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A Construction of Maslov-Type Index for Paths of 2 × 2 Symplectic Matrices

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Abstract: In this article, we construct a kind of Maslov-type index for general paths of 2×2 symplectic matrices that have two arbitrary endpoints. Our method is consistent and direct no matter whether the starting point of the path is an identity or not, which is different from those regarding the Conley–Zehnder–Long index of symplectic paths starting from an identity and Long's Maslov-type index of symplectic path segments. In addition, we compare this index with the Conley–Zehnder–Long index.

Keywords: Maslov index; Conley–Zehnder–Long index; second-order symplectic path

MSC: 53D12; 57R20; 37J11

1. Introduction

The Maslov index was invented and mathematically formalized in the 1960s by physicist V. P. Maslov as well as mathematician V. I. Arnold in their seminal works [1,2]. Such an index is an invariant for oriented closed curves in a Lagrangian submanifold of a symplectic manifold. In 1984, Conley and Zehnder [3,4] defined a kind of Maslov-type index, called the Conley–Zehnder index, for non-degenerate paths of symplectic matrices and non-degenerate Hamiltonian periodic solutions. The Conley–Zehnder index has become an important tool for studying other mathematical problems. For instance, in symplectic geometry, the Conley–Zehnder index of non-degenerate 1-periodic orbits of a Hamiltonian diffeomorphism can be regarded as the degree of Hamiltonian Floer homology [5–11]. Robbin–Salamon [12] also studied the Maslov index from the index of Lagrangian path pairs. For degenerate symplectic paths, Maslov-type indices were first constructed by Long [13] and Viterbo [14] independently in 1990. Zhu–Long [15] also studied spectral flow for paths of admissible operators in Banach space and studied some related properties. Then, representing great progress, iterative formulae for Maslov-type indices were established by Long [16,17], which can be applied for investigating various problems related to periodic solutions, closed characteristics, etc. (see [18,19] and the references therein).

Denote by $Sp(2n, \mathbb{R})$ the set of $2n \times 2n$ symplectic matrices and

$$\mathcal{P}(2n, \mathbb{R}) := \{\Phi : [0,1] \to Sp(2n, \mathbb{R}) \text{ is continuous}\}$$

the space of the general paths of symplectic matrices. Recall that, for paths of $2n \times 2n$ symplectic matrices, the Conley–Zehnder–Long index, denoted by $\mu_{CZL}(\Phi)$, is defined for symplectic (matrix) paths starting from identity I_{2n}. For the non-degenerate case, i.e., $\det(\Phi(1) - I_2) \neq 0$, the index is also named after Conley–Zehnder, denoted by $\mu_{CZ}(\Phi)$. For the degenerate case, i.e., $\det(\Phi(1) - I_2) = 0$, the index is also named after Long, denoted by $\mu_L(\Phi)$. Roughly speaking, the idea of constructing a Conley–Zehnder index is to construct

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a mapping from a non-degenerate symplectic path Φ to a path on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$, calculate the rotation number of this mapping, and then construct an integer-valued index by extending the path appropriately (see Definition 6). Long's index is defined by first perturbing the degenerate path near the terminal point $\Phi(1)$ into non-degenerate ones and then taking the infimum of all the Conley–Zehnder indices of all these non-degenerate paths. For a path starting from an arbitrary symplectic matrix, Long also provides a definition, called Long's symplectic path segment index (SPS-index), which is defined indirectly as the difference regarding the Long indices of two paths starting from an identity.

Recently, Zhong and the second named author [20] provided another method of constructing a Maslov-type index for paths of 2×2 orthogonal symplectic matrices. Their method is consistent and direct no matter if the starting point of the path is identity I_2 or not. That new index can be used to explain the geometric intersection number of a pair of Lagrangian paths and is related to the Cappell–Lee–Miller index [21]. However, the singular set of 2×2 orthogonal symplectic matrices they consider in [20] is $\{I_2, -I_2\}$, which is different from the one, i.e., $Sp_1^0(2,\mathbb{R}) := \{M \in Sp(2,\mathbb{R}) \mid \det(I_2 - M) = 0\}$, in previous works [3,13] for the case $n = 1$.

In this paper, for the special case $n = 1$, i.e., paths of 2×2 symplectic matrices, we show another consistent construction of a Maslov-type index using the ordinary singular set $Sp_1^0(2,\mathbb{R})$. The main idea of our construction is adopted from [20]. It includes the first *orthogonalization* at both endpoints of a path and a subsequent *global perturbation* to obtain a modified path, for which an *extension* can be appropriately taken. Then, we can calculate the rotation numbers of that modified path as well as the extending path and construct an integer-valued index. The detailed construction will be introduced in Section 4 (in particular, see (14)). Our main results are the following:

Theorem 1. *For any $\Phi \in \mathcal{P}(2,\mathbb{R})$, there exists a Maslov-type index $\mu(\Phi)$, which is an integer and well defined. The index satisfies the following properties:*

(1) **Homotopy invariant:**
If Φ and Ψ are homotopic with the same endpoints, then $\mu(\Phi) = \mu(\Psi)$.

(2) **Vanishing:** *If $\nu_t(\Phi) = \dim Ker(I - \Phi(t))$ is constant, then $\mu(\Phi) = 0$.*

(3) **Catenation:** $\forall\, 0 < a < 1$,
$$\mu(\Phi) = \mu(\Phi|_{[0,a]}) + \mu(\Phi|_{[a,1]}).$$

We remark that these properties above are analogous to the ones in the axioms of Long's SPS index (see Corollary 10 on page 148 of [18]).

Now, we come to compare $\mu(\Phi)$ with the Conley–Zehnder–Long index $\mu_{CZL}(\Phi)$ for a symplectic path Φ that satisfies $\Phi(0) = I_2$. For a degenerate path, by Theorem 7.3 of [16], up to a similarity transformation, one can express any symplectic matrix M with an eigenvalue of 1 as $N(1,b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where $b = 0$, or ± 1.

Theorem 2. *If $\Phi \in \mathcal{P}(2,\mathbb{R})$ satisfies $\Phi(0) = I_2$, then*

$$\mu(\Phi) = \mu_{CZL}(\Phi) + \chi(\Phi(1)),$$

if one of the eigenvalues of $\Phi(1)$ is in $(0,1)$ or $\Phi(1)$ with eigenvalue 1 becomes $N_1(1,-1)$ by symplectic matrix similar transformation (8), and then $\chi(\Phi(1)) = 0$. In other cases, $\chi(\Phi(1)) = 1$.

2. Preliminaries

In this section, we introduce some definitions and results that we use in the article.

2.1. Symplectic Matrix and Symplectic Path

Definition 1. *Let ω_0 be the standard bilinear form on $\mathbb{R}^{2n}$, satisfying the following:*

$$(1)\ \forall\, \xi, \eta \in \mathbb{R}^{2n},\ \omega_0(\xi, \eta) = -\omega_0(\eta, \xi),$$
$$(2)\ \text{If } \forall\, \xi \in \mathbb{R}^{2n},\ \omega_0(\xi, \eta) = 0,\ \text{then } \eta = 0,$$

where

$$\omega_0(\xi_1, \xi_2) = (J_n \xi_1)^T \xi_2,\ J_n = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}.$$

The space $(\mathbb{R}^{2n}, \omega_0)$ is called the standard symplectic space.

Definition 2. *Let $M \in \mathbb{R}^{2n\times 2n}$. M is called a symplectic matrix if it satisfies*

$$M^T J_n M = J_n.$$

The symplectic matrix M can be considered as a symplectic isomorphism on $\mathbb{R}^{2n}$, which satisfies $\omega_0(\xi_1, \xi_2) = \omega_0(M\xi_1, M\xi_2)$. The set of all symplectic isomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ with composition can be viewed as a group [18,22], called symplectic group, and is denoted by $Sp(2n, \mathbb{R})$.

Definition 3. *A continuous map*

$$\Phi : [0, 1] \to Sp(2n, \mathbb{R})$$

is called a **symplectic path** *in $Sp(2n, \mathbb{R})$.*

If $\Phi(0) = \Phi(1)$, then Φ is called a symplectic loop. If two symplectic paths Φ, Ψ satisfy $\Phi(1) = \Psi(0)$, then they have a catenation defined by

$$\Phi\#\Psi(t) := \begin{cases} \Phi(2t), & 0 \le t < \frac{1}{2}, \\ \Psi(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

We continue to introduce some properties regarding 2×2 symplectic matrices. The following proposition holds (also see [18,22]).

Proposition 1. *For a two-order symplectic matrix M, denote the set of all eigenvalues of M by $\sigma(M)$. We have the following:*

(1) *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$, and then $M^T J_2 M = J_2 \Leftrightarrow \det M = ad - bc = 1$.*

(2) *$\sigma(M) = \{\lambda | \lambda^2 - tr(M)\lambda + 1 = 0\}$, where $tr(M)$ is the trace of M. If $\lambda \in \sigma(M) \subset \mathbb{C}$, then $\lambda^{-1} \in \sigma(M)$; i.e., M has the pair $\{\lambda, \lambda^{-1}\}$ of eigenvalues. If $\lambda \in \sigma(M) \cap \mathbb{S}^1 \subset \mathbb{C}$, then M has the pair $\{\lambda, \bar{\lambda}\}$ of eigenvalues. However, M only has two eigenvalues. If $\lambda = \bar{\lambda}$, then $\lambda \in \mathbb{R}$. If $\lambda^{-1} = \bar{\lambda}$, then $\lambda \in \mathbb{S}^1$. Thus, the eigenvalues of M are in the unit circle $\mathbb{S}^1$ or $\mathbb{R}\backslash\{0\}$.*

We remark that higher-order symplectic matrices satisfy the same necessary condition as in the first property, and the eigenvalues of higher-order symplectic matrices also exist in pairs, but they are not necessarily lying in $\mathbb{S}^1$ or $\mathbb{R}\backslash\{0\}$ as in the second property above.

2.2. The Rotation Number of a Symplectic Path

Now, we introduce the first kind eigenvalue [8] of a 2×2 symplectic matrix M, and we have the following definition.

Definition 4. *$\lambda \in \sigma(M)$ is called the* **first kind eigenvalue** *of M if it satisfies one of the following conditions:*

$$(1)\ \lambda = \pm 1 \text{ or } |\lambda| < 1; \qquad (2)\ \lambda \in \mathbb{S}^1 \backslash \{\pm 1\} \text{ and } Im(\omega_0(\overline{\xi}, \xi)) > 0,$$

where ξ is the eigenvector corresponding to λ.

Here, we also call ± 1 the first kind eigenvalue of M, which is different from [8]. The number of the first kind eigenvalues of M is one. When the double eigenvalues of M are 1 (or -1), we only regard one 1 (or -1) as the first kind eigenvalue. Then, we can define a map $\rho : Sp(2, \mathbb{R}) \to \mathbb{S}^1$ as

$$\rho(M) = \frac{\lambda}{|\lambda|}. \tag{1}$$

For any symplectic path $\Phi : [0, 1] \to Sp(2, \mathbb{R})$, the map $\rho(\Phi)$ is continuous, and then there exists a continuous map $\alpha : [0, 1] \to \mathbb{R}$ such that

$$\rho(\Phi(t)) = e^{i\pi\alpha(t)}. \tag{2}$$

Definition 5. *Define the* **rotation number** *as*

$$\Delta(\Phi) = \alpha(1) - \alpha(0). \tag{3}$$

Let Φ and Ψ be two paths in $Sp(2, \mathbb{R})$; we call Φ and Ψ homotopic if there exists a continuous map $H(t, s)$ on $[0, 1] \times [0, 1]$ such that

$$H(t, 0) = \Phi(t),\ H(t, 1) = \Psi(t).$$

If a loop in $Sp(2, \mathbb{R})$ is homotopic to a point, then we say this loop is contractible.

Proposition 2. *Δ has the following properties:*

(1) *If Φ is a symplectic loop, then $\Delta(\Phi) \in \mathbb{Z}$. In particular, if Φ is contractible, then*

$$\Delta(\Phi) = 0. \tag{4}$$

(2) *If $0 < a < 1$, then*

$$\Delta(\Phi) = \Delta(\Phi([0, a])) + \Delta(\Phi([a, 1])). \tag{5}$$

(3) *If Φ, Ψ are two homotopic symplectic paths with fixed end points, then*

$$\Delta(\Phi) = \Delta(\Psi). \tag{6}$$

Proof.

(1) Since $\rho(\Phi(0)) = \rho(\Phi(1))$, then $e^{i\pi(\alpha(1) - \alpha(0))} = 1$ and we have

$$\Delta(\Phi) = \alpha(1) - \alpha(0) \in \mathbb{Z}.$$

If Φ is contractible, then $\rho(\Phi(t))$ is contractible on $\mathbb{S}^1$, and then $\Delta(\Phi) = 0$.

(2)

$$\begin{aligned}\Delta(\Phi) &= \alpha(1) - \alpha(0) \\ &= (\alpha(1) - \alpha(a)) + (\alpha(a) - \alpha(0)) \\ &= \Delta(\Phi([0,a])) + \Delta(\Phi([a,1])).\end{aligned}$$

(3) Since Φ and Ψ have the same end points, then

$$\Phi\#(-\Psi(t)) := \begin{cases} \Phi(2t), & 0 \le t < \frac{1}{2}, \\ \Psi(2-2t), & \frac{1}{2} \le t \le 1, \end{cases}$$

is a contractible loop. Here, we denote by $-\Psi(t) := \Psi(1-t)$ for $t \in [0,1]$, which implies that $-\Psi$ is the path with reverse direction of Ψ. It follows from (4) and (5) that

$$\Delta(\Phi) - \Delta(\Psi) = \Delta(\Phi\# - \Psi) = 0$$

and then $\Delta(\Phi) = \Delta(\Psi)$.

□

3. Conley–Zehnder–Long Index

We use the following notations, which were first introduced by [18]:

$$\begin{aligned} Sp_1^0(2,\mathbb{R}) &:= \{M \in Sp(2,\mathbb{R}) |\ \det(I-M) = 0\}, \\ Sp_1^*(2,\mathbb{R}) &:= \{M \in Sp(2,\mathbb{R}) |\ \det(I-M) \neq 0\}, \\ Sp_1^{\pm}(2,\mathbb{R}) &:= \{M \in Sp(2,\mathbb{R}) |\ \pm\det(I-M) < 0\}. \end{aligned}$$

For any matrix $M \in Sp(2,\mathbb{R})$, by the polar decomposition, M can be written in the form

$$M = \begin{pmatrix} r & z \\ z & (1+z^2)/r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $(r,\theta,z) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}$, $\mathbb{R}^+ = \{r \in \mathbb{R} | r > 0\}$, $\mathbb{S}^1 = \mathbb{R}\backslash(2\pi\mathbb{R} - \pi)$, and (r,θ,z) is uniquely determined by M. Define

$$Sp_{1,\pm}^0(2,\mathbb{R}) = \{(r,\theta,z) \in Sp_1^0(2,\mathbb{R}) | \pm\sin\theta > 0\}.$$

In this section, we recall the construction of Conley–Zehnder–Long index for a symplectic path as

$$\Phi : [0,1] \to Sp(2n,\mathbb{R}),\ \Phi(0) = I. \tag{7}$$

A path such as (7) is called a non-degenerate path if it satisfies $\det(I - \Phi(1)) \neq 0$ and called a degenerate path if it satisfies $\det(I - \Phi(1)) = 0$. By Theorem 3.1 of [8], the symplectic path Φ corresponds to a path $\rho(\Phi)$ on $\mathbb{S}^1$ and yields a number $\Delta(\Phi)$ (3) but not always an integer. If Φ is a non-degenerate path, the idea of constructing the Conley–Zehnder index is that one has to find a suitable extension γ for Φ such that $\Delta(\Phi) + \Delta(\gamma)$ is an integer.

In particular, when $n = 1$, the end point of the extension can be $W^+ = -I_2$ or $W^- = diag\{2, \frac{1}{2}\}$. Since $W^+ = -I_2$ and $W^- = diag\{2, \frac{1}{2}\}$ are in different connected components of $Sp_1^*(2,\mathbb{R})$, then we define the extension path as follows

$$\gamma : [0,1] \to Sp_1^*(2,\mathbb{R}),\ \ \gamma(0) = \Phi(1),\ \ \gamma(1) \in \{W^+, W^-\}.$$

Then, we have the following definition:

Definition 6 (Conley–Zehnder index)**.** *For any non-degenerate path Φ, the Conley–Zehnder index for Φ is defined by*

$$\mu_{CZ}(\Phi) = \Delta(\Phi) + \Delta(\gamma).$$

According to Lemma 3.2 of [8], every loop in $Sp_1^*(2n, \mathbb{R})$ is contractible. If we choose another extension γ', by Proposition 2, $\Delta(\gamma' \# - \gamma) = 0$ and hence $\Delta(\gamma) = \Delta(\gamma')$, and then $\mu_{CZ}(\Phi)$ is independent of the choice regarding γ. So, it is well defined.

For degenerate path (i.e., $\det(I - \Phi(1)) = 0$), we first introduce the **normal forms** of two-order symplectic matrices with eigenvalue 1, which are taken from [18].

$$N_1(1,b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ b = 0\ or\ \pm 1.$$

Next, we introduce the perturbations of normal forms, which are from [18]. By direct computation, we obtain

$$N_1(1,b) \in \begin{cases} Sp^0_{1,+}(2,\mathbb{R}), & for\ b < 0, \\ Sp^0_{1,-}(2,\mathbb{R}), & for\ b > 0. \end{cases}$$

Since $N_1(1,1) \in Sp^0_{1,-}(2,\mathbb{R})$, there exists $\theta_0 > 0$ such that

$$\sigma(N_1(1,1)R(\theta)) \subset \begin{cases} \mathbb{R}^+, & if\ 0 < \theta \leq \theta_0, \\ \mathbb{S}^1 \backslash \mathbb{R}, & if\ 0 < -\theta \leq \theta_0. \end{cases}$$

Since $N_1(1,-1) \in Sp^0_{1,+}(2,\mathbb{R})$, there exists $\theta_0 > 0$ such that

$$\sigma(N_1(1,-1)R(\theta)) \subset \begin{cases} \mathbb{S}^1 \backslash \mathbb{R}, & if\ 0 < \theta \leq \theta_0, \\ \mathbb{R}^+, & if\ 0 < -\theta \leq \theta_0. \end{cases}$$

Note that $N_1(1,0) = I_2$ and

$$\sigma(N_1(1,0)R(\theta)) \subset \mathbb{S}^1 \backslash \{1\},$$

if and only if $\theta \neq 0 \bmod 2\pi$.

According to Theorem 7.3 of [16], one has the following

Proposition 3. *For any $M \in Sp^0_1(2,\mathbb{R})$, there exists $P \in Sp(2,\mathbb{R})$ such that*

$$PMP^{-1} = N_1(1,b). \tag{8}$$

Then, for the degenerate path $\Phi(t)$ ($\Phi(1)$ has eigenvalue 1), there exists $P \in Sp(2,\mathbb{R})$ such that $P\Phi(1)P^{-1} = N_1(1,b)$. For any $s \in [-1,1]$, one can define the rotational perturbation path of Φ as

$$\Phi_s(t) = \Phi(s,t) := \Phi(t)P^{-1}R(sp(t)\theta)P, \tag{9}$$

where $\theta > 0$, $p(t) = 0$ for $0 \leq t \leq t_0 \leq 1$, $\dot{p}(t) \leq 0$ for $0 \leq t \leq 1$, $\dot{p}(1) = 0$, $p(1) = 1$ and $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. When t_0 is sufficiently close to 1, $\Phi(s,t)$ converges to $\Phi(t)$ as

$s \to 0$. Fix $s \neq 0$, and $\Phi(s, \cdot)$ is a non-degenerate path that has the Conley–Zehnder index. Then, Long's Maslov-type index can be provided in two ways as follows.

Definition 7 (Long index)**.** *For any degenerate path Φ and $s \in (0,1]$, the Long index is defined by*

$$\mu_L(\Phi) := \mu_{CZ}(\Phi_{-s}),$$

where Φ_{-s} is defined in (9).

The above definition can be found in Definition 5.4.2 in Long's book [18] for the special case $\omega = 1$. Another more intuitive and equivalent definition is

Definition 8 (Long index)**.** *For any degenerate path Φ, the Long index is defined by*

$$\mu_L(\Phi) := \inf\{\mu_{CZ}(\Psi)\},$$

where Ψ is any non-degenerate path and C^0-close to Φ.

The above definition can be found in Definition 6.1.13 in Long's book [18] for the special case $\omega = 1$. The equivalence between Definitions 7 and 8 is proved in Section 6.1 of [18].

4. Construction of the Index

In this section, we will introduce our index μ. This index is constructed by orthogonalization, global perturbation, and extension. Define

$$\mathcal{P}(2, \mathbb{R}) := \{\Phi : [0,1] \to Sp(2, \mathbb{R}) \text{ is continuous}\}.$$

4.1. Orthogonalization

For $\forall\ M \in Sp(2, \mathbb{R})$, the characteristic polynomial of M is $\lambda^2 - tr(M)\lambda + 1$. The eigenvalues of M are $\lambda_1 = \frac{tr(M)+\sqrt{(tr(M))^2-4}}{2}$ and $\lambda_2 = \frac{tr(M)-\sqrt{(tr(M))^2-4}}{2}$. Then, we can construct an orthogonal symplectic matrix from M as $O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $\cos\theta = \frac{tr(M)}{2}$.

We can choose a symplectic path $\Psi(t)$ such that it starts at M and ends at O and satisfies $\Delta(\Psi(t)) = 0$. By (1) and (2), at two end points of Ψ, we have

$$e^{i\pi\alpha(0)} = \rho(\Psi(0)) = \rho(\Psi(1)) = e^{i\pi\alpha(1)}.$$

Then, $\Delta(\Psi) = \alpha(1) - \alpha(0) = 2k$ $(k \in \mathbb{Z})$. If $k = 0$, then $\Psi(t)$ is the path we desired to achieve the normalization of eigenvalues. Otherwise, one can construct a loop $\Psi'(t)$ as

$$\begin{pmatrix} \cos(\theta - 2k\pi t) & -\sin(\theta - 2k\pi t) \\ \sin(\theta - 2k\pi t) & \cos(\theta - 2k\pi t) \end{pmatrix}.$$

Then, the catenation $\Psi\#\Psi'$ also starts at M and ends at O and satisfies $\Delta(\Psi\#\Psi') = 2k - 2k = 0$.

For $\Phi(0),\ \Phi(1) \in Sp(2, \mathbb{R})$, they are corresponding to two orthogonal symplectic matrices O_1 and O_2, where O_1 and O_2 have the same form as O. Then, we can choose the tails of Φ as $\beta_j : [0,1] \to Sp(2, \mathbb{R}), j = 1, 2$ such that

$$\beta_1(0) = O_1,\ \ \beta_1 = \Phi(0),\ \ \beta_2(0) = \Phi(1),\ \ \beta_2(1) = O_2,$$

satisfying $\Delta(\beta_j) = 0$. This means that we can also add two tails β_1 and β_2 to Φ with the general endpoints and do not change the rotation number.

Then, we define

$$\Phi^{\#}(t) := \begin{cases} \beta_1(3t), & 0 \le t < \frac{1}{3}, \\ \Phi(3t-1), & \frac{1}{3} \le t < \frac{2}{3}, \\ \beta_2(3t-2), & \frac{2}{3} \le t \le 1, \end{cases} \tag{10}$$

and we call (10) the **orthogonalization** of Φ at the two endpoints.

Lemma 1. *For $\Phi \in \mathcal{P}(2,\mathbb{R})$, the rotation number (3) is independent of the choice regarding orthogonalizations, i.e., $\Delta(\Phi^{\#}) = \Delta(\Phi)$.*

Proof. By (5) and $\Delta(\beta_1) = \Delta(\beta_2) = 0$, we have

$$\Delta(\Phi^{\#}) = \Delta(\beta_1) + \Delta(\Phi) + \Delta(\beta_2) = \Delta(\Phi).$$

This completes the proof. □

4.2. Global Perturbation

To deal with the symplectic paths whose endpoints are in the degenerate set $Sp_1^0(2,\mathbb{R})$, we need the following lemma:

Lemma 2. *Given a $\Phi \in \mathcal{P}(2,\mathbb{R})$, for its orthogonalization $\Phi^{\#}$, there exists a sufficiently small $\theta \ge 0$ such that the endpoints of $\Phi^{\#}$ under the perturbation are not in the degenerate set; i.e., both $e^{-\theta J}\Phi^{\#}(0)$ and $e^{-\theta J}\Phi^{\#}(1)$ have no eigenvalues equal to 1, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

Proof. For $\Phi^{\#}(0)$ with the form as (10), let $0 \le \theta < 2\pi$ and choose a sufficiently small θ_0 such that $0 < \theta_0 < \theta$, and then

$$\det(I_2 - e^{-\theta_0 J}\Phi^{\#}(0)) = [1 - \cos(\theta - \theta_0)]^2 + \sin^2(\theta - \theta_0) \neq 0.$$

Thus, $e^{-\theta J}\Phi^{\#}(0)$ has no eigenvalues equal to 1. For $e^{-\theta J}\Phi^{\#}(1)$, we have a similar result such that there exists θ_0' such that $e^{\theta_0' J}\Phi^{\#}(1)$ has no eigenvalues equal to 1, and we choose $\theta = \min\{\theta_0, \theta_0'\}$ and the lemma holds. In particular, if $\Phi^{\#}(0)$ and $\Phi^{\#}(1)$ have no eigenvalues equal to 1, we can choose $\theta = 0$. This completes the proof. □

In order to deal with the symplectic paths with endpoints in the degenerate set, we consider the global perturbation

$$\Phi^{\#}_{\theta}(t) := e^{-\theta J}\Phi^{\#}(t), \quad \theta \ge 0, \tag{11}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We call (11) the **global perturbation** of $\Phi^{\#}$ with the rotation angle of θ. By Lemma 2, there exists a sufficiently small $\theta \ge 0$ such that

$$\Phi^{\#}_{\theta}(0),\ \Phi^{\#}_{\theta}(1) \notin Sp_1^0(2,\mathbb{R}).$$

Lemma 3. *For $\Phi \in \mathcal{P}(2,\mathbb{R})$, the rotation number is invariant under the operations of orthogonalizations and sufficiently small global perturbations, i.e.,*

$$\Delta(\Phi^{\#}_{\theta}) = \Delta(\Phi).$$

Proof. From Lemma 1, we know $\Delta(\Phi^{\#}) = \Delta(\Phi)$. So, we only need to prove $\Delta(\Phi_\theta^{\#}) = \Delta(\Phi^{\#})$. Define

$$\eta_0(t) = e^{-\theta t J}\Phi^{\#}(0),\ \eta_1(t) = e^{-\theta t J}\Phi^{\#}(1).$$

We construct a homotopic map

$$H(t,s) = \eta_0([0,s])\#\,\Phi^{\#}_{\theta s}\#\,(-\eta_1([0,s]))(t),\ \ 0 \le t \le 1,\ 0 \le s \le 1,$$

which satisfies $H(t,0) = \Phi^{\#}(t)$ and $H(t,1) = \eta_0\#\,\Phi^{\#}_\theta(t)\#\,(-\eta_1)(t)$. Then, $\Phi^{\#}$ is homotopic to $\eta_0\#\,\Phi^{\#}_\theta(t)\#\,(-\eta_1)$, and, by (5) and (6), we obtain

$$\Delta(\Phi^{\#}) = \Delta(\Phi^{\#}_\theta) + \Delta(\eta_0) - \Delta(\eta_1).$$

Then, we only need to show $\Delta(\eta_0) = \Delta(\eta_1)$. For a second-order orthogonal symplectic matrix $\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}$, let

$$\begin{aligned} \eta(t) :&= e^{-\theta t J}\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta t)\cos\theta_0 + \sin(\theta t)\sin\theta_0 & \sin(\theta t)\cos\theta_0 - \cos(\theta t)\sin\theta_0 \\ \cos(\theta t)\sin\theta_0 - \sin(\theta t)\cos\theta_0 & \cos(\theta t)\cos\theta_0 + \sin(\theta t)\sin\theta_0 \end{pmatrix}. \end{aligned}$$

By calculation, we obtain $\rho(\eta(t)) = e^{-i\theta t}$. By (2) and (3), the rotation number $\Delta(\eta) = -\frac{\theta}{\pi}$. Since $\Phi^{\#}(0)$ and $\Phi^{\#}(1)$ are orthogonal symplectic matrices, then $\Delta(\eta_0) = \Delta(\eta_1) = -\frac{\theta}{\pi}$. Thus, $\Delta(\Phi^{\#}_\theta) = \Delta(\Phi^{\#})$. This completes the proof. □

4.3. Extension

In order to prove that it is independent of the choice regarding extension, we need the following lemma, which is a corollary of Lemma 1.7 of [4] and Lemma 3.2 of [8].

Lemma 4. *If Φ is a loop in $Sp_1^*(2,\mathbb{R})$, then the rotation number of Φ is equal to zero, i.e., $\Delta(\Phi) = 0$.*

Proof. For a loop $\Phi(t) \in Sp_1^*(2,\mathbb{R})$ with $\Phi(0) = \Phi(1)$. By (1) and (2), at two end points of Φ, we have

$$e^{i\pi\alpha(0)} = \rho(\Phi(0)) = \rho(\Phi(1)) = e^{i\pi\alpha(1)}.$$

Then, $\Delta(\Phi) = \alpha(1) - \alpha(0) = 2k$ ($k \in \mathbb{Z}$). Since the first kind eigenvalue $\Phi(t)$ cannot cross 1, then $|\alpha(1) - \alpha(0)| < 2\pi$. So, $\Delta(\Phi(t)) = \alpha(t) - \alpha(0) = 0$. □

Then, we consider the extension of $\Phi^{\#}_\theta$ and set

$$A := \Phi^{\#}_\theta(0), \quad B := \Phi^{\#}_\theta(1) \tag{12}$$

and the first kind eigenvalue of A and B belong to $\mathbb{S}^1\backslash\{1\}$. Since $\det(I-A)\det(I-B) > 0$, A and B are in the same connected components of $Sp_1^*(2,\mathbb{R})$. Thus, we can choose the **extension** path for $\Phi^{\#}_\theta$ as

$$\beta : [0,1] \to Sp_1^*(2,\mathbb{R}),\ \beta(0) = B,\ \beta(1) = A. \tag{13}$$

If we choose another extension β', then $\beta'\# - \beta$ is a loop in $Sp_1^*(2,\mathbb{R})$. It follows from Lemma 4 that $\Delta(\beta'\# - \beta) = 0$ and hence $\Delta(\beta') = \Delta(\beta)$, so $\Delta(\beta)$ is independent of the choice regarding β. Then, we define the Maslov-type index as

Definition 9. *For any $\Phi \in \mathcal{P}(2,\mathbb{R})$, the Maslov-type index is defined by*

$$\mu(\Phi) := \Delta(\Phi_\theta^\#) + \Delta(\beta). \tag{14}$$

We will show in Section 5 that $\mu(\Phi)$ is also independent of the choice regarding a sufficiently small θ so that it is well defined.

5. Proof of Main Results

In this section, we provide the proofs of the main results.

5.1. Proof of Theorem 1

Proof. By (14), $\mu(\Phi)$ is defined by $\Delta(\Phi_\theta^\#) + \Delta(\beta)$; let

$$\Phi^*(t) := \begin{cases} \Phi_\theta^\#(2t), & 0 \le t < \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1, \end{cases}$$

and then $\Phi^*(0) = \Phi^*(1)$ (see (12) and (13)). From (1) and (2), we have

$$\rho(\Phi^*(t)) = e^{i\pi\alpha(t)} = \frac{\lambda(t)}{|\lambda(t)|},$$

where $\lambda(t)$ is the first kind eigenvalue of $\Phi^*(t)$. Because $\lambda(0) = \lambda(1)$, then

$$\Delta(\Phi^*) = \alpha(1) - \alpha(0) = 2k(k \in \mathbb{Z}).$$

Thus, $\mu(\Phi)$ is an integer.

Recall $\Phi^\#(t)$ is the path after orthogonalization of $\Phi(t)$, and $\Phi_\theta^\#(t)$ is the modified path after conducting global perturbation with rotation angle θ to $\Phi^\#(t)$. Let A, B be the endpoints of $\Phi_\theta^\#(t)$. Denote $\Phi_{\theta'}^\#(t)$ as the path $\Phi^\#(t)$ after global perturbation with another rotation angle θ'. Let A', B' be the endpoints of $\Phi_{\theta'}^\#(t)$. Denote AA' as the path from A to A' in $Sp_1^*(2,\mathbb{R})$. Because we make a slight perturbation, then AA', BB' are in the same connected components of $Sp_1^*(2,\mathbb{R})$ and

$$|\theta| < \delta,\ |\theta'| < \delta,\ \Delta(AA') < \frac{\epsilon}{4},\ \Delta(BB') < \frac{\epsilon}{4}.$$

By the continuity of ρ,

$$|\Delta(\Phi_\theta^\#) - \Delta(\Phi_{\theta'}^\#)| < \frac{\epsilon}{2}.$$

Let β be the extension of AB and β' be the extension of $A'B'$. Because β, β', AA', BB' are in the same connected component of $Sp_1^*(2,\mathbb{R})$, by Lemma 4, we have

$$\Delta(\beta') + \Delta(A'A) + \Delta(-\beta) + \Delta(BB') = 0.$$

Then,

$$|\Delta(\beta') - \Delta(\beta)| = |\Delta(AA') + \Delta(B'B)| \le |\Delta(AA')| + |\Delta(BB')| < \frac{\epsilon}{2},$$

and

$$\begin{aligned}|\Delta(\Phi^\#_\theta)+\Delta(\beta)-\Delta(\Phi^\#_{\theta'})-\Delta(\beta')| &= |\Delta(\Phi^\#_\theta)-\Delta(\Phi^\#_{\theta'})+\Delta(\beta)-\Delta(\beta')| \\ &\le |\Delta(\Phi^\#_\theta)-\Delta(\Phi^\#_{\theta'})|+|\Delta(\beta)-\Delta(\beta')| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Since $\Delta(\Phi^\#_\theta)+\Delta(\beta)$ and $\Delta(\Phi^\#_{\theta'})+\Delta(\beta')$ are integers, we see

$$\Delta(\Phi^\#_\theta)+\Delta(\beta)=\Delta(\Phi^\#_{\theta'})+\Delta(\beta').$$

So, ,$\mu(\Phi)$ does not depend on the choice in rotation angle, which means it is well defined. Then, we prove the three properties that our index satisfies.

(1) First, we prove **Homotopy invariant:** Since Φ and Ψ are homotopic with the same endpoints, by (6), $\Delta(\Phi)=\Delta(\Psi)$. From Lemma 3, $\Delta(\Phi^\#_\theta)=\Delta(\Phi)$ and $\Delta(\Psi^\#_\theta)=\Delta(\Psi)$, and then $\Delta(\Phi^\#_\theta)=\Delta(\Psi^\#_\theta)$. Since our index is well defined, it is independent of the choice regarding orthogonalization and perturbation of a sufficiently small angle. Φ and Ψ have the same endpoints, so we can choose the same orthogonalization and perturbation, and then $\Phi^\#_\theta(t)$ and $\Psi^\#_\theta(t)$ have the same endpoints. According to our extension rule (13), β_Φ and β_Ψ have the same endpoints. Because β_Φ and β_Ψ are in the same connected components of $Sp_1^*(2,\mathbb{R})$, by Lemma 4, $\Delta\big(\beta_\Phi\#(-\beta_\Psi)\big)=0$, and then $\Delta(\beta_\Phi)=\Delta(\beta_\Psi)$. Thus, $\mu(\Phi)=\mu(\Psi)$.

(2) Next, we prove **Vanishing:** Since $\nu_t(\Phi)=\dim Ker(I-\Phi(t))$ is constant.

   (i) If $\nu_t(\Phi)=0$, i.e., the first kind eigenvalue of $\Phi(t)$ is not equal to 1. If the first kind eigenvalue path of $\Phi(t)$ is in $\mathbb{S}^1\backslash\{1\}\cup(-1,0)$, we can choose the first kind eigenvalue path of orthogonalization to not cross 1, and then $\Phi^\#_\theta\#\beta$ is a loop in $Sp_1^*(2,\mathbb{R})$. By Lemma 4, $\Delta(\Phi^\#_\theta\#\beta)=0$. From (5), $\Delta(\Phi^\#_\theta)+\Delta(\beta)=0$. If the path lies in the real interval $(0,1)$, which does not contribute to rotation number, then $\Delta(\Phi)=0$. According to Lemma 3, $\Delta(\Phi^\#_\theta)=\Delta(\Phi)=0$. After orthogonalization, $\lambda|_{\Phi^\#(0)}=\lambda|_{\Phi^\#(1)}=1$, and, after perturbation, $\lambda|_{\Phi^\#_\theta(0)}=\lambda|_{\Phi^\#_\theta(1)}=e^{-i\theta}$, and then the endpoints of the path after orthogonalization and perturbation are the same. By (13), we can choose a loop as the extension in $Sp_1^*(2,\mathbb{R})$. According to Lemma 4, $\Delta(\beta)=0$.

   (ii) If $\nu_t(\Phi)\neq 0$, i.e., the first kind eigenvalue of $\Phi(t)$ is 1. By (1), $\rho(\Phi(t))\equiv 1$. From (2) and (3), $\Delta(\Phi)=0$. According to Lemma 3, $\Delta(\Phi^\#_\theta)=\Delta(\Phi)=0$. Since the endpoints of the path after orthogonalization and perturbation are the same, in a similar way with (i), $\Delta(\beta)=0$. Thus, $\mu(\Phi)=0$.

(3) Next, we prove **Catenation:** Choosing a proper small enough θ, $\Phi^\#_\theta(a)$ is non-degenerate. Using (14), note that the definition of the index from 0 to a is same as from 0 to 1, and we have

$$\mu(\Phi([0,a]))+\mu(\Phi([a,1]))=\Delta(\Phi^\#_\theta|_{[0,a]})+\Delta(\beta_1)+\Delta(\Phi^\#_\theta|_{[a,1]})+\Delta(\beta_2).$$

By (5),

$$\Delta(\Phi^\#_\theta)=\Delta(\Phi^\#_\theta|_{[0,a]})+\Delta(\Phi^\#_\theta|_{[a,1]}).$$

Since our index is independent of the choice regarding extension, we can choose $\beta=\beta_2\#\beta_1$, by (5),

$$\Delta(\beta)=\Delta(\beta_1)+\Delta(\beta_2).$$

Then,

$$\mu(\Phi)=\mu(\Phi([0,a]))+\mu(\Phi([a,1]))(0<a<1).$$

□

5.2. Proof of Theorem 2

Proof. (i) We first consider a non-degenerate path $\Phi \in \mathcal{P}(2,\mathbb{R})$, i.e., $\Phi(0) = I$ and $\det(I - \Phi(1)) \neq 0$. By (14), $\mu(\Phi) = \Delta(\Phi^{\#}_{\theta}) + \Delta(\beta)$. By Lemma 3, $\Delta(\Phi^{\#}_{\theta}) = \Delta(\Phi)$, and then the difference between $\mu(\Phi)$ and $\mu_{CZ}(\Phi) = \Delta(\Phi) + \Delta(\gamma)$ depends on the different extensions starting from $\Phi^{\#}_{\theta}(1)$ or $\Phi(1)$; we take for two constructions. We discuss the contribution of the first kind eigenvalue of $\Phi(1)$ to $\mu(\Phi)$ and $\mu_{CZ}(\Phi)$; all cases are as follows:

(1) Suppose the first kind eigenvalue λ of $\Phi(1)$ is in $(0,1)$. For $\mu_{CZ}(\Phi)$, the first kind eigenvalue path of extension is from $\lambda|_{\Phi(1)}$ to $\frac{1}{2}$, while $\lambda|_{\gamma(t)} \neq 1$, and then $\Delta(\gamma) = 0$. For our construction, after orthogonalization, $\lambda|_{\Phi^{\#}(0)} = \lambda|_{\Phi^{\#}(1)} = 1$, and, after perturbation, $\lambda|_{\Phi^{\#}_{\theta}(0)} = \lambda|_{\Phi^{\#}_{\theta}(1)} = e^{-i\theta}$, and then the endpoints of the path after orthogonalization and perturbation are the same. By (13), we can choose a loop as the extension in $Sp^*_1(2,\mathbb{R})$. According to Lemma 4, $\Delta(\beta) = 0$. Thus, $\mu(\Phi) = \mu_{CZ}(\Phi)$.

(2) Suppose the first kind eigenvalue λ of $\Phi(1)$ is in $(-1,0)$ or $\mathbb{S}^1 \backslash \{1\}$. One can ignore possible part of this path lying in the real interval $(-1,0)$ since such a part does not contribute to rotation number. Then, we only need to consider the first kind eigenvalue λ of $\Phi(1)$ is in S^1. For $\mu_{CZ}(\Phi)$, the first kind eigenvalue path of extension is from $\lambda|_{\Phi(1)}$ to -1, while $\lambda|_{\gamma(t)} \neq 1$. For our construction $\Delta(\Phi^{\#}_{\theta}) + \Delta(\beta)$, after orthogonalization and perturbation, $\lambda|_{\Phi^{\#}_{\theta}(0)} = e^{-i\theta}, \lambda|_{\Phi^{\#}_{\theta}(1)} = e^{-i\theta}\lambda|_{\Phi^{\#}(1)}$, the first kind eigenvalue path of extension is from $\lambda|_{\Phi^{\#}_{\theta}(1)}$ to $\lambda|_{\Phi^{\#}_{\theta}(0)}$. Since $\Delta(\beta)$ is independent of the choice regarding β, we can choose the extension from $e^{-i\theta}\lambda|_{\Phi^{\#}(1)}$ to $-e^{-i\theta}$ first, then from $-e^{-i\theta}$ to $e^{-i\theta}$. The rotation number of the path from $e^{-i\theta}\lambda|_{\Phi^{\#}(1)}$ to $-e^{-i\theta}$ is equal to $\Delta(\gamma)$, and the rotation number of the path from $-e^{-i\theta}$ to $e^{-i\theta}$ is equal to 1. Thus, $\mu(\Phi) = \mu_{CZ}(\Phi) + 1$.

(ii) Now, we consider a degenerate path $\Phi(t) \in \mathcal{P}(2,\mathbb{R})$, i.e., $\Phi(0) = I, \det(I - \Phi(1)) = 0$. For $\mu_L(\Phi)$, the degenerate path is turned into a non-degenerate path $\Phi(s,t)$ after rotation perturbation (9). By Definition 7, $\mu_L(\Phi) = \mu_{CZ}(\Phi(-s,t))$. Next, we calculate the first kind eigenvalues of $\Phi(-s,1) = \Phi(1)P^{-1}R(-s\theta)P$. By Proposition 3, we have $P\Phi(-s,1)P^{-1} = P\Phi(1)P^{-1}R(-s\theta) = N_1(1,b)R(-s\theta)$. Let $\theta' = -s\theta$, and then $\theta' < 0$ and sufficiently small. So, we only need to calculate the first kind eigenvalues of $N_1(1,b)R(\theta')$.

When $b = -1$, $N_1(1,-1)R(\theta') = \begin{pmatrix} \cos\theta' - \sin\theta' & -\sin\theta' - \cos\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$. The first kind eigenvalue of $N_1(1,-1)R(\theta')$ is

$$\lambda_{-1} = \frac{(2\cos\theta' - \sin\theta') - \sqrt{(2\cos\theta' - \sin\theta')^2 - 4}}{2}, \quad 0 < \lambda_{-1} < 1.$$

According to case (1) of non-degenerate path,

$$\mu(\Phi) = \mu_{CZ}(\Phi(-s,t)).$$

When $b = 0$, $N_1(1,0)R(\theta') = \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$. The first kind eigenvalue of $N_1(1,0)R(\theta')$ is $\lambda_0 = e^{i\theta'}$, $\lambda_0 \in \mathbb{S}^1 \backslash \{1\}$. According to the above case (2) of non-degenerate path,

$$\mu(\Phi) = \mu_{CZ}(\Phi(-s,t)) + 1.$$

When $b = 1$,

$$N_1(1,1)R(\theta') = \begin{pmatrix} \cos\theta' + \sin\theta' & \cos\theta' - \sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix},$$

the first kind eigenvalue of $N_1(1,1)R(\theta')$ is $\lambda_1 = \frac{(2\cos\theta'+\sin\theta')-\sqrt{(2\cos\theta'+\sin\theta')^2-4}}{2}$, $\lambda_1 \in \mathbb{S}^1\backslash\{1\}$. According to case (2) of non-degenerate path,

$$\mu(\Phi) = \mu_{CZ}(\Phi(-s,t)) + 1.$$

In conclusion,

$$\mu(\Phi) = \mu_{CZL}(\Phi) + \chi(\Phi(1)),$$

if one of the eigenvalues of $\Phi(1)$ is in $(0,1)$ or $\Phi(1)$ with eigenvalue 1 becomes $N_1(1,-1)$ by symplectic matrix similar transformation, then $\chi(\Phi(1)) = 0$. In other cases, $\chi(\Phi(1)) = 1$. $\square$

6. Conclusions

In the discussion above, we constructed a new Maslov-type index for general paths of 2×2 symplectic matrices. We considered the same singular set as usual, while our method, which includes the main three steps of orthogonalization, global perturbation, and appropriate extension, is consistent and much different from those used by Conley–Zehnder and Long. We compared the index we defined with the Conley–Zehnder–Long index for the case of dimension 2. We suggest that the higher-dimensional generalization of our construction should be studied in the future. This new Maslov-type index is also suggested to be applied to the related problems regarding estimates of periodic solutions of Hamiltonian systems, closed characteristics, and closed geodesics, etc.

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