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Noether's Problem for p -Groups with Abelian Normal Subgroups and Central p -Powers

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Abstract: This paper addresses Noether's problem for p -groups G , having an abelian normal subgroup of index p , under the condition $G^p = \{g^p : g \in G\} \leq Z(G)$ —the center of G . We prove that the fixed field $K(G) = K(x(g) : g \in G)^G$ is rational over K in such cases, focusing on both the classification and structural analysis of these groups. Our results extend existing work by removing restrictive assumptions and providing a refined understanding of p -groups and their representations.

Keywords: Noether's problem; the rationality problem; nilpotent groups; p -groups; metabelian groups

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1. Introduction

Let K be a field and G be a finite group which acts on the rational function field $K(x(g) : g \in G)$ by K -automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Noether's problem then asks whether the fixed field $K(G) = K(x(g) : g \in G)^G$ is rational (=purely transcendental) over K . It is related to the inverse Galois problem, to the existence of generic G -Galois extensions over k , and to the existence of versal G -torsors over k -rational field extensions (see [1,2] and ([3], 33.1, p. 86)). Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. Fischer's theorem is a starting point for investigating Noether's problem for finite abelian groups in general.

Theorem 1 (Fischer ([2], Theorem 6.1)). *Let G be a finite abelian group of exponent e . Assume that (i) either $\text{char } K = 0$ or $\text{char } K > 0$ with $\text{char } K \nmid e$, and (ii) K contains a primitive e -th root of unity. Then, $K(G)$ is rational over K .*

Swan's paper [2] also gives a survey of many results related to Noether's problem for abelian groups. It is considered that, although not completely, Noether's problem for abelian groups is largely solved. On the other hand, the systematic study of Noether's problem for non-abelian groups is a more difficult process, and it is far from complete. Typically, each group is investigated individually and the answer depends on the generators and relations in the group. Naturally, the next stage is the investigation of Noether's problem for metabelian groups.

Our goal is to find the answer for a special class of metabelian groups. Namely, we consider p -groups G , having an abelian normal subgroup of index p , under the condition

$G^p = \{g^p : g \in G\} \leq Z(G)$ —the center of G . Since there is no known classification of these groups, we also prove several classification theorems. This is a very large class of groups. For example, all non-abelian groups of orders p^3 and p^4 are of this type. The reader can use James' classification [4] to see which groups of orders p^5 and p^6 are also of this type.

Let us proceed with a short summary of the known results for metabelian groups related to the groups we are going to discuss. Saltman [5] found examples of metabelian groups G of order p^9 such that $\mathbb{C}(V)^G$ is not stably rational over $\mathbb{C}$. His main method was an application of the unramified cohomology group $H_{nr}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. Bogomolov [6] proved that $H_{nr}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$B_0(G) = \bigcap_A \ker\{\text{res}_G^A : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups). The group $B_0(G)$ is a subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$, and Kunyavskiĭ [7] called it the *Bogomolov multiplier* of G . Thus, the vanishing of the Bogomolov multiplier is an obstruction to Noether's problem. It is worth mentioning that the vanishing of the Bogomolov multiplier is not a sufficient condition for the positive answer to Noether's problem, as is shown in [8].

Definition 1. We say that G is an AEC group if G has an abelian normal subgroup H such that the quotient group G/H is cyclic, or equivalently, G is an abelian extension of a cyclic group; i.e., we have the exact sequence $1 \rightarrow H \rightarrow G \rightarrow C_n \rightarrow 1$ for some cyclic group C_n .

Bogomolov ([6], Lemma 4.9) proved that if G is an AEC group, then $B_0(G) = 0$. On the other hand, Noether's problem for such groups is still far from being solved. However, a number of partial results have been obtained recently. We list several of them.

Theorem 2 (Kang [9], Theorem 1.5). Let G be a metacyclic p -group with exponent p^e , and let K be any field such that (i) $\text{char } K = p$, or (ii) $\text{char } K \neq p$ and K contains a primitive p^e -th root of unity. Then, $K(G)$ is rational over K .

Theorem 3 (Haeuslein [10]). Let K be a field and G be a finite group. Assume that (i) G contains an abelian normal subgroup H so that G/H is cyclic of prime order p , (ii) $\mathbb{Z}[\zeta_p]$ is a unique factorization domain, and (iii) $\zeta_{p^e} \in K$ where e is the exponent of G . If $G \rightarrow \text{GL}(V)$ is any finite-dimensional linear representation of G over K , then $K(V)^G$ is rational over K .

Theorem 4 (Hajja [11]). Let K be a field and G be a finite group. Assume that (i) G contains an abelian normal subgroup H so that G/H is cyclic of order n , (ii) $\mathbb{Z}[\zeta_n]$ is a unique factorization domain, and (iii) K is algebraically closed with $\text{char } K = 0$. If $G \rightarrow \text{GL}(V)$ is any finite-dimensional linear representation of G over K , then $K(V)^G$ is rational over K .

Theorem 5 ([12], Theorem 1.4). Let K be a field and G be a finite group. Assume that (i) G contains an abelian normal subgroup H so that G/H is cyclic of order n , (ii) $\mathbb{Z}[\zeta_n]$ is a unique factorization domain, and (iii) $\zeta_e \in K$ where e is the exponent of G . If $G \rightarrow \text{GL}(V)$ is any finite-dimensional linear representation of G over K , then $K(V)^G$ is rational over K .

Note that those integers n for which $\mathbb{Z}[\zeta_n]$ is a unique factorization domain are determined by Masley and Montgomery.

Theorem 6 (Masley and Montgomery [13]). $\mathbb{Z}[\zeta_n]$ is a unique factorization domain if and only if $1 \leq n \leq 22$, or $n = 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 40, 42, 45, 48, 50, 54, 60, 66, 70, 84, 90$.

Therefore, Theorem 5 holds only for primes p such that $1 \leq p \leq 19$. Recently, the first author of the present paper gave a positive answer to Noether's problem for a specific class of AEC p -groups.

Theorem 7 (Michailov [14], Theorem 1.8). Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1} , and let G be of exponent p^e . Choose any $\alpha \in G$ such that α generates G/H , i.e., $\alpha \notin H, \alpha^p \in H$. Denote $H(p) = \{h \in H : h^p = 1, h \notin H^p\}$, and assume that $[H(p), \alpha] \subset H(p) \cup \{1\}$. Denote by $G_{(i)} = [G, G_{(i-1)}]$ the lower central series for $i \geq 1$ and $G_{(0)} = G$. Let the p -th lower central subgroup $G_{(p)}$ be trivial. Assume that (i) $\text{char } K = p > 0$, or (ii) $\text{char } K \neq p$ and K contains a primitive p^e -th root of unity. Then $K(G)$ is rational over K .

In the main result of this paper we will show that the condition $[H(p), \alpha] \subset H(p) \cup \{1\}$ can be dropped. More generally, in Section 4 we prove the following.

Theorem 8. For any prime p , let G be an AEC p -group such that $G^p = \{g^p : g \in G\} \leq Z(G)$. Denote by p^e the exponent of G . Assume that (i) $\text{char } K = p > 0$, or (ii) $\text{char } K \neq p$ and K contains a primitive p^e -th root of unity. Then, $K(G)$ is rational over K .

The main idea to prove our results is to find a faithful G -subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K . The subspace W is obtained as an induced representation from H . Then, by various linearization methods, we reduce the problem to verifying some combinatorial identities.

In order to find a suitable faithful G -subspace W , we need a 'rough' classification of all p -groups $G = \langle H, \alpha \rangle$, where H is a normal abelian subgroup and $\alpha^p \in H$, under the assumption that the p -th lower central series $G_{(p)}$ is trivial, or equivalently (according to Lemma 3), $G^p \leq Z(G)$. This classification is performed in Section 2.

Standing notations. Let us introduce some notations which will be used henceforth:

The cyclic group of order n we denote by C_n .

The subgroups $G_{(0)} = G$ and $G_{(i)} = [G, G_{(i-1)}]$ for $i \geq 1$ are called the lower central series of G .

$G = \langle H, \alpha \rangle$, where H is an abelian normal subgroup of G and $\alpha^p \in H$.

$H^p = \{h^p : h \in H\}$.

$H(p) = \{h \in H : h^p = 1, h \notin H^p\}$.

$Z(p) = \{h \in H : h^p = 1, h \notin H(p)\}$.

For any $\beta \in H$, define $N_G(\beta) = \langle \alpha^{-x} \beta \alpha^x : 0 \leq x \leq p-1 \rangle$, the normalizer of $\langle \beta \rangle$ in G .

2. On the Classification of AEC p -Groups

Bender determined in an old paper [15] some basic formulas and properties concerning the p -groups G , having an abelian normal subgroup H of index p . We have not been made aware of a more thorough examination of these groups since then. Our goal in this section is to make a characterization of this class of p -groups with the additional condition $G_{(p)} = \{1\}$. This is achieved in terms of the ways in which we can decompose H into a direct product of groups that are either normal in G or normal modulo $Z(p)$.

As is well known, every abelian group decomposes as a direct product of cyclic groups. The problem is that not every one of them is normal in G . Thus, our goal turns out to be a description of the possible normalizers of these cyclic groups in G . Although this characterization is not a strict classification in terms of generators and relations, it still

provides, in our view, enough information both for clarifying the structure of these groups from a group-theoretic point of view, and for solving Noether's problem.

Let us also point out that some results describing the normalizers up to isomorphism are given in papers [14,16]. However, these results hold under some restrictive assumptions such as $[H(p), \alpha] \subset H(p) \cup \{1\}$ from [14]. In this paper, we are going to assume no additional restrictions, except $G_{(p)} = \{1\}$. We begin with two Lemmas, which will give us the tools we will need to calculate the normalizers and replace the initial generators of H with 'better' ones. Note that in the statements, we do not assume that $G_{(p)} = \{1\}$.

Lemma 1. *Let p be a prime, let $n \geq 2$, and let G be a group of order p^n with a normal abelian subgroup H of order p^{n-1} . Then, the commutator chains are subject to the following conditions:*

- (1) *For any $\beta_1 \in H, \beta_1 \notin Z(G)$, there exist $\beta_2, \dots, \beta_k \in H$ for some $k \geq 2$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq k-1$ and $\beta_k \neq 1$ is central. We call k the length of the commutator chain, starting with β_1 .*
- (2) *The normalizer $N_G(\beta_1)$ is generated by $\beta_1, \beta_2, \dots, \beta_k$. Moreover, if $G_{(p)} = \{1\}$, then $\beta_2, \dots, \beta_k$ are independent, and $\beta_2^p = \dots = \beta_k^p = 1$. If $\beta_1^p = 1$, then $\beta_1, \beta_2, \dots, \beta_k$ are also independent.*

Proof. (1) Since G is nilpotent, there exist $\beta_2, \dots, \beta_k \in H$ for some $k \geq 2$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq k-1$ and $\beta_k \neq 1$ is central.

(2) We have $\beta_2 = [\beta_1, \alpha] = \alpha^{-0}\beta_1^{-1}\alpha^0 \cdot \alpha^{-1}\beta_1\alpha \in N_G(\beta_1)$, and for any $x : 3 \leq x \leq k$, we make the inductive assumption that $\beta_{x-1} = [\beta_1, \alpha^{x-2}]h$ for $h \in \langle \alpha^{-y}\beta_1\alpha^y : 0 \leq y \leq x-3 \rangle \leq N_G(\beta_1)$. In the following computations, we are going to use the well-known commutator identities about multiplication rules and inverse elements, and the fact that H is abelian. More specifically, we use the rules $[x, zy] = [x, y][x, z][[x, z], y]$ and $[xz, y] = [x, y][[x, y], z][z, y]$. Clearly, when $x, z \in H$, the latter rule becomes $[xz, y] = [x, y][z, y]$.

First, we have $[\beta_1, \alpha^{x-1}] = [\beta_1, \alpha][\beta_1, \alpha^{x-2}][[\beta_1, \alpha^{x-2}], \alpha]$. Further, $[[\beta_1, \alpha^{x-2}], \alpha] = [\beta_{x-1}h^{-1}, \alpha] = \beta_x[h^{-1}, \alpha] = [\beta_1, \alpha^{x-1}][\beta_1, \alpha^{x-2}]^{-1}[\beta_1, \alpha]^{-1}$. Hence, we can find $\beta_x = [\beta_1, \alpha^{x-1}]h_1 \in N_G(\beta_1)$, where $h_1 = [\beta_1, \alpha^{x-2}]^{-1}[\beta_1, \alpha]^{-1}[h, \alpha] \in \langle \alpha^{-y}\beta_1\alpha^y : 0 \leq y \leq x-2 \rangle \leq N_G(\beta_1)$. Since $N_G(\beta_1)$ is the minimal normal subgroup of G containing β_1 , we obtain $N_G(\beta_1) = \langle \beta_1, \dots, \beta_k \rangle$.

From $[\beta_j, \alpha] = \beta_{j+1}$, it follows the well-known formula

$$\alpha^{-p}\beta_1\alpha^p = \beta_1\beta_2^{(p)}\beta_3^{(p)} \cdots \beta_p^{(p)}\beta_{p+1},$$

where we put $\beta_{k+1} = \dots = \beta_{p+1} = 1$. Since α^p is in H , we obtain the formula

$$\beta_2^{(p)}\beta_3^{(p)} \cdots \beta_k^{(p)} = 1.$$

Hence, $(\beta_2 \cdot \prod_{j=3}^k \beta_j^{a_j})^p = 1$ for some integers a_j . It is not hard to see that this identity is impossible if the order of β_2 is greater than p . Indeed, from $\beta_2^p = \prod_{j=3}^k \beta_j^{-a_j p}$, it follows that

$$\begin{aligned} \beta_3^p &= [\beta_2^p, \alpha] = \prod_{j=4}^k \beta_j^{-a_{j-1}p}, \\ \beta_4^p &= [\beta_3^p, \alpha] = \prod_{j=5}^k \beta_j^{-a_{j-2}p}, \\ &\vdots \\ \beta_{k-1}^p &= [\beta_{k-2}^p, \alpha] = \beta_k^{-a_{j-(k-3)}p}, \end{aligned}$$

$$\beta_k^p = [\beta_{k-1}^p, \alpha] = [\beta_k^{-a_{j-(k-3)}p}, \alpha] = 1.$$

Therefore, $\beta_k^p = \beta_{k-1}^p = \cdots = \beta_3^p = \beta_2^p = 1$.

Next, if we suppose that $\beta_2, \dots, \beta_k$ are dependent, then $\prod_{i=2}^k \beta_i^{a_i} = 1$, where at least two $a_i \not\equiv 0 \pmod{\text{ord}(\beta_i) = p}$. Denote $\ell = \min\{i : a_i \not\equiv 0 \pmod{p}\}$. From $\gcd(a_\ell, p) = 1$, it follows that we can find $b_\ell : a_\ell b_\ell \equiv 1 \pmod{p}$, so we can express $\beta_\ell = \prod_{i=\ell+1}^k \beta_i^{c_i}$ for some $0 \leq c_i \leq p-1$ and at least one $c_i \not\equiv 0$. Then, we have

$$\begin{aligned} \beta_{\ell+1} &= [\beta_\ell, \alpha] = \prod_{i=\ell+2}^k \beta_i^{c_{i-1}}, \\ \beta_{\ell+2} &= [\beta_{\ell+1}, \alpha] = \prod_{i=\ell+3}^k \beta_i^{c_{i-2}}, \\ &\vdots \\ \beta_{k-1} &= [\beta_{k-2}, \alpha] = \beta_k^{c_{i-(k-\ell-1)}}, \\ \beta_k &= [\beta_{k-1}, \alpha] = [\beta_k^{c_{i-(k-\ell-1)}}, \alpha] = 1, \end{aligned} \tag{1}$$

which is a contradiction.

Finally, assume that $\beta_1^p = 1$ and $\beta_1, \beta_2, \dots, \beta_k$ are dependent. From the above argument, $\beta_2, \dots, \beta_k$ are independent, so $\prod_{i=1}^k \beta_i^{a_i} = 1$ for $a_1 \not\equiv 0 \pmod{p}$ and at least one more $a_i \not\equiv 0 \pmod{p}$, where $i \neq 1$. Now, we can express $\beta_1 = \prod_{i=2}^k \beta_i^{c_i}$, for some $0 \leq c_i \leq p-1$ and at least one $c_i \not\equiv 0$. Then, we can apply Equation (1) for $\ell = 1$ to obtain a contradiction. $\square$

Lemma 2. Let p be a prime, let $n \geq 2$, and let G be a group of order p^n with a normal abelian subgroup H of order p^{n-1} . Then, the normalizers are subject to the following conditions:

- (1) $N_G(\beta_1) = N_G(\beta'_1)$ for $\beta'_1 \in N_G(\beta_1)$ if and only if $\beta'_1 = \prod_{i=1}^k \beta_i^{x_i}$ for some integers x_i , where $\gcd(x_1, p) = 1$.
- (2) Let $\beta = \prod_{i=2}^k \beta_i^{x_i}$ for some integers x_i , such that $\gcd(x_{i_0}, p) = 1$, where $i_0 = \min\{i : x_i \neq 0\}$. Then, β appears in a commutator chain starting with a generator β'_1 ; i.e., there exist $\beta'_1, \dots, \beta'_{i_0} = \beta$ such that $[\beta'_j, \alpha] = \beta'_{j+1}$, for $1 \leq j \leq i_0 - 1 \leq k-1$, and $N_G(\beta_1) = N_G(\beta'_1)$. In particular, if $\beta_2^p = 1$, then every element $\beta \in \langle \beta_2, \dots, \beta_k \rangle$ appears in a commutator chain starting with a generator β'_1 such that $N_G(\beta_1) = N_G(\beta'_1)$.

Proof. (1) Let $\beta'_1 = \prod_{i=1}^k \beta_i^{x_i}$ for some integers x_i , where $\gcd(x_1, p) = 1$. From (2), it follows that $N_G(\beta_1) \geq N_G(\beta'_1)$. On the other hand,

$$\begin{aligned} [\beta'_1, \alpha] &= \prod_{i=1}^k [\beta_i, \alpha]^{x_i} = \prod_{i=2}^k \beta_i^{x_{i-1}} \in N_G(\beta'_1), \\ [\beta'_1, \alpha^2] &= [\beta'_1, \alpha][\beta'_1, \alpha][[\beta'_1, \alpha], \alpha] \equiv \left[\prod_{i=2}^k \beta_i^{x_{i-1}}, \alpha \right] \equiv \\ &\prod_{i=3}^k \beta_i^{x_{i-2}} \pmod{N_G(\beta'_1)} \Rightarrow \prod_{i=3}^k \beta_i^{x_{i-2}} \in N_G(\beta'_1), \\ &\vdots \\ [\beta'_1, \alpha^{k-2}] &\equiv \beta_{k-1}^{x_1} \beta_k^{x_2} \pmod{N_G(\beta'_1)} \Rightarrow \beta_{k-1}^{x_1} \beta_k^{x_2} \in N_G(\beta'_1), \\ [\beta'_1, \alpha^{k-1}] &\equiv \beta_k^{x_1} \pmod{N_G(\beta'_1)} \Rightarrow \beta_k^{x_1} \in N_G(\beta'_1). \end{aligned}$$

Since $\gcd(x_1, p) = 1$, from the above equations, we can express consecutively $\beta_k \in N_G(\beta'_1), \beta_{k-1} \in N_G(\beta'_1), \dots, \beta_1 \in N_G(\beta'_1)$. Therefore, $N_G(\beta_1) \leq N_G(\beta'_1)$.

Conversely, if $N_G(\beta_1) = N_G(\beta'_1)$, we have $\beta'_1 = \prod_{i=1}^k \beta_i^{x_i}$ and $\beta_1 = \prod_{i=1}^k \beta_i^{y_i}$ for $\beta'_{i+1} = [\beta'_i, \alpha]$ and some integers x_i, y_i . Therefore, $\beta_1 = \beta_1^{x_1 y_1} \prod_{i=2}^k \beta_i^{x_i y_1} \prod_{i=2}^k \beta_i^{y_i}$, which is possible only if $x_1 y_1 \equiv 1 \pmod{\text{ord}(\beta_1)}$, and hence, $\gcd(x_1, p) = 1$.

(2) We can write $\beta = \prod_{i=i_0}^k \beta_i^{x_i} = \prod_{i=i_0}^k [\beta_{i-1}^{x_i}, \alpha] = [\prod_{i=i_0}^k \beta_{i-1}^{x_i}, \alpha]$. Put $\beta'_{i_0-1} = \prod_{i=i_0}^k \beta_{i-1}^{x_i}$, so $[\beta'_{i_0-1}, \alpha] = \beta$. If $i_0 = 2$, then $N_G(\beta_1) = N_G(\beta'_1)$ (according to (3)) and $[\beta'_1, \alpha] = \beta$. If $i_0 > 2$, then $\beta'_{i_0-1} = [\beta'_{i_0-2}, \alpha]$ for $\beta'_{i_0-2} = \prod_{i=i_0}^k \beta_{i-2}^{x_i}$. It is clear now that proceeding in this way, we will obtain at the end a generator β'_1 such that $N_G(\beta_1) = N_G(\beta'_1)$ and $[\beta'_j, \alpha] = \beta'_{j+1}$ for $1 \leq j \leq i_0 - 1$ and $\beta'_{i_0} = \beta$.

Finally, if $\beta_2^p = 1$, then from $[\beta_j^p, \alpha] = \beta_{j+1}^p$, it follows that $\beta_3^p = \dots = \beta_k^p = 1$. Clearly, every non-trivial element $\beta \in \langle \beta_2, \dots, \beta_k \rangle$ can be written as $\beta = \prod_{i=2}^k \beta_i^{x_i}$ for some integers x_i , such that $\gcd(x_i, p) = 1$. $\square$

In the following Lemma, we show that for AEC p -groups G , the condition $G_{(p)} = \{1\}$ is equivalent to $G^p \leq Z(G)$.

Lemma 3. For any prime p , G is an AEC p -group with $G^p \leq Z(G)$, if and only if G has an abelian normal subgroup H of index p with $G_{(p)} = \{1\}$.

Proof. First, assume that G is an AEC p -group with $G^p \leq Z(G)$. Let G be an H extension of a cyclic group $\langle \alpha \rangle$. Therefore, $\mathcal{H} = \langle H, \alpha^p \rangle$ is abelian, and the index of $\mathcal{H}$ in G is p . Now, suppose that $G_{(p)} \neq \{1\}$. Then, we can assume that there exist $\beta_1, \dots, \beta_{p+1} \in \mathcal{H}$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq p$ and $\beta_{p+1} \neq 1$. From the identity

$$\alpha^{-p} \beta_1 \alpha^p = \beta_1 \beta_2^{\binom{p}{1}} \beta_3^{\binom{p}{2}} \dots \beta_p^{\binom{p}{p-1}} \beta_{p+1},$$

it follows that

$$\beta_2^{\binom{p}{1}} \beta_3^{\binom{p}{2}} \dots \beta_p^{\binom{p}{p-1}} \beta_{p+1} = 1.$$

Hence, β_2 will have an order greater than p . Indeed, if we suppose that $\beta_2^p = 1$, then from the commutator rules, it follows that $\beta_3^p = \dots = \beta_p^p = 1$, so $\beta_{p+1} = 1$, which is a contradiction. Therefore, β_2 has an order greater than p . On the other hand, from the condition $\mathcal{H}^p \leq Z(G)$, it follows that $\beta_1^p = \alpha^{-1} \beta_1^p \alpha = (\beta_1 \beta_2)^p$, and hence, $\beta_2^p = 1$, which is a contradiction. Therefore, $G_{(p)} = \{1\}$.

Now, assume that G has an abelian normal subgroup H of index p , $G = \langle H, \alpha : \alpha^p \in H \rangle$ and $G_{(p)} = \{1\}$. Let β_1 be any non-central element of H . Since $G_{(p)} = \{1\}$, there exist $\beta_2, \dots, \beta_k \in H$ for some $k : 2 \leq k \leq p$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq k-1$ and $\beta_k \neq 1$ is central. From Lemma 1(2), we know that $\beta_k^p = \beta_{k-1}^p = \dots = \beta_3^p = \beta_2^p = 1$. Then, from $[\beta_1, \alpha] = \beta_2$, it follows that $[\beta_1^p, \alpha] = \beta_2^p = 1$, i.e., $\beta_1^p \in Z(G)$. Since β_1 was arbitrary element of H , we obtain $G^p \leq Z(G)$. $\square$

Next, we are going to consider the special case when H is an elementary abelian group. In this case, we can completely determine the decomposition of H into a direct product of groups which are normal in G . This is enough to solve Noether's problem by constructing the induced representation from each direct factor.

Theorem 9. Let p be prime, let $n \geq 2$, and let G be a group of order p^n with an abelian normal subgroup $H \simeq (C_p)^{n-1}$, the elementary abelian group of order p^{n-1} . Then, H is subject to the following conditions:

- (1) There exist elements $\gamma_1, \dots, \gamma_s \in H$ (for some $s \leq n - 1$) such that $H \simeq N_G(\gamma_1) \times \dots \times N_G(\gamma_s)$;
- (2) For any $i : 1 \leq i \leq s$, either $N_G(\gamma_i) = \langle \gamma_i \rangle \leq Z(G)$, or there exists $k_i : 2 \leq k_i \leq p$ such that $N_G(\gamma_i) \simeq (C_p)^{k_i}$, and independent generators $\beta_{i1}, \dots, \beta_{ik_i} \in N_G(\gamma_i)$, such that $\beta_{i1} = \gamma_i, [\beta_{ij}, \alpha] = \beta_{ij+1}$ for $1 \leq j \leq k_i - 1$ and β_{ik_i} is central in G .

Proof. First, let us decompose H as a direct product of cyclic groups: $H = \langle \gamma_1 \rangle \times \dots \times \langle \gamma_{n-1} \rangle$. Observe that for any i , $\langle \gamma_i \rangle \simeq C_p$ is normal in G if and only if γ_i is central in G . Indeed, if $\langle \gamma_i \rangle$ is normal in G , then $\alpha^{-1}\gamma_i\alpha = \gamma_i^x$ for some $x : \gcd(x, p) = 1$. Then, $\alpha^{-p}\gamma_i\alpha^p = \gamma_i^{x^p} = \gamma_i^x$ by Fermat's little theorem. Since α^p is in H , $\gamma_i^x = \gamma_i$.

Next, let us see that there exists a generator, say γ_1 , such that $\gamma_1 \notin [H, \alpha]$. Indeed, if we suppose that $\gamma_i \in [H, \alpha]$ for all i , from the commutation rule $[a, \alpha][b, \alpha] = [ab, \alpha]$, we obtain that any element $\gamma = \prod_i \gamma_i^{a_i} \in H$ is in $[H, \alpha]$. But then, we will obtain infinite commutator series $[\alpha_{j+1}, \alpha] = \alpha_j, \alpha_1 = \gamma_1, j = 1, 2, \dots$, which is a contradiction, since G being a p -group is nilpotent.

If all elements of H are central, then G is abelian and $s = n - 1$. Now, for any $i : 1 \leq i \leq n - 1$ such that γ_i is not central in G , put $\beta_{i1} = \gamma_i$. Since H is of exponent p , from Lemma 3, it follows that $G_{(p)} = \{1\}$. According to Lemma 1(2), we have $N_G(\gamma_i) = \langle \beta_{i1}, \dots, \beta_{ik_i} \rangle$, where $[\beta_{ij}, \alpha] = \beta_{ij+1}$, for $1 \leq j \leq k_i - 1 \leq p - 1$. We also have that $\beta_{ik_i} \neq 1$ is central and $\beta_{i1}, \dots, \beta_{ik_i}$ are independent generators of $N_G(\gamma_i)$.

Next, without loss of generality, we may assume that γ_1 has the maximal length of the commutator chain, i.e., $k_1 \geq k_j$ for $2 \leq j \leq n - 1$. Put $\beta_1 = \beta_{11} = \gamma_1, \beta_2 = \beta_{12}, \dots, \beta_{k_1} = \beta_{1k_1}$. According to Lemma 1(2), the elements $\beta_1, \dots, \beta_{k_1}$ are independent generators of $N_G(\gamma_1)$. If $k_1 < n - 1$, then there exists another generator, say $\gamma_2 \notin N_G(\gamma_1)$. We can also assume that γ_2 is not central; otherwise, $H = N_G(\gamma_1) \times \langle \gamma_2 \rangle \times \dots \times \langle \gamma_s \rangle$ and the process is completed. Put $\beta_{k_1+1} = \beta_{21} = \gamma_2, \beta_{k_1+2} = \beta_{22}, \dots, \beta_{k_1+k_2} = \beta_{2k_2}$. Again, the elements $\beta_{k_1+1}, \dots, \beta_{k_1+k_2}$ are independent generators of $N_G(\gamma_2)$. However, it is possible that $\beta_{k_1+1}, \dots, \beta_{k_1+k_2}$ are dependent modulo $N_G(\gamma_1)$, i.e., $\prod_{i=1}^{k_1+k_2} \beta_i^{x_i} = 1$ for $x_i : 0 \leq x_i \leq p - 1$ such that $x_{i_0} \neq 0, x_{j_0} \neq 0$ for some $i_0, j_0 : 1 \leq i_0 \leq k_1, k_1 + 1 \leq j_0 \leq k_1 + k_2$. If we suppose that $x_{j_1} \neq 0$ for some $j_1 \neq j_0, k_1 + 1 \leq j_1 \leq k_1 + k_2$, we will obtain a contradiction. Indeed, denote $\bar{H} = H/N_G(\gamma_1)$ and $\bar{G} = G/N_G(\gamma_1)$. Then, $\bar{H} = H/N_G(\gamma_1)$ is again the elementary abelian p -group and the index of $\bar{H}$ in $\bar{G}$ is p . So, we can apply Lemma 1(2) again and see that the images of $\beta_{k_1+1}, \dots, \beta_{k_1+k_2}$ in $\bar{H}$ must be independent, which is impossible, since we have that the image of $\prod_{i=k_1+1}^{k_1+k_2} \beta_i^{x_i}$ is the identity, where $x_{j_0} \neq 0, x_{j_1} \neq 0$ for some $j_1 \neq j_0, k_1 + 1 \leq j_1 \leq k_1 + k_2$.

Thus, the only possibility is that there exists $\ell_2 \leq k_2$ such that $\beta_{k_1+1}, \dots, \beta_{k_1+\ell_2-1} \notin N_G(\gamma_1)$, but $\beta_{k_1+\ell_2} \in N_G(\gamma_1)$. According to Lemma 2 (2), $\beta_{2\ell_2} = \beta_{k_1+\ell_2}$ appears in a commutator chain starting with a generator γ'_1 such that $N_G(\gamma_1) = N_G(\gamma'_1)$; i.e., there exist $\beta'_{11}, \dots, \beta'_{1\ell_1} \in N_G(\gamma'_1)$, such that $\beta'_{11} = \gamma'_1, [\beta'_{1j}, \alpha] = \beta'_{1j+1}$ for $1 \leq j \leq \ell_1 - 1$ and $\beta'_{1\ell_1} = \beta_{2\ell_2}$ for some $\ell_1 \leq k_1$. Notice that $k_1 - \ell_1 = k_2 - \ell_2$, because after $\beta_{2\ell_2}$, the two commutator chains coincide. Since we assumed that $k_1 \geq k_2$, we obtain $\ell_1 \geq \ell_2$. Define $\gamma'_2 = \beta_{1\ell_1-\ell_2+1}^{-1}\gamma_2$ and $\beta'_{21} = \gamma'_2, [\beta'_{2j}, \alpha] = \beta'_{2j+1}$ for $1 \leq j \leq \ell_2 - 1$. Therefore, $\beta'_{2\ell_2} = 1$, and it is not hard to see that $\beta_1, \dots, \beta_{k_1}, \beta'_{21}, \dots, \beta'_{2\ell_2-1}$ are independent generators of the whole subgroup $N_G(\gamma_1)N_G(\gamma'_2)$. Indeed, $\beta'_{1\ell_1-\ell_2+1}, \dots, \beta'_{1\ell_1} \in N_G(\gamma_1) = N_G(\gamma'_1)$ and $\gamma_2 = \beta_{21}, \dots, \beta_{2\ell_2-1} = \beta_{k_1+\ell_2-1} \notin N_G(\gamma_1)$.

If $k_1 + \ell_2 - 1 < n - 1$, then there exists another generator, say $\gamma_3 \notin N_G(\gamma_1)N_G(\gamma'_2)$, and we may proceed in a similar manner until we finish the generators of H . We are going to see that this process will give us the decomposition $H = N_G(\gamma_1) \times N_G(\gamma'_2) \times \dots \times N_G(\gamma'_s)$ for some independent generators $\gamma_1, \gamma'_2, \dots, \gamma'_s$ of direct cyclic factors.

Assume that we performed the replacement of the generators $\gamma_2, \dots, \gamma_t$ with $\gamma'_2, \dots, \gamma'_t$ for some $t : 2 \leq t \leq s$, so that the generators β_{1j}, β'_{ij} are independent, where $\beta'_{i1} = \gamma'_i, [\beta'_{ij}, \alpha] = \beta'_{ij+1}$ for $1 \leq j \leq \ell_i - 1, 2 \leq i \leq t$. We have just shown how to accomplish this for $t = 2$. Next, we are going to accomplish this for the $(t + 1)$ -th generator.

Let $\gamma_{t+1} \notin N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$ be a generator such that its commutator chain is dependent modulo $N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$. Equivalently, we can write $\prod_{i=2}^t \prod_{j=1}^{\ell_i} \beta'^{x_{ij}}_{ij} \cdot \prod_{j=1}^{k_{t+1}} \beta^{x_{t+1j}}_{t+1j} \in N_G(\gamma_1)$ for some $x_{t+1j_0} \neq 0$, where $\beta_{t+11} = \gamma_{t+1}, \beta_{t+1j+1} = [\beta_{t+1j}, \alpha]$. If we suppose that $x_{t+1j_0} \neq 0$ and $x_{t+1j_1} \neq 0$ for some $1 \leq j_0 < j_1 \leq k_{t+1}$, we will obtain a contradiction with Lemma 1(2), if we consider the images of β_{t+1j} in $H/N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$. Thus, the only possibility is that there exists $\ell_{t+1} \leq k_{t+1}$ such that $\beta_{t+1j} \notin N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$ for $1 \leq j \leq \ell_{t+1} - 1$, but $\beta_{t+1\ell_{t+1}} = \beta \prod_{i=2}^t \prod_{j=1}^{\ell_i} \beta'^{y_{ij}}_{ij}$, where $\beta \in N_G(\gamma_1)$ and $0 \leq y_{ij} \leq p - 1$.

Now, we show that we can replace the generator γ_{t+1} in order to obtain $\beta = 1$. Indeed, if $\beta \neq 1$, according to Lemma 2 (2), β appears in a commutator chain starting with a generator γ''_1 such that $N_G(\gamma_1) = N_G(\gamma''_1)$; i.e., there exist $\beta''_{11}, \dots, \beta''_{1\ell'_1} \in N_G(\gamma''_1)$, such that $\beta''_{11} = \gamma''_1, [\beta''_{1j}, \alpha] = \beta''_{1j+1}$ for $1 \leq j \leq \ell'_1 - 1$ and $\beta''_{1\ell'_1} = \beta$ for some $\ell'_1 \leq k_1$. Notice that $k_1 - \ell'_1 \leq k_{t+1} - \ell_{t+1}$. (Here, we might have an inequality, when $\beta_{t+1\ell_{t+1}+k_1-\ell'_1} \notin Z(G)$.) Since we assumed that $k_1 \geq k_{t+1}$, we obtain $\ell'_1 \geq \ell_{t+1}$. Define $\gamma'_{t+1} = \beta''_{1\ell'_1-1} \gamma_{t+1}$, and $\beta'_{t+11} = \gamma'_{t+1}, [\beta'_{t+1j}, \alpha] = \beta'_{t+1j+1}$ for $1 \leq j \leq \ell_{t+1} - 1$. Therefore, $\beta'_{t+1\ell_{t+1}} = \prod_{i=2}^t \prod_{j=1}^{\ell_i} \beta'^{y_{ij}}_{ij}$.

Similarly, if we suppose that $\gamma = \prod_{j=1}^{\ell_i} \beta'^{y_{ij}}_{2j} \neq 1$ and ℓ_2 is the maximal length of the commutator chains of the new generators, we can replace the generator γ'_{t+1} in order to obtain $\gamma = 1$. To perform this replacement, we use a suitable element from $N_G(\gamma'_2)$, so there is no danger of bringing back the elements from $N_G(\gamma_1)$. We can continue this process until we obtain a generator $\gamma^{(t-1)}_{t+1}$ such that $\beta^{(t-1)}_{t+1\ell_{t+1}} = 1$ and $\beta^{(t-1)}_{t+11} = \gamma^{(t-1)}_{t+1}, [\beta^{(t-1)}_{t+1j}, \alpha] = \beta^{(t-1)}_{t+1j+1} \notin N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$ for $1 \leq j \leq \ell_{t+1} - 2$. In this way, we have found a generator $\gamma^{(t-1)}_{t+1} \notin N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$ such that its commutator chain is independent modulo $N_G(\gamma_1)N_G(\gamma'_2) \cdots N_G(\gamma'_t)$. The process is complete. $\square$

We can summarize the latter theorem into the following table:

Direct normal factors
$\langle \gamma_i \rangle \simeq C_p \leq Z(G)$
$N_G(\gamma_i) \simeq (C_p)^{k_i}$

In the following theorem, we find a decomposition of an arbitrary abelian group H as a direct product of subgroups that are either normal or normal mod $Z(p)$ in G .

Theorem 10. Let p be prime and let G be a group of order p^n for $n \geq 2$ with an abelian normal subgroup H of order p^{n-1} . Choose any $\alpha \in G$ such that α generates G/H , i.e., $\alpha \notin H, \alpha^p \in H$. Assume also that the p -th lower central subgroup $G_{(p)}$ is trivial. Then, there exist elements $\alpha_1, \dots, \alpha_s \in H$ such that the following conditions are satisfied:

- (i) $H = N_G(\alpha_1) \cdots N_G(\alpha_s)$; i.e., H is a product (not necessarily direct) of the normalizers of $\alpha_1, \dots, \alpha_s$.
- (ii) $H = \langle \alpha_1 \rangle \times \cdots \times \langle \alpha_s \rangle \times \cdots \times \langle \alpha_r \rangle$ for some $r \geq s$. Denote $\text{ord}(\alpha_j) = p^{a_j}$ for $1 \leq j \leq r$.
- (iii) For any $i : 1 \leq i \leq s$, if α_i is in $Z(G)$, then $N_G(\alpha_i) = \langle \alpha_i \rangle$; if α_i is not in $Z(G)$, then there exist $k_i : 2 \leq k_i \leq p$ and generators $\alpha_{i1}, \dots, \alpha_{ik_i} \in N_G(\alpha_i)$, such that $\alpha_{i1} = \alpha_i, [\alpha_{ij}, \alpha] =$

- α_{ij+1} for $1 \leq j \leq k_i - 1$, α_{ik_i} is in the center $Z(G)$. Moreover, if α_{i2} is not in $Z(G)$, then $\langle \alpha_{i2}, \dots, \alpha_{ik_i-1} \rangle \leq H(p)$ and $\alpha_{i2}, \dots, \alpha_{ik_i}$ are independent.
- (iv) For any $j : 1 \leq j \leq s$ such that $\alpha_{j2} \in H(p)$, put $\mathcal{H}_j = \langle \alpha_{j2}, \dots, \alpha_{jk_j} \rangle$, if $\alpha_{jk_j} \in H(p)$; $\mathcal{H}_j = \langle \alpha_{j2}, \dots, \alpha_{jk_j-1} \rangle$, if $\alpha_{jk_j} \notin H(p)$; $\mathcal{H}_j = \langle \alpha_j, \mathcal{H}_j \rangle$. For any $j : 1 \leq j \leq s$ such that $\alpha_{j2} \notin H(p)$, put $\mathcal{H}_j = \langle \alpha_j \rangle$. Then, $H \simeq \mathcal{H}_1 \times \dots \times \mathcal{H}_s$. Moreover, if $\alpha_{jk_j} \in H(p)$, then $\mathcal{H}_j = N_G(\alpha_j) \simeq C_{p^{a_j}} \times (C_p)^{k_j-1}$ —normal in G ; if $\alpha_{jk_j} \notin H(p)$ then $\mathcal{H}_j \simeq C_{p^{a_j}} \times (C_p)^{k_j-2}$ —normal mod $Z(p)$ in G , where $2 \leq k_j \leq p$.

Proof. If $H^p = \{1\}$; i.e., H is a direct product of cyclic groups of order p , we can apply Theorem 9. Henceforth, we assume that there is a generator of H of order p^a for $a > 1$. If $H(p) = \emptyset$, i.e., any generator α_i is of order p^{a_i} for $a_i > 1$, then clearly, $\mathcal{H}_j = \langle \alpha_j \rangle$ for $1 \leq j \leq r$, and all conditions are satisfied. Henceforth, we also assume that there is a generator of H of order p .

Step I. In this step, we are going to show that there exist $\alpha_1, \dots, \alpha_s \in H$ such that conditions (i)–(iii) are satisfied. First, let us decompose H as a direct product of cyclic groups: $H = \langle \alpha_1 \rangle \times \dots \times \langle \alpha_r \rangle$. If all generators of H are central in G , then G is abelian and our theorem is obvious. So, we will assume that there exists at least one generator which is not central. For any $i : 1 \leq i \leq r$ such that α_i is not central in G , define $\alpha_{i1} = \alpha_i$. Since $G_{(p)} = \{1\}$, there exist $\alpha_{i2}, \dots, \alpha_{ik_i} \in H$ for some $k_i : 2 \leq k_i \leq p$ such that $[\alpha_{ij}, \alpha] = \alpha_{ij+1}$, where $1 \leq j \leq k_i - 1$ and $\alpha_{ik_i} \neq 1$ is central. According to Lemma 1(2), $\alpha_{i2}, \dots, \alpha_{ik_i}$ are independent generators of order p . According to Lemma 3, $H^p \leq Z(G)$, so if α_{i2} is not in $Z(G)$, then $\langle \alpha_{i2}, \dots, \alpha_{ik_i-1} \rangle \leq H(p)$; if $\alpha_{i2} = 1$; i.e., α_{i1} is in $Z(G)$, then $N_G(\alpha_i) = \langle \alpha_i \rangle$.

Next, among all generators of H , we can choose a subset of generators that have the maximal length k_i of their commutator chain. If there is more than one generator in this chosen set, we choose a generator with the smallest order. Denote this generator by α_1 . Thus, we have $k_1 \geq k_i$ for all $i : 1 \leq i \leq r$ and $\text{ord}(\alpha_1) \leq \text{ord}(\alpha_i)$ for all i such that $k_i = k_1$.

Similarly, if $H \neq N_G(\alpha_1)$, we may choose generators $\alpha_i \in H \setminus N_G(\alpha_1) \dots N_G(\alpha_{i-1})$, for $2 \leq i \leq s \leq r$ (after eventual renumbering), such that $H = N_G(\alpha_1)N_G(\alpha_2) \dots N_G(\alpha_s)$. Therefore, conditions (i) – (iii) are satisfied.

Step II. In this step, we are going to show that there exist new generators $\alpha'_2, \dots, \alpha'_s \in H$ such that $\mathcal{H}_1 \cap (\mathcal{H}_2' \dots \mathcal{H}_s') = \{1\}$, where $\mathcal{H}_i'$ corresponds to $\mathcal{H}_i$ for α'_i . In light of the above considerations, we may assume that $k_1 \geq k_2 \geq \dots \geq k_s \geq 1$ and if $k_i = k_j$ for some $i < j$, then $\text{ord}(\alpha_i) \leq \text{ord}(\alpha_j)$. We can also assume that α_2 is not central; otherwise, all the remaining generators are also central (because of $k_1 = k_2 = \dots = k_s = 1$), so $H = \mathcal{H}_1 \times \mathcal{H}_2 \times \dots \times \mathcal{H}_s$, where $\mathcal{H}_i = \langle \alpha_i \rangle$, $r = s$ and the process is completed.

According to (iii), the elements $\alpha_{i2}, \alpha_{i3}, \dots \in \mathcal{H}_i$ are independent generators of $N_G(\alpha_i)$ for $1 \leq i \leq s$. However, it is possible that $\alpha_{21}, \alpha_{22}, \dots, \alpha_{2k_2}$ are dependent modulo $N_G(\alpha_1)$, i.e., $\prod_{j=1}^{k_2} \alpha_{2j}^{x_j} \in N_G(\alpha_1) \setminus \{1\}$ for $x_i : 0 \leq x_i \leq p - 1$. If we suppose that $x_{j_1} \neq 0$ and $x_{j_0} \neq 0$ for some $1 \leq j_0 < j_1 \leq k_2$, we will obtain a contradiction with Lemma 1(2), if we consider the images of α_{2j} in $H/N_G(\alpha_1)$. Thus, the only possibility is that there exists $\ell_2 \leq k_2$ such that $\alpha_{2\ell_2} \in N_G(\alpha_1) \setminus \{1\}$, and if $\ell_2 > 1$, then $\alpha_{21}, \dots, \alpha_{2\ell_2-1} \notin N_G(\alpha_1) \setminus \{1\}$.

We can write $\alpha_{2\ell_2} = \beta\gamma$, where $\beta \in \mathcal{H}_1$ and $\gamma \in \langle \alpha_1^{p^{a_1-1}} \rangle$. According to Lemma 2(2), β appears in a commutator chain starting with a generator α'_1 such that $N_G(\alpha_1) = N_G(\alpha'_1)$; i.e., there exist $\alpha'_{11}, \dots, \alpha'_{1\ell_1} \in N_G(\alpha'_1)$, such that $\alpha'_{11} = \alpha'_1$, $[\alpha'_{1j}, \alpha] = \alpha'_{1j+1}$ for $1 \leq j \leq \ell_1 - 1$ and $\alpha'_{1\ell_1} = \beta$ for some $\ell_1 \leq k_1$. Notice that $k_1 - \ell_1 = k_2 - \ell_2$, because after β , the two commutator chains coincide. Since we assumed that $k_1 \geq k_2$, we obtain $\ell_1 \geq \ell_2$. Define $\alpha'_2 = \alpha'_{1\ell_1-\ell_2+1}\alpha_2$ and $\alpha'_{21} = \alpha'_2$, $[\alpha'_{2j}, \alpha] = \alpha'_{2j+1}$ for $1 \leq j \leq \ell_2 - 1$. Therefore, $\alpha'_{2\ell_2} = \gamma$, and if we define $\mathcal{H}_2'$ in the same way as $\mathcal{H}_2$ for the new generator α'_2 , then $\mathcal{H}_1 \cap \mathcal{H}_2' = \{1\}$. Notice also that if $\ell_1 = \ell_2$, then $\alpha'_2 = \alpha_1^{-1}\alpha_2$ and $\text{ord}(\alpha'_2) = \text{ord}(\alpha_2)$, since we assumed that $\text{ord}(\alpha'_1) = \text{ord}(\alpha_1) \leq \text{ord}(\alpha_2)$. This means that the change of the generator α_2 with α'_2 is

‘valid’, in the sense that both $\langle \alpha_1 \rangle$ and $\langle \alpha'_2 \rangle$ remain direct factors in the decomposition of H as a direct product of cyclic groups.

If there exists another generator, say $\alpha_3 \notin N_G(\alpha_1)N_G(\alpha'_2)$, we may proceed in a similar manner. Namely, suppose that for some $t \geq 2$, there exist generators $\alpha'_2, \dots, \alpha'_t$ such that $\alpha'_{i1} = \alpha'_i, [\alpha'_{ij}, \alpha] = \alpha'_{ij+1}$ and $\{\alpha'_{ij} : 1 \leq j \leq \ell_i - 1, 2 \leq i \leq t\}$ are generators of H with the property that for $\mathcal{H}'_i = N_G(\alpha'_i) \cap H(p)$, $2 \leq i \leq t$, then $\mathcal{H}_1 \cap (\mathcal{H}'_2 \cdots \mathcal{H}'_t) = \{1\}$. We proved this assertion for $t = 2$, and we will show that it holds for $t + 1$.

Assume that $\alpha_{t+1} \notin N_G(\alpha_1)N_G(\alpha'_2) \cdots N_G(\alpha'_t)$ and $\prod_{i=2}^t \prod_{j=1}^{\ell_i-1} \alpha'_{ij} \cdot \prod_{j=1}^{k_{t+1}} \alpha_{t+1j}^{x_{t+1j}} \in N_G(\alpha_1) \setminus \{1\}$. If we suppose that $x_{t+1j_0} \neq 0$ and $x_{t+1j_1} \neq 0$ for some $1 \leq j_0 < j_1 \leq k_{t+1}$, we will obtain a contradiction with Lemma 1(2), if we consider the images of α_{t+1j} in $H/N_G(\alpha_1)N_G(\alpha'_2) \cdots N_G(\alpha'_t)$. Thus, the only possibility is that there exists $\ell_{t+1} \leq k_{t+1}$ such that $\alpha_{t+1\ell_{t+1}} = \beta\gamma \prod_{i=2}^t \prod_{j=1}^{\ell_i-1} \alpha'_{ij}^{y_{ij}}$, where $\beta \in \mathcal{H}_1, \gamma \in \langle \alpha_1^{p^{a_1-1}} \rangle$ and if $\ell_{t+1} > 1$, then the elements $\alpha_{t+11}, \dots, \alpha_{t+1\ell_{t+1}-1}$ are not in $N_G(\alpha_1)N_G(\alpha'_2) \cdots N_G(\alpha'_t)$.

According to Lemma 2(2), β appears in a commutator chain starting with a generator α''_1 such that $N_G(\alpha_1) = N_G(\alpha''_1)$; i.e., there exist $\alpha''_{11}, \dots, \alpha''_{1\ell'_1} \in N_G(\alpha''_1)$, such that $\alpha''_{11} = \alpha''_1, [\alpha''_{1j}, \alpha] = \alpha''_{1j+1}$ for $1 \leq j \leq \ell'_1 - 1$ and $\alpha''_{1\ell'_1} = \beta$ for some $\ell'_1 \leq k_1$. Notice that $k_1 - \ell'_1 \leq k_{t+1} - \ell_{t+1}$. (Here, we might have an inequality, when $\alpha'_{t+1\ell_{t+1}+k_1-\ell'_1} \notin Z(G)$.) Since we assumed that $k_1 \geq k_{t+1}$, we obtain $\ell'_1 \geq \ell_{t+1}$. Define $\alpha'_{t+1} = \alpha''_{1\ell'_1-\ell_{t+1}+1} \alpha_{t+1}$ and $\alpha'_{t+1j} = \alpha'_{t+1}, [\alpha'_{t+1j}, \alpha] = \alpha'_{t+1j+1}$ for $1 \leq j \leq \ell_{t+1} - 1$. Therefore, $\alpha'_{t+1\ell_{t+1}} = \prod_{i=2}^t \prod_{j=1}^{\ell_i-1} \alpha'_{ij}^{y_{ij}} \in N_G(\alpha_2) \cdots N_G(\alpha_t)$, and using the inductive assumption, we obtain that $\mathcal{H}_1 \cap (\mathcal{H}'_2 \cdots \mathcal{H}'_{t+1}) = \{1\}$.

We can continue this process until we finish the generators of H . Thus, we will obtain finally that $H = N_G(\alpha_1)N_G(\alpha_2) \cdots N_G(\alpha_s)$ for some new generators denoted again by $\alpha_2, \dots, \alpha_s$ of direct cyclic factors such that $\mathcal{H}_1 \cap (\mathcal{H}_2 \cdots \mathcal{H}_s) = \{1\}$.

Step III. We apply Step II consecutively for $N_G(\alpha_2) \cdots N_G(\alpha_s), N_G(\alpha_3) \cdots N_G(\alpha_s), \dots, N_G(\alpha_s)$. In this way, we obtain new generators, denoted again by $\alpha_1, \dots, \alpha_s$ of direct cyclic factors such that $\mathcal{H}_i \cap (\mathcal{H}_{i+1} \cdots \mathcal{H}_s) = \{1\}$ for $1 \leq i \leq s - 1$. This implies that

$$\mathcal{H}_i \cap \prod_{j \neq i} \mathcal{H}_j = \{1\}, \quad \text{for } 1 \leq i \leq s. \quad (2)$$

Finally, for $1 \leq j \leq s$, such that $\alpha_{j2} \in H(p)$, put $\mathcal{H}_j = \langle \alpha_j, \mathcal{H}'_j \rangle$. For any $j : 1 \leq j \leq s$ such that $\alpha_{j2} \notin H(p)$, put $\mathcal{H}_j = \langle \alpha_j \rangle$. As we pointed out in Step II, each time we replace a generator with a new one, there is a one-to-one correspondence between the original decomposition of H as a direct product of cyclic groups and the new one. Then, from (2), it follows that $H \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_s$. Moreover, if $\alpha_{jk_j} \in H(p)$, then $\mathcal{H}_j = N_G(\alpha_j) \simeq C_{p^{a_j}} \times (C_p)^{k_j-1}$ —normal in G ; if $\alpha_{jk_j} \notin H(p)$, then $\mathcal{H}_j \simeq C_{p^{a_j}} \times (C_p)^{k_j-2}$ —normal mod $Z(p)$ in G . $\square$

As a straightforward application of Theorem 10(iv), we obtain the following:

Corollary 1. Let G be a group of the type described in Theorem 10. Put $\overline{H} = H/Z(p)$ and $\overline{G} = G/Z(p)$. Then, $\overline{H} \simeq \overline{\mathcal{H}}_1 \times \cdots \times \overline{\mathcal{H}}_t$, where each $\overline{\mathcal{H}}_j = N_{\overline{G}}(\overline{\alpha}_j)$ is normal in $\overline{G}$ and is isomorphic to one of the following types:

- (i) $\overline{\mathcal{H}}_j = \langle \overline{\alpha}_j \rangle \simeq C_{p^{b_j}}$ for $\overline{\alpha}_j \in Z(\overline{G})$ and $b_j \geq 1$.
- (ii) $\overline{\mathcal{H}}_j \simeq N_{\overline{G}}(\overline{\alpha}_j) \simeq C_{p^{b_j}} \times (C_p)^{k_j-1}$ is the group generated by some element $\overline{\alpha}_1$ and its commutator chain $[\overline{\alpha}_j, \alpha] = \overline{\alpha}_{j+1}$, where $1 \leq j \leq k_j - 1$. Moreover, $\alpha_2, \dots, \alpha_{k_1}$ are independent generators of order p .

3. Generalities

We list several results which will be used in the sequel.

Theorem 11 ([17], Theorem 1). *Let G be a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over a field L such that*

- (i) *for any $\sigma \in G, \sigma(L) \subset L$;*
- (ii) *the restriction of the action of G to L is faithful;*
- (iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in \text{GL}_m(L)$ and $B(\sigma)$ is $m \times 1$ matrix over L . Then, there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ so that $L(x_1, \dots, x_m)^G = L^G(z_1, \dots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq m$.

Theorem 12 ([18], Theorem 3.1). *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L . Assume that, for any $\sigma \in G, \sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then, $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 13 (Kuniyoshi [19], Theorem 1.7). *If $\text{char} K = p > 0$ and G is a finite p -group, then $K(G)$ is rational over K .*

The following Lemma can be extracted from some proofs in [20,21].

Lemma 4. *Let $\langle \tau \rangle$ be a cyclic group of order $n > 1$, acting on $K(v_1, \dots, v_{n-1})$, the rational function field of $n - 1$ variables over a field K such that*

$$\tau : v_1 \mapsto v_2 \mapsto \dots \mapsto v_{n-1} \mapsto (v_1 \dots v_{n-1})^{-1} \mapsto v_1.$$

If K contains a primitive n th root of unity ξ , then $K(v_1, \dots, v_{n-1}) = K(s_1, \dots, s_{n-1})$, where $\tau : s_i \mapsto \xi^i s_i$ for $1 \leq i \leq n - 1$.

Moreover, in [14], we generalized Kang's argument from ([21], Case 5, Step II), obtaining the following:

Lemma 5. *Let $k > 1$, let p be any prime, and let $\langle \alpha \rangle$ be a cyclic group of order p , acting on $K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \leq i \leq p - 1)$, the rational function field of $k(p - 1)$ variables over a field K such that*

$$\alpha : y_{j1} \mapsto y_{j2} \mapsto \dots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \dots y_{jp-1})^{-1}, \quad \text{for } 1 \leq j \leq k.$$

Assume that $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \leq i \leq p - 1) = K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \leq i \leq p - 1)$, where for any $j : 1 \leq j \leq k$ and for any $i : 1 \leq i \leq p - 1$, the variable v_{ji} is a monomial in the variables $y_{1i}, y_{2i}, \dots, y_{ki}$. Assume also that the action of α on $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \leq i \leq p - 1)$ is given by

$$\alpha : v_{j1} \mapsto v_{j1} v_{j2}^p, v_{j2} \mapsto v_{j3} \mapsto \dots \mapsto v_{jp-1} \mapsto A_j \cdot (v_{j1} v_{j2}^{p-1} v_{j3}^{p-2} \dots v_{jp-1}^2)^{-1},$$

for $1 \leq j \leq k$,

where A_j is some monomial in $v_{1i}, \dots, v_{j-1i}$ for $2 \leq j \leq k$ and $A_1 = 1$. If K contains a primitive p -th root of unity ζ , then $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \leq i \leq p-1) = K(s_{1i}, s_{2i}, \dots, s_{ki} : 1 \leq i \leq p-1)$ where $\alpha : s_{ji} \mapsto \zeta^i s_{ji}$ for $1 \leq j \leq k, 1 \leq i \leq p-1$.

4. Proof of Theorem 8

The key idea is to find a faithful G -subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K . The subspace W is obtained as an induced representation from H . Then, by various linearization methods given in Section 3, we reduce the problem to verifying some combinatorial identities.

If $\text{char } K = p > 0$, we can apply Theorem 13. Therefore, we will assume that $\text{char } K \neq p$. Let V be a K -vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$ where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus, $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Case I. Here, we assume that H is not the elementary abelian group; i.e., it has some generators α_i of order p^{a_i} for $a_i > 1, 1 \leq i \leq k$. We are going to use the description of such groups given in Theorem 10. In particular, we assume that $H = \langle \alpha_1 \rangle \times \dots \times \langle \alpha_r \rangle$.

Step I.1. Define $X_1, X_2, \dots, X_r \in V^*$ by

$$X_j = \sum_{\ell_1, \dots, \ell_r} x \left(\prod_{i \neq j} \alpha_i^{\ell_i} \right), \quad \text{for } 1 \leq j \leq r, 0 \leq \ell_i \leq \text{ord}(\alpha_i) - 1.$$

Note that $\alpha_i \cdot X_j = X_j$ for $j \neq i$. Let $\zeta_{p^{a_i}} \in K$ be a primitive p^{a_i} -th root of unity for $1 \leq i \leq k$, and let ζ be a primitive p -th root of unity. Define $Y_1, Y_2, \dots, Y_r \in V^*$ by

$$Y_i = \sum_{m=0}^{p^{a_i}-1} \zeta_{p^{a_i}}^{-m} \alpha_i^m \cdot X_i, \quad Y_j = \sum_{m=0}^{p-1} \zeta^{-m} \alpha_j^m \cdot X_j$$

for $1 \leq i \leq k$ and $k+1 \leq j \leq r$.

It follows that

$$\begin{aligned} \alpha_i : Y_i &\mapsto \zeta_{p^{a_i}} Y_i, \quad Y_j \mapsto Y_j, \quad \text{for } j \neq i \text{ and } 1 \leq i \leq k \\ \alpha_j : Y_j &\mapsto \zeta Y_j, \quad Y_i \mapsto Y_i, \quad \text{for } i \neq j \text{ and } k+1 \leq j \leq r. \end{aligned}$$

Thus, $U = \bigoplus_{1 \leq j \leq r} K \cdot Y_j$ is a faithful representation space of the subgroup H .

Next, define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \leq j \leq r, 0 \leq i \leq p-1$. Then, $W = \bigoplus_{j,i} K \cdot x_{ji} \subset V^*$ is the induced representation of G obtained from H . Denote by $Z(p) = \langle \alpha_1^{p^{a_1}-1}, \dots, \alpha_k^{p^{a_k}-1} \rangle$ the group generated by all elements of order p which are not in $H(p)$. Note that

$$\alpha_i^{p^{a_i}-1} : Y_i \mapsto \zeta Y_i, \quad Y_j \mapsto Y_j, \quad \text{for } j \neq i \text{ and } 1 \leq i \leq k.$$

Hence, $U^{Z(p)} = (\bigoplus_{1 \leq j \leq k} K \cdot Y_j^p) (\bigoplus_{k+1 \leq j \leq r} K \cdot Y_j)$. Since $Z(p) \leq Z(G)$, we have that $W^{Z(p)} = \bigoplus_{j,i} K \cdot y_{ji}$, where $y_{ji} = \alpha^i \cdot Y_j^p$ for $1 \leq j \leq k, 0 \leq i \leq p-1$ and $y_{ji} = \alpha^i \cdot Y_j$ for $k+1 \leq j \leq r, 0 \leq i \leq p-1$.

Step I.2. Put $\bar{H} = H/Z(p)$, $\bar{G} = G/Z(p)$, and $\bar{W} = W^{Z(p)}$. Since $Z(p)$ is normal in G , we have $W^G = \bar{W}^{\bar{G}}$. Thus, by Theorem 11, it suffices to show that $\bar{W}^{\bar{G}}$ is rational over K . According to Corollary 1, we can decompose $\bar{H} \simeq \bar{\mathcal{H}}_1 \times \dots \times \bar{\mathcal{H}}_t$, as a direct product of normal subgroups $\bar{\mathcal{H}}_i$ of $\bar{G}$.

Next, we are going to show that we can decompose $\bar{W} = \bigoplus_{i=1}^t \bar{W}_k$ in such a way that $\bar{\mathcal{H}}_i$ acts trivially on $\bigoplus_{k \neq i} \bar{W}_k$. If we assume that $\alpha_1, \dots, \alpha_l$ are the generators of $\bar{\mathcal{H}}_1$, we can write the following: $\bar{W} = \bigoplus_{j,i} K \cdot y_{ji} = \bar{W}_1 \oplus \bar{W}_2$, where $\bar{W}_1 = \bigoplus_{1 \leq j \leq l, 0 \leq i \leq p-1} K \cdot y_{ji}$ contains all y_{ji} s on which $\bar{\mathcal{H}}_1$ acts in a non-trivial way, and $\bar{W}_2$ contains all y_{ji} s on which

$\overline{\mathcal{H}}_1$ acts trivially. Indeed, since $\overline{\mathcal{H}}_1$ is normal in $\overline{G}$, we have that for any $h \in \overline{\mathcal{H}}_1$, there exists $h_1 \in \overline{\mathcal{H}}_1$ such that $h\alpha^i = \alpha^i h_1$. This means that $\overline{\mathcal{H}}_1$ acts on the same set of y_{ji} s for $1 \leq j \leq \ell, i : 0 \leq i \leq p-1$, and $\overline{\mathcal{H}}_1$ acts trivially on y_{ji} s for $\ell+1 \leq j \leq r, i : 0 \leq i \leq p-1$. The same process can be repeated for $\overline{\mathcal{H}}_2, \dots, \overline{\mathcal{H}}_t$ non-trivially.

Finally, for the fixed field, we have $\overline{W}^{\overline{G}} = (\overline{W}^{\overline{H}})^{\langle \alpha \rangle} = ((\dots (\overline{W}^{\overline{\mathcal{H}}_1})^{\overline{\mathcal{H}}_2} \dots)^{\overline{\mathcal{H}}_t})^{\langle \alpha \rangle} = ((\dots (\overline{W}_1^{\overline{\mathcal{H}}_1} \oplus_{2 \leq j \leq t} \overline{W}_j)^{\overline{\mathcal{H}}_2} \dots)^{\overline{\mathcal{H}}_t})^{\langle \alpha \rangle} = \dots = \oplus_{1 \leq j \leq t} (\overline{W}_j^{\overline{\mathcal{H}}_j})^{\langle \alpha \rangle}$. Therefore, we need to calculate $\overline{W}_j^{\overline{\mathcal{H}}_j}$ when $\overline{\mathcal{H}}_j$ is isomorphic to any of the two types, described in Corollary 1. For brevity of notation, we will consider the isomorphism types of $\overline{\mathcal{H}}_1$.

Step I.2.1. Let $\overline{\mathcal{H}}_1$ be of type (i), i.e., $\overline{\mathcal{H}}_1 = \langle \overline{\alpha}_1 \rangle \simeq C_{p^{b_1}}$ for $\overline{\alpha}_1 \in Z(\overline{G})$ and $b_1 \geq 1$. In the notations of Step I.1, for the corresponding generator $\alpha_1 \in H$, we have

$$\alpha_1 : Y_1 \mapsto \zeta_{p^{a_1}} Y_1, Y_j \mapsto Y_j, \text{ for } j \neq 1.$$

If $a_1 > 1$, then $b_1 = a_1 - 1$ and $y_{1i} = \alpha^i \cdot Y_1^p$. If $a_1 = 1$, then $b_1 = a_1$ and $y_{1i} = \alpha^i \cdot Y_1$. Then, $\overline{W}_1 = \oplus_i K \cdot y_{1i}$ and the actions of $\overline{\mathcal{H}}_1$ and α are as follows:

$$\begin{aligned} \overline{\alpha}_1 : y_{1i} &\mapsto \zeta_{p^{b_1}} y_{1i}, \text{ for } 0 \leq i \leq p-1, \\ \alpha : y_{10} &\mapsto y_{11} \mapsto \dots \mapsto y_{1p-1} \mapsto \zeta_{p^{b_1}}^b y_{10}, \end{aligned}$$

where b is some integer such that $0 \leq b < p^{b_1}$. (In fact, b is determined by the decomposition of $\alpha^p = \overline{\alpha}_1^b \cdot \beta$ for some $\beta \in \overline{H}_2 \times \dots \times \overline{H}_t$.)

Next, for $1 \leq i \leq p-1$, define $y_i = y_{1i}/y_{1i-1}$. Thus, $\overline{W}_1 = K(y_{10}, y_i : 1 \leq i \leq p-1)$ and for every $g \in \overline{G}$

$$g \cdot y_{10} \in K(y_i : 1 \leq i \leq p-1) \cdot y_{10},$$

while the subfield $K(y_i : 1 \leq i \leq p-1)$ is invariant by the action of $\overline{G}$.

From Theorem 12, it follows that if $K(y_i : 1 \leq i \leq p-1)^{\overline{G}}$ is rational over K , so is $K(y_{10}, y_i : 1 \leq i \leq p-1)^{\overline{G}}$ over K . The action of α is

$$\alpha : y_1 \mapsto y_2 \mapsto \dots \mapsto y_{p-1} \mapsto \zeta_{p^{b_1}}^b (y_1 \dots y_{p-1})^{-1}.$$

Since K contains a primitive p^e -th root of unity ζ_{p^e} where p^e is the exponent of G , K contains as well the root of unity $\zeta_{p^{b_1+1}}^b$, and we may replace the variables y_i by $y_i/\zeta_{p^{b_1+1}}^b$ so that we obtain a more convenient action of α without changing the actions of $\overline{\alpha}_1$. Namely, we may assume that

$$\alpha : y_1 \mapsto y_2 \mapsto \dots \mapsto y_{p-1} \mapsto (y_1 y_2 \dots y_{p-1})^{-1}.$$

The action of $\overline{\alpha}_1$ is trivial:

$$\overline{\alpha}_1 : y_i \mapsto y_i, \text{ for } 1 \leq i \leq p-1.$$

According to Lemma 4, we can linearize the action of α .

Step I.2.2. $\overline{\mathcal{H}}_1 \simeq N_G(\overline{\alpha}_1) \simeq C_{p^{b_1}} \times (C_p)^{k_1-1}$ is the group generated by some element $\overline{\alpha}_1$ and its commutator chain $[\overline{\alpha}_j, \alpha] = \overline{\alpha}_{j+1}$, where $1 \leq j \leq k_1-1$. According to Corollary 1(2), $\overline{\alpha}_j$ s are all generators of order p of $\overline{H}$ for $2 \leq j \leq k_1$ and only the last $\overline{\alpha}_{k_1}$ is in the center of $\overline{G}$.

In the notations of Step I.1, if $a_1 > 1$, then $b_1 = a_1 - 1$ and define $Z_1 = Y_1^p$; if $a_1 = 1$, then $b_1 = a_1 = 1$ and define $Z_1 = Y_1$; for $2 \leq j \leq k_1$, define $Z_j = Y_j$. Then, for the generators $\bar{\alpha}_j \in \bar{\mathcal{H}}_1$, we have

$$\begin{aligned}\bar{\alpha}_1 &: Z_1 \mapsto \zeta_{p^{b_1}} Z_1, Z_j \mapsto Z_j, \text{ for } j \neq 1, \\ \bar{\alpha}_j &: Z_j \mapsto \zeta Z_j, Z_i \mapsto Z_i, \text{ for } i \neq j \text{ and } 2 \leq j \leq k_1.\end{aligned}$$

Next, define $y_{1i} = \alpha^i \cdot Z_1, y_{ji} = \alpha^i \cdot Z_j$ for $2 \leq j \leq k_1$. Then, $\bar{W}_1 = \bigoplus_{1 \leq j \leq k_1, 0 \leq i \leq p-1} K \cdot y_{ji}$.

From $[\bar{\alpha}_j, \alpha] = \bar{\alpha}_{j+1}$, it follows that

$$\alpha^{-i} \bar{\alpha}_j \alpha^i = \bar{\alpha}_j \bar{\alpha}_{j+1}^{(i)} \bar{\alpha}_{j+2}^{(i)} \cdots \bar{\alpha}_{k_1}^{(k_1-i)},$$

so we obtain the actions

$$\begin{aligned}\bar{\alpha}_1 &: y_{1i} \mapsto \zeta_{p^{b_1}} y_{1i}, y_{ji} \mapsto \zeta^{(j-1)} y_{ji}, \text{ for } 2 \leq j \leq k_1, 0 \leq i \leq p-1 \\ \bar{\alpha}_j &: y_{\ell i} \mapsto y_{\ell i}, y_{mi} \mapsto \zeta^{(m-j)} y_{mi}, \text{ for } 2 \leq j \leq k_1, 1 \leq \ell \leq j-1, j \leq m \leq k_1, \\ &\text{and } 1 \leq i \leq p-1.\end{aligned}$$

The action of α is

$$\begin{aligned}\alpha &: y_{10} \mapsto y_{11} \mapsto \cdots \mapsto y_{1p-1} \mapsto \zeta_{p^{b_1}}^{b_1} y_{10}, \\ &y_{j0} \mapsto y_{j1} \mapsto \cdots \mapsto y_{jp-1} \mapsto \zeta_{p^{b_j}}^{b_j} y_{j0}, \text{ for } 2 \leq j \leq k_1,\end{aligned}$$

where b_j s and b are determined by the decomposition of $\alpha^p = \bar{\alpha}_1^b \cdot \prod_{2 \leq j \leq k_1} \bar{\alpha}_j^{b_j} \cdot \beta$ for some $\beta \in \bar{H}_2 \times \cdots \times \bar{H}_t$.)

Next, for $1 \leq j \leq k_1$ and for $1 \leq i \leq p-1$, define $z_{ji} = y_{ji}/y_{ji-1}$. Thus, $\bar{W}_1 = K(y_{j0}, z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)$ and for every $g \in \bar{G}$,

$$g \cdot y_{j0} \in K(z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1) \cdot y_{j0}, \text{ for } 1 \leq j \leq k_1,$$

while the subfield $K(z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)$ is invariant by the action of $\bar{G}$.

From Theorem 12, it follows that if $K(z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)^{\bar{G}}$ is rational over K , so is $K(y_{j0}, z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)^{\bar{G}}$ over K . The action of α is

$$\begin{aligned}\alpha &: z_{11} \mapsto z_{12} \mapsto \cdots \mapsto z_{1p-1} \mapsto \zeta_{p^{b_1}}^{b_1} (z_{11} \cdots z_{1p-1})^{-1}, \\ &z_{j1} \mapsto z_{j2} \mapsto \cdots \mapsto z_{jp-1} \mapsto \zeta_{p^{b_j}}^{b_j} (z_{j1} \cdots z_{jp-1})^{-1}, \text{ for } 2 \leq j \leq k_1.\end{aligned}$$

If $b \not\equiv 0 \pmod{p^{b_1}}$, then from the decomposition of α^p , it follows that K contains the root of unity $\zeta_{p^{b_1+1}}^b$. (Recall that K contains a primitive p^e -th root of unity ζ_{p^e} where p^e is the exponent of G .) The same argument applies for b_j s such that $b_j \not\equiv 0 \pmod{p}$. Then, we may replace the variables z_{1i} by $Z_{1i} = z_{1i}/\zeta_{p^{b_1+1}}^b$ and z_{ji} by $Z_{ji} = z_{ji}/\zeta_{p^{b_j}}^{b_j}$ so that we obtain a more convenient action of α without changing the actions of $\bar{\alpha}_j$ s. Namely, calculations show us that

$$\alpha : Z_{j1} \mapsto Z_{j2} \mapsto \cdots \mapsto Z_{jp-1} \mapsto (Z_{j1} \cdots Z_{jp-1})^{-1}, \text{ for } 1 \leq j \leq k_1.$$

Using the well-known binomial formula $\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$ with the boundary values $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{-1} = 0$, for the new variables, we obtain the actions

$$\begin{aligned}\bar{\alpha}_1 : Z_{1i} &\mapsto Z_{1i}, Z_{ji} \mapsto \zeta^{\binom{i-1}{j-2}} Z_{ji}, \text{ for } 2 \leq j \leq k_1 \text{ and } 1 \leq i \leq p-1, \\ \bar{\alpha}_j : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{mi} \mapsto \zeta^{\binom{i-1}{m-j-1}} Z_{mi}, \text{ for } 2 \leq j \leq k_1, 1 \leq \ell \leq j-1, j \leq m \leq k_1, \\ &\text{and } 1 \leq i \leq p-1.\end{aligned}$$

Note that $\bar{\alpha}_{k_1}$ acts trivially on all Z_{ji} s, so we continue looking for the invariant field of $\langle \bar{\alpha}_{k_1-1} \rangle$. From the above actions, we obtain

$$\bar{\alpha}_{k_1-1} : Z_{\ell i} \mapsto Z_{\ell i}, Z_{k_1 i} \mapsto \zeta Z_{k_1 i}, \text{ for } 1 \leq \ell \leq k_1-1 \text{ and } 1 \leq i \leq p-1.$$

Define $u_{k_1 1} = Z_{k_1 1}^p$, $u_{k_1 i} = Z_{k_1 i} / Z_{k_1 i-1}$ for $2 \leq i \leq p-1$. Then, $K(Z_{ji}, u_{k_1 i} : 1 \leq j \leq k_1-1, 1 \leq i \leq p-1) = K(Z_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)^{\langle \bar{\alpha}_{k_1-1} \rangle}$. We have the following actions:

$$\begin{aligned}\bar{\alpha}_j : u_{k_1 i} &\mapsto \zeta^{\binom{i-2}{k_1-j-2}} u_{k_1 i}, \text{ for } 2 \leq i \leq p-1 \text{ and } 1 \leq j \leq k_1-2, \\ \alpha : u_{k_1 2} &\mapsto u_{k_1 3} \mapsto \cdots \mapsto u_{k_1 p-1} \mapsto (u_{k_1 1} u_{k_1 2}^{p-1} u_{k_1 3}^{p-2} \cdots u_{k_1 p-1}^2)^{-1} \mapsto \\ &u_{k_1 1} u_{k_1 2}^{p-2} u_{k_1 3}^{p-3} \cdots u_{k_1 p-2}^2 u_{k_1 p-1}.\end{aligned}$$

Next, we continue looking for the invariant field of $\langle \bar{\alpha}_{k_1-2} \rangle$. We have the action

$$\begin{aligned}\bar{\alpha}_{k_1-2} : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{k_1-1 i} \mapsto \zeta Z_{k_1-1 i}, \text{ for } 1 \leq \ell \leq k_1-2 \text{ and } 1 \leq i \leq p-1, \\ u_{k_1 1} &\mapsto u_{k_1 1}, u_{k_1 i} \mapsto \zeta u_{k_1 i}, \text{ for } 2 \leq i \leq p-1.\end{aligned}$$

Define $v_{k_1-1 1} = Z_{k_1-1 1}^p$, $v_{k_1-1 i} = Z_{k_1-1 i} / Z_{k_1-1 i-1}$, $v_{k_1 1} = u_{k_1 1}$ and $v_{k_1 i} = u_{k_1 i} Z_{k_1-1 i}^{-1}$ for $2 \leq i \leq p-1$. Then, $K(Z_{ji}, v_{mi} : 1 \leq j \leq k_1-2, k_1-1 \leq m \leq k_1, 1 \leq i \leq p-1) = K(Z_{ji}, u_{k_1 i} : 1 \leq j \leq k_1-1, 1 \leq i \leq p-1)^{\langle \bar{\alpha}_{k_1-2} \rangle}$. We have the following actions:

$$\begin{aligned}\bar{\alpha}_j : v_{k_1-1 i} &\mapsto \zeta^{\binom{i-2}{k_1-1-j-2}} v_{k_1-1 i}, v_{k_1 i} \mapsto \zeta^{-\binom{i-2}{k_1-1-j-2}} v_{k_1 i}, \text{ for } 2 \leq i \leq p-1 \text{ and } \\ &1 \leq j \leq k_1-3, \\ \alpha : v_{k_1-1 2} &\mapsto v_{k_1-1 3} \mapsto \cdots \mapsto v_{k_1-1 p-1} \mapsto (v_{k_1-1 1} v_{k_1-1 2}^{p-1} v_{k_1-1 3}^{p-2} \cdots v_{k_1-1 p-1}^2)^{-1} \mapsto \\ &v_{k_1-1 1} v_{k_1-1 2}^{p-2} v_{k_1-1 3}^{p-3} \cdots v_{k_1-1 p-2}^2 v_{k_1-1 p-1}, \\ v_{k_1 2} &\mapsto v_{k_1 3} \mapsto \cdots \mapsto v_{k_1 p-1} \mapsto A_{k_1} \cdot (v_{k_1 1} v_{k_1 2}^{p-1} v_{k_1 3}^{p-2} \cdots v_{k_1 p-1}^2)^{-1},\end{aligned}$$

where A_{k_1} is some monomial in $v_{k_1-1 i}$ for $1 \leq i \leq p-1$. Indeed,

$$\begin{aligned}\alpha : v_{k_1 p-1} &= u_{k_1 p-1} Z_{k_1-1 p-1}^{-1} \mapsto (u_{k_1 1} u_{k_1 2}^{p-1} u_{k_1 3}^{p-2} \cdots u_{k_1 p-1}^2)^{-1} Z_{k_1-1 1} \cdots Z_{k_1-1 p-1} = \\ &v_{k_1 1}^{-1} v_{k_1 2}^{-(p-1)} Z_{k_1-1 2}^{-(p-1)} v_{k_1 3}^{-(p-2)} Z_{k_1-1 3}^{-(p-2)} \cdots v_{k_1 p-1}^{-2} Z_{k_1-1 p-1}^{-2} Z_{k_1-1 1} \cdots Z_{k_1-1 p-1} = \\ &Z_{k_1-1 1} Z_{k_1-1 2}^{-p+2} Z_{k_1-1 3}^{-p+3} \cdots Z_{k_1-1 p-1}^{-1} \cdot (v_{k_1 1} v_{k_1 2}^{p-1} v_{k_1 3}^{p-2} \cdots v_{k_1 p-1}^2)^{-1} = \\ &Z_{k_1-1 1}^A \cdot B \cdot (v_{k_1 1} v_{k_1 2}^{p-1} v_{k_1 3}^{p-2} \cdots v_{k_1 p-1}^2)^{-1},\end{aligned}$$

where B is some monomial in $v_{k_1-1 i}$ for $2 \leq i \leq p-1$ and $A = 1 - [(p-2) + (p-3) + \cdots + (p - (p-1))] = -(2+3+\cdots+p-2) = -\frac{p(p-3)}{2}$, so $Z_{k_1-1 1}^A = v_{k_1-1 1}^{-\frac{p-3}{2}}$.

It becomes clear now that if we continue in the same way, we will find the fixed field of all α_j s. However, in order to obtain a precise inductive assumption, we need one more step. So, let us find the invariant field of $\langle \bar{\alpha}_{k_1-3} \rangle$. We have the action

$$\begin{aligned}\bar{\alpha}_{k_1-3} : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{k_1-2i} \mapsto \zeta Z_{k_1-2i}, \text{ for } 1 \leq \ell \leq k_1-3 \text{ and } 1 \leq i \leq p-1, \\ v_{k_1-11} &\mapsto v_{k_1-11}, v_{k_1-1i} \mapsto \zeta v_{k_1-1i}, \text{ for } 2 \leq i \leq p-1 \\ v_{k_11} &\mapsto v_{k_11}, v_{k_1i} \mapsto \zeta^{-1} v_{k_1i}, \text{ for } 2 \leq i \leq p-1.\end{aligned}$$

Define $w_{k_1-21} = Z_{k_1-21}^p, w_{k_1-2i} = Z_{k_1-2i}/Z_{k_1-2i-1}, w_{k_1-11} = v_{k_1-11}, w_{k_1-1i} = v_{k_1-1i} Z_{k_1-2i}^{-1}, w_{k_11} = v_{k_11}$ and $w_{k_1i} = v_{k_1i} Z_{k_1-2i}$ for $2 \leq i \leq p-1$. Then, $K(Z_{ji}, w_{mi} : 1 \leq j \leq k_1-3, k_1-2 \leq m \leq k_1, 1 \leq i \leq p-1) = K(Z_{ji}, v_{mi} : 1 \leq j \leq k_1-2, k_1-1 \leq m \leq k_1, 1 \leq i \leq p-1)^{\langle \bar{\alpha}_{k_1-3} \rangle}$.

We can now make the inductive assumption that after we find the fixed field $K(Z_{ji}, U_{mi} : 1 \leq j \leq k_1-s+1, k_1-s+2 \leq m \leq k_1, 1 \leq i \leq p-1)$ of $\bar{\alpha}_{k_1-1}, \dots, \bar{\alpha}_{k_1-s+1}$, for any $j : 2 \leq j \leq k_1-s$, we have the actions

$$\begin{aligned}\bar{\alpha}_j : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{mi} \mapsto \zeta^{\binom{i-1}{m-j-1}} Z_{mi}, \text{ for } 1 \leq \ell \leq j-1, j \leq m \leq k_1-s+1, \\ &\text{and } 1 \leq i \leq p-1, \\ U_{k_1-s+2i} &\mapsto \zeta^{\binom{i-2}{k_1-s+2-j-2}} U_{k_1-s+2i}, \text{ for } 2 \leq i \leq p-1, \\ U_{k_1-s+3i} &\mapsto \zeta^{-\binom{i-2}{k_1-s+2-j-2}} U_{k_1-s+3i}, \text{ for } 2 \leq i \leq p-1, \\ &\vdots \\ U_{k_1i} &\mapsto \zeta^{(-1)^s \binom{i-2}{k_1-s+2-j-2}} U_{k_1i}, \text{ for } 2 \leq i \leq p-1, \\ \alpha : U_{k_1-s+22} &\mapsto U_{k_1-s+23} \mapsto \dots \mapsto U_{k_1-s+2p-1} \mapsto \\ &(U_{k_1-s+21} U_{k_1-s+22}^{p-1} U_{k_1-s+23}^{p-2} \dots U_{k_1-s+2p-1}^2)^{-1} \mapsto \\ &U_{k_1-s+21} U_{k_1-s+22}^{p-2} U_{k_1-s+23}^{p-3} \dots U_{k_1-s+2p-2}^2 U_{k_1-s+2p-1}, \\ U_{m2} &\mapsto U_{m3} \mapsto \dots \mapsto U_{mp-1} \mapsto A_m \cdot (U_{m1} U_{m2}^{p-1} U_{m3}^{p-2} \dots U_{mp-1}^2)^{-1}, \\ &\text{for } k_1-s+3 \leq m \leq k_1,\end{aligned}$$

where A_m s are some monomials in $U_{k_1-s+2i}, \dots, U_{m-1i}$ for $1 \leq i \leq p-1$. In particular, we have

$$\begin{aligned}\bar{\alpha}_{k_1-s} : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{k_1-s+1i} \mapsto \zeta Z_{k_1-s+1i}, \text{ for } 1 \leq \ell \leq k_1-s \text{ and } 1 \leq i \leq p-1, \\ U_{k_1-s+21} &\mapsto U_{k_1-s+21}, U_{k_1-s+2i} \mapsto \zeta U_{k_1-s+2i}, \text{ for } 2 \leq i \leq p-1, \\ U_{k_1-s+31} &\mapsto U_{k_1-s+31}, U_{k_1-s+3i} \mapsto \zeta^{-1} U_{k_1-s+3i} \text{ for } 2 \leq i \leq p-1, \\ &\vdots \\ U_{k_11} &\mapsto U_{k_11}, U_{k_1i} \mapsto \zeta^{(-1)^s} U_{k_1i}, \text{ for } 2 \leq i \leq p-1.\end{aligned}$$

Define $V_{k_1-s+11} = Z_{k_1-s+11}^p, V_{k_1-s+1i} = Z_{k_1-s+1i}/Z_{k_1-s+1i-1}, V_{k_1-s+21} = U_{k_1-s+21}, V_{k_1-s+2i} = U_{k_1-s+2i} Z_{k_1-s+1i}^{-1}, V_{k_1-s+31} = U_{k_1-s+31}, V_{k_1-s+3i} = U_{k_1-s+3i} Z_{k_1-s+1i}, \dots, V_{k_11} = U_{k_11}, V_{k_1i} = U_{k_1i} Z_{k_1-s+1i}^{(-1)^s}$ for $2 \leq i \leq p-1$. Then, $K(Z_{ji}, V_{mi} : 1 \leq j \leq k_1-s, k_1-s+1 \leq m \leq k_1, 1 \leq i \leq p-1) = K(Z_{ji}, U_{mi} : 1 \leq j \leq k_1-s+1, k_1-s+2 \leq m \leq k_1, 1 \leq i \leq p-1)^{\langle \bar{\alpha}_{k_1-s} \rangle}$. Using again the formula $\binom{i-1}{m} - \binom{i-2}{m} = \binom{i-2}{m-1}$, for any $j : 2 \leq j \leq k_1-s-1$, we obtain the actions

$$\begin{aligned}
\bar{\alpha}_j : Z_{\ell i} &\mapsto Z_{\ell i}, Z_{mi} \mapsto \zeta^{\binom{i-1}{m-j-1}} Z_{mi}, \text{ for } 1 \leq \ell \leq j-1, j \leq m \leq k_1-s, \\
&\text{and } 1 \leq i \leq p-1, \\
V_{k_1-s+1i} &\mapsto \zeta^{\binom{i-2}{k_1-s+1-j-2}} V_{k_1-s+1i}, \text{ for } 2 \leq i \leq p-1, \\
V_{k_1-s+2i} &\mapsto \zeta^{\binom{i-2}{k_1-s+1-j-2}} V_{k_1-s+2i}, \text{ for } 2 \leq i \leq p-1, \\
&\vdots \\
V_{k_1i} &\mapsto \zeta^{(-1)^{s-1} \binom{i-2}{k_1-s+1-j-2}} V_{k_1i}, \text{ for } 2 \leq i \leq p-1, \\
\alpha : V_{k_1-s+12} &\mapsto V_{k_1-s+13} \mapsto \cdots \mapsto V_{k_1-s+1p-1} \mapsto \\
&(V_{k_1-s+11} V_{k_1-s+12}^{p-1} V_{k_1-s+13}^{p-2} \cdots V_{k_1-s+1p-1}^2)^{-1} \mapsto \\
&V_{k_1-s+11} V_{k_1-s+12}^{p-2} V_{k_1-s+13}^{p-3} \cdots V_{k_1-s+1p-2}^2 V_{k_1-s+1p-1}, \\
V_{m2} &\mapsto V_{m3} \mapsto \cdots \mapsto V_{mp-1} \mapsto A_m \cdot (V_{m1} V_{m2}^{p-1} V_{m3}^{p-2} \cdots V_{mp-1}^2)^{-1}, \\
&\text{for } k_1-s+2 \leq m \leq k_1,
\end{aligned}$$

where A_m s are some monomials in $V_{k_1-s+1i}, \dots, V_{m-1i}$ for $1 \leq i \leq p-1$. This proves the inductive assumption, so we need to show only that the action of α can be linearized.

Indeed, after all new variables are introduced, we have $K(W_{mi} : 1 \leq m \leq k_1, 1 \leq i \leq p-1) = K(Z_{mi} : 1 \leq m \leq k_1, 1 \leq i \leq p-1)^{\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{k_1} \rangle}$ and the action of α is given by

$$\begin{aligned}
\alpha : W_{m1} &\mapsto W_{m1} W_{m2}^p, W_{m2} \mapsto W_{m3} \mapsto \cdots \mapsto W_{mp-1} \mapsto \\
&A_m \cdot (W_{m1} W_{m2}^{p-1} W_{m3}^{p-2} \cdots W_{mp-1}^2)^{-1},
\end{aligned}$$

where A_m is some monomial in $W_{1i}, \dots, W_{m-1i}$ for $2 \leq m \leq k_1$ and $A_1 = 1$. From Lemma 5, it follows that the action of α on $K(W_{ji} : 1 \leq j \leq k_1, 1 \leq i \leq p-1)$ can be linearized.

Case II. Suppose that H is the elementary abelian p -group. According to Theorem 9, H can be decomposed as a direct product $H \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_s$ of normal subgroups $\mathcal{H}_i$ in G such that either $\mathcal{H}_i \simeq N_G(\gamma_i) \simeq C_p \leq Z(G)$, or $\mathcal{H}_i \simeq N_G(\gamma_i) \simeq (C_p)^{k_i}$ is the group generated by some element γ_i and its commutator chain $\beta_{i1} = \gamma_i, [\beta_{ij}, \alpha] = \beta_{ij+1}$ for $1 \leq i \leq s, k_2 \geq 2, 1 \leq j \leq k_i - 1$ and β_{ik_i} is central in G .

Step II.1. Define $X_{ij} \in V^*$ for $1 \leq i \leq s, 1 \leq j \leq k_i$ by

$$X_{ij} = \sum_{m_{11}, \dots, m_{ik_i}} x \left(\prod_{(c,d) \neq (i,j)} \beta_{cd}^{m_{cd}} \right),$$

where $0 \leq m_{cd} \leq p-1$. Note that $\beta_{cd} \cdot X_{ij} = X_{ij}$ for $(c,d) \neq (i,j)$. Let $\zeta \in K$ be a primitive p -th root of unity. Next, define $Y_{ij} \in V^*$ for $1 \leq i \leq s, 1 \leq j \leq k_i$ by

$$Y_{ij} = \sum_{r=0}^{p-1} \zeta^{-r} \beta_{ij}^r \cdot X_{ij}.$$

It follows that

$$\beta_{ij} : Y_{ij} \mapsto \zeta Y_{ij}, Y_{cd} \mapsto Y_{cd}, \text{ for } (c,d) \neq (i,j) \text{ and } 1 \leq i \leq s, 1 \leq j \leq k_i.$$

Thus, $\bigoplus_{ij} K \cdot Y_{ij}$ is a representation space of the subgroup H .

Next, define $x_{ij\ell} = \alpha^\ell \cdot Y_{ij}$ for $1 \leq i \leq s, 1 \leq j \leq k_i, 0 \leq \ell \leq p-1$. Then, $W = \bigoplus_{i,j,\ell} K \cdot x_{ij\ell} \subset V^*$ is the induced representation of G obtained from H .

Step II.2. Define $W_i = \bigoplus_{j,\ell} K \cdot x_{ij\ell}$. Then, $W = \bigoplus_{i=1}^s W_i$, where $\mathcal{H}_i$ acts trivially on $\bigoplus_{k \neq i} W_k$. Indeed, since $\mathcal{H}_1 = N_G(\beta_{11})$ is normal in G , we have that for any $h \in \mathcal{H}_1$ there exists $h_1 \in \mathcal{H}_1$ such that $h\alpha^\ell = \alpha^\ell h_1$. This means that $\mathcal{H}_1$ acts on $x_{1j\ell}$ s non-trivially, and acts trivially on $x_{ij\ell}$ s for any $i \neq 1$. It follows that $W^G = (W^H)^{\langle \alpha \rangle} = ((\dots (W^{\mathcal{H}_1})^{\mathcal{H}_2} \dots)^{\mathcal{H}_s})^{\langle \alpha \rangle} = ((\dots (W_1^{\mathcal{H}_1} \oplus_{2 \leq j \leq s} W_j)^{\mathcal{H}_2} \dots)^{\mathcal{H}_s})^{\langle \alpha \rangle} = \dots = \bigoplus_{1 \leq j \leq s} (W_j^{\mathcal{H}_j})^{\langle \alpha \rangle}$. Therefore, we need to calculate $W_j^{\mathcal{H}_j}$. For brevity of notation, we will consider the isomorphism types of $\mathcal{H}_1$.

Step II.3. Let $\mathcal{H}_1$ be of type I, i.e., $\mathcal{H}_1 = \langle \alpha_1 \rangle \simeq C_p$ for $\alpha_1 \in Z(G)$. In the notations of Step II.1, we have

$$\alpha_1 : Y_{11} \mapsto \zeta Y_{11}, Y_{ij} \mapsto Y_{ij}, \text{ for } (i, j) \neq (1, 1).$$

This action, however, is a particular case of Step I.2.1 for $b_1 = a_1 = 1$.

Step II.4. $\mathcal{H}_1 \simeq N_G(\alpha_i) \simeq (C_p)^{k_1}$ is the group generated by $\alpha_1 = \beta_{11}$ and its commutator chain $[\alpha_j, \alpha] = \alpha_{j+1}$, where $1 \leq j \leq k_1 - 1$ and α_{k_1} is in the center of G .

In the notations of Step II.1, we have

$$\alpha_j : Y_{1j} \mapsto \zeta Y_{1j}, Y_{cd} \mapsto Y_{cd}, \text{ for } (c, d) \neq (1, j) \text{ and } 1 \leq j \leq k_1.$$

This action, however, is a particular case of Step I.2.2 for $b_1 = a_1 = 1$. The process is completed.

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