, the proof for I_- being analogous.

Let $(u_n)_n$ in $X_{\alpha,\beta}$ be such that

$$|I_+(u_n)| \le M_1 \tag{35}$$

for some $M_1 > 0$ and all $n \ge 1$, and

$$(1+\|u_n\|)I'_+(u_n) \to 0 \text{ in } X'_{\alpha,\beta} \text{ as } n \to \infty. \tag{36}$$

From (36), we have

$$|I'_+(u_n)h| \le \frac{\varepsilon_n h}{1+\|u_n\|}$$

for every $h \in X_{\alpha,\beta}$ and with $\varepsilon_n \to 0$ as $n \to \infty$, that is

$$\begin{aligned}\Big|\alpha \int_\Omega |Du_n|^{p-1}Du_n \cdot Dh\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{J_p(u_n(x)-u_n(y))(h(x)-h(y))}{|x-y|^{N+ps}}\,dxdy \\ + \int_\Omega |u_n|^{p-2}u_n h\,dx - \int_\Omega f(x,(u_n)_+)h\,dx\Big| \le \frac{\varepsilon_n h}{1+\|u_n\|}.\end{aligned} \tag{37}$$

Taking $h = (u_n)_-$ in (37), we obtain

$$\begin{aligned}\Big|\alpha \int_\Omega |D(u_n)_-|^p\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{J_p(u_n(x)-u_n(y))((u_n)_-(x)-(u_n)_-(y))}{|x-y|^{N+ps}}\,dxdy \\ + \int_\Omega |(u_n)_-|^p\,dx\Big| \le \varepsilon_n\end{aligned} \tag{38}$$

Observing that, similarly to (17), we have

$$|(u_n)_-(x)-(u_n)_-(y)|^p \le J_p(u_n(x)-u_n(y))((u_n)_-(x)-(u_n)_-(y)), \tag{39}$$

thus, from (38), we obtain

$$\|(u_n)_-\|^p \le \varepsilon_n,$$

and, so,

$$(u_n)_- \to 0 \text{ in } X'_{\alpha,\beta} \text{ as } n \to \infty. \tag{40}$$

Now, taking $h = (u_n)_+$ in (37), we obtain

$$\begin{aligned}-\alpha \int_\Omega |D(u_n)_+|^p\,dx - \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{J_p(u_n(x)-u_n(y))((u_n)_+(x)-(u_n)_+(y))}{|x-y|^{N+ps}}\,dxdy \\ - \int_\Omega |(u_n)_+|^p\,dx + \int_\Omega f(x,(u_n)_+)(u_n)_+\,dx \le \varepsilon_n.\end{aligned} \tag{41}$$

From (35), we have

$$\begin{aligned}\alpha \int_\Omega |Du_n|^p\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dxdy \\ + \int_\Omega |u_n|^p\,dx - p\int_\Omega F(x,(u_n)_+)\,dx \le pM_1\end{aligned}$$

for $M_1 > 0$ and $n \geq 1$, which, together with (38), gives

$$\alpha \int_\Omega |D(u_n)_+|^p \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))((u_n)_+(x) - (u_n)_+(y))}{|x-y|^{N+ps}} \, dxdy + \int_\Omega |(u_n)_+|^p \, dx - p \int_\Omega F(x, (u_n)_+) \, dx \leq M_2 \tag{42}$$

for some $M_2 > 0$ and all $n \geq 1$. Adding (41) to (42), we obtain

$$\int_\Omega f(x, (u_n)_+)(u_n)_+ \, dx - p \int_\Omega F(x, (u_n)_+) \, dx \leq M_3$$

for some $M_3 > 0$ and all $n \geq 1$, that is

$$\int_\Omega \sigma(x, (u_n)_+) \, dx \leq M_3. \tag{43}$$

Now, we want to prove that $((u_n)_+)_n$ is bounded in $X_{\alpha,\beta}$, and, for this, we argue by contradiction. Passing to a subsequence if necessary, we assume that $\|(u_n)_+\| \to \infty$ as $n \to \infty$. Defining $y_n = (u_n)_+ / \|(u_n)_+\|$, we can assume that

$$y_n \rightharpoonup y \text{ in } X_{\alpha,\beta} \text{ and } y_n \to y \text{ in } L^q(\Omega) \tag{44}$$

for every $q \in [p, p^*)$ and for some $y \geq 0$.

First, we treat the case $y \neq 0$. We define the set

$$Z(y) = \{x \in \Omega : \, y(x) = 0\},$$

so that $|\Omega \setminus Z(y)| > 0$ and $(u_n)_+ \to \infty$ for a.e. $x \in \Omega \setminus Z(y)$ as $n \to \infty$. By hypothesis (f_2), we have

$$\lim_{n\to\infty} \frac{F(x, (u_n)_+(x))}{\|(u_n)_+\|^p} = \lim_{n\to\infty} \frac{F(x, (u_n)_+(x))}{(u_n)_+(x)^p} y_n(x)^p = +\infty$$

for almost every $x \in \Omega \setminus Z(y)$. On the other hand, by Fatou's Lemma

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x, (u_n)_+(x))}{\|(u_n)_+\|^p} \, dx \leq \liminf_{n\to\infty} \int_\Omega \frac{F(x, (u_n)_+(x))}{\|(u_n)_+\|^p} \, dx,$$

which leads to

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, (u_n)_+(x))}{\|(u_n)_+\|^p} \, dx = +\infty. \tag{45}$$

From (35), we have

$$-\frac{1}{p}\|u_n\|^p + \int_\Omega F(x, (u_n)_+(x)) \, dx \leq M_1$$

for all $n \geq 1$.

Recalling that $\|u_n\|^p \leq 2^{p-1}(\|(u_n)_+\|^p + \|(u_n)_-\|^p)$, from (40), we obtain

$$-\frac{2^{p-1}}{p}\|(u_n)_+\|^p + \int_\Omega F(x, (u_n)_+(x)) \, dx \leq M_4$$

for some $M_4 > 0$. Dividing by $\|(u_n)_+\|^p$,

$$-\frac{2^{p-1}}{p} + \int_\Omega \frac{F(x, (u_n)_+(x))}{\|(u_n)_+\|^p} \, dx \leq \frac{M_4}{\|(u_n)_+\|^p}.$$

Passing to the limit, we have

$$\limsup_{n\to\infty} \int_\Omega \frac{F(x,(u_n)_+(x))}{\|(u_n)_+\|^p}\,dx \le \frac{2^{p-1}}{p},$$

which is in contradiction with (45), and this concludes the case $y \neq 0$.

Now, we deal with the case $y \equiv 0$. We consider the continuous functions $\phi_n : [0,1] \to \mathbb{R}$, defined as $\phi_n(t) = I_+(t(u_n)_+)$ with $t \in [0,1]$ and $n \ge 1$. So, we can define t_n such that

$$\phi_n(t_n) = \max_{t\in[0,1]} \phi_n(t). \tag{46}$$

Now, if $\lambda > 0$, we set $v_n := (p\lambda)^{\frac{1}{p}} y_n \in X_{\alpha,\beta}$. From (44), $v_n \to 0$ in $L^q(\Omega)$ for all $q \in [1, p^*)$. Performing some integration, from (f_1), we have

$$\left| \int_\Omega F(x, v_n(x))\,dx \right| \le \int_\Omega a(x)|v_n(x)|\,dx + C\int_\Omega |v_n(x)|^r\,dx$$

for some $C > 0$, which implies that

$$\lim_{n\to\infty} \int_\Omega F(x, v_n(x))\,dx = 0. \tag{47}$$

Since $\|(u_n)_+\| \to \infty$, there exists $n_0 \ge 1$ such that $(p\lambda)^{\frac{1}{p}}/\|(u_n)_+\| \in (0,1)$ for all $n \ge n_0$. Then, from (46),

$$\phi_n(t_n) \ge \phi_n\left(\frac{(p\lambda)^{\frac{1}{p}}}{\|(u_n)_+\|}\right)$$

for all $n \ge n_0$. So,

$$I_+(t_n(u_n)_+) \ge I_+((p\lambda)^{\frac{1}{p}} y_n) = \lambda\|y_n\|^p - \int_\Omega F(x, v_n(x))\,dx.$$

Then, (47) implies that

$$I_+(t_n(u_n)_+) \ge \lambda\|y_n\|^p + o(1),$$

and, since λ is arbitrary, we have

$$\lim_{n\to\infty} I_+(t_n(u_n)_+) = +\infty. \tag{48}$$

We observe that $0 \le t_n(u_n)_+ \le (u_n)_+$ for all $n \ge 1$; so, from (f_3), we know that

$$\int_\Omega \sigma(x, t_n(u_n)_+)\,dx \le \theta\sigma(x,(u_n)_+)\,dx + \|\beta^*\|_1 \tag{49}$$

for all $n \geq 1$. Clearly, $I_+(0) = 0$. In addition, by (38), we have

$$\begin{aligned} I_+((u_n)_+) &= \frac{1}{p}\int_\Omega |Du_n|^p dx - \frac{1}{p}\int_\Omega |D(u_n)_-|^p dx \\ &+ \frac{1}{p}\int_\Omega |u_n|^p dx - \frac{1}{p}\int_\Omega |(u_n)_-|^p dx - \int_\Omega F(x,((u_n)_+)\,dx \\ &+ \frac{1}{p}\iint_{\{x,y\in\mathcal{Q}:u(x)>0,u(y)>0\}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy \\ &- \frac{1}{p}\iint_{\{x,y\in\mathcal{Q}:u(x)\leq 0,u(y)\leq 0\}} \frac{|(u_n)_-(x)-(u_n)_-(y)|^p}{|x-y|^{N+ps}}dxdy \\ &- \frac{2}{p}\iint_{\{x,y\in\mathcal{Q}:u(x)>0,u(y)\leq 0\}} \frac{|(u_n)_+(x)+(u_n)_-(y)|^p}{|x-y|^{N+ps}}dxdy \\ &= I_+(u_n) - \frac{2}{p}\iint_{\{x,y\in\mathcal{Q}:u(x)>0,u(y)\leq 0\}} \frac{|(u_n)_+(x)+(u_n)_-(y)|^p}{|x-y|^{N+ps}}dxdy + o(1) \\ &\leq I_+(u_n) + o(1) \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. By (35), we obtain that $I_+((u_n)_+) \leq M_7$ for some $M_7 > 0$ and all $n \in \mathbb{N}$. Together with (48), this implies that $t_n \in (0,1)$ for all $n \geq n_1 \geq n_0$. Since t_n is a maximum point, we have

$$\begin{aligned} 0 &= t_n\phi_n'(t_n) \\ &= +\frac{\beta}{2}\iint_{\mathcal{Q}} \frac{J_p(t_nu_n(x)-t_nu_n(y))(t_n(u_n)_+(x)-t_n(u_n)_+(y))}{|x-y|^{N+ps}}\,dxdy \\ &+ \alpha\int_\Omega |Dt_n(u_n)_+|^p\,dx + \int_\Omega |t_n(u_n)_+|^p\,dx - \int_\Omega f(x,t_n(u_n)_+)t_n(u_n)_+\,dx. \end{aligned}$$

As in (17), we have

$$|t_n(u_n)_+(x) - t_n(u_n)_+(y))|^p \leq J_p(t_nu_n(x)-t_nu_n(y))(t_n(u_n)_+(x)-t_n(u_n)_+(y)),$$

so that

$$\|t_n(u_n)_+\|^p - \int_\Omega f(x,t_n(u_n)_+)t_n(u_n)_+\,dx \leq 0. \tag{50}$$

Adding (50) to (49), we obtain

$$\|t_n(u_n)_+\|^p - p\int_\Omega F(x,t_n(u_n)_+)\,dx \leq \theta\int_\Omega \sigma(x,(u_n)_+)\,dx + \|\beta^*\|_1,$$

that is

$$pI_+(t_n(u_n)_+) \leq \theta\int_\Omega \sigma(x,(u_n)_+)\,dx + \|\beta^*\|_1.$$

Hence, from (48), we obtain

$$\lim_{n\to\infty} \sigma(x,(u_n)_+)\,dx = +\infty. \tag{51}$$

By combining (43) and (51), we obtain a contradiction, which concludes the case $y \equiv 0$.

In conclusion, we have proved that $((u_n)_+)_n$ is bounded in $X_{\alpha,\beta}$, and from (40), we know that $(u_n)_n$ is bounded in $X_{\alpha,\beta}$. Then, we can assume

$$u_n \rightharpoonup u \text{ in } X_{\alpha,\beta} \text{ and } u_n \to u \text{ in } L^q(\Omega) \tag{52}$$

with $q \in [1, p^*)$ and for some $u \in X_{\alpha,\beta}$. Taking $h = u_n - u$ in (37), we have

$$\begin{aligned} \|u_n\|^p - \alpha \int_\Omega |Du_n|^{p-2} Du_n \cdot Du \, dx - \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+ps}} \, dxdy \\ - \int_\Omega |u_n|^{p-2} u_n u \, dx - \int_\Omega f(x, (u_n)_+)(u_n - u) \, dx = \varepsilon_n. \end{aligned} \tag{53}$$

From (f_1) and (52), we know that

$$\lim_{n\to\infty} \int_\Omega |f(x, (u_n)_+)(u_n - u)| \, dx = 0.$$

Passing to the limit in (53), we obtain

$$\begin{aligned} \lim_{n\to\infty} \Bigg(\|u_n\|^p - \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+ps}} \, dxdy \\ - \alpha \int_\Omega |Du_n|^{p-2} Du_n \cdot Du \, dx - \int_\Omega |u_n|^{p-2} u_n u \, dx \Bigg) = 0. \end{aligned}$$

The (S) property, see Proposition 4, implies that $\|u_n\| \to \|u\|$; so, from the uniform convexity of $X_{\alpha,\beta}$ (as $1 < p < \infty$), we know that $u_n \to u$ in $X_{\alpha,\beta}$ as $n \to \infty$. Then, I_+ satisfies the (C) condition, which concludes the proof. □

We can now give the proof of Theorem 3.

Proof of Theorem 3. We want to apply the Mountain Pass Theorem to I_+. From Proposition 5, we know that I_+ satisfies the (C) condition, so we only have to verify the geometric conditions.

From (f_1) and (f_4), for every $\varepsilon > 0$, there exists C_ε such that

$$F(x,t) \le \frac{\varepsilon}{p}|t|^p + C_\varepsilon |t|^r$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Then,

$$\begin{aligned} I_+(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(x, u_+) \, dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{\varepsilon}{p}\|u\|_p^p - C_\varepsilon \|u\|_r^r \\ &\ge \frac{1 - \varepsilon C_1}{p}\|u\|^p - C_2 \|u\|^r \end{aligned}$$

for some $C_1, C_2 > 0$. From this, we obtain that, if $\|u\| = \rho$ is small enough, then

$$\inf_{\|u\|=\rho} I_+(u) > 0.$$

Now, we take $u \in X_{\alpha,\beta}$ with $u > 0$ and $t > 0$; then,

$$\begin{aligned} I_+(u) &= \frac{t^p}{p}\|u\|^p - \int_\Omega F(x, tu) \, dx \\ &= \frac{t^p}{p}\|u\|^p - t^p \int_\Omega \frac{F(x, tu)}{|tu|^p} u^p \, dx. \end{aligned}$$

By Fatou's Lemma,

$$\int_\Omega \liminf_{t\to\infty} \frac{F(x, tu)}{|tu|^p} u^p \, dx \le \liminf_{t\to\infty} \int_\Omega \frac{F(x, tu)}{|tu|^p} u^p \, dx,$$

so, from (f_2), we know that

$$\lim_{t\to\infty}\int_\Omega \frac{F(x,tu)}{|tu|^p}u^p\,dx = +\infty.$$

Then,

$$\lim_{t\to\infty} I_+(tu) = -\infty,$$

therefore, there exists $e \in X_{\alpha,\beta}$ such that $\|e\| > \rho$ and $I_+(e) < 0$.

Now, we can apply the Mountain Pass Theorem (see, e.g., [37]) to I_+ and obtain a nontrivial critical point u. In particular, we have

$$\begin{aligned}0 &= \alpha\int_\Omega |Du_-|^p\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}}\frac{J_p(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+ps}}\,dxdy\\ &\quad+\int_\Omega |u_-|^p\,dx - \int_\Omega f(x,u_+)u_-\,dx\\ &= \alpha\int_\Omega |Du_-|^p\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}}\frac{J_p(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+ps}}\,dxdy\\ &\quad+\int_\Omega |u_-|^p\,dx.\end{aligned}$$

By (39), we have

$$|u_-(x)-u_-(y)|^p \le J_p(u(x)-u(y))(u_-(x)-u_-(y)),$$

so that

$$0 \ge \|u_-\|^p,$$

and, thus, $u_- \equiv 0$. Then, $I_+(u) = I(u)$, and, so, $u \ge 0$ is a solution of (32). Arguing in the same way for I_-, we can find a nontrivial negative solution of (32). □

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Appendix A. Details of Example 1

In this appendix, we give more details on the computations of Example 1. Thus, as in Example 1, we let $N = 1$, $p = 2$, and $\Omega = (-1,1)$ and define $u : (-1,1) \to \mathbb{R}$ as

$$u(x) = x(x-1)^2(x+1)^2.$$

Clearly,

$$\lim_{x\to -1^+} u'(x) = \lim_{x\to 1^-} u'(x) = 0.$$

Then, in agreement with (30), for every $x > 1$, we define

$$g_s(x) := \frac{\displaystyle\int_{-1}^{1} \frac{y(y-1)^2(y+1)^2}{(x-y)^{1+2s}}\,dy}{\displaystyle\int_{-1}^{1} \frac{dy}{(x-y)^{1+2s}}}. \tag{A1}$$

Let us compute the integrals above. First,

$$\int_{-1}^{1} \frac{dy}{(x-y)^{\gamma}} = \begin{cases} \dfrac{1}{\gamma-1}\left[(x-1)^{1-\gamma} - (x+1)^{1-\gamma}\right] & \text{if } \gamma \neq 1 \\ \\ \log\dfrac{x+1}{x-1} & \text{if } \gamma = 1. \end{cases} \tag{A2}$$

In order to compute

$$\int_{-1}^{1} \frac{y^5 - 2y^3 + y}{(x-y)^{1+2s}}\,dy$$

we use the identity

$$\begin{aligned} \int_{-1}^{1} \frac{y^n}{(x-y)^{\gamma}}\,dy &= \int_{-1}^{1} \frac{y^n - x^n}{(x-y)^{\gamma}}\,dy + x^n \int_{-1}^{1} \frac{dy}{(x-y)^{\gamma}} \\ &= -\sum_{k=0}^{n-1} x^k \int_{-1}^{1} \frac{y^{n-k-1}}{(x-y)^{\gamma-1}}\,dy + x^n \int_{-1}^{1} \frac{dy}{(x-y)^{\gamma}}. \end{aligned} \tag{A3}$$

When $s \neq 1/2$, by using (A3) (multiple times if needed), we obtain

$$\int_{-1}^{1} \frac{y}{(x-y)^{1+2s}}\,dy = -\int_{-1}^{1} \frac{dy}{(x-y)^{2s}} + x\int_{-1}^{1} \frac{dy}{(x-y)^{1+2s}},$$

$$\begin{aligned} \int_{-1}^{1} \frac{y^3}{(x-y)^{1+2s}}\,dy = &-\int_{-1}^{1} \frac{dy}{(x-y)^{2s-2}} + 3x\int_{-1}^{1} \frac{dy}{(x-y)^{2s-1}} \\ &- 3x^2\int_{-1}^{1} \frac{dy}{(x-y)^{2s}} + x^3\int_{-1}^{1} \frac{dy}{(x-y)^{1+2s}} \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^{1} \frac{y^5}{(x-y)^{1+2s}}\,dy = &-\int_{-1}^{1} \frac{dy}{(x-y)^{2s-4}} + 5x\int_{-1}^{1} \frac{dy}{(x-y)^{2s-3}} - 10x^2\int_{-1}^{1} \frac{dy}{(x-y)^{2s-2}} \\ &+ 10x^3\int_{-1}^{1} \frac{dy}{(x-y)^{2s-1}} - 5x^4\int_{-1}^{1} \frac{dy}{(x-y)^{2s}} + x^5\int_{-1}^{1} \frac{dy}{(x-y)^{1+2s}} \end{aligned}$$

Accordingly, by (A2), we can compute both the integrals in (A1) to obtain

$$\begin{aligned} g_s(x) = \frac{2s(x-1)^{2s}(x+1)^{2s}}{(x+1)^{2s}-(x-1)^{2s}}\Bigg[& -\frac{1}{5-2s}\Big((x+1)^{5-2s}-(x-1)^{5-2s}\Big) \\ & +\frac{5x}{4-2s}\Big((x+1)^{4-2s}-(x-1)^{4-2s}\Big) \\ & -\frac{10x^2-2}{3-2s}\Big((x+1)^{3-2s}-(x-1)^{3-2s}\Big) \\ & +\frac{10x^3-6x}{2-2s}\Big((x+1)^{2-2s}-(x-1)^{2-2s}\Big) \\ & -\frac{5x^4-6x^2+1}{1-2s}\Big((x+1)^{1-2s}-(x-1)^{1-2s}\Big) \\ & -\frac{x^5-2x^3+x}{2s}\Big((x+1)^{-2s}-(x-1)^{-2s}\Big)\Bigg], \end{aligned} \tag{A4}$$

When $s = 1/2$, by (A2) and (A3), we obtain

$$g_{\frac{1}{2}}(x) := 5x^5 - \frac{28}{3}x^3 + \frac{13}{3}x - \frac{1}{2}(5x^6 - 11x^4 + 7x^2 - 1)\log\frac{x+1}{x-1}.$$

Thus, we can extend u in $\mathbb{R}$ as

$$u(x) = \begin{cases} -g_s(-x) & x < -1 \\ x(x-1)^2(x+1)^2 & -1 \le x \le 1 \\ g_s(x) & x > 1 \end{cases}$$

so that, by construction, $\mathscr{N}_s u(x) = 0$ for any $x \in \mathbb{R} \setminus [-1,1]$.

We observe that, extending u in this way, we have $u \in C(\mathbb{R})$, which is in agreement with [22] (Prosition 5.2). Moreover, one can compute

$$\lim_{|x|\to+\infty} g_s(x) = 0,$$

which is in agreement with [22] (Proposition 3.13), that is, $u(x)$ at infinity tends to its integral mean in $(-1,1)$, namely

$$\lim_{|x|\to+\infty} u(x) = \frac{1}{2}\int_{-1}^{1} u(x)\,dx = 0.$$

However, computing the first derivative of $g_s(x)$, we see that

$$\lim_{x\to 1^+} u'(x) = \lim_{x\to -1^-} u'(x) = \begin{cases} +\infty & s \in (0, \frac{1}{2}), \\ \frac{4}{3} & s = \frac{1}{2}, \\ 0 & s \in (\frac{1}{2}, 1), \end{cases}$$

so $u \notin C^1(\mathbb{R})$ if $s \in (0, \frac{1}{2}]$.

Now, we compute $u''(x)$ and $(-\Delta)^s u(x)$ for any $x \in (-1,1)$. Indeed,

$$u''(x) = 20x^3 - 12x$$

for any $x \in (-1,1)$. Moreover,

$$\begin{aligned}(-\Delta)^s u(x) &= PV \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\,dy \\ &= \int_{-\infty}^{-1} \frac{x(x-1)^2(x+1)^2 + g_s(-y)}{|x-y|^{1+2s}}\,dy \\ &\quad + PV \int_{-1}^{1} \frac{x(x-1)^2(x+1)^2 - y(y-1)^2(y+1)^2}{|x-y|^{1+2s}}\,dy \\ &\quad + \int_{1}^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\,dy.\end{aligned} \tag{A5}$$

To compute

$$PV \int_{-1}^{1} \frac{x(x-1)^2(x+1)^2 - y(y-1)^2(y+1)^2}{|x-y|^{1+2s}}\,dy$$

we take $\varepsilon > 0$ small enough and consider, separately, the integrals

$$\int_{-1}^{x-\varepsilon} \frac{x(x-1)^2(x+1)^2 - y(y-1)^2(y+1)^2}{(y-x)^{1+2s}}\,dy \tag{A6}$$

and

$$\int_{x+\varepsilon}^{1} \frac{x(x-1)^2(x+1)^2 - y(y-1)^2(y+1)^2}{(x-y)^{1+2s}}\,dy. \tag{A7}$$

Exploiting (A2) and (A3), we can compute the integrals in (A6) and (A7). Then, adding them up and taking the limit for $\varepsilon \to 0$, (A5) becomes

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{5-2s}\left[(1+x)^{5-2s} - (1-x)^{5-2s}\right] \\ &\quad - \frac{5x}{4-2s}\left[(1+x)^{4-2s} + (1-x)^{4-2s}\right] \\ &\quad + \frac{10x^2-2}{3-2s}\left[(1+x)^{3-2s} - (1-x)^{3-2s}\right] \\ &\quad - \frac{10x^3-6x}{2-2s}\left[(1+x)^{2-2s} + (1-x)^{2-2s}\right] \\ &\quad + \frac{5x^4-6x^2+1}{1-2s}\left[(1+x)^{1-2s} - (1-x)^{1-2s}\right] \\ &\quad + \int_{1}^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\,dy \\ &\quad + \int_{-\infty}^{-1} \frac{x(x-1)^2(x+1)^2 + g_s(-y)}{|x-y|^{1+2s}}\,dy\end{aligned} \tag{A8}$$

for $s \neq 1/2$, while

$$\begin{aligned}(-\Delta)^{\frac{1}{2}} u(x) &= -8x^3 + \frac{20}{3}x + (5x^4 - 6x^2 + 1)\log\frac{1+x}{1-x} \\ &\quad + \int_{1}^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_{\frac{1}{2}}(y)}{|x-y|^2}\,dy \\ &\quad + \int_{-\infty}^{-1} \frac{x(x-1)^2(x+1)^2 + g_{\frac{1}{2}}(-y)}{|x-y|^2}\,dy.\end{aligned} \tag{A9}$$

With (A8) and (A9) in mind, we can define, for any $s \in (0,1)$ and $x \in (-1,1)$,

$$f_s(x) := -20x^3 + 12x + (-\Delta)^s u(x).$$

As a consequence, we have that u is a solution of

$$\begin{cases} -u''(x) + (-\Delta)^s u(x) = f_s(x) & x \in (-1,1), \\ u'(x) = 0 & x \in \{-1,1\}, \\ \mathcal{N}_s u = 0 & x \in \mathbb{R} \setminus [-1,1]. \end{cases}$$

Notice that, in this setting, such a problem corresponds to (5) with $\alpha = \beta = 1$.

Now, we want to check that $f_s(x) \in L^q((-1,1))$ for any $q \geq 1$. First, the non-integral part of $f_s(x)$ is bounded in $(-1,1)$. Thus, we claim that

$$\int_1^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\, dy \text{ belongs to } L^q((-1,1)) \text{ for any } q \geq 1. \tag{A10}$$

To this aim, we fix $\varepsilon > 0$. We recall that, since $\mathcal{N}_s u = 0$ in $\mathbb{R} \setminus [-1,1]$, we have

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^\infty((0,1))} < +\infty.$$

Thus, recalling (31) and (A1), we obtain

$$\begin{aligned} \left| \int_{1+\varepsilon}^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\, dy \right| &\leq 2\|u\|_{L^\infty(\mathbb{R})} \int_{1+\varepsilon}^{+\infty} \frac{1}{(y-x)^{1+2s}}\, dy \\ &= \frac{\|u\|_{L^\infty(\mathbb{R})}}{s} (1-x+\varepsilon)^{-2s} \\ &\leq \frac{\|u\|_{L^\infty(\mathbb{R})}}{s\varepsilon^{2s}}. \end{aligned}$$

Therefore,

$$\int_{-1}^{1} \left| \int_{1+\varepsilon}^{+\infty} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\, dy \right|^q dx \leq 2 \left(\frac{\|u\|_{L^\infty(\mathbb{R})}}{s\varepsilon^{2s}} \right)^q < +\infty. \tag{A11}$$

On the other hand, if $y \in (1, 1+\varepsilon)$, we have

$$|g_s(y)| \leq c_\varepsilon (y-1)^{2s}$$

for some $c_\varepsilon > 0$. Indeed, for this estimate, it is enough to note that $5y^4 - 6y^2 + 1 = (y-1)(y+1)(5y^2-1)$ and $y^5 - 2y^3 + y = y(y-1)^2(y+1)^2$, so that all the functions in square brackets in (A4) are bounded when $y \in (1, 1+\varepsilon)$. As a consequence, for every $x \in (-1,1)$, we have

$$\begin{aligned} \left| \int_1^{1+\varepsilon} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\, dy \right| &\leq |x|(x-1)^2(x+1)^2 \int_1^{1+\varepsilon} \frac{1}{(y-x)^{1+2s}}\, dy \\ &\quad + c_\varepsilon \int_1^{1+\varepsilon} \frac{(y-1)^{2s}}{(y-x)^{1+2s}}\, dy =: I_1(x) + I_2(x). \end{aligned}$$

First, we clearly have

$$I_1(x) = \frac{|x|(x-1)^2(x+1)^2}{2s} [(1-x)^{-2s} - (1-x+\varepsilon)^{-2s}]$$

which is bounded in $(-1,1)$, and, so, $I_1 \in L^q((-1,1))$ for any $q \geq 1$. Moreover, since $x \in (-1,1)$, we have

$$I_2(x) \leq c_\varepsilon \int_1^{1+\varepsilon} \frac{(y-x)^{2s}}{(y-x)^{1+2s}}\, dy = c_\varepsilon(\log(1-x+\varepsilon) - \log(1-x)).$$

Recalling the inequality

$$|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q) \text{ for any } a, b \in \mathbb{R}, \tag{A12}$$

we obtain that, for any $q \geq 1$,

$$\int_{-1}^1 |I_2(x)|^q\, dx \leq c_2 \int_{-1}^1 (|\log(1-x+\varepsilon)|^q + |\log(1-x)|^q)\, dx < +\infty,$$

for some $c_2 = c_2(s,q,\varepsilon) > 0$.

Combining the integrability of I_1 and I_2 with (A12), we obtain

$$\int_{-1}^1 \left| \int_1^{1+\varepsilon} \frac{x(x-1)^2(x+1)^2 - g_s(y)}{|x-y|^{1+2s}}\, dy \right|^q dx \leq c_3 \int_{-1}^1 (|I_1(x)|^q + |I_2(x)|^q)\, dx < +\infty \tag{A13}$$

for every $q \geq 1$ and for some $c_3 = c_3(s,q,\varepsilon)$. Finally, (A11) and (A13), together with (A12), imply the claim in (A10).

A similar argument shows that

$$\int_{-\infty}^{-1} \frac{x(x-1)^2(x+1)^2 + g_s(-y)}{|x-y|^{1+2s}}\, dy \text{ belongs to } L^q((-1,1)) \text{ for any } q \geq 1.$$

This concludes the proof that $f_s(x) \in L^q((-1,1))$ for any $q \geq 1$.

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