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Backward Continuation of the Solutions of the Cauchy Problem for Linear Fractional System with Deviating Argument

Hristo Kiskinov¹ , Mariyan Milev² , Milena Petkova¹  and Andrey Zahariev^{1,*} ¹ Faculty of Mathematics and Informatics, University of Plovdiv, 4000 Plovdiv, Bulgaria; kiskinov@uni-plovdiv.bg (H.K.); milenapetkova@uni-plovdiv.bg (M.P.)² Faculty of Economics and Business Administration, Sofia University, 1504 Sofia, Bulgaria; m.milev@feb.uni-sofia.bg

* Correspondence: zandrey@uni-plovdiv.bg

Abstract: Fractional calculus provides tools to model systems with memory effects; when coupled with delays, they model process histories inspired by two independent sources—the memory of the fractional derivative and the impact conditioned by the delays. This work considers a Cauchy (initial) problem for a linear delayed system with derivatives in Caputo’s sense of incommensurate order, distributed delays, and piecewise initial functions. For this initial problem, we study the important problem of the backward continuation of its solutions. We consider the backward continuation of the solutions as a problem of the renewal of a process with aftereffect under given final observation. Sufficient conditions for backward continuation of the solutions of these systems have been obtained. As application, a formal (Lagrange) adjoint system for the studied homogeneous system is introduced, and using the backward continuation, it is proved that for this system there exists a unique matrix solution called by us as the formal adjoint fundamental matrix, which can play the same role as the fundamental matrix in the forward case.

Keywords: distributed delay; linear fractional system; backward continuation

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1. Introduction

Fractional calculus and fractional differential equations have been intensively investigated in the last decades through the point of view as important tools for modeling real-world phenomena. It is well known that many natural systems can be represented more accurately via the memory data included through fractional derivative formulation. For valuable information in details on fractional calculus theory and fractional differential equations, we recommend the monographs of Kilbas et al. [1] and Podlubny [2]. Regarding fractional differential equations of distributed order, we refer to Jiao et al. [3]. The theme of fractional impulsive differential equations and their applications are considered in the monograph of Stamova and Stamov [4].

However, when the fractional differential equations are coupled with delays, then the dependence on the past evolution history of the processes described by such equations is inspired by two sources. The first of them is the impact conditioned by the delays, and the other one is the mentioned impact conditioned from the availability of the Volterra-type integral in the definitions of the fractional derivatives, i.e., the memory of the fractional derivative. It must be noted that the first of them (conditioned by the delays) is independent

from the derivative type (integer or fractional), but the impact of both memory sources cause some difficulties at least in the technical aspect but not only in that regard.

As in the integer case (systems with first-order derivatives) and in view of applications, it is natural to consider the backward continuation of the solutions as a problem of renewal of a process with aftereffect under given final observation. We emphasize that this formulation is equivalent from the mathematical point of view to the problem of forward prolongation of the solutions of fractional differential equations with a deviating argument of the advanced type. The main problem is that in following this concept in the fractional case, we have two main obstacles to overcome. The first one is that we are losing the key properties of the first-order derivative as locality, symmetry, and the simple rule of Leibniz. The second one is that a shift of the lower terminal in general leads to different results that are in contrast to the case of the deterministic systems described with ordinary differential equations. Moreover, the problem of the continuation of solutions has features not encountered in ordinary differential equations, as demonstrated by Myshkis [5]. From other monographs considering the same themes as those studied in the present article for integer- (first-) order delayed systems, we recommend Halanay [6], Lunel et al. [7], and Kolmanovskii et al. [8].

The above-mentioned obstacles partially explain the absence of results concerning the backward continuation of the solutions of delayed fractional equations or systems. As far as we know, this article is the first one of its kind containing such results for fractional systems.

In the present work, it is considered an initial (Cauchy) problem (IP) for a linear delayed system with derivatives in Caputo's sense of incommensurate order, distributed delays, and piecewise initial functions. The motivation for studying such distributed delay systems is because this type of delay includes, as a special case, all types of delays (it follows from the Riesz theorem applied to the Krasovskii functional), and in this sense, it is most appropriate to obtain results valid for all types of delays. For this IP, we consider the backward continuation of the solutions as a problem of renewal of a process with aftereffect under a given final observation. Following this concept, we have obtained sufficient conditions for backward continuation of the solutions of these systems. The proposed conditions correspond to those that guarantee the same result in the case of same-class systems with integer orders of differentiation. It is shown with an example that the introduced additional conditions for the backward continuation of the solutions of the Cauchy problem are essential and cannot be weakened in the general case. Furthermore, we introduce a formal (Lagrange) adjoint system for the studied homogeneous system, and then as an application of the backward continuation, it is proved that for this system there exists a unique matrix solution called by us as the formal adjoint fundamental matrix, which can play the same role as the fundamental matrix in the forward case.

The paper is organized as follows. In Section 2, as usual, we recall the definitions of the left and right side Riemann–Liouville fractional integrals and derivatives, Caputo fractional derivatives, and some of their properties. In the same section is also presented the statement of the problem, as well as some necessary comments and results used later. Section 3 is devoted to the adaption of an approach (only used so far for linear delayed systems with integer-order derivatives) to the studied linear systems with distributed delays with Caputo-type fractional derivatives of incommensurate order for establishing the sufficient conditions for backward continuation of the solutions of these systems. It is shown with an example that the introduced additional condition for the backward continuation of the solutions of the Cauchy problem is essential and cannot be weakened. In Section 4, we introduce a formal (Lagrange) adjoint system for the studied homogeneous system, and using the proved in Section 3 theorem for the backward continuation, we establish that for the adjoint system there exists a unique matrix solution called by us the

formal adjoint fundamental matrix, which plays the same role as the fundamental matrix in the forward case. Section 5 contains the conclusion and comments, as well as some ideas for future research.

2. Preliminaries and Problem Statement

To avoid possible mistakes, we recall below the notations used, as well as the definitions of Riemann–Liouville and Caputo fractional derivatives with some descriptions of their properties. For more detailed information, we recommend the books [1,2]. Furthermore, we also consider some auxiliary results, which are needed for our exposition below.

For $a, b \in \mathbb{R}$ and $f \in L_1^{loc}(\mathbb{R}, \mathbb{R})$, where with L_1^{loc} is denoted as the linear space of all locally Lebesgue integrable functions, the left- and right-sided fractional integral operators, as well as the Riemann–Liouville and Caputo fractional derivatives of order $q \in (0, 1)$, are defined by

$$(I_{a+}^q f)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s) ds, \quad {}^{RL}D_{a+}^q f(t) = \frac{d}{dt} (I_{a+}^{1-q} f(t)), t > a;$$

$$(I_{b-}^q f)(t) = \frac{1}{\Gamma(q)} \int_t^b (s-t)^{q-1} f(s) ds, \quad {}^{RL}D_{b-}^q f(t) = -\frac{d}{dt} (I_{b-}^{1-q} f(t)), t < b;$$

$${}^CD_{a+}^q f(t) = {}^{RL}D_{a+}^q [f(s) - f(a)](t) = {}^{RL}D_{a+}^q f(t) - \frac{f(a)}{\Gamma(1-q)} (t-a)^{-q}, t > a;$$

$${}^CD_{b-}^q f(t) = {}^{RL}D_{b-}^q [f(s) - f(b)](t) = {}^{RL}D_{b-}^q f(t) - \frac{f(b)}{\Gamma(1-q)} (b-t)^{-q}, t < b.$$

Below, we will use the following notations: $\mathbb{R}_+ = (0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty)$; $J_{-h} = [-h, \infty)$, $J_a = [a, \infty)$; $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_k \in (0, 1)$, where $k \in \langle n \rangle = \{1, \dots, n\}$; $\mathbb{R}^{n \times n}$ denotes the real linear space of the square matrices with dimension, n , $I, \Theta \in \mathbb{R}^{n \times n}$ are the identity and zero matrix, respectively, and $\mathbf{0} \in \mathbb{R}^n$ denotes the zero vector-column.

Let $W : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, $W(t, \theta) = \{w_{kj}(t, \theta)\}_{k,j=1}^n$, with BV^{loc} denote as the linear space of the matrix-valued functions $W(t, \theta)$ with bounded variation in θ on every compact interval $[c, d] \subset \mathbb{R}$ for every $t \in J_a$, i.e., $\text{Var}_{[c,d]} W(t, \cdot) = \sum_{k,j=1}^n \text{Var}_{[c,d]} w_{kj}(t, \cdot)$, and $|W(t, \theta)| = \sum_{k,j=1}^n |w_{kj}(t, \theta)|$. For $Y(t) = (y_1(t), \dots, y_n(t))^T : J_a \rightarrow \mathbb{R}^n$, $\beta = (\beta_1, \dots, \beta_n)$, $\beta_k \in [-1, 1]$, $k \in \langle n \rangle$ we use the notations $I_\beta(Y(t)) = \text{diag}(y_1^{\beta_1}(t), \dots, y_n^{\beta_n}(t))$.

Let $a \in \mathbb{R}$ be an arbitrary number. Consider for $t > a$ and arbitrary $h > 0$ the inhomogeneous fractional linear delayed system with Caputo-type derivatives of incommensurate orders and distributed delays

$$D_{a+}^\alpha x(t) = \mathbf{U}(t, x_t) + f(t) = \int_{-h}^0 [d_\theta U(t, \theta)] x(t + \theta) + f(t), \quad (1)$$

and the corresponding homogeneous system

$$D_{a+}^\alpha x(t) = \mathbf{U}(t, x_t) = \int_{-h}^0 [d_\theta U(t, \theta)] x(t + \theta), \quad (2)$$

where $D_{a+}^\alpha = \text{diag}({}^CD_{a+}^{\alpha_1}, \dots, {}^CD_{a+}^{\alpha_n})$, $D_{a+}^\alpha x(t) = (D_{a+}^{\alpha_1} x_1(t), \dots, D_{a+}^{\alpha_n} x_n(t))^T$ (the symbol ${}^\top$ means transposition), $D_{a+}^{\alpha_k} x_k(t) = {}^CD_{a+}^{\alpha_k} x_k(t)$, $k \in \langle n \rangle = \{1, \dots, n\}$; $x : [a, \infty) \rightarrow \mathbb{R}^n$; $x(t) = (x_1(t), \dots, x_n(t))^T$, $x_t(\theta) = x(t + \theta)$, $-h \leq \theta \leq 0$; $f : \mathbb{R} \rightarrow \mathbb{R}^n$, $f(t) = (f_1(t), \dots, f_n(t))^T$, $f \in L_1^{loc}(\mathbb{R}, \mathbb{R}^n)$, and $\mathbf{U}(t, \cdot) : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n$ define a linear continuous operator (vector-valued functional) for any $t \in \mathbb{R}$.

Introduce also the initial condition for each $\phi \in \mathbf{PC}$ as follows:

$$x_a(\theta) = x(a + \theta) = \phi(\theta) \quad \text{for} \quad -h \leq \theta \leq 0, \quad (3)$$

where $\mathbf{C} = C([-h, 0], \mathbb{R}^n)$ and $\mathbf{PC} = PC([-h, 0], \mathbb{R}^n)$ are the Banach spaces of all continuous and piecewise continuous vector functions $\phi: [-h, 0] \rightarrow \mathbb{R}^n$, $\phi = (\phi_1, \dots, \phi_n)$, $h > 0$, respectively, with norm

$$\|\phi\| = \sum_{k=1}^n \sup_{s \in [-h, 0]} |\phi_k(s)| < \infty.$$

The integral representation in the right sides of (1) and (2) follows from the Riesz representation theorem, $U(t, \theta) = U_j(t, \theta) + U_{ac}(t, \theta) + U_s(t, \theta) \in BV^{loc}$, where $U_j(t, \theta)$, $U_{ac}(t, \theta)$, and $U_s(t, \theta)$ are in θ the jump, the absolutely continuous and singular parts, respectively, in its Lebesgue decomposition, $U_j(t, \theta) = \sum_{i \in \langle m \rangle_0} U_j^i(t, \theta)$, $i \in \langle m \rangle_0 = \{0, 1, \dots, m\}$, $m \in \mathbb{N}$ and the integral in (1) is understood in the Lebesgue–Stieltjes sense for any $\phi \in \mathbf{PC}$.

Remark 1. The initial condition (3) means that for the function $t \rightarrow x(t)$, $t \in J_a$ is considered as a prolongation from the right of the function $t \rightarrow \phi(t - a)$ for $t \in J_a$.

Definition 1 ([6–8]). We say that for the kernel $U: \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n}$, the Conditions (S) are fulfilled if for each $i \in \langle m \rangle_0$, $m \in \mathbb{N}$, and any $(t, \theta) \in \mathbb{R}^2$ the following conditions hold:

(S1) The functions $U_j^i(t, \theta)$, $U_{ac}(t, \theta)$, and $U_s(t, \theta)$ are measurable in $(t, \theta) \in \mathbb{R}^2$, continuous from the left in θ on $(-h, 0)$, normalized so that $U_j^i(t, \theta) = U_{ac}(t, \theta) = U_s(t, \theta) = \Theta$ for $\theta \geq 0$, $U_j^i(t, \theta) = U_j^i(t, -\sigma_i)$, $U_{ac}(t, \theta) = U_{ac}(t, -\sigma_i)$, and $U_s(t, \theta) = U_s(t, -\sigma_i)$ for $\theta \leq -\sigma_i$, where $\sigma_i(t) \in C(\mathbb{R}, [0, \sigma_i])$, $\sigma_i = \max_{t \in \mathbb{R}} \sigma_i(t)$, and $0 = \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_m = h$.

(S2) The functions $U_j^i(t, \theta)$, $U_{ac}(t, \theta) \in BV^{loc}$ for any fixed $t \in \mathbb{R}$, $|Var_{[-h, 0]} U_j^i(t, \cdot)|$, $|Var_{[-h, 0]} U_{ac}(t, \cdot)|$, $|Var_{[-h, 0]} U_s(t, \cdot)|$ are locally bounded in $\mathbb{R}$, and

$$Var_{[-h, 0]} U_j^i(t, \cdot), Var_{[-h, 0]} U_{ac}(t, \cdot), Var_{[-h, 0]} U_s(t, \cdot) \in L_1^{loc}(\mathbb{R}, \mathbb{R}^{n \times n}).$$

(S3) $U_j^i(t, \theta) = \{a_{kl}^i(t) H(\theta + \sigma_i(t))\}_{k,l=1}^n$, where $H(t)$ is the Heaviside function, and

$$A^m(t) = \{a_{kl}^m(t)\}_{k,l=1}^n \in C(\mathbb{R}, \mathbb{R}^{n \times n}).$$

The matrix $\frac{\partial U_{ac}(t, \theta)}{\partial \theta} = \tilde{A}(t, \theta) \in L_1^{loc}(\mathbb{R}^2, \mathbb{R}^{n \times n})$ and is locally bounded.

(S4) [9] For each $t^* \in J_a$, the following relations hold:

$$\lim_{t \rightarrow t^*} \int_{-h}^0 |U^i(t, \theta) - U^i(t^*, \theta)| d\theta = 0, \quad \lim_{t \rightarrow t^*} \int_{-h}^0 |U_{ac}(t, \theta) - U_{ac}(t^*, \theta)| d\theta = 0,$$

$$\lim_{t \rightarrow t^*} \int_{-h}^0 |U_s(t, \theta) - U_s(t^*, \theta)| d\theta = 0.$$

For all $\phi \in \mathbf{PC}$, we denote the set of its all jump points by S_ϕ and assume that the sets $S_\phi^i = \{t \in J_a | t - \sigma_i(t) \in S_\phi\}$ do not have limit points.

Note that for any $\phi \in \mathbf{PC}$, the set S_ϕ is at most finite, and from Condition (S4), it follows that the sets $S_\phi^i(b) = \{t \in [a, a + b] | t - \sigma_i(t) \in S_\phi\}$ are also finite or empty for all $i \in \langle m \rangle$.

Remark 2. In many works, instead of Condition **(S4)**, the authors suppose that the kernels $U_j^i(t, \theta)$ are continuous in t , which excludes from consideration the important for the applications case of systems with variable concentrated delays (i.e., the system (1) can have only constant delays).

Definition 2. The vector function $x(t; a, \phi, f) = (x_1(t), \dots, x_n(t))^T$ is a solution of the IP (1), (3) in J_a for some $\phi \in \mathbf{PC}$ if $x(t; a, \phi, f) \in C(J_a, \mathbb{R}^n)$ satisfies the system (1) for all $t \in (a, \infty)$ and the initial condition (3) for each $t \in [a - h, a]$. (When misunderstanding is not possible, we will write only $x(t)$).

Consider the following auxiliary system:

$$x(t) = \phi(0) + I_{-1}(\Gamma(\alpha)) \left(\int_a^t I_{\alpha-1}(t-s) \int_{-h}^0 [d_\theta U(s, \theta)] x(s+\theta) ds + \int_a^t I_{\alpha-1}(t-s) f(s) ds \right), \quad (4)$$

where $I_{-1}(\Gamma(\alpha)) = \text{diag}(\Gamma^{-1}(\alpha_1), \dots, \Gamma^{-1}(\alpha_n))$.

The system (4) written in detail for $k \in \langle n \rangle$ has the following form:

$$x_k(t) = \phi_k(0) + \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k-1} \left(\sum_{i \in \langle m \rangle_0} \left(\sum_{j \in \langle n \rangle} \int_{-h}^0 x_j(s+\theta) d_\theta u_{k,j}^i(s, \theta) ds \right) \right) + \int_a^t (t-s)^{\alpha_k-1} f_k(s) ds.$$

Definition 3. The vector function $x(t; a, \phi, f) = (x_1(t), \dots, x_n(t))^T$ is a solution of the IP (4), (3) in J_a for some $\phi \in \mathbf{PC}$ if $x(t; a, \phi, f) \in C(J, \mathbb{R}^n)$ satisfies the system (4) for all $t \in (a, \infty)$ and the initial condition (3) for each $t \in [a - h, a]$.

Introduce the matrix $\tilde{U}(t, \theta) = U(t, \theta + 0) - U(t, \theta - 0)$ for $t \in \mathbb{R}$ and $\theta \in [-h, 0]$.

Definition 4 ([7]). The matrix $U(t, \theta)$ is called atomic at some $\bar{\theta} \in [-h, 0]$ if $\det \tilde{U}(t, \bar{\theta}) \neq 0$ for all $t \in \mathbb{R}$.

Remark 3. It is well known that the matrix $U(t, \theta)$ satisfying the Conditions **(S)** defines a Lebesgue–Stieltjes measure. From the jump points in θ of the jump part in its Lebesgue decomposition, it follows that at these points the measure will have atoms, and as consequence, the system will possess concentrated delays (variable or/and constant). So, if in its Lebesgue decomposition the jump part is missing, then the corresponding system will be without concentrated delays.

For any arbitrary fixed number $s \in J_a$, consider the following matrix IP:

$$D_{a+}^\alpha C(t, s) = \int_{-h}^0 [d_\theta U(t, \theta)] C(t + \theta, s), \quad t \in (s, \infty); \quad (5)$$

$$C(s, s) = I, C(t, s) = \Theta \quad \text{for } t \in (-\infty, s). \quad (6)$$

Definition 5 ([10]). The matrix-valued function $t \rightarrow C(t, s) = (c^1(t, s), \dots, c^n(t, s)) = \{c_{jk}(t, s)\}_{k,j=1}^n$ is called a solution of the IP (5), (6) for any fixed $s \in J_a$ if $C(\cdot, s) : [s, \infty) \rightarrow \mathbb{R}^{n \times n}$ is continuous for $t \in [s, \infty)$ and satisfies the matrix Equation (5) on $t \in (s, \infty)$, as well as the initial condition (6) too.

In our exposition below, we need the following results summarized in the next theorem.

Theorem 1 ([10–12]). Let the following conditions hold:

1. Conditions **(S)** hold on $J_a \times \mathbb{R}$.

2. The function $f \in L_1^{\text{loc}}(J_a, \mathbb{R}^n)$ is locally bounded.

Then, the following statements hold:

(i) Then, every solution $x(t)$ of IP (1), (3) is a solution of the IP (4), (3) and vice versa.

(ii) For every initial function $\phi \in \mathbf{PC}$, the IP (5), (6) has a unique solution for $t \in (s, \infty)$, with $x_t(\theta) = x(t + \theta) = \phi(\theta)$ and $-h \leq \theta \leq 0$ for any fixed $s \in J_a$.

Remark 4. The existence and the uniqueness of $C(t, s)$ for $s \geq a$ has been established in [12]. Furthermore, for every $t \in J_a$, the function $C(t, \cdot) : [a, t] \rightarrow \mathbb{R}^{n \times n}$ is locally bounded and Lebesgue measurable in s , and $C(t, s)$ is absolutely continuous in t on every compact subinterval of J_a for every fixed $s \in J_a$ (see [13]).

3. Backward Continuation

In this section, we study the possibility of a backward continuation of the solutions of the IP (1), (3) via an new appropriate definition, since the fractional derivatives are not symmetric and in contrast with the case of systems with integer- (first-) order derivatives. It must be noted that the definition introduced by us for the fractional case coincides with the definition used in the integer case (see [7,8]), where the order of differentiation is the integer (first) derivative.

To construct a backward problem (BwP) of the IP (1), (3), let us consider the system

$$-D_{a-}^{\alpha} y(t) = \int_{-h}^0 [d_{\theta} U(t, \theta)] y(t + \theta) + f(t), \quad t < a, \quad (7)$$

where $D_{a-}^{\alpha} = \text{diag}(^C D_{a-}^{\alpha_1}, \dots, ^C D_{a-}^{\alpha_n})$, $D_{a-}^{\alpha} y(t) = (D_{a-}^{\alpha_1} y_1(t), \dots, D_{a-}^{\alpha_n} y_n(t))^{\top}$, $D_{a-}^{\alpha_k} y_k(t) = ^C D_{a-}^{\alpha_k} y_k(t)$, $k \in \langle n \rangle$; $y : (-\infty, a) \rightarrow \mathbb{R}^n$; $y(t) = (y_1(t), \dots, y_n(t))^{\top}$, $y_t(\theta) = y(t + \theta)$, $-h \leq \theta \leq 0$; with the following final condition for $\phi \in \mathbf{PC}$ being

$$y_a(\theta) = y(a + \theta) = \phi(\theta) \quad \text{for} \quad -h \leq \theta \leq 0. \quad (8)$$

Applying the operator I_{a-}^{α} to both sides of (7) and using formula 2.4.44 in [1], we obtain

$$y(t) = \phi(0) + I_{-1}(\Gamma(\alpha)) \left(\int_t^a I_{\alpha-1}(s-t) \int_{-h}^0 [d_{\theta} U(s, \theta)] y(s + \theta) ds + \int_t^a I_{\alpha-1}(s-t) f(s) ds \right), \quad (9)$$

Definition 6. The vector function $y(t; a, \phi, f) = (y_1(t), \dots, y_n(t))^{\top} : (-\infty, a) \rightarrow \mathbb{R}^n$ is a solution of the backward problem (BwP) (7), (8) for some $\phi \in \mathbf{PC}$ on the interval $[a - h - M, a]$ for arbitrary $M > 0$ (or on $(-\infty, a]$) if $y(t; a, \phi, f) \in C([a - h - M, a - h], \mathbb{R}^n)$ ($y(t; a, \phi, f) \in C((-\infty, a - h], \mathbb{R}^n)$) satisfies the system (7) for $t \in (a - h - M, a]$ ($t \in (-\infty, a]$) and at the same time $y_a(\theta; a, \phi, f) = \phi(\theta)$ for $-h \leq \theta \leq 0$ (i.e., the final condition (8)).

Remark 5. As in the case for the left-sided fractional derivatives, for the right-sided fractional derivatives, by applying the operator I_{a-}^{α} to both sides of (7) in the same way, we can conclude that each solution in the sense of Definition 6 of the BwP (7), (8) is a solution in the same sense of the BwP (9), (8) and vice versa. The result is obvious, and that is why the proof of this statement will be omitted.

Remark 6. Note that the left- and right-sided fractional derivatives are nonlocal and not symmetric in contrast to the first derivative, which is local and symmetric at every point where it exists. But in the case when all $\alpha_k = 1$, $k \in \langle n \rangle$, we have that $-^C D_{a-}^{\alpha} y(t) = \frac{dy(t)}{dt} = ^C D_{a+}^{\alpha} y(t)$ (see formula 2.4.14 in [1]) and hence, the introduced definition coincides with the definition for backward continuation of the solution in the integer case (see [7]). Furthermore, as in the integer case, the Definition 6 means that the backward continuation is a prolongation for $t < a$ of the solution for

$t \geq a$ of the IP (1), (3) for some $\phi \in \mathbf{PC}$, and in addition ϕ as a part of the prolongation, it must satisfy the system (8) for $t + a \in [-h, 0]$. Hence, in both cases (integer and fractional), we obtain a solution for all $t \in \mathbb{R}$. The additional assumption that $\phi \in \mathbf{C}$ in virtue of Definition 6 extends the interval of continuity of the solution, i.e., $y(t; a, \phi, f) \in C([a - h - M, a], \mathbb{R}^n)$. Moreover, we call the condition (8) (with the same function as in the initial condition (3)) final from the physical point of view, since the considered problem is treated as a renewal of a process with an aftereffect under the given final observation.

Remark 7. We emphasize that Definition 6 requires that for $t \in [a - h, a]$, the final condition (8) $y(t; a, \phi, f) = \phi(t - a)$ must be fulfilled, and at the same time, $y(t; a, \phi, f)$ must satisfy the system (7), which in general is not fulfilled for arbitrary $\phi \in \mathbf{PC}$. So, as in the integer case, the backward continuation is not possible for all $\phi \in \mathbf{PC}$ (even and for all $\phi \in \mathbf{C}$), but it is only possible for those which satisfy some appropriate additional conditions. It must also be noted that the function $f(t)$ and the kernel $U(t, \theta)$ in (1) and (7) are the same (they coincide).

As was mentioned above in view of the applications, following the approach in the integer case (first-order derivatives), it is natural to consider the backward continuation of the solutions as a problem of renewal of a process with aftereffect the under given final observation. This formulation is equivalent from the mathematical point of view to the problem of forward prolongation of the solutions of fractional differential equations with a deviating argument of the advanced type, but this concept leads also to some complications, which need to be overcome.

Lemma 1. Let the following conditions hold:

1. Conditions (S) hold on $\mathbb{R} \times \mathbb{R}$.
2. The function $f \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ is locally bounded.

Then, BwP (7), (8) possess a unique solution $y(t; a, \phi, f) \in C([a - h - M, a - h], \mathbb{R}^n)$ with an interval of existence $[a - h - M, a]$ only when the final function $\phi \in \mathbf{PC}$ satisfies the following relations:

$$\begin{aligned} -D_{a-}^{\alpha} \phi(0) &= \int_{-h}^0 [d_{\theta} U(a, \theta)] \phi(\theta) + f(a), \\ -D_{a-}^{\alpha} \phi(-h) &= -U(a - h, -h) \phi(-h) + f(a - h). \end{aligned} \quad (10)$$

Proof. Let $y(t; a, \phi, f): (-\infty, a) \rightarrow \mathbb{R}^n$ be the solution of the BwP (7), (8) in the interval $[a - h - M, a]$.

Then, $y(t; a, \phi, f) \in C([a - h - M, a - h], \mathbb{R}^n)$, $y_a(\theta; a, \phi, f) = \phi(\theta)$, and $y(t; a, \phi, f)$ for $t \in [a - h, a]$ satisfy the system

$$-D_{a-}^{\alpha} y(t) = \int_{-h}^0 [d_{\theta} U(t, \theta)] y(t + \theta) + f(t)$$

and hence, for $t + a \in [-h, 0]$, we have

$$-D_{a-}^{\alpha} y(t + a) = \int_{-h}^0 [d_{\theta} U(t + a, \theta)] y(t + a + \theta) + f(t + a). \quad (11)$$

Let us define $y(t + a) = \bar{\phi}(t) + S(t)$ for $t \in (-\infty, 0]$, where $\bar{\phi}(t) = \phi(t)$, $t \in [-h, 0]$, $\bar{\phi}(t) = \phi(-h)$, $t \in (-\infty, -h]$ and then from (8) for $t \in [-h, 0]$, it follows that $S(t) = \mathbf{0}$ ($S_0 = S(\theta) = \mathbf{0}$, $\theta \in [-h, 0]$). Since for any $t \in [-h, 0]$, we have $-D_{a-}^{\alpha} y(t + a) = -D_{a-}^{\alpha} \phi(t)$, we obtain the $y(t)$ that satisfies (11) for $t \in [a - h, a]$ if and only if when for any $t \in [-h, 0]$, the function $S(t)$ satisfies the system:

$$\begin{aligned}
 -D_{a-}^{\alpha}\phi(t) + S(t) &= -D_{a-}^{\alpha}\phi(t) = \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + S(t+\theta) + f(t+a) \\
 &= \int_{-h}^0 [d_{\theta}U(t+a, \theta)]S(t+\theta) + \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + f(t+a).
 \end{aligned} \tag{12}$$

For any $t \in [-h, 0]$ from (8) and (12), it follows that

$$-D_{a-}^{\alpha}\phi(t) = \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + f(t+a) \tag{13}$$

and then the needed relations (10) follow from (13) for $t = 0$ and $t = -h$, respectively. $\square$

Let us for any final function $\phi \in \mathbf{PC}$ denote by $t_{\phi}^1 \in [-\sigma_1, 0)$ its largest jump point if it exists or $t_{\phi}^1 = -\sigma_1$ if there are no jump points.

Theorem 2 (Local backward continuation). *Let the following conditions hold:*

1. The initial function $\phi \in \mathbf{PC}$ satisfies the relation (10) and $\det A^m(t) \neq 0$, $t \in \mathbb{R}$ (i.e., the matrix $U(t, \theta)$ in Definition 4 is atomic at $\theta = -h$ for any $t \in \mathbb{R}$).
2. Conditions (S) hold on $\mathbb{R} \times \mathbb{R}$ and the function $f \in C(\mathbb{R}, \mathbb{R}^n)$.
3. The kernel $U(t, \theta)$ has the form $U(t, \theta) = \sum_{i \in \langle m \rangle_0} A^i(t)H(\theta + \sigma_i(t)) + \int_{-h}^{\theta} \tilde{A}(t, s)ds$ (i.e.,

$U_s(t, \theta) \equiv \Theta$ in the Lebesgue decomposition).

Then, the BwP (7), (8) possess a unique solution

$$y(t; a, \phi, f) \in C([a-h-h_*, a-h], \mathbb{R}^n)$$

with an interval of existence $[a-h-h_*, a]$, where $h_* \in (0, \min(|t_{\phi}^1|, h - \sigma_{m-1}))$.

Proof. Let $\phi \in \mathbf{PC}$ satisfying the relation (10) be arbitrary. Then, as in the proof of Lemma 1, we can conclude that the function $y(t; a, \phi, f) : (-\infty, a) \rightarrow \mathbb{R}^n$ is a solution of BwP (9), (8) in $[a-h-h_*, a]$ if and only if $y_a(\theta, a, \phi, f) = \phi(\theta)$ for $\theta \in [-h, 0]$ and $y(t+a)$ satisfies the system (11) for any $t \in [-h, 0]$.

As above, let $y(t+a) = \bar{\phi}(t) + S(t)$ for $t \in (-\infty, 0]$, and hence, $y(t+a)$ satisfies (11) for any $t \in [-h, 0]$ if and only if when for any $t \in [-h, 0]$, the function $S(t)$ with $S_0 = \mathbf{0}$ satisfies the system

$$-D_{a-}^{\alpha}\phi(t) = \int_{-h}^0 [d_{\theta}U(t+a, \theta)]S(t+\theta) + \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + f(t+a). \tag{14}$$

From equality (14), it follows that for $t \in [-h, 0]$, we have

$$\begin{aligned}
 A^m(t+a)S(t-h) &= \\
 &- \left[\int_{-h}^0 [d_{\theta}U(t+a, \theta)]S(t+\theta) + D_{a-}^{\alpha}\phi(t) + \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + f(t+a) \right]
 \end{aligned} \tag{15}$$

and hence, according condition 3, the system (15) can be rewritten in the form

$$\begin{aligned}
 S(t-h) &= -A^m(t+a)^{-1} \left[\int_{-h}^0 [d_{\theta}U(t+a, \theta)]S(t+\theta) \right] \\
 &- A^m(t+a)^{-1} \left[D_{a-}^{\alpha}\phi(t) + \int_{-h}^0 [d_{\theta}U(t+a, \theta)]\bar{\phi}(t+\theta) + f(t+a) \right].
 \end{aligned} \tag{16}$$

So, we obtain that the equality (11) for $t \in [a-h, a]$ is equivalent to the equalities $S(t) = \mathbf{0}$ and (16) for $t \in [-h, 0]$.

Define the set

$$\mathbf{D}^{h*} = \{S : [-h - h_*, 0] \rightarrow \mathbb{R}^n \mid S \in C([-h - h_*, -h], \mathbb{R}^n); S(t) = \mathbf{0}, t \in [-h, 0]\}$$

endowed with the metric function

$$d^{h*}(S, \bar{S}) = \sum_{k \in \langle n \rangle} \sup_{t \in [-h - h_*, 0]} e^{-\lambda(t+h)} |s_k(t) - \bar{s}_k(t)|,$$

for any $\lambda \in \mathbb{R}_+$ and $S = (s_1(t), \dots, s_n(t))^\top, \bar{S} = (\bar{s}_1(t), \dots, \bar{s}_n(t))^\top \in \mathbf{D}^{h*}$.

It is clear that $\mathbf{D}^{h*}$ is a complete metric space with respect to the introduced metric function for any $\lambda \in \mathbb{R}_+$. For any $S \in \mathbf{D}^{h*}$ and $t \in [-h_*, 0]$ ($t - h \in [-h - h_*, -h]$) define the operator $\mathfrak{J}S(t) = (\mathfrak{J}s_1(t), \dots, \mathfrak{J}s_n(t))^\top$ via the following equality:

$$\begin{aligned} \mathfrak{J}S(t-h) = & -A^m(t+a)^{-1} \left[\int_{-h^+}^0 [d_\theta U(t+a, \theta)] S(t+\theta) \right] \\ & - A^m(t+a)^{-1} \left[D_{a-}^\alpha \phi(t) + \int_{-h}^0 [d_\theta U(t+a, \theta)] \bar{\phi}(t+\theta) + f(t+a) \right]. \end{aligned} \quad (17)$$

with the additional condition

$$\mathfrak{J}S(t) = S(t) = \mathbf{0}, t \in [-h, 0]. \quad (18)$$

First, we will prove that $\mathfrak{J}\mathbf{D}^{h*} \subset \mathbf{D}^{h*}$.

From Lemma 1 in [9] and Conditions (S), it follows that, since for any $t \in [-h_*, 0]$ the function $S(t+\theta)$ is continuous, the function $\bar{\phi}(t)$ is continuous for $t \in [-h - h_*, h]$, and the expression in the second brackets in (17) is equal to zero for $t \in [-h_*, 0]$; then, the functions $t \rightarrow \int_{-h}^0 [d_\theta U(t, \theta)] S(t+\theta)$ and the expression in the second brackets in (17) are continuous too. Since $A^m(t) \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ and $f \in C(\mathbb{R}, \mathbb{R}^n)$, then the right side of the system (17) is also continuous in the interval $t \in [-h_*, 0]$. Then from (17), taking into account the first relation in (10) and that $S_0 = S(\theta) = \mathbf{0}, \theta \in [-h, 0]$, we obtain for $t \rightarrow -h - 0$ that

$$\mathfrak{J}S(-h-0) = -A^m(0)^{-1} \left[\int_{-h^+}^0 [d_\theta U(a, \theta)] S(\theta) + D_{a-}^\alpha \phi(0) + \int_{-h}^0 [d_\theta U(a, \theta)] \bar{\phi}(\theta) + F(a) \right] = \mathbf{0}.$$

Condition (18) implies that $\mathfrak{J}S(-h+0) = \mathbf{0}$, and hence, $\mathfrak{J}S \in \mathbf{D}^{h*}$. Thus, we have that $\mathfrak{J}\mathbf{D}^{h*} \subset \mathbf{D}^{h*}$.

Denote by

$$\begin{aligned} \ell(t) = & D_{a-}^\alpha \phi(t) + \int_{-h}^0 [d_\theta U(t+a, \theta)] \bar{\phi}(t+\theta) + f(t+a), \\ A_h = & \max \left(\sup_{t \in [-2h, 0]} |A^m(t)|^{-1}, \sup_{t \in [-2h, 0], \theta \in [-h, 0]} |\tilde{A}(t, \theta)| \right) \end{aligned}$$

and then rewrite (17) in the form

$$\begin{aligned} \mathfrak{J}S(t-h) = & -A^m(t+a)^{-1} \int_{-h^+}^0 [d_\theta U(t+a, \theta)] S(t+\theta) + \ell(t) \\ = & -A^m(t+a)^{-1} \left(\sum_{i \in \langle m-1 \rangle_0} A^i(t+a) S(t-\sigma_i(t)) + \int_{-h}^0 S(t+\theta) \tilde{A}(t+a, \theta) d\theta \right) + \ell(t). \end{aligned} \quad (19)$$

Since for any $t \in [-h_*, 0]$, we have that $\sum_{i \in \langle m-1 \rangle_0} A^i(t+a)S(t-\sigma_i(t)) = \mathbf{0}$ then finally from (19), we obtain

$$\mathfrak{J}S(t-h) = -A^m(t+a)^{-1} \int_{-h}^0 S(t+\theta) \tilde{A}(t+a, \theta) d\theta + \ell(t). \quad (20)$$

Then for any $S, \bar{S} \in \mathbf{D}^{h*}$ using (20), we have that

$$\begin{aligned} |\mathfrak{J}S(t-h) - \mathfrak{J}\bar{S}(t-h)| &\leq A_h \left| \int_{-h}^0 S(t+\theta) - \bar{S}(t+\theta) \tilde{A}(t+a, \theta) d\theta \right| \\ &\stackrel{t+\theta=s}{\leq} A_h \left| \int_{t-h}^t e^{\lambda(s+h)} e^{-\lambda(s+h)} S(s) - \bar{S}(s) \tilde{A}(t+a, s-t) ds \right| \\ &\leq \frac{A_h^2 h}{\lambda} e^{\lambda(t+h)} d_{\lambda}^{h*}(S, \bar{S}) \end{aligned} \quad (21)$$

and hence, for $\lambda \geq 2A_h^2 h$ we obtain that $d_{\lambda}^{h*}(S, \bar{S}) \leq \frac{1}{2} d_{\lambda}^{h*}(S, \bar{S})$, i.e., the operator $\mathfrak{J}$ is contractive in $\mathbf{D}^{h*}$ and has a unique fixed point $\tilde{S}(t) \in \mathbf{D}^{h*}$.

Thus, the function $y(t+a) = \phi(t) + \tilde{S}(t)$ is the unique solution of the BwP (7), (8), with an interval of existence $[a-h-h_*, a]$ and $y(t; a, \phi, f) \in C([a-h-h_*, a-h], \mathbb{R}^n)$. $\square$

Let $\phi \in \mathbf{C}$ be arbitrary and $y(t; a, \phi, f) \in C((t_y, a], \mathbb{R}^n)$, $\bar{y}(t; a, \phi, f) \in C((t_{\bar{y}}, a], \mathbb{R}^n)$, $a-h > t_{\bar{y}}$, t_y are two solutions of the IP (7), (8) with intervals of existences $(t_y, a]$ and $(t_{\bar{y}}, a]$.

Definition 7 ([8]). The solution $Y(t; a, \phi, f)$ will be called a continuation of the solution $\bar{Y}(t; a, \phi, f)$ if from the relation $(t_{\bar{y}}, a] \subset (t_y, a]$, it follows that $y(t; a, \phi, f) \equiv \bar{y}(t; a, \phi, f)$, $t \in (t_{\bar{y}}, a]$. The solution $y_{\max}(t; a, \phi, f)$ of the BwP (7), (8), with an interval of existence $(t_{\max}, a]$ is called maximal if it is a continuation of any other solution of the BwP (7), (8).

Theorem 3. Let the conditions of Theorem 2 hold and let for some $\phi \in \mathbf{PC}$ the BwP (7), (8) possess a unique solution.

Then for this $\phi \in \mathbf{PC}$, the interval of existence of the maximal solution of the BwP (7), (8) cannot be a closed interval.

Proof. Assume the contrary that there exists $\phi \in \mathbf{PC}$ satisfying the relations (10) such that the maximal solution $y_M(t; a, \phi, f)$ of the BwP (7), (8) is finite and closed from the left interval of existence $[t_M, a]$. Since according Theorem 2 we have that $t_M \leq a-h-h_*$, then $[t_M, a-h] \neq \emptyset$. To obtain a contradiction, we need to construct a prolongation of the solution $y_M(t; a, \phi, f)$. Define the final function as follows: $\phi_*(t - (t_M + h)) = y_M(t)$ for $t \in J_0 = [t_M, t_M + h]$, and hence, for $t \in [-h, 0]$ we have that $y_M(t + t_M + h) = \phi_*(t)$. We will seek a solution $\bar{y}(t; t_M + h, \bar{\phi}_*, f)$ of the system (7) in the interval $J_{-h_*} = [t_M - h_*, t_M + h]$ which satisfies the following final condition

$$\bar{y}_{t_M+h}(\theta) = \bar{y}(t_M + h + \theta) = \phi_*(\theta), \quad \theta \in [-h, 0]. \quad (22)$$

As above, define $\bar{\phi}_*(t) = \phi_*(t)$ for $t \in [-h, 0]$ and $\bar{\phi}_*(t) = \phi_*(-h)$ when $t \leq -h$. Similarly, we seek the solution in the form $\bar{y}(t + t_M + h) = \bar{\phi}_*(t) + S(t)$, where $S(t) \in \mathbf{D}^{-h*}$ (defined in Theorem 2). We will prove that (7) possess a unique solution $\bar{y}(t; t_M + h, \bar{\phi}_*, f)$ with an interval of existence J_{-h_*} , which satisfies the final condition (22). It is clear that $\bar{y}(t + t_M + h)$ satisfies (11) for any $t \in [-h, 0]$ if and only if $S(t) = \mathbf{0}$ for $t \in [-h, 0]$ and for any $t \in [-h_*, 0]$ the function $S(t)$ satisfies the following system:

$$\begin{aligned}
S(t-h) &= -A^m(t+t_M+h)^{-1} \int_{-h^+}^0 [d_\theta U(t+t_M+h, \theta)] S(t+\theta) \\
&\quad - A^m(t+t_M+h)^{-1} \left[D_{a-}^\alpha \phi_*(\theta) + \int_{-h}^0 [d_\theta U(t+t_M+h, \theta)] \bar{\phi}_*(t+\theta) + f(t+t_M+h) \right].
\end{aligned} \tag{23}$$

Using (23), we can define the operator $(\mathfrak{J}S)(t)$ for any $S(t) \in \mathbf{D}^{-h^*}$ and $t \in [-h_*, 0]$ via the equality

$$\begin{aligned}
\mathfrak{J}S(t-h) &= -A^m(t+t_M+h)^{-1} \int_{-h^+}^0 [d_\theta U(t+t_M+h, \theta)] S(t+\theta) + \ell^*(t) \\
&= -A^m(t+t_M+h)^{-1} \left(\sum_{i \in \langle m-1 \rangle_0} A^i(t+t_M+h) S(t-\sigma_i(t)) + \int_{-h}^0 S(t+\theta) \tilde{A}(t+t_M+h, \theta) d\theta \right) \\
&\quad + \ell^*(t),
\end{aligned} \tag{24}$$

where

$$\ell^*(t) = -A^m(t+t_M+h)^{-1} \left(D_{a-}^\alpha \phi_*(t) + \int_{-h}^0 [d_\theta U(t+t_M+h, \theta)] \bar{\phi}_*(t+\theta) + f(t+t_M+h) \right)$$

and the additional condition $(\mathfrak{J}S)(t-h) = \mathbf{0}$, $t \in [-h, 0]$.

Then, using the notation

$$A_h^* = \max \left(\sup_{t \in [t_M-h, t_M+h]} |A^m(t)|^{-1}, \sup_{t \in [t_M-h, t_M+h], \theta \in [-h, 0]} |\tilde{A}(t, \theta)| \right)$$

in the same way as in Theorem 2, we establish that for $\lambda \geq 2(A_h^*)^2 h$, the operator $\mathfrak{J}$ is contractive in $\mathbf{D}^{h^*}$ and has a unique fixed point $\tilde{S}(t) \in \mathbf{D}^{h^*}$. Thus, the function $\bar{y}(t+t_M+h) = \bar{\phi}_*(t) + \tilde{S}(t)$ is the unique solution of (7), which satisfies the final condition (22) with an interval of existence in the interval $J_{-h_*} = [t_M - h_*, t_M + h]$.

So, under our assumption that the interval of existence of the maximal solution $y_M(t; a, \phi, f)$ is closed from the left, we have constructed a prolongation of $y_M(t; a, \phi, f)$, which is a contradiction. $\square$

Corollary 1 (Global backward continuation). *Let the conditions of Theorem 2 hold.*

Then, for any $\phi \in \mathbf{PC}$ satisfying the relations (10), the BwP (7), (8) possess a unique solution $Y^\infty(t; a, \phi, f) \in C((-\infty, a], \mathbb{R}^n)$ with an interval of existence $(-\infty, a]$.

Proof. Let $\phi \in \mathbf{PC}$ satisfying the relations (10) be arbitrary. Then, according to Theorem 2 and Theorem 3, the interval of existence of the maximal solution $y_M(t; a, \phi, f)$ has the form $J_M = (t_M, a]$ and $J_M \neq \emptyset$.

Let us assume the contrary that J_M is finite, i.e., $t_{\max} > -\infty$, and then $y_M(t; a, \phi, f)$ satisfies the system (7) for $t \in (t_{\max}, a]$; hence, from (9) it follows that $y_M(t; a, \phi, f)$ for any $t \in (t_{\max}, a]$ satisfies the system

$$y_M(t) = \phi(0) + I_{-1}\Gamma(\alpha) \left[\int_a^t I_{\alpha-1}(s-t) \int_{-h}^0 [d_\theta U(s, \theta)] y_M(s+\theta) ds + \int_a^t I_{\alpha-1}(s-t) f(s) ds \right]. \tag{25}$$

Since the right side of (25) is a continuous function for $t \in (t_{\max}, a]$, then taking the limit for $t \rightarrow t_M + 0$, we obtain that $y_M(t; a, \phi, f)$ satisfies (25) at $t = t_M$ too. Therefore, we have constructed a continuation of $y_M(t; a, \phi, f)$, which is a contradiction. Thus, $t_M = -\infty$, and the proof is completed. $\square$

Remark 8. Since condition 1 of Theorem 2 is not present in the theorems for forward continuation of the solutions, it is worthy to establish that this condition is essential for the backward continuation of the solutions, and it is not introduced as necessary because of the choice of proof technique of Theorem 2. The next example shows that condition 3 of Theorem 2 is essential for its validity.

Example 1. Let $a = 0$, $h = 1$, $\alpha \in (0, 1)$, $U_{ac}(t, \theta), U_s(t, \theta) \equiv \Theta$, $U_d(t, \theta) = a(t)H(\theta + 1)$, $-1 \leq \theta \leq 0$ (that implies that the example has one constant delay), $\phi(t) \in \mathbf{C} = C([-1, 0], \mathbb{R})$, $b(t) \in C(\mathbb{R}, \mathbb{R})$, and consider the backward problem

$$-D_{0-}^{\alpha} y(t) = b(t)y(t-1), \quad t < 0; \quad (26)$$

$$y(t) = \phi(t), \quad t \in [-1, 0]. \quad (27)$$

Let $\gamma \in (0, 1)$, and if $b(\bar{t}) \neq 0$ for $\bar{t} \in (-\gamma, 0]$ and since $U_d(t, \theta)$ is atomic at $\theta = -1$, then the condition 1 of Theorem 2 holds for any $\bar{t} \in (-\gamma, 0]$. From (26) for each $\bar{t} \in (-\gamma, 0]$, we have that $y(\bar{t}-1) = -\frac{1}{b(\bar{t})}D_{0-}^{\alpha} y(\bar{t})$, and hence, using (27) for any $t = \bar{t}-1 \in (\gamma-1, -1]$, we obtain the equation

$$y(t) = -\frac{1}{b(t+1)}D_{0-}^{\alpha} \phi(t+1). \quad (28)$$

Note that the condition $\det A^m(t) \neq 0$, $t \in \mathbb{R}$ (i.e., the matrix $U(t, \theta)$ to be atomic at $\theta = -h$ for $t \in \mathbb{R}$) in the considered equation for $h = 1$ is equivalent to $b(t) \neq 0$ for any $t \in \mathbb{R}$, which is a necessary and sufficient condition to obtain the system (28) from (26). Then, to have a backward continuation (to can define $y(t)$ as unique solution of the BwP (26), (27)), we need to establish that $y(t) \in C((-1-\gamma, 0], \mathbb{R})$, and hence, it is necessary to check the continuity of $y(t)$ at $t = -1$. Since $\phi(t) \in \mathbf{C} = C([-1, 0], \mathbb{R})$, then from (27), it follows that $y(-1+0) = y(-1) = \phi(-1)$. From (28), we obtain that $y(-1) = -\frac{1}{b(0)}D_{0-}^{\alpha} \phi(0)$, and therefore, the validity of the equality

$$\phi(-1) = -\frac{1}{b(0)}D_{0-}^{\alpha} \phi(0), \quad (29)$$

is a necessary and sufficient condition for the continuation $y(t)$ to be continuous at $t = -1$. So, the initial function must satisfy the system (26) at least at $t = 0$.

From the example, it is clear that if $b(\bar{t}) \neq 0$ for $\bar{t} \in (-\infty, 0]$, then we have a backward continuation with an interval of existence $(-\infty, 0]$. So, this part of condition 3 of Theorem 2, namely, $\det A^m(t) \neq 0$, $t \in \mathbb{R}$, is essential for its validity. Via a simple comparison with the corresponding equality in the integer case ($\alpha = 1$), namely,

$$\phi(-1) = -a^{-1}(0)\phi'(0), \quad (30)$$

we see that when the function ϕ is only continuous and nondifferentiable at $t = 0$, the equality (29) is a nonlocal condition, and equality (30) is a local condition, which in fact implies that the backward continuation in the fractional case is a more difficult problem compared to the integer case.

Remark 9. It must be noted that in the partial case when there exists $\varepsilon > 0$ such that $\phi(t) \in AC((-\varepsilon, 0], \mathbb{R})$ with a locally bounded first derivative in some left vicinity of $t = 0$ (or $\phi(t) \in C^1((-\varepsilon, 0], \mathbb{R})$ as it is required in the integer case), then for $D_{0-}^{\alpha} \phi(0)$ (see [1], formula 2.4.21), we have that $D_{0-}^{\alpha} \phi(0) = 0$, and hence, (29) implies that $\phi(-1) = 0$, which condition can replace in this case condition (29).

4. Application

It is well known that the adjoint systems play a prominent role, especially in the representation of the solutions and their application in stability and control theory. In addition, the adjoint theory also appears in systems with a periodical right side and is useful in linear boundary problems too.

The main goal of this section is to introduce a formal (Lagrange) adjoint system for the homogeneous system (2) and then as an application of the backward continuation to prove that for this system there exists a unique matrix solution called by us the formal adjoint fundamental matrix, which can play the same role as the fundamental matrix of (2).

The predominant type of mathematical models using (linear) systems with integer- or fractional-order derivatives and time lag known to be useful in the applications is the case when the kernel $U(t, \theta)$ has a Lebesgue decomposition without a singular part, i.e., it has the form

$$U(t, \theta) = \sum_{i \in \langle m \rangle_0} U_j^i(t, \theta) + U_{ac}(t, \theta) = \sum_{i \in \langle m \rangle_0} A^i(t) H(\theta + \sigma_i) + \int_0^\theta \tilde{A}(t, \xi) d\xi, \quad t \in \mathbb{R}. \quad (31)$$

Everywhere below, we will assume that the kernel $U(t, \theta)$ is defined via (31), and hence, the system (2) obtains the form

$$-D_{a+}^\alpha x(t) = \int_{-h}^0 [d_\theta U(t, \theta)] x(t + \theta) = \sum_{i \in \langle m \rangle_0} A^i(t) x(t - \sigma_i) + \int_{-h}^0 \tilde{A}(t, \theta) x(t + \theta) d\theta. \quad (32)$$

For arbitrary fixed $t \in \mathbb{R}$, $s \leq t$, and $z(s) = (z^1(s), \dots, z^n(s)) : \mathbb{R} \rightarrow \mathbb{R}^{n*}$, we introduce the formal (Lagrange) adjoint system with kernel

$$U_{adj}(s, \theta) = \sum_{i \in \langle m \rangle_0} A^i(s + \sigma_i) H(\theta - \sigma_i) + \int_0^\theta \tilde{A}(s - \xi, \xi) d\xi, \quad (33)$$

and then the adjoint system obtains the form

$$\begin{aligned} -D_{t-}^\alpha z(s) &= \int_{-h}^0 z(s - \theta) d_\theta U_{adj}(s, \theta) \\ &= \sum_{i \in \langle m \rangle_0} z(s + \sigma_i) A^i(s + \sigma_i) + \int_{-h}^0 z(s - \theta) \tilde{A}(s - \theta, \theta) d\theta. \end{aligned} \quad (34)$$

Let $t \in \mathbb{R}$ be arbitrarily fixed, and by applying the operator I_{t-}^α to both sides of (34) and using formula 2.4.44 in [12] for any $s \leq t - h$, we obtain the formal (Lagrange) adjoint system as a Volterra-type integral equation:

$$\begin{aligned} z(s) &= I_{-1} \Gamma(\alpha) \int_s^{t-h} I_{\alpha-1}(\tau - s) \left(\sum_{i \in \langle m \rangle_0} z(\tau + \sigma_i) A^i(\tau + \sigma_i) + \int_{-h}^0 z(\tau - \eta) \tilde{A}(\tau - \eta, \eta) d\eta \right) d\tau \\ &\quad + \text{const.} \end{aligned} \quad (35)$$

Let us introduce the Banach space:

$$\mathbf{C}_0 = \{ \xi : [-h, 0] \rightarrow \mathbb{R}^n \mid \xi(t) = \xi(-h), t \in (-\infty, 0], \xi \in BV^{loc}([-h, 0], \mathbb{R}^n) \}$$

with norm $\|\xi\| = |Var_{[-h, 0]} \xi(\cdot)|$.

For arbitrary $\zeta \in \mathbf{C}_0$, introduce a final condition in the form:

$$z_t(\theta; t - h, \zeta) = z(t + \theta; t - h, \zeta) = \zeta(\theta) \text{ for } -h \leq \theta \leq 0, \quad z(s; t - h, \zeta) = \mathbf{0}^\top \text{ for } s \geq t. \quad (36)$$

Definition 8. The vector function $z(s; t - h, \varsigma) = (z_1(s), \dots, z_n(s)) : (-\infty, t] \rightarrow \mathbb{R}^{n*}$ is a solution of the final problem (FP) (35), (36) (respectively FP (34), (36)) for some $\varsigma \in \mathbf{C}_0$ on the interval $(-\infty, t - h]$, if $z(s; t - h, \varsigma) \in C((-\infty, t - h], \mathbb{R}^{n*})$ satisfies the final condition (36) and the system (35) (respectively (34)) for all $s \in (-\infty, t - h]$.

To achieve our goal to obtain a formal adjoint fundamental matrix whose rows are solutions of the FP (35), (36), we introduce the final functions $\varsigma_k(\theta)$ for the final condition (36) for the system (35) for any $k \in \langle n \rangle$ as follows: $\varsigma_k(\theta) = \mathbf{0}^\top$, $\theta > -h$, $\varsigma_k(-h) = I_k$, where I_k denotes the k th row of the identity matrix I in $\mathbb{R}^n$. Then, for arbitrary $k \in \langle n \rangle$, the system (35) and the final condition (36) obtain the form:

$$\begin{aligned} z(s) &= \varsigma_k(0) + \\ &+ I_{-1}\Gamma(\alpha) \int_s^t I_{-1}(\tau - s) \left(\sum_{i \in \langle m \rangle_0} z(\tau + \sigma_i) A^i(\tau + \sigma_i) + \int_{-h}^0 z(\tau - \eta) \tilde{A}(\tau - \eta, \eta) d\eta \right) d\tau \\ &\equiv \mathbf{0}^\top. \end{aligned} \quad (37)$$

$$z_t(\theta; t, \varsigma) = z(t + \theta; t, \varsigma) = \varsigma(\theta), -h \leq \theta \leq 0; z(s; t, \varsigma_k) = \mathbf{0}^\top \quad \text{when } s \geq t. \quad (38)$$

Theorem 4. Let the following conditions hold:

1. Conditions (S) hold on $\mathbb{R} \times \mathbb{R}$.
2. $\det A^m(s) \neq 0$, $s \in \mathbb{R}$ (i.e., the matrix $U_{adj}(s, \theta)$ (see Definition 4) is atomic at $\theta = -h$ for any $s \in \mathbb{R}$).

Then, for any fixed $t \in \mathbb{R}$ and $k \in \langle n \rangle$, the FP (37), (38) with final function $\varsigma_k(s)$ has a unique solution $z(s; t, \varsigma_k) = (z_k^1(s), \dots, z_k^n(s))$ with an interval of existence $(-\infty, t]$.

Proof. To apply Theorem 2 and Corollary 1, it is necessary to check that all conditions of Theorem 2 hold for the FP (37), (38). As first since for any fixed $t \in \mathbb{R}$, the right side of (37) is a continuous function, then condition 1 of Theorem 2 holds. It is clear that the kernel defined via (33) satisfies condition 2 of Theorem 2. It is clear that the final function $\varsigma_k(s)$ satisfies the system (37) for any $s \in [t - h, t]$. Thus, $\varsigma_k(s)$ also satisfies the relations (10) with end point $t \in \mathbb{R}$ (it must be replaced in (10) only the point a with t), and hence, we can rewrite (37) in the form

$$\begin{aligned} z(s) &= \varsigma_k(0) + \\ &+ I_{-1}\Gamma(\alpha) \int_s^t I_{-1}(\tau - s) \left(\sum_{i \in \langle m \rangle_0} z(\tau + \sigma_i) A^i(\tau + \sigma_i) + \int_{-h}^0 z(\tau - \eta) \tilde{A}(\tau - \eta, \eta) d\eta \right) d\tau \\ &\equiv \mathbf{0}^\top. \end{aligned} \quad (39)$$

Then according condition 2 of Theorem 2 and Corollary 1, the system (39) possess a unique solution $z(s; t, \varsigma_k) = (z_k^1(s, t), \dots, z_k^n(s, t))$, which satisfies the final condition (38) and has an interval of existence $(-\infty, t]$. $\square$

The matrix $Z(s, t) = \{z_k^j(s, t)\}_{k, j \in \langle n \rangle}$, where the rows $z(s; t, \varsigma_k) = (z_k^1(s, t), \dots, z_k^n(s, t))$ are solutions of the FP (37), (38) for any $k \in \langle n \rangle$, will be called a formal adjoint fundamental matrix.

In order to explore its properties, we state below an open problem for the fractional delayed systems related to the fundamental and the adjoint fundamental matrices.

Open problem: Let $t, a \in \mathbb{R}$ with $t > a$ be arbitrary fixed, $Z(s, t)$ be a solution of the FP (37), (38) with interval of existence $(-\infty, t]$, and $C(t, s)$ satisfies the IP (5), (6) for $t \in [s, \infty)$. Prove or disprove that for any $s \in [a, t]$ the relation $C(t, s) = Z(s, t)$ holds.

It must be noted that the statement above is true for the delayed systems of first (integer) order.

5. Comments and Conclusions

In this work, we studied the possibility of a backward continuation of the solutions of the Cauchy (Initial) problem for a class of delayed fractional linear system with Caputo-type derivatives of incommensurate order.

In our point of view, the problem of the backward continuation of the solutions must be considered as a problem of renewal of a process with aftereffect under the given final observation. This formulation is equivalent from the mathematical point of view to a problem of forward prolongation of the solutions of fractional differential equations with a deviating argument of an advanced type.

As far as we know, this article is the first one containing backward continuation for fractional systems.

The backward continuation of the solutions in this article was studied in the case of systems with distributed-type delays and piecewise initial functions. As main result, we have obtained sufficient conditions for global backward continuation of the solutions of the considered systems. The proposed conditions correspond to the conditions, which guarantee the same result for these systems in the case of an integer order of differentiation. It was shown with an example that the introduced additional conditions for the backward continuation (not present in the theorems for forward continuation of the solutions) are essential and cannot be weakened in the general case.

Note that the methods used by the proof of the backward continuation are not natural generalizations of methods from ordinary differential equations, inclusive of the technical details and techniques.

But the key properties of the integer case of derivatives as locality and symmetry are not only a technical problem but are related to the inner nature of the described processes. Taking into account the geometric and physical interpretation of the fractional integration and differentiation (see [14]), we can conclude that the fractional models are closer to the point of view developed in the relativistic theory of the real worlds phenomena.

In the last section, we have introduced a formal (Lagrange) adjoint system of the studied homogeneous system and as an application of our backward continuation result, it was proved that the adjoint system with the identity matrix as initial function possess a unique matrix solution, called by us as the formal adjoint fundamental matrix, which can play the same role as the fundamental matrix in the forward case.

At the end of the last section, we have stated one of many open problems related to the further research of the properties introduced by us regarding a formal adjoint system. Our technical point of view is that the adequate development of the adjoint theory for delayed fractional systems needs an approach that is essentially based of the of the adjoint theory in functional analysis. It seems that in the future, via this approach, many different results under more general conditions can be obtained in a simpler way.

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