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Pseudo-Normality and Pseudo-Tychonoffness of Topological Groups

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Abstract: It is common knowledge that any topological group that satisfies the lowest separation axiom, T_0 , is immediately Hausdorff and completely regular; however, this is not the case for normality. This motivates us to introduce the concept of pseudo-normal groups along with pseudo-Tychonoff topological groups as generalizations of the normality and Tychonoffness of topological groups, respectively. We show that every pseudo-normal (resp. pseudo-Tychonoff) topological group is normal (resp. Tychonoff). Generally, the reverse implication of the latter does not hold. Then, we discuss their main properties in detail. To clarify these properties, we provide some examples. Finally, we establish some other results.

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1. Introduction

The theory of topological groups is rich in terms of applications as well as its significant conclusions. The major purpose of our research on topological groups is to show how an algebraic structure introduced into a space can change or improve its topological features.

A topological group is a group equipped with a topology such that multiplication and inversion in the group are continuous. By L.S. Pontryagin's result in [1], every topological group satisfying the T_1 -separation axiom is a completely regular space. In approximately 1935, A.N. Kolmogorov asked whether every Hausdorff (equivalently, T_1) topological group is a normal topological space. In 1941, A.A. Markov presented a construction of free topological groups and used it to demonstrate the existence of a number of Hausdorff topological groups that are not normal spaces. In fact, Markov's results imply that every Tychonoff space is homeomorphic to a closed subspace of a Hausdorff topological group.

A further study showed that normality is very fragile, even when restricted to the class of Hausdorff topological groups. For example, it was shown in [2] that the product of two normal topological groups can fail to be normal, even if one of the factors is the group $\mathbb{R}$ of real numbers with its usual Euclidean topology or the compact torus group $\mathbb{T}$. The properties of topological groups have been widely used in the study of topology, analysis, and category theory (see [3–7]). In this work, we define a new type of normality independent from the other kinds of normality in a topological group, which we call

pseudo-normal. We also present a new type of Tychonoffness in a topological group, which we call pseudo-Tychonoffness.

The structure of this work is as follows. In Section 2, we recall the basic concepts and findings that make this work self-contained. In Section 3, we define the concept of pseudo-normal topological groups as a generalization of the normality of topological groups. Then, we discuss their main properties and provide some examples. In Section 4, we define the pseudo-Tychonoffness of topological groups as a generalization of Tychonoffness and discuss their main properties. To clarify these properties, we provide some examples. In Section 5, we present some other results on pseudo-normality and pseudo-Tychonoffness with respect to ω -narrow, $\mathbb{R}$ -factorizable, and $\mathcal{N}$ -factorizable topological groups.

2. Preliminaries

We recall some basic notions and facts that are used in this article. A T_4 space is a normal T_1 space, while a Tychonoff space is a completely regular T_1 space. For a topological group G , we distinguish between the normality of the group and the topology in this work. When we mean that the set A is normal with respect to the group G , we denote it as $A \trianglelefteq G$; otherwise, we mean that A is normal with respect to the topology on G . $\mathbb{N}$, $\mathbb{R}$, ω , and Λ denote the set of non-negative integers, the additive group of real numbers, the first infinite ordinal, and the indexing set, respectively. A topological group S is said to be totally bounded if for each neighborhood V of the identity, a finite number of translates of V covers S (see [8]). A topological group S that is pseudo-compact as a topological space is totally bounded in the sense that for each nonempty open $V \subseteq S$, there is a finite $F \subseteq S$ such that $S = FV$ (see [9]). A continuous homomorphism of a topological group is a group homomorphism that is also continuous. A topological isomorphism of a topological group is a group isomorphism that is also a homeomorphism. The infinite product of a topological group with the direct product group and the product topology is a topological group.

A Hausdorff topological group S is called $\mathbb{R}$ -factorizable [10] if, for every continuous, real-valued function α on S , there exists a continuous homomorphism $\beta : S \rightarrow T$ onto a second-countable T such that $\alpha = \gamma \circ \beta$ for some continuous, real-valued function $\gamma : T \rightarrow \mathbb{R}$. Replacing the second-countability of T in the definition of $\mathbb{R}$ -factorizability with metrizability or normality, one obtains, respectively, the classes of $\mathcal{M}$ -factorizable [11] and $\mathbb{N}$ -factorizable [12] groups. A space S is Fréchet if, for any subset B of S and any $s \in \overline{B}$, there exists a sequence $(d_n)_{n \in \mathbb{N}}$ of points of B such that $d_n \rightarrow s$ (see [13]).

3. Pseudo-Normal Topological Groups

Definition 1. A topological group S is called pseudo-normal if there exists a normal topological group T and a one-to-one function ϕ from S onto T such that for each pseudo-compact, non-trivial subgroup $V \subseteq S$, the restriction function $\phi|_V$ from V onto $\phi(V)$ is a topological isomorphism.

Obviously, any normal topological group is pseudo-normal simply by taking $G = T$ and f to be the identity function. However, a pseudo-normal topological group does not need to be normal, as shown in the example below.

Example 1. According to a theorem by A.H. Stone, which states the product of uncountably many non-compact metric spaces is never normal, the space $\mathbb{N}^{\omega_1}$ is not normal, where $\mathbb{N}$ is the discrete set of non-negative integers. Since $\mathbb{N}$ and $\mathbb{Z}$ with the discrete topology are homeomorphic, $\mathbb{Z}^{\omega_1}$ cannot be normal. The set of real numbers, $\mathbb{R}$, with the usual topology, forms a normal topological group under addition. Consider the identity function $i : \mathbb{Z}^{\omega_1} \rightarrow \mathbb{R}$, which is a one-to-one and onto function. If K is any infinite pseudo-compact, non-trivial subgroup of $\mathbb{Z}^{\omega_1}$, according to Remark 3.8 in [14], K is a (pseudo-compact infinite) nondiscrete subgroup. Thus, $\mathbb{Z}^{\omega_1}$ is a pseudo-normal group but not normal.

However, we have the following possible converse:

Theorem 1. *Each pseudo-normal, pseudo-compact topological group is normal.*

Proof. Suppose that S is a pseudo-normal topological group. Then, there exists a normal topological group T and a bijective function $\phi : S \rightarrow T$ such that for each pseudo-compact subgroup V of S , $\phi|_V : V \rightarrow \phi(V)$ is a topological isomorphism. Since S is pseudo-compact and a group, then S is topologically isomorphic to T . Hence, S is normal. $\square$

Corollary 1. *Each pseudo-normal, pseudo-compact Hausdorff topological group is T_4 .*

Proposition 1. *No pseudo-normal topological group is pseudo-compact and not normal.*

Proof. Suppose that S is a pseudo-normal topological group. Then, by the pseudo-normality of S , there exists a normal topological group T and a bijective function $\phi : S \rightarrow T$ such that for each pseudo-compact, non-trivial subgroup V of S , $\phi|_V : V \rightarrow \phi(V)$ is a topological isomorphism. Since S is a pseudo-compact, non-trivial group, S is topologically isomorphic to T . Hence, S is normal, which contradicts the fact that S is not normal. Therefore, S is not pseudo-normal. $\square$

The following result shows that the class of pseudo-normal topological groups is very wide.

Proposition 2. *Every Hausdorff topological group is a quotient group of a pseudo-normal group.*

Proof. By ([5], Theorem 7.6.18), every Hausdorff topological group H is a quotient group of a Hausdorff paracompact topological group G . Since regular paracompact spaces are normal, the group G is pseudo-normal. $\square$

Propositions 1 and 2 may suggest that every Hausdorff topological group is a pseudo-normal topological group. We refute this hypothesis in Example 2 below. Therefore, pseudo-normal topological groups constitute a proper subclass of Hausdorff topological groups.

Example 2. By ([12], Proposition 3.2), let K be any compact metrizable topological group with $|K| > 1$. Denote by H the group ΣK^{ω_1} , the Σ -product of ω_1 copies of the group K considered a subgroup of the product K^{ω_1} (see also [5], p. 55). Then, the topological group $S = H \times K^{\omega_1}$ is countably compact. Hence, $S = H \times K^{\omega_1}$ is pseudo-compact and fails to be a normal space. Therefore, by Proposition 1, S is not pseudo-normal.

Corollary 2. *The product of two pseudo-normal topological groups may fail to be pseudo-normal.*

Proof. In Example 2, $S = H \times K^{\omega_1}$ is not a pseudo-normal topological group. However, according to a theorem by H. Corson in [15], the space H is normal; hence, H is a pseudo-normal topological group. Also, K^{ω_1} is compact; hence, K^{ω_1} is normal since any compact topological group is normal. So, K^{ω_1} is pseudo-normal. $\square$

Theorem 2. *The product of two pseudo-normal topological groups S_1 and S_2 is pseudo-normal if the product of the normal topological groups that witness the pseudo-normality of S_1 and S_2 is normal.*

Proof. Let S_1 and S_2 be pseudo-normal topological groups. Let R_1 and $\phi_1 : S_1 \rightarrow R_1$ be witnesses of the pseudo-normality of S_1 , and let R_2 and $\phi_2 : S_2 \rightarrow R_2$ be witnesses of the pseudo-normality of S_2 . Consider the function $\phi : S_1 \times S_2 \rightarrow R_1 \times R_2$, defined by $\phi(s_1, s_2) = (\phi_1(s_1), \phi_2(s_2))$ for each $(s_1, s_2) \in S_1 \times S_2$. It is obvious that ϕ is a bijective function. From the hypothesis, $R_1 \times R_2$ is a normal topological group. Let K be any pseudo-compact, non-trivial subgroup of $S_1 \times S_2$. Since the continuous image of a pseudo-compact space is pseudo-compact, the projection map $\pi_1(K)$ is a pseudo-compact subset of S_1 , and $\pi_2(K)$ is a pseudo-compact subset of S_2 . Thus, $\phi_1 \upharpoonright_{\pi_1(K)} : \pi_1(K) \rightarrow \phi_1(\pi_1(K))$ is a topological isomorphism and $\phi_2 \upharpoonright_{\pi_2(K)} : \pi_2(K) \rightarrow \phi_2(\pi_2(K))$ is a topological isomorphism. Since the product of two homeomorphisms is a homeomorphism [13], and the product of two isomorphisms is an isomorphism, the product of two topological isomorphisms is a topological isomorphism. Since

$$\phi \upharpoonright_{\pi_1(K) \times \pi_2(K)} = (\phi_1 \upharpoonright_{\pi_1(K)}) \times (\phi_2 \upharpoonright_{\pi_2(K)}),$$

and

$$(\phi_1(\pi_1(K))) \times (\phi_2(\pi_2(K))) = \phi(\pi_1(K) \times \pi_2(K)),$$

then,

$$\phi \upharpoonright_{\pi_1(K) \times \pi_2(K)} : \pi_1(K) \times \pi_2(K) \rightarrow \phi(\pi_1(K) \times \pi_2(K)).$$

Since $K \subseteq \pi_1(K) \times \pi_2(K)$ and the restriction of a topological isomorphism is a topological isomorphism, we conclude that $\phi \upharpoonright_K : K \rightarrow \phi(K)$ is a topological isomorphism. Hence, $S_1 \times S_2$ is pseudo-normal. $\square$

Theorem 3. *If S is a pseudo-compact, pseudo-normal topological group, any function witnessing its pseudo-normality is continuous.*

Proof. Suppose that S is a pseudo-compact, pseudo-normal topological group. By pseudo-normality, there exists a normal topological group T and a one-to-one correspondence $\phi : S \rightarrow T$ such that the restriction $\phi \upharpoonright_B : B \rightarrow \phi(B)$ is a topological isomorphism for each pseudo-compact, non-trivial subgroup $B \subseteq S$. Since S is a pseudo-compact group, $\phi = \phi \upharpoonright_S$ is continuous. $\square$

Theorem 4. *If S is a pseudo-normal Fréchet topological group and $\phi : S \rightarrow T$ witnesses the pseudo-normality of S , ϕ is continuous.*

Proof. Suppose that S is a pseudo-normal Fréchet topological group and $\phi : S \rightarrow T$ witnesses the pseudo-normality of S . Take $B \subseteq S$ and pick $t \in \phi(\overline{B})$. This implies that there exists a unique $s \in S$ such that $\phi(s) = t$, and thus $s \in \overline{B}$. Since S is Fréchet, there exists a sequence $(d_n) \subseteq B$ such that $d_n \rightarrow s$. The sequence $K = \{s\} \cup \{d_n : n \in \mathbb{N}\}$ of S is compact by being pseudo-compact, so $\phi \upharpoonright_K : K \rightarrow \phi(K)$ is a topological isomorphism. Let $W \subseteq T$ be any open neighborhood of t . Then, $W \cap \phi(K)$ is open in the subspace $\phi(K)$ containing t . Since $\phi(\{d_n : n \in \mathbb{N}\}) \subseteq \phi(K) \cap \phi(B)$ and $W \cap \phi(K) \neq \emptyset$, we have $W \cap \phi(B) \neq \emptyset$. Hence, $t \in \phi(B)$, and thus $\phi(\overline{B}) \subseteq \phi(B)$. Therefore, ϕ is continuous. $\square$

Since any first-countable space is Fréchet, we conclude the following corollary:

Corollary 3. *If S is a pseudo-normal first-countable topological group and $\phi : S \rightarrow T$ witnesses the pseudo-normality of S , then ϕ is continuous.*

Theorem 5. *If S is a T_0 topological group in which each pseudo-compact subgroup of S is finite, then S is pseudo-normal.*

Proof. Let S be a T_0 topological group such that each pseudo-compact subgroup of S is finite. Take T to be the discrete topological group and $i : S \rightarrow T$ to be the identity function. Let K be any pseudo-compact, non-trivial subgroup of S . Since S is a T_0 topological group, it follows that S is T_1 , and by the hypothesis, K is finite. Thus, K is discrete since K is T_1 and finite. Therefore, $i|_K : K \rightarrow i(K)$ is a topological isomorphism, and so S is pseudo-normal. $\square$

Theorem 6. *If S is a T_0 topological group and the identity element $\{e\}$ has a countable basis of neighborhoods, then S is pseudo-normal.*

Proof. Suppose that $\{e\}$ has a countable basis of neighborhoods. By using translations, each element possesses a countable basis of neighborhoods, which implies that S is first-countable. Since each T_0 topological group is Hausdorff, then by Theorem 1 in [16], S is metrizable. Hence, S is pseudo-normal since any metrizable topological group is normal. $\square$

4. Pseudo-Tychonoff Topological Groups

Definition 2. *A topological group S is called pseudo-Tychonoff if there exists a Tychonoff topological group T and a one-to-one function ϕ from S onto T such that for each pseudo-compact subgroup $V \subseteq S$, the restriction of ϕ to V is a topological isomorphism.*

Clearly, any T_0 topological group is T_3 . Also, any topological group is completely regular and hence $T_{3.5}$ (Tychonoff). The following example shows that any pseudo-Tychonoff topological group does not necessarily have to be a Tychonoff topological group.

Example 3. *Consider $L^2(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{C} : f \text{ is measurable and } \int_{\mathbb{R}} \|f(t)\|^2 dt < \infty\}$ with the topology induced by the semi-norm $\|f\|_2$. If f and g differ on a set of measure zero, they cannot be separated, as f and g are in the same open sets. Thus, $L^2(\mathbb{R})$ is not a T_0 space. Hence, $L^2(\mathbb{R})$ is not a Tychonoff topological group. On the other hand, the only pseudo-compact subgroup of $L^2(\mathbb{R})$ is the identity, which implies that $L^2(\mathbb{R})$ is a pseudo-Tychonoff topological group.*

In general, a pseudo-normal topological group does not necessarily have to be a pseudo-Tychonoff topological group, and the converse does not necessarily hold, as shown in the examples below.

Example 4. *In Example 2, $S = H \times K^{\omega_1}$ is not a pseudo-normal topological group. However, S is a Tychonoff topological group since S is Hausdorff. Therefore, S is pseudo-Tychonoff.*

Example 5. *Suppose that $(\mathbb{R}, \mathcal{I})$ is the indiscrete topological space on $\mathbb{R}$ (see [17]). Then, $(\mathbb{R}, +, \mathcal{I})$ is an indiscrete topological group. Since $(\mathbb{R}, \mathcal{I})$ is a normal topological group, $(\mathbb{R}, +, \mathcal{I})$ is a pseudo-normal topological group. Conversely, assume that $(\mathbb{R}, +, \mathcal{I})$ is a pseudo-Tychonoff topological group. Since $(\mathbb{R}, +, \mathcal{I})$ is pseudo-compact, and by the pseudo-Tychonoffness definition, $(\mathbb{R}, +, \mathcal{I})$ is a Tychonoff topological group, which leads to a contradiction. Hence, $(\mathbb{R}, +, \mathcal{I})$ cannot be pseudo-Tychonoff.*

Theorem 7. *Each pseudo-Tychonoff, pseudo-compact topological group is Tychonoff.*

Proof. By applying the same argument used in Theorem 1, we can obtain the result. $\square$

Theorem 8. *If S is a pseudo-Tychonoff–Fréchet topological group, then any function that witnesses its pseudo-Tychonoffness is continuous.*

Proof. If S is a Tychonoff topological group, then by the definition of pseudo-Tychonoffness, the function that witnesses its pseudo-Tychonoffness is a topological isomorphism, and this implies continuity. Suppose that S is not a Hausdorff topological group. In this case, we can apply the same method used in Theorem 4 to obtain the desired result. $\square$

Theorem 9. *The pseudo-Tychonoff topological group property is hereditary with respect to subgroups.*

Proof. Let V be a nonempty subgroup of a pseudo-Tychonoff topological group S . Then, V is a topological group with respect to the subspace topology. Pick a bijective function ϕ from S onto a Tychonoff topological group T such that $\phi|_U : U \rightarrow \phi(U)$ is a topological isomorphism for each pseudo-compact subgroup $U \subseteq S$. Let $B = \phi(V) \subseteq T$, and then B is a Tychonoff topological group, being a subspace of T . Now, we have that $\phi|_V : V \rightarrow \phi(V) = B$ is a bijective function. Since each pseudo-compact subgroup in V is a pseudo-compact subgroup in S and $(\phi|_V)|_U = \phi|_U$, we conclude that V is a pseudo-Tychonoff topological group. $\square$

It is clear that the product of T_0 topological groups is Tychonoff and thus a pseudo-Tychonoff topological group.

Theorem 10. *The pseudo-Tychonoff topological groups that are not T_0 are productive.*

Proof. Let S_γ be a pseudo-Tychonoff topological group for each $\gamma \in \Lambda$. Pick a Tychonoff group T_γ and a bijective function $g_\gamma : S_\gamma \rightarrow T_\gamma$ such that $g_\gamma|_{H_\gamma} : H_\gamma \rightarrow g_\gamma(H_\gamma)$ is a topological isomorphism for each pseudo-compact subgroup H_γ of S_γ . Since T_γ is a Tychonoff topological group for each $\gamma \in \Lambda$, the Cartesian product $\prod_{\gamma \in \Lambda} T_\gamma$ is a Tychonoff topological group. Define $g : \prod_{\gamma \in \Lambda} S_\gamma \rightarrow \prod_{\gamma \in \Lambda} T_\gamma$ by $g((s_\gamma : \gamma \in \Lambda)) = (g_\gamma(s_\gamma) : \gamma \in \Lambda)$ for each $\gamma \in \Lambda$, and then g is bijective. Let $H \subseteq \prod_{\gamma \in \Lambda} S_\gamma$ be each pseudo-compact subgroup, and let p_γ be the usual projection. Then, $p_\gamma(H) \subseteq S_\gamma$ is a pseudo-compact subgroup because any usual projection map is continuous, open, and onto, and the continuous image of a pseudo-compact group is pseudo-compact. Also, the usual projection map is a topological group homomorphism. Now, we have that $H \subseteq \prod_{\gamma \in \Lambda} p_\gamma(H) = H'$ is a pseudo-compact subgroup. Hence, $g|_{H'} = \prod_{\gamma \in \Lambda} g_\gamma|_{p_\gamma(H)}$ is a topological isomorphism. Thus, $g|_H$ is a topological isomorphism because the restriction of a topological isomorphism is a topological isomorphism. $\square$

Theorem 11. *If $\prod_{\gamma \in \Lambda} S_\gamma$ is a pseudo-Tychonoff topological group, then S_γ is a pseudo-Tychonoff topological group for all γ in Λ .*

Proof. Since S_γ is a topological group for all γ in Λ with respect to the subspace topology, and by Theorem 9, the pseudo-Tychonoff topological group property is hereditary, so S_γ is a pseudo-Tychonoff topological group for all γ . $\square$

Theorem 12. *If S_β is a pseudo-Tychonoff topological group for all $\beta \in \Lambda$, then $\bigoplus_{\beta \in \Lambda} S_\beta$ is a pseudo-Tychonoff topological group.*

Proof. If S_β is a Tychonoff topological group for all $\beta \in \Lambda$, we are done. Suppose that S_β is not a Tychonoff topological group for some $\beta \in \Lambda$. Let S_β be a pseudo-Tychonoff topological group for all $\beta \in \Lambda$. Then, by pseudo-Tychonoffness, take a Tychonoff topological group H_β for all $\beta \in \Lambda$, and a bijective function $\phi_\beta : S_\beta \rightarrow H_\beta$ such that $\phi_\beta|_{C_\beta} : C_\beta \rightarrow \phi_\beta(C_\beta)$ is a topological isomorphism for each pseudo-compact subgroup C_β of S_β . Since Tychonoff topology is an additive property (see [13] 2.2.7), consider the sum function ([13] 2.2.E) $\bigoplus_{\beta \in \Lambda} \phi_\beta : \bigoplus_{\beta \in \Lambda} S_\beta \rightarrow \bigoplus_{\beta \in \Lambda} H_\beta$, defined by $\bigoplus_{\beta \in \Lambda} \phi_\beta(g) = \phi_\alpha(g)$ if $g \in S_\alpha, \alpha \in \Lambda$. Now, a subspace $C \subseteq \bigoplus_{\beta \in \Lambda} S_\beta$ is a pseudo-compact subgroup if and only if the set $\Lambda_0 = \{\beta \in \Lambda : C \cap S_\beta \neq \emptyset\}$ is finite and $C \cap S_\beta$ is a pseudo-compact subgroup in S_β for each $\beta \in \Lambda_0$. If $C \subseteq \bigoplus_{\beta \in \Lambda} S_\beta$ is a pseudo-compact subgroup, then $(\bigoplus_{\beta \in \Lambda} \phi_\beta)|_C$ is a topological isomorphism because $(\phi_\beta)|_{C \cap S_\beta}$ is a topological isomorphism for all $\beta \in \Lambda_0$. $\square$

Theorem 13. Every pseudo-Tychonoff–Fréche–Lindelöf topological group S is pseudo-normal.

Proof. Suppose that S is a pseudo-Tychonoff–Fréchet–Lindelöf topological group. Then, there exists a Tychonoff topological group T and a one-to-one corresponding function $\phi : S \rightarrow T$ such that $\phi|_H : H \rightarrow \phi(H)$ is a topological isomorphism for each pseudo-compact subgroup H of S . By Theorem 8, ϕ is continuous. Since the continuous image of a Lindelöf space is Lindelöf, T is Lindelöf. Thus, T is regular, so Lindelöf implies that T is normal. Hence, S is a pseudo-normal topological group. $\square$

5. Other Results

Theorem 14. Let H be a closed subgroup of a pseudo-normal topological group S and $H \trianglelefteq S$. If the group H and the quotient group S/H are pseudo-compact, then the topological group S is normal.

Proof. By ([14], Theorem 2.2), S is pseudo-compact, and by the definition of pseudo-normality, we conclude that S is normal. $\square$

Since the quotient group S/V is Hausdorff if and only if V is closed in a topological group S , and any Hausdorff locally compact topological group is normal, we obtain the theorems below.

Theorem 15. Let V be a closed subgroup of a locally compact topological group S . If $V \trianglelefteq S$, then S/V is a pseudo-normal quotient group.

Proof. Since V is a closed subgroup of a locally compact topological group S and $V \trianglelefteq S$, S/V is a Hausdorff locally compact quotient group. Hence, S/V is a normal topological group, and by Definition 1, S/V is a pseudo-normal quotient group. $\square$

Theorem 16. Let S be a topological group S . If V and S/V are Hausdorff locally compact topological groups, then S is pseudo-normal.

Proof. Since V and S/V are locally compact, S is a locally compact topological group and normal. Hence, S is pseudo-normal. $\square$

Theorem 17. Let H be an open subset of a Hausdorff topological group S and $H \trianglelefteq S$. Then, H and S/H are pseudo-Tychonoff topological groups.

Proof. Suppose that H is an open subset of a Hausdorff topological group S and $H \trianglelefteq S$. Since Hausdorffness is hereditary, and since $H \trianglelefteq S$, H is a Tychonoff topological group, and hence H is pseudo-Tychonoff. Regarding the other request, we have that S/H is a quotient topological group. By the openness of H , we have that $\{sH\}$ is open for all $s \in S$. Hence,

S/H is a discrete group. Since any discrete group is T_4 , S/H is a Tychonoff topological group. Therefore, S/H is pseudo-Tychonoff. $\square$

Recall that a simple group S is a non-trivial group in which the only normal subgroups are the identity and the group itself (normal implies normal with respect to the group).

Theorem 18. *Let S be a pseudo-normal (resp. pseudo-Tychonoff) topological simple group. If $K \trianglelefteq S$, then S/K is pseudo-normal (resp. pseudo-Tychonoff).*

Proof. Let S/K be a topological group and $K \trianglelefteq S$. Since S is a topological simple group, then K is equal to either S or the identity element $\{e\}$. Hence, S/S or $S/\{e\}$ are pseudo-normal (resp. pseudo-Tychonoff) topological groups. Therefore, S/K is pseudo-normal (resp. pseudo-Tychonoff). $\square$

Theorem 19. *Let K be a closed compact subset of a topological group S . If $K \trianglelefteq S$, then S/K is pseudo-normal (resp. pseudo-Tychonoff).*

Proof. Let K be a closed compact subset of a topological group S and $K \trianglelefteq S$. Then, S/K is a Hausdorff compact topological group, and thus S/K is normal. Hence, S/K is pseudo-normal. Regarding the other request, S/K is a T_4 topological group since S/K is compact and T_2 . Hence, S/K is pseudo-Tychonoff. $\square$

Since any component of the identity element of a topological group space is a closed and normal subgroup (normal with respect to the group), we have the following corollary:

Corollary 4. *Let C be a compact component of the identity of a topological group S , and then S/C is a pseudo-normal (resp. pseudo-Tychonoff) topological group.*

Theorem 20. *Let H be an open subset of the topological group S and $K \trianglelefteq S$. If K is a subgroup of S containing H , then K/H is pseudo-normal (resp. pseudo-Tychonoff).*

Proof. Since H is open, S/H is a discrete group. Since S/H and K/H have identical topologies, K/H is a discrete group, and so K/H is a T_4 group. Therefore, K/H is pseudo-normal (resp. pseudo-Tychonoff). $\square$

Theorem 21. *Let S be a topological Abelian group, V be a symmetric neighborhood of the identity element e , and $K = \bigcup_{n \geq 1} V^n$. Then, S/K is pseudo-normal (resp. pseudo-Tychonoff).*

Proof. Let S be a topological Abelian group, V be a symmetric neighborhood of the identity element e , and $K = \bigcup_{n \geq 1} V^n$. Then, K is a closed and open normal subgroup of S (see [6]). Hence, S/K is pseudo-normal (resp. pseudo-Tychonoff). $\square$

Obviously, from the definitions,

$$\mathbb{R} - \text{factorizable} \Rightarrow \mathcal{M} - \text{factorizable} \Rightarrow \mathcal{N} - \text{factorizable},$$

none of the above implications is reversible.

Since any Hausdorff topological group is pseudo-Tychonoff, we have the following corollary:

Corollary 5. *Any $\mathbb{R}$ -factorizable (resp. $\mathcal{M}$ -factorizable, $\mathcal{N}$ -factorizable) is pseudo-Tychonoff.*

In general, the converse of the above corollary need not be true as shown in the following example:

Example 6. In ([12], Example 3.5), let D be a discrete Boolean group with $|D| = \omega_1$. Denote by G the topological product $D \times \mathbb{Z}^{\omega_1}$, where $\mathbb{Z}$ is the discrete group of integers. Then, G is a Hausdorff topological Abelian group, but not $\mathcal{N}$ -factorizable. By the Hausdorffness of G , we have that G is pseudo-Tychonoff since any Tychonoff topological group is pseudo-Tychonoff. However, G is neither $\mathbb{R}$ -factorizable, nor $\mathcal{M}$ -factorizable, nor $\mathcal{N}$ -factorizable.

Proposition 3. Let S be a pseudo-compact, pseudo-normal Hausdorff topological group. If normal topological groups that witness the pseudo-normality of S are second-countable, then S is $\mathbb{R}$ -factorizable.

Proof. Suppose that α is any continuous, real-valued function on S . By the pseudo-normality of S , there exists a normal topological group T and a bijective function $f : S \rightarrow T$ such that for each pseudo-compact, non-trivial subgroup V of S , $f|_V : V \rightarrow f(V)$ is a topological isomorphism. Since S is a pseudo-compact, non-trivial group, $f = f|_S : S \rightarrow f(S) = T$ is a topological isomorphism. Hence, T is Hausdorff, and from the hypothesis, T is second-countable. Now, there exists a continuous, onto homomorphism $f : S \rightarrow T$, where T is Hausdorff and second-countable, and a continuous, real-valued function $\gamma = \alpha \circ f^{-1} : T \rightarrow \mathbb{R}$ such that $\alpha = \gamma \circ f$. Therefore, S is $\mathbb{R}$ -factorizable. $\square$

Proposition 4. Let S be a pseudo-compact, pseudo-normal Hausdorff topological group. If normal topological groups that witness the pseudo-normality of S are metrizable, then S is $\mathcal{M}$ -factorizable.

Proof. By applying the same argument used in Proposition 3, we can obtain the result. $\square$

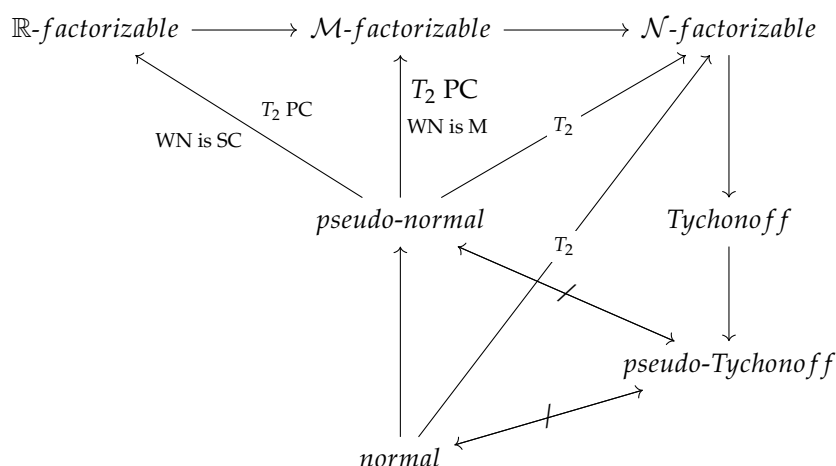
Proposition 5. Let S be a pseudo-compact, pseudo-normal Hausdorff topological group. Then, S is $\mathcal{N}$ -factorizable.

Proof. Suppose that α is any continuous, real-valued function on S . By the pseudo-normality of S , there exists a normal topological group T and a bijective function $f : S \rightarrow T$ such that for each pseudo-compact, non-trivial subgroup V of S , $f|_V : V \rightarrow f(V)$ is a topological isomorphism. Since S is a pseudo-compact, non-trivial group, $f = f|_S : S \rightarrow f(S) = T$ is a topological isomorphism. Hence, T is a Hausdorff topological group. Now, there exists a Hausdorff normal topological group T ; a continuous, onto homomorphism $f : S \rightarrow T$; and a continuous, real-valued function $\gamma = \alpha \circ f^{-1} : T \rightarrow \mathbb{R}$ such that $\alpha = \gamma \circ f$. Therefore, S is $\mathcal{N}$ -factorizable. $\square$

Here is an example to explain that the converse of Propositions 3–5 need not be true.

Example 7. By [12] (Proposition 3.2.), the topological group $S = H \times K^{\omega_1}$ is countably compact and $\mathbb{R}$ -factorizable, and by Example 2, $S = H \times K^{\omega_1}$ is not a pseudo-normal topological group. Since any $\mathbb{R}$ -factorizable topological group is $\mathcal{M}$ -factorizable, and any $\mathcal{M}$ -factorizable topological group is $\mathcal{N}$ -factorizable, we obtain the desired result.

In the following diagram, for convenience, WN denotes a topological group that witnesses pseudo-normality. The notions T_2 , SC , PC , and M represent the properties of Hausdorff, second-countable, pseudo-compact, and metrizable topological groups, respectively.



None of these implications are reversible, as shown in Examples 2 and 4–7. Other examples can be found in [10–12].

6. Conclusions and Future Work

In this paper, we have defined the pseudo-normal and pseudo-Tychonoffness topological group spaces as a generalization of normality and Tychonoffness, respectively. Then, we discussed their main properties in detail. To elucidate these properties, we provided some examples. Toward the end of this work, we provided some results belonging to these properties. Pseudo-normality and pseudo-Tychonoffness are purely topological group properties that acquire specific features, which are of great interest for investigation. The results presented may aid in a more thorough understanding of the current situation and in the development of topological groups. We have provided further findings on pseudo-normality and pseudo-Tychonoffness with regard to ω -narrow, $\mathbb{R}$ -factorizable, and $\mathcal{N}$ -factorizable topological groups.

Our future work will concentrate on further topological group notions related to pseudo-normal and pseudo-Tychonoff topological group properties.

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