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Existence of Bounded Solutions for a Class of Degenerate Fourth-Order Elliptic Equations with Convection Terms

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Abstract: This paper deals with the existence of bounded and locally Hölder continuous weak solutions of the following nonlinear fourth-order Dirichlet problem: $\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \left[A_\alpha(x,u,\mathrm{D}^1 u, \mathrm{D}^2 u) - E_\alpha(x) \, |u|^{\lambda(p_\alpha - 1)} \, \operatorname{sign} u \right] = f$ in Ω, where the coefficients A_α satisfy a strengthened degenerate coercivity condition.

Keywords: high-order equations; nonlinear elliptic problems; degenerate coercivity; convection term; boundedness; Hölder continuity

MSC: 35J25; 35J30; 35J70; 35B45

1. Introduction

In the present article, we consider the following nonlinear fourth-order equation with convection terms:

$$\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \left[A_\alpha(x,u,\mathrm{D}^1 u, \mathrm{D}^2 u) - E_\alpha(x) \, |u|^{\lambda(p_\alpha - 1)} \, \operatorname{sign} u \right] = f \quad \text{in } \Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open bounded set, $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index with non-negative integer components and length $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $\mathrm{D}^\alpha u(x) = \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$, $\mathrm{D}^h u = \left\{ \mathrm{D}^\alpha u : |\alpha| = h \right\}$, for $h = 1, 2$.

The coefficients A_α satisfy the following coercivity condition:

$$\sum_{|\alpha|=1,2} A_\alpha(x,\eta,\xi)\xi_\alpha \geq \nu_1 \sum_{|\alpha|=1,2} \frac{|\xi_\alpha|^{p_\alpha}}{(1+|\eta|)^{\theta(p_\alpha - 1)}}, \tag{2}$$

where θ, λ and p_α are real numbers such that $0 \leq \theta < 1$, $0 < \lambda < 1$ and

$$p_\alpha = \begin{cases} q & \text{if } |\alpha| = 1 \\ p & \text{if } |\alpha| = 2 \end{cases} \tag{3}$$

with $1 < p < \frac{N}{2}$, $2p < q < N$. The terms E_α, $|\alpha| = 1, 2$, and f are a vector field and a function, respectively, satisfying suitable summability assumptions.

The coercivity condition (2) was introduced by I. V. Skrypnik in 1978 in [1] for the case $\theta = 0$ in the framework of higher-order equations (with $E_\alpha \equiv 0$, $|\alpha| = 1, 2$) in order to prove

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the boundedness and Holder continuity of the solutions. It turns out that this condition is stronger than the one usually considered for nonlinear fourth-order elliptic equations, i.e.,

$$\sum_{|\alpha|=2} A_\alpha(x,\xi)\xi_\alpha \geq \nu_1 \sum_{|\alpha|=2} |\xi_\alpha|^p - \nu_2 \sum_{|\alpha|<2} |\xi_\alpha|^{s_\alpha} - f(x), \tag{4}$$

which, unfortunately, does not even ensure the boundedness of the solutions (see well-known counterexamples in [2–4]) unless $2p > N$ (as a consequence of Sobolev's embedding theorem) or $2p = N$ (see [5]) or $N - 2p$ is sufficiently small (see [6]).

Operators satisfying condition (2) with $\theta = 0$ have been studied in connection with many other questions, such as L^1-theory, the qualitative properties of the solutions, and removable singularities in the degenerate and non-degenerate cases in [7–12]. Moreover, existence and regularity results for the solutions of a class of nonlinear fourth-order equations, with the principal part satisfying (2) with $\theta = 0$ and lower-order terms having so-called "natural growth", were obtained in [13–17].

In dealing with the boundary-value problem related to Equation (1), two difficulties arise. Firstly, the nonlinear fourth-order operator

$$u \to \sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \left[A_\alpha(x,u,\mathrm{D}^1 u, \mathrm{D}^2 u) \right]$$

under condition (2), though well defined, is not coercive on $W_0^{1,q}(\Omega) \cap W_0^{2,p}(\Omega)$ when u is large. Moreover, due to the presence of the convection term, the operator

$$u \to \sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \left[A_\alpha(x,u,\mathrm{D}^1 u, \mathrm{D}^2 u) - E_\alpha \, |u|^{\lambda(p_\alpha - 1)} \operatorname{sign}(u) \right]$$

may be not coercive even if $\theta = 0$, unless we assume that the sizes of the norm $||E_\alpha||_{L^{\frac{N}{|\alpha|(p_\alpha-1)}}(\Omega)}$, $|\alpha| = 1, 2$, are sufficiently small, or $\sum_{|\alpha|=1,2} \mathrm{D}^\alpha \left[E_\alpha \right] = 0$.

To overcome the lack of coercivity, we approximate our problem with a sequence of homogeneous non-degenerate Dirichlet problems, and we prove an L^∞ a priori estimate on the approximating solutions, which, in turn, implies an a priori estimate in the energy space. Once this has been accomplished, a compactness result for the approximating solutions allows us to find a bounded weak solution of the boundary-value problem related to Equation (1), which is also locally Hölder continuous.

In the framework of the second-order Dirichlet problem, the model problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^\theta} - E(x) \, |u|^\lambda \operatorname{sign} u \right) = f & \text{in } \Omega \quad \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

was studied in [18], where the authors obtained existence and regularity results under various assumptions on f (see [19,20] in the case $E \equiv 0$).

It is well known that when dealing with fourth-order equations, some difficulties arise. As a matter of fact, the corresponding weak formulation contains terms related to second-order derivatives of test functions that must be correctly taken into account.

This article is organized as follows. In Section 2, we formulate the hypotheses and state the results. In Section 3, we present some auxiliary lemmas, and we prove a priori estimates that will be used in the proof of the boundedness theorem. In Section 4, we provide proof of the existence of a bounded solution. Finally, the Hölder continuity of bounded solutions is proved in Section 5.

2. Preliminaries and Statement of Results

Let Ω be an open bounded set in $\mathbb{R}^N$ with $N \geq 3$. We denote by $N(2)$ the number of different multi-indices α such that $|\alpha| = 1, 2$.

Let $A_\alpha(x, \eta, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^{N(2)} \to \mathbb{R}$, with $|\alpha| = 1, 2$, be Carathéodory functions (i.e., $A_\alpha(\cdot, \eta, \xi)$ is measurable on Ω for every $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^{N(2)}$, and $A_\alpha(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^{N(2)}$ for almost every $x \in \Omega$), satisfying the following structural conditions for almost every $x \in \Omega$, every $\eta \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^{N(2)}$, $\xi \neq \xi'$:

$$\sum_{|\alpha|=1,2} A_\alpha(x, \eta, \xi)\xi_\alpha \geq \nu_1 \sum_{|\alpha|=1,2} \frac{|\xi_\alpha|^{p_\alpha}}{(1+|\eta|)^{\theta(p_\alpha-1)}}, \tag{5}$$

$$\sum_{|\alpha|=1,2} |A_\alpha(x, \eta, \xi)|^{\frac{p_\alpha}{p_\alpha-1}} \leq \nu_2 \Big[\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big], \tag{6}$$

$$\sum_{|\alpha|=1,2} \big[A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi')\big](\xi_\alpha - \xi'_\alpha) > 0, \tag{7}$$

where ν_1, ν_2 are positive constants, $0 \leq \theta < 1$, and p_α, $|\alpha| = 1, 2$, is defined by (3). Let $E_\alpha(x) : \Omega \to \mathbb{R}^N$, $|\alpha| = 1$, and $E_\alpha(x) : \Omega \to \mathbb{R}^{N^2}$, $|\alpha| = 2$, be vector fields such that

$$|E_\alpha| \in L^{r_\alpha}(\Omega) \quad \text{with } r_\alpha > \frac{N}{|\alpha|(p_\alpha - 1)}, \ |\alpha| = 1, 2 \tag{8}$$

and

$$f \in L^t(\Omega) \quad \text{with } t > N/q. \tag{9}$$

We set

$$W^{1,q}_{2,p}(\Omega) = W^{1,q}(\Omega) \cap W^{2,p}(\Omega),$$

and

$$\mathring{W}^{1,q}_{2,p}(\Omega) = W^{1,q}_0(\Omega) \cap W^{2,p}_0(\Omega).$$

Assumptions (5)–(9) allow us to give the following definition.

Definition 1. *A weak solution of the problem*

$$\begin{cases} \displaystyle\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \big[A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) - E_\alpha\, |u|^{\lambda(p_\alpha-1)} \operatorname{sign} u\big] = f & \text{in } \Omega \\ \mathrm{D}^\alpha u = 0, \ |\alpha| = 0, 1 & \text{on } \partial\Omega \end{cases} \tag{10}$$

is a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that the integral identity

$$\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u)\mathrm{D}^\alpha v \, \mathrm{d}x = \sum_{|\alpha|=1,2} \int_\Omega E_\alpha |u|^{\lambda(p_\alpha-1)-1} u \, \mathrm{D}^\alpha v \, \mathrm{d}x + \int_\Omega f v \, \mathrm{d}x \tag{11}$$

holds for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$.

Our first result states the existence of a bounded weak solution of (10).

Theorem 1. *Let us suppose that conditions (5)–(9) are satisfied, and*

$$0 \leq \theta < \frac{q-p}{p(q-1)} \quad \text{and} \quad \theta + \lambda < 1. \tag{12}$$

Then, there exists a weak solution u of problem (10), in the sense of Definition 1, such that

$$\|u\|_{L^\infty(\Omega)} \leq M \tag{13}$$

where $M > 0$ is a constant depending on θ, N, q, p, ν_1, ν_2, $|\Omega|$, $\|E_\alpha\|_{L^{r_\alpha}(\Omega)}$, $|\alpha| = 1, 2$ and $||f||_{L^t(\Omega)}$.

Furthermore, under the same assumptions as Theorem 1, the local Hölder continuity of any weak solution $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ holds, as stated in the following theorem.

Theorem 2. *Let us suppose that conditions (5)–(9) and (12) are satisfied. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ be a bounded solution of problem (10). Then, there exists $\rho \in (0,1)$, depending on θ, N, q, p, ν_1, ν_2, $|\Omega|$, $\|E_\alpha\|_{L^{r_\alpha}(\Omega)}$, $|\alpha| = 1, 2$, and $||f||_{L^t(\Omega)}$ and on $||u||_{L^\infty(\Omega)}$, such that $u \in C^{0,\rho}_{loc}(\Omega)$, and, for any domain $\Omega' \subset\subset \Omega$, we have*

$$|u(x) - u(y)| \leq C|x-y|^\rho \quad \text{for any } x, y \in \Omega'$$

where C is a positive constant depending on the same parameters as ρ, and $d' = \operatorname{dist}(\Omega', \partial\Omega)$.

Remark 1. *Assumption (9) on f, required in Theorems 1 and 2, is the same one that yields the existence of bounded and Hölder continuous solutions for fourth-order equations satisfying the degenerate condition (5) and $E_\alpha \equiv 0$, $|\alpha| = 1, 2$ (see [21]), as well as for non-degenerate (i.e., $\theta = 0$) fourth-order equations and $E_\alpha \equiv 0$, $|\alpha| = 1, 2$ (see [22]). In the latter case, examples of unbounded solutions of Equation (1) with $f \in L^{\frac{N}{q}}(\Omega)$ and $f \notin L^{\frac{N}{q}+\varepsilon}(\Omega)$, for every $\varepsilon > 0$, are constructed in [23]. On the other hand, hypothesis (9) is also required to prove the boundedness and Hölder continuity of the solutions of Equation (1), where $\lambda = 1$ and the coefficients $A_\alpha(x, \eta, \xi)$ satisfy condition (5) with $\theta = 0$ (see [24]).*

3. Auxiliary Lemmas and a Priori Estimates

We begin this section by recalling an algebraic lemma according to Serrin (see Lemma 2 in [25]).

Lemma 2. *Let χ be a positive exponent and a_i, β_i, $i = 1, \ldots, m$, be two sets of m real numbers such that $0 < a_i < +\infty$ and $0 < \beta_i < \chi$. Suppose that z is a positive number satisfying the inequality*

$$z^\chi \leq \sum_{i=1}^{m} a_i z^{\beta_i}.$$

Then,

$$z \leq C \sum_{i=1}^{m} a_i^{\gamma_i},$$

where C depends only on m, χ, β_i and $\gamma_i = \frac{1}{\chi - \beta_i}$, $i = 1, \ldots, m$.

Moreover, we present a slightly modified version of Stampacchia's well-known lemma (see [26]), whose proof is contained in [19,22]. See also [27] for new generalizations.

Lemma 3. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that*

$$\phi(h) \leq \frac{c_0}{(h-k)^\nu} k^{\tau\nu} [\phi(k)]^{1+\mu}, \quad \text{for all } h > k \geq k_0 > 0,$$

for some positive constants c_0 and k_0, with $\nu > 0$, $0 \leq \tau < 1$ and $\mu > 0$. Then, there exists $k^* > 0$, depending on c_0, τ, ν, μ and k_0, such that $\phi(k^*) = 0$.

Finally, we recall the following lemma, which ensures that the composition of a C^2 function $\zeta(s)$ with a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ belongs to $\mathring{W}^{1,q}_{2,p}(\Omega)$.

Lemma 4 ([28] Proposition 9.5)**.** *Let $\zeta \in C^2(\mathbb{R})$ be a function with bounded derivatives ζ' and ζ'' such that $\zeta(0) = 0$. If $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$, then*

$$\zeta(u) \in \mathring{W}^{1,q}_{2,p}(\Omega),$$

and for each multi-index α such that $|\alpha| = 1, 2$, the following assertion holds:

$$D^\alpha \zeta(u) = \zeta'(u) D^\alpha u + R_{|\alpha|}(u) \tag{14}$$

where, for i.e., $x \in \Omega$,

$$R_{|\alpha|}(u) = \begin{cases} 0 & \text{if} \quad |\alpha| = 1 \\ \zeta''(u) \displaystyle\sum_{|\beta|=|\gamma|=1} D^\beta u D^\gamma u & \text{if} \quad |\alpha| = 2. \end{cases}$$

Given $n \in \mathbb{N}$, let $T_n(s)$ be the truncation function defined by

$$T_n(s) = \begin{cases} s & \text{if } |s| \leq n \\ \\ n \operatorname{sign}(s) & \text{if } |s| > n. \end{cases}$$

Following the technique already used in [19,20] in the framework of second-order elliptic equations, let us define the following Dirichlet problems:

$$\begin{cases} u_n \in \mathring{W}^{1,q}_{2,p}(\Omega), \\ \displaystyle\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), D^1 u_n, D^2 u_n) D^\alpha v \, dx \\ \quad = \displaystyle\sum_{|\alpha|=1,2} \int_\Omega E_\alpha \frac{u_n}{\frac{1}{n} + |u_n|} \frac{|u_n|^{\lambda(p_\alpha - 1)}}{1 + \frac{1}{n}|u_n|^{\lambda(p_\alpha - 1)}} D^\alpha v \, dx + \displaystyle\int_\Omega f v \, dx \\ \qquad\qquad \text{for every } v \in \mathring{W}^{1,q}_{2,p}(\Omega). \end{cases} \tag{15}$$

By Schauder's fixed-point theorem, there exists a solution $u_n \in \mathring{W}^{1,q}_{2,p}(\Omega)$ of problem (15). Moreover, since, for every fixed $n \in \mathbb{N}$,

$$\frac{u_n}{\frac{1}{n} + |u_n|} \frac{|u_n|^{\lambda(p_\alpha - 1)}}{1 + \frac{1}{n}|u_n|^{\lambda(p_\alpha - 1)}} \leq n \quad \text{for i.e., } x \in \Omega$$

under conditions (8) and (9), every u_n is bounded thanks to the boundedness result of [22] (see also [14]).

The next two lemmas are crucial for passing to the limit in the approximating problems (15). The first one concerns the equi-boundedness of the sequence $\{u_n\}$.

Lemma 5. *Assume that conditions (5)–(9) and (12) are satisfied. Let u_n be a solution of problem (15) for every $n \in \mathbb{N}$. Then, there exists a positive constant M, depending only on θ, λ, N, q, p, ν_1, ν_2, $|\Omega|$, $\|E_\alpha\|_{L^{r_\alpha}(\Omega)}$, $|\alpha| = 1, 2$ and $||f||_{L^t(\Omega)}$, such that*

$$||u_n||_{L^\infty(\Omega)} \leq M \quad \text{for every } n \in \mathbb{N}. \tag{16}$$

Proof. Given $k \geq 1$ and $\sigma > 2$, let us consider the function

$$\zeta(s) = |s|^{\theta(q-1)} \left[|s| - k\right]_+^\sigma \operatorname{sign}(s).$$

Due to the boundedness of u_n and thanks to Lemma 4, $\zeta(u_n)$ is an admissible test function in (15). Moreover,

$$\mathrm{D}^\alpha \zeta(u_n) = \left[\sigma + \theta(q-1)\frac{\left[|u_n| - k\right]_+}{|u_n|}\right] |u_n|^{\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma-1} \mathrm{D}^\alpha u_n + R_{|\alpha|}(u_n) \tag{17}$$

where, for i.e., $x \in \Omega$,

$$\begin{cases} R_{|\alpha|}(u_n) = 0 & \text{if } |\alpha| = 1 \\ |R_{|\alpha|}(u_n)| \leq \sigma^2\theta^2 q^2 \, |u_n|^{\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma-2} \displaystyle\sum_{|\beta|=1} |\mathrm{D}^\beta u_n|^2 & \text{if } |\alpha| = 2. \end{cases} \tag{18}$$

Note that we have used the inequality

$$0 \leq \frac{\left[|u_n| - k\right]_+}{|u_n|} \leq 1, \qquad \text{for any } k \geq 0. \tag{19}$$

We test the integral identity (15) with $v = \zeta(u_n)$. Using condition (5) and inequality (19), we obtain

$$\begin{aligned} &\nu_1\sigma \textstyle\sum_{|\alpha|=1,2} \displaystyle\int_\Omega \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha} \, |u_n|^{\theta(q-1)}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}} \left[|u_n| - k\right]_+^{\sigma-1} \mathrm{d}x \\ &\leq \textstyle\sum_{|\alpha|=1,2} \displaystyle\int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) \Big[\sigma \, |u_n|^{\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma-1} \\ &\qquad + \theta(q-1)|u_n|^{\theta(q-1)-1} \left[|u_n| - k\right]_+^{\sigma}\Big] \mathrm{D}^\alpha u_n \, \mathrm{d}x \\ &\leq c\Big\{ \textstyle\sum_{|\alpha|=2} \displaystyle\int_\Omega |A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n)||R_{|\alpha|}(u_n)| \, \mathrm{d}x \\ &\qquad + \textstyle\sum_{|\alpha|=1,2} \displaystyle\int_\Omega |E_\alpha| \, |u_n|^{\lambda(p_\alpha-1)+\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma-1} |\mathrm{D}^\alpha u_n| \, \mathrm{d}x \\ &+ \textstyle\sum_{|\alpha|=2} \displaystyle\int_\Omega |E_\alpha| \, |u_n|^{\lambda(p_\alpha-1)} |R_{|\alpha|}(u_n)| \, \mathrm{d}x + \displaystyle\int_\Omega |f| \, |u_n|^{\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma} \mathrm{d}x \Big\} \\ &= c\Big\{ I_1 + J_\alpha + I_2 + \displaystyle\int_\Omega |f| \, |u_n|^{\theta(q-1)} \left[|u_n| - k\right]_+^{\sigma} \mathrm{d}x \Big\}. \end{aligned} \tag{20}$$

From now on, we denote by c a positive constant that does not depend on n (namely, it may depend on N, $|\Omega|$, θ, λ, σ, p, q, ν_1, ν_2 and $\|E_\alpha\|_{L^{r_\alpha}(\Omega)} \, \|f\|_{L^t(\Omega)}$) and whose value may vary from line to line.

Let us evaluate the integrals on the right-hand side of (20).

We observe that the following inequality holds:

$$1 + |T_n(u_n)| \leq 1 + |u_n|, \quad \text{for any } n \in \mathbb{N}. \tag{21}$$

Using Young's inequality with the exponents p_α and $\frac{p_\alpha}{p_\alpha - 1}$, for all $\tau > 0$, we obtain

$$J_\alpha \le \tau \sum_{|\alpha|=1,2} \int_\Omega \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}\ |u_n|^{\theta(q-1)}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x + C(c,\tau) \sum_{|\alpha|=1,2} \int_\Omega |E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}} \big(1+|u_n|\big)^{\lambda p_\alpha+q\theta} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x. \tag{22}$$

The growth condition (6), estimate (18) and, again, Young's inequality with the exponents $\frac{p}{p-1}$, $\frac{q}{2}$ and $\delta=\frac{pq}{q-2p}$ give us, for all $\tau>0$ and $\sigma>1+\delta$,

$$\begin{aligned} I_1 &\le c\int_\Omega \Big[\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u_n|^{p_\alpha}\Big]^{\frac{p-1}{p}} |u_n|^{\theta(q-1)} \big[|u_n|-k\big]_+^{\sigma-2} \sum_{|\alpha|=1} |\mathrm{D}^\alpha u_n|^2\,\mathrm{d}x \\ &\le \tau \sum_{|\alpha|=1,2} \int_\Omega \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}\ |u_n|^{\theta(q-1)}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\quad + C(c,\tau)\Big[\int_\Omega \big(1+|u_n|\big)^{\theta(q-1)\delta} \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x\Big]. \end{aligned} \tag{23}$$

and

$$\begin{aligned} I_2 &\le c\textstyle\sum_{|\alpha|=2}\int_\Omega |E_\alpha|\,|u_n|^{\lambda(p-1)+\theta(q-1)} \big[|u_n|-k\big]_+^{\sigma-2} \sum_{|\alpha|=1} |\mathrm{D}^\alpha u_n|^2\,\mathrm{d}x \\ &\le \tau \textstyle\sum_{|\alpha|=1}\int_\Omega \frac{|\mathrm{D}^\alpha u_n|^q\ |u_n|^{\theta(q-1)}}{(1+|T_n(u_n)|)^{\theta(q-1)}} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\quad + C(c,\tau)\Big[\textstyle\sum_{|\alpha|=2}\int_\Omega |E_\alpha|^{\frac{p}{p-1}} \big(1+|u_n|\big)^{\lambda p+q\theta} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\quad + \textstyle\int_\Omega \big(1+|u_n|\big)^{\theta(q+\delta)} \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x\Big]. \end{aligned} \tag{24}$$

From (20), thanks to (22)–(24) and choosing a suitable $0<\tau<1$, we deduce

$$\begin{aligned} &\int_\Omega \sum_{|\alpha|=1,2} \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}\ |u_n|^{\theta(q-1)}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\le c\Big\{ \sum_{|\alpha|=1,2}\int_\Omega |E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}} \big(1+|u_n|\big)^{(\lambda+\theta)q} \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\qquad + \int_\Omega \big(1+|u_n|\big)^{\theta(q-1)\delta} \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x + \int_\Omega |f|\big[|u_n|-k\big]_+^{\sigma}\,\mathrm{d}x\Big\} \end{aligned} \tag{25}$$

(note that $(q-1)\delta>q+\delta$).

Taking into account (21) and the inequality

$$1+|u_n| \le 2\big(k+\big[|u_n|-k\big]_+\big) \quad \text{if } k\ge 1,$$

from (25), we obtain

$$\begin{aligned} &\Big(\tfrac12\Big)^{\theta(q-1)} \textstyle\sum_{|\alpha|=1,2}\int_\Omega |\mathrm{D}^\alpha u_n|^{p_\alpha}\big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x \\ &\le c\Big\{ \textstyle\sum_{|\alpha|=1,2}\int_\Omega |E_\alpha|^{p'_\alpha}\ \big[|u_n|-k\big]_+^{\sigma-1+(\lambda+\theta)q}\,\mathrm{d}x \\ &\qquad + \textstyle\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta+\theta(q-1)\delta}\,\mathrm{d}x + \int_\Omega |f|\big[|u_n|-k\big]_+^{\sigma+\theta(q-1)}\,\mathrm{d}x \\ &\quad + \textstyle\sum_{|\alpha|=1,2} k^{(\lambda+\theta)q}\int_\Omega |E_\alpha|^{p'_\alpha}\ \big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x + k^{\theta(q-1)\delta}\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x \\ &\qquad + k^{\theta(q-1)}\textstyle\int_\Omega |f|\big[|u_n|-k\big]_+^{\sigma}\,\mathrm{d}x\Big\}, \end{aligned} \tag{26}$$

for any $k \geq 1$.

By Sobolev's embedding theorem and the above inequality, we obtain

$$\begin{aligned}
&\Big[\int_\Omega \Big|\big[|u_n|-k\big]_+^{\left(\frac{\sigma-1}{q}+1\right)}\Big|^{q^*}\,dx\Big]^{\frac{q}{q^*}} \\
&\quad\leq c\Big\{\textstyle\sum_{|\alpha|=1,2}\int_\Omega |E_\alpha|^{p'_\alpha}\big[|u_n|-k\big]_+^{\sigma-1+(\lambda+\theta)q}\,dx \\
&\qquad+\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta+\theta(q-1)\delta}\,dx+\int_\Omega |f|\big[|u_n|-k\big]_+^{\sigma+\theta(q-1)}\,dx \\
&\quad+\textstyle\sum_{|\alpha|=1,2}k^{(\lambda+\theta)q}\int_\Omega |E_\alpha|^{p'_\alpha}\big[|u_n|-k\big]_+^{\sigma-1}\,dx+k^{\theta(q-1)\delta}\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,dx \\
&\qquad+k^{\theta(q-1)}\int_\Omega |f|\big[|u_n|-k\big]_+^{\sigma}\,dx\Big\}
\end{aligned} \tag{27}$$

for every $k \geq 1$ and $\sigma > 1+q$ (note that if θ and λ verify condition (12), then $1+q > 1+(\lambda+\theta)q$ and $1+q > 1+\delta+\theta(q-1)\delta$) .

We denote by $A_n(k)$ the level set of $|u_n|$, that is,

$$A_n(k) = \big\{x \in \Omega : |u_n(x)| > k\big\}$$

and we denote by $|A_n(k)|$ the N-dimensional Lebesgue measure of $A_n(k)$.

Let $\gamma = \big(\sigma-1+q\big)\frac{q^*}{q}$. Applying Hölder's inequality to (27) and taking into account assumptions (8), (9) and (12), we obtain

$$\begin{aligned}
&\Big[\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\Big]^{\frac{q}{q^*}} \\
&\quad\leq c\Big\{\textstyle\sum_{|\alpha|=1,2}\|E_\alpha\|_{r_\alpha}^{p'_\alpha}\left(\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\right)^{\frac{\sigma-1+(\lambda+\theta)q}{\gamma}}\cdot|A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1+(\lambda+\theta)q}{\gamma}} \\
&\qquad+\left(\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\right)^{\frac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma}}\cdot|A_n(k)|^{1-\frac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma}} \\
&\qquad+\|f\|_t\left(\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\right)^{\frac{\sigma+\theta(q-1)}{\gamma}}\cdot|A_n(k)|^{1-\frac{1}{t}-\frac{\sigma+\theta(q-1)}{\gamma}} \\
&\quad+k^{(\lambda+\theta)q}\textstyle\sum_{|\alpha|=1,2}\|E_\alpha\|_{r_\alpha}^{p'_\alpha}\left(\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\right)^{\frac{\sigma-1}{\gamma}}\cdot|A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1}{\gamma}} \\
&\qquad+k^{\theta(q-1)\delta}\left(\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\right)^{\frac{\sigma-1-\delta}{\gamma}}\cdot|A_n(k)|^{1-\frac{\sigma-1-\delta}{\gamma}} \\
&\qquad+k^{\theta(q-1)}\|f\|_t\Big[\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx\Big]^{\frac{\sigma}{\gamma}}\cdot|A_n(k)|^{1-\frac{1}{t}-\frac{\sigma}{\gamma}}\Big\}.
\end{aligned} \tag{28}$$

Now, we set

$$\varphi(u_n) = \int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,dx$$

and

$$\chi = \tfrac{q}{q^*},$$

$$\beta_1 = \tfrac{\sigma-1+(\lambda+\theta)q}{\gamma},\ \beta_2 = \tfrac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma},\ \beta_3 = \tfrac{\sigma+\theta(q-1)}{\gamma},$$

$$\beta_4 = \tfrac{\sigma-1}{\gamma},\ \beta_5 = \tfrac{\sigma-1-\delta}{\gamma},\ \beta_6 = \tfrac{\sigma}{\gamma}.$$

Then, inequality (28) becomes

$$\begin{aligned}\big[\varphi(u_n)\big]^{\chi} \le c\Big\{ & \big[\varphi(u_n)\big]^{\beta_1}|A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1+(\lambda+\theta)q}{\gamma}} \\ & +\big[\varphi(u_n)\big]^{\beta_2}|A_n(k)|^{1-\frac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma}} + \big[\varphi(u_n)\big]^{\beta_3}|A_n(k)|^{1-\frac{1}{t}-\frac{\sigma+\theta(q-1)}{\gamma}} \\ & +k^{(\lambda+\theta)q}\big[\varphi(u_n)\big]^{\beta_4}|A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1}{\gamma}} + k^{\theta(q-1)\delta}\big[\varphi(u_n)\big]^{\beta_5}|A_n(k)|^{1-\frac{\sigma-1-\delta}{\gamma}} \\ & +k^{\theta(q-1)}\big[\varphi(u_n)\big]^{\beta_6}|A_n(k)|^{1-\frac{1}{t}-\frac{\sigma}{\gamma}}\Big\}.\end{aligned} \tag{29}$$

Thanks to assumption (12), the exponents β_1–β_6 are less than χ. Therefore, we can apply Lemma 2 to the previous inequality, with

$$\begin{aligned} a_1 &= |A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1+(\lambda+\theta)q}{\gamma}}, \quad a_2 = |A_n(k)|^{1-\frac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma}}, \\ a_3 &= |A_n(k)|^{1-\frac{1}{t}-\frac{\sigma+\theta(q-1)}{\gamma}}, \quad a_4 = k^{(\lambda+\theta)q}|A_n(k)|^{1-\frac{p'_\alpha}{r_\alpha}-\frac{\sigma-1}{\gamma}}, \\ a_5 &= k^{\theta(q-1)\delta}|A_n(k)|^{1-\frac{\sigma-1-\delta}{\gamma}}, \quad a_6 = k^{\theta(q-1)}|A_n(k)|^{1-\frac{1}{t}-\frac{\sigma}{\gamma}}, \end{aligned}$$

and we obtain

$$\int_\Omega \big[|u_n|-k\big]_+^{\gamma}\,\mathrm{d}x \le c\sum_{i=1}^{6} k^{\tau_i\gamma}\,|A_n(k)|^{\gamma_i}, \tag{30}$$

where

$$\begin{aligned} \gamma_1 &= \tfrac{\gamma}{1-(\lambda+\theta)}\Big(1-\tfrac{p'_\alpha}{r_\alpha}-\tfrac{\sigma-1+(\lambda+\theta)q}{\gamma}\Big), \\ \gamma_2 &= \tfrac{\gamma}{q+\delta-\theta(q-1)\delta}\Big(1-\tfrac{\sigma-1-\delta+\theta(q-1)\delta}{\gamma}\Big) \\ \gamma_3 &= \tfrac{\gamma}{(q-1)(1-\theta)}\Big(1-\tfrac{1}{t}-\tfrac{\sigma+\theta(q-1)}{\gamma}\Big), \\ \gamma_4 &= \tfrac{\gamma}{q}\Big(1-\tfrac{p'_\alpha}{r_\alpha}-\tfrac{\sigma-1}{\gamma}\Big) \\ \gamma_5 &= \tfrac{\gamma}{q+\delta}\Big(1-\tfrac{\sigma-1-\delta}{\gamma}\Big) \\ \gamma_6 &= \tfrac{\gamma}{q-1}\Big(1-\tfrac{1}{t}-\tfrac{\sigma}{\gamma}\Big) \end{aligned}$$

and

$$\begin{aligned} &\tau_1 = 0, && \tau_2 = 0, && \tau_3 = 0, \\ &\tau_4 = (\lambda+\theta), && \tau_5 = \tfrac{\theta(q-1)\delta}{q+\delta}, && \tau_6 = \theta. \end{aligned}$$

Under assumptions of $0 \le \theta < \frac{q-p}{p(q-1)}$ and $\lambda+\theta<1$, every number τ_i, $i=1,\dots,6$, is less than 1; furthermore, hypotheses (8), (9) and (12) ensure that it is possible to choose a large enough σ such that $\gamma_i > 1$ for any $i=1,\dots,6$.

Now, for every $h > k \ge 1$, with $\big[|u_n|-k\big]_+ \ge h-k$ on $A_n(h)$, from (30), we have

$$|A_n(h)| \le \sum_{i=1}^{6} \frac{c\,k^{\tau_i\gamma}}{(h-k)^{\gamma}}|A_n(k)|^{\gamma_i}.$$

Hence, there exists i, $i=1,\dots,6$, such that

$$\frac{1}{6}|A_n(h)| \le \frac{c\,k^{\tau_i\gamma}}{(h-k)^{\gamma}}|A_n(k)|^{\gamma_i}.$$

Therefore, using Lemma 3, we conclude that there exist two positive constants k^* and d_0 (independent of n) such that

$$|A_n(k)| = 0 \qquad \text{for every } k \ge k^* + d_0.$$

Hence,

$$\|u_n\|_{L^\infty(\Omega)} \le M \quad \text{for every } n \in \mathbb{N} \tag{31}$$

with $M = k^* + d_0$, and the proof of the lemma is complete. □

The next lemma deals with the boundedness of u_n in the energy space $\mathring{W}^{1,q}_{2,p}(\Omega)$.

Lemma 6. *Let hypotheses (5)–(9) be satisfied. Then, there exists a positive constant C, depending only on θ, N, q, p, ν_1, ν_2, $|\Omega|$, $\|E_\alpha\|_{L^{r_\alpha}(\Omega)}$, $|\alpha| = 1, 2$, and $||f||_{L^t(\Omega)}$, such that*

$$||u_n||_{\mathring{W}^{1,q}_{2,p}(\Omega)} \le C \text{ for every } \ n \in \mathbb{N}. \tag{32}$$

Proof. Using $v = u_n$ as a test function in the integral identity (15) and applying the ellipticity condition (5), the growth condition (6) and Young's inequality, we have

$$\begin{aligned}
&\nu_1 \sum_{|\alpha|=1,2} \int_\Omega \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\,\mathrm{d}x \\
&\le \tau \sum_{|\alpha|=1,2} \int_\Omega \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\,\mathrm{d}x + C(\tau) \sum_{|\alpha|=1,2} \int_\Omega |E_\alpha|^{p'_\alpha}(1+|u_n|)^{\lambda q+\theta}\,\mathrm{d}x \\
&\qquad + \int_\Omega |f|\,|u_n|\,\mathrm{d}x.
\end{aligned}$$

Therefore, choosing a suitable $0 < \tau < \nu_1$ and taking into account (31), from the above inequality, we obtain

$$\int_\Omega \sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u_n|^{p_\alpha}\,\mathrm{d}x \le c \sum_{|\alpha|=1,2} (M+1)^{(\lambda+\theta)q+1} \Big\{ \int_\Omega |E_\alpha|^{p'_\alpha}\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x \Big\}.$$

and the lemma follows. □

4. Existence of Bounded Solution

From Lemma 5 and Lemma 6, we infer that there exists a subsequence of $\{u_n\}$, not relabeled, and a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ such that

(a) $\{u_n\}$ converges to u weakly in $\mathring{W}^{1,q}_{2,p}(\Omega)$ and almost everywhere in Ω;
(b) $\{u_n\}$ converges to u weakly* in $\mathrm{L}^\infty(\Omega)$;
(c) $\{u_n\}$ converges to u strongly in $\mathrm{W}^{1,p}(\Omega)$;
(d) $\{u_n\}$ converges to u strongly in $\mathrm{L}^q(\Omega)$.

To pass to the limit in the approximating problems (15), first, we need to prove that the sequences $\{\mathrm{D}^\alpha u_n\}$, $|\alpha| = 1, 2$, are convergent almost everywhere in Ω.

To this aim, we exploit the following compactness result, whose proof is in [13].

Lemma 7. *Assume that hypotheses (5)–(7) hold and let $\{z_n\}$ be a sequence of functions such that*

$$z_n \to z \ \text{ in } \mathring{W}^{1,q}_{2,p}(\Omega) \text{ weakly and, i.e., in } \Omega \tag{33}$$

and

$$\begin{aligned}
\lim_{n\to+\infty} \sum_{|\alpha|=1,2} \int_\Omega \Big[&A_\alpha(x, z_n, \mathrm{D}^1 z_n, \mathrm{D}^2 z_n) \\
&- A_\alpha(x, z, \mathrm{D}^1 z, \mathrm{D}^2 z) \Big] \mathrm{D}^\alpha [z_n - z]\,\mathrm{d}x = 0.
\end{aligned} \tag{34}$$

Then, $\{z_n\}$ is relatively compact in the strong topology of $\mathring{W}^{1,q}_{2,p}(\Omega)$.

We take $v = u_n - u$ as a test function in the weak formulation of problem (15), and we obtain

$$\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x$$
$$= \sum_{|\alpha|=1,2} \int_\Omega E_\alpha \frac{u_n}{\frac{1}{n} + |u_n|} \frac{|u_n|^{\lambda(p_\alpha - 1)}}{1 + \frac{1}{n}|u_n|^{\lambda(p_\alpha - 1)}} \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x + \int_\Omega f_n [u_n - u] \, \mathrm{d}x.$$

Using (31), from the above equality, it follows that

$$\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x$$
$$\leq (1 + M)^{\lambda(q-1)} \sum_{|\alpha|=1,2} \int_\Omega E_\alpha \, \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x + \int_\Omega f \, [u_n - u] \, \mathrm{d}x \tag{35}$$

and therefore,

$$\sum_{|\alpha|=1,2} \int_\Omega \Big[A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) - A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) \Big] \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x$$
$$\leq (1 + M)^{\lambda(q-1)} \sum_{|\alpha|=1,2} \int_\Omega E_\alpha \, \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x$$
$$+ \sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) \, \mathrm{D}^\alpha [u_n - u] \, \mathrm{d}x + \int_\Omega f \, [u_n - u] \, \mathrm{d}x \tag{36}$$

The sums on the right-hand side of (36) tend to zero as n tends to $+\infty$ since $\{u_n\}$ converges to u weakly in $\mathring{W}^{1,q}_{2,p}(\Omega)$, and $A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u)$ and E_α belong to $L^{p'_\alpha}(\Omega)$, $|\alpha| = 1, 2$, thanks to (6) and (8). The last integral converges to zero as n tends to $+\infty$ because $\{u_n\}$ converges to u weakly* in $L^\infty(\Omega)$.

On the other hand, the boundedness of $\|u_n\|_{L^\infty}$ implies $T_n(u_n) = u_n$ for sufficiently large n; therefore, the hypotheses of Lemma 7 are satisfied, and we conclude that, up to a subsequence,

$$u_n \to u \quad \text{strongly in } \mathring{W}^{1,q}_{2,p}(\Omega).$$

Proof of Theorem 1. Let $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ be a test function in (15).

Taking into account that $T_n(u_n) = u_n$ for $n > M$ and using the growth condition (6) and the strong convergence of $\{u_n\}$ to u in $\mathring{W}^{1,q}_{2,p}(\Omega)$, we have

$$\lim_{n \to +\infty} \sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) \mathrm{D}^\alpha v \, \mathrm{d}x$$
$$= \sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) \mathrm{D}^\alpha v \, \mathrm{d}x.$$

Now, we define the sequence $\{Y^\alpha_n\}$, where

$$Y^\alpha_n(x) = E_\alpha(x) \frac{u_n(x)}{\frac{1}{n} + |u_n(x)|} \frac{|u_n(x)|^{\lambda(q-1)}}{1 + \frac{1}{n}|u_n(x)|^{\lambda(q-1)}} : \Omega \to \mathbb{R}^N, \text{ if } |\alpha| = 1,$$

and

$$Y_n^\alpha(x) = E_\alpha(x)\frac{u_n(x)}{\frac{1}{n}+|u_n(x)|}\,\frac{|u_n(x)|^{\lambda(p-1)}}{1+\frac{1}{n}|u_n(x)|^{\lambda(p-1)}} : \Omega \to \mathbb{R}^{N^2}, \text{ if } |\alpha| = 2.$$

Thanks to the convergence of $\{u_n(x)\}$ to $u(x)$, i.e., $x \in \Omega$, the sequence $\{Y_n^\alpha(x)\}$ converges to $E_\alpha(x)\,|u(x)|^{\lambda(p_\alpha-1)-1}\,u(x)$, $|\alpha| = 1,2$ i.e., $x \in \Omega$.

Furthermore, by exploiting (31), we deduce that

$$\left|E_\alpha(x)\frac{u_n(x)}{\frac{1}{n}+|u_n(x)|}\,\frac{|u_n(x)|^{\lambda(p_\alpha-1)}}{1+\frac{1}{n}|u_n(x)|^{\lambda(p_\alpha-1)}}\right| \le M^{\lambda(p_\alpha-1)}\,|E_\alpha(x)|,\ |\alpha| = 1,2$$

for any $x \in \Omega$. Hence, by the Lebesgue Theorem,

$$Y_n^\alpha \to E_\alpha\,|u|^{\lambda(p_\alpha-1)-1}\,u \quad \text{strongly in } L^{p'_\alpha}(\Omega),\ |\alpha| = 1,2.$$

We can pass to the limit as $n \to +\infty$ in the weak formulation (15), and we find that u is a weak solution of problem (10), in the sense of Definition 1.

Finally, the almost-everywhere convergence of $\{u_n(x)\}$ to $u(x)$ in Ω and (31) yield the estimate

$$\|u\|_{L^\infty(\Omega)} \le M.$$

This completes the proof of Theorem 1. □

5. Proof of Theorem 2

We begin this section by recalling the following lemma according to Skripnik (see [29]). Let $y \in \mathbb{R}^N$ be a fixed point. Given $R > 0$, we denote by $B_R(y)$ an open ball centered on y with radius R.

Lemma 8. *Let $g \in W^{1,1}(B_R(y))$. Assume that there exists a measurable subset $G \subset B_R(y)$ and two positive constants C' and C'' such that*

$$\text{meas}(G) \ge C'R^N, \qquad \max_G |g| \le C''.$$

Then, there exists a positive constant C, independent of R, such that

$$\int_{B_R(y)} |g|\,\mathrm{d}x \le CR\left(\int_{B_R(y)} \sum_{|\beta|=1} |\mathrm{D}^\beta g|\,\mathrm{d}x + R^{N-1}\right).$$

Let $\Omega' \subset\subset \Omega$ be any strictly interior sub-region of Ω and $d' = \text{dist}(\Omega', \partial\Omega)$.

Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of problem (10), such that

$$|u(x)| \le M \quad \text{for i.e., } x \in \Omega. \tag{37}$$

We fix $x_0 \in \Omega'$, and for any R such that $0 < R < \dfrac{d'}{4}$, we set

$$\omega_1(R) = \operatorname*{ess\,inf}_{B_R(x_0)} u(x), \quad \omega_2(R) = \operatorname*{ess\,sup}_{B_R(x_0)} u(x), \quad \omega(R) = \omega_2(R) - \omega_1(R).$$

By virtue of Lemma 4.8 pag. 66 of [30], Theorem 2 will be proved as long as the inequality

$$\omega(R) \le \chi\omega(2R) + R^r \tag{38}$$

holds, where $0 < \chi < 1$, depending only on known parameters and independent of R, and r is a positive number such that the following inequalities hold:

$$|\alpha|p_\alpha + r(q - p_\alpha) \le q,\ |\alpha| = 1, 2, \qquad r\Big(q + \frac{pq}{q-2p}\Big) \le q,$$
$$r < \min\Big\{1 - \frac{N}{qt}, \frac{q-2p}{q-p}\Big\}. \tag{39}$$

To prove (38), we fix $0 < R < \min\{1, \frac{d'}{4}\}$, and we consider the sets

$$G_1(R) = \Big\{x \in B_R(x_0) \,:\, u(x) \le \frac{\omega_1(R) + \omega_2(R)}{2}\Big\}, \quad G_2(R) = B_R(x_0) \setminus G_1(R).$$

At least one of the following inequalities holds:

$$\operatorname{meas}\big(G_1(2R)\big) \ge \frac{1}{2}\operatorname{meas}\big(B_{2R}(x_0)\big), \tag{40}$$

or

$$\operatorname{meas}\big(G_2(2R)\big) \ge \frac{1}{2}\operatorname{meas}\big(B_{2R}(x_0)\big). \tag{41}$$

If inequality (40) holds, we define the function $v_0 : \Omega \to \mathbb{R}$ as follows:

$$v_0(x) = \begin{cases} 1 + \log \frac{2\omega}{\omega_2 - u(x) + R^r} & \text{if } x \in B_{2R}(x_0) \\ 0 & \text{if } x \in \Omega \setminus B_{2R}(x_0), \end{cases} \tag{42}$$

where $\omega = \omega(2R)$, $\omega_2 = \omega_2(2R)$. Otherwise, if inequality (41) holds, the function v_0 is defined by

$$v_0(x) = \begin{cases} 1 + \log \frac{2\omega}{u(x) - \omega_1 + R^r} & \text{if } x \in B_{2R}(x_0) \\ 0 & \text{if } x \in \Omega \setminus B_{2R}(x_0), \end{cases}$$

where $\omega_1 = \omega_1(2R)$.

For the sake of simplicity, we assume that (40) holds, so v_0 is the function defined in (42).

Our aim is to prove that

$$\operatorname*{ess\,sup}_{B_{2R}(x_0)} v_0(x) \le M' \tag{43}$$

with M' depending on N, p, q, θ, $||u||_{L^\infty(\Omega)}$, $||f||_{L^t(\Omega)}$ and $||E_\alpha||_{L^{r_\alpha}(\Omega)}$, with $|\alpha| = 1, 2$ and d'.

As a matter of fact, if (43) holds, then $u(x) \le \omega_2(2R) - 2\mathrm{e}^{-M'}\omega(2R) + R^r$ for i.e., $x \in B_{2R}(x_0)$; therefore, $\omega(R) \le (1 - 2\mathrm{e}^{-M'})\omega(2R) + R^r$, which gives inequality (38) with $\chi = 1 - 2\mathrm{e}^{-M'}$.

We can assume that

$$\omega(2R) \ge \frac{\mathrm{e}R^r}{2} \tag{44}$$

which implies $v_0(x) \ge 1$, i.e., in $B_{2R}(x_0)$; otherwise, inequality (38) holds.

We fix a function $\psi_0 \in C_0^\infty(\mathbb{R})$ such that

$$\begin{aligned} &0 \le \psi_0(s) \le 1 && \text{for every } s \in \mathbb{R}, \\ &\psi_0(s) = 1 && \text{if } |s| \le 1, \\ &\psi_0(s) = 0 && \text{if } |s| \ge 2. \end{aligned} \tag{45}$$

For any $x \in \Omega$, we set $\psi(x) = \psi_0\left(\frac{|x-x_0|}{R}\right)$ and

$$v(x) = \frac{[v_0(x)]^k}{(\omega_2 - u + R^r)^{q-1}}\psi^s(x) \tag{46}$$

with $k \geq 0$, $s \geq q$. Simple calculations show that $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, and the following assertion holds:

$$\mathrm{D}^\alpha v = \frac{k\,[v_0]^{k-1}}{(\omega_2 - u + R^r)^q}\mathrm{D}^\alpha u\psi^s + \frac{(q-1)\,[v_0]^k}{(\omega_2 - u + R^r)^q}\mathrm{D}^\alpha u\psi^s + R_{|\alpha|}(u)$$

where, for i.e., $x \in \Omega$,

$$\begin{cases} |R_{|\alpha|}(u)| \leq c\dfrac{s\,[v_0]^k\psi^{s-1}}{(\omega_2 - u + R^r)^{q-1}}\,\dfrac{1}{R} & \text{if } |\alpha| = 1 \\ \\ |R_{|\alpha|}(u)| \leq c\,\dfrac{(k+s+1)^2\,[v_0]^k\psi^{s-2}}{(\omega_2 - u + R^r)^{q-1}}\Big\{\dfrac{1}{R^2} + \displaystyle\sum_{|\beta|=1}\dfrac{|\mathrm{D}^\beta u|^2\,\psi^2}{(\omega_2 - u + R^r)^2}\Big\} & \text{if } |\alpha| = 2 \end{cases} \tag{47}$$

Now, we test the integral identity (11) with v. By using (5), (6) and (37), we obtain

$$\begin{aligned} &\frac{(q-1)}{(1+M)^{\theta(q-1)}}\int\limits_\Omega \frac{[v_0]^k}{(\omega_2 - u + R^r)^q}\Big\{\sum_{|\alpha|=1,2}|\mathrm{D}^\alpha u|^{p_\alpha}\Big\}\psi^s\,\mathrm{d}x \leq \\ &\leq c(k+1)\Big\{J_1 + J_2 + J_3 + \int\limits_\Omega \frac{|f|\,[v_0]^k}{(\omega_2 - u + R^r)^{q-1}}\psi^s\,\mathrm{d}x\Big\} \end{aligned} \tag{48}$$

where

$$J_1 = \sum_{|\alpha|=1,2}\int\limits_\Omega |E_\alpha|\,|u|^{\lambda(p_\alpha-1)}\frac{[v_0]^k|\mathrm{D}^\alpha u|}{(\omega_2 - u + R^r)^q}\psi^s\,\mathrm{d}x, \tag{49}$$

$$J_2 = \sum_{|\alpha|=1,2}\int\limits_\Omega |E_\alpha|\,|u|^{\lambda(p_\alpha-1)}|R_{|\alpha|}(u)|\,\mathrm{d}x, \tag{50}$$

and

$$J_3 = \sum_{|\alpha|=1,2}\int\limits_\Omega |A_\alpha(x, \mathrm{D}^1u, \mathrm{D}^2u)|\,|R_{|\alpha|}(u)|\,\mathrm{d}x. \tag{51}$$

Let us estimate J_1. By using Young's inequality with $\varepsilon > 0$ and taking into account (37), we derive

$$\begin{aligned} J_1 \leq\ &\varepsilon\int\limits_\Omega \frac{[v_0]^k}{(\omega_2 - u + R^r)^q}\Big\{\sum_{|\alpha|=1,2}|\mathrm{D}^\alpha u|^{p_\alpha}\Big\}\psi^s\,\mathrm{d}x \\ &+ C(c,\varepsilon)(M+1)^q\sum_{|\alpha|=1,2}\int\limits_\Omega \frac{[v_0]^k\,|E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}}}{(\omega_2 - u + R^r)^q}\psi^s\,\mathrm{d}x. \end{aligned} \tag{52}$$

In order to estimate J_2, we use the inequalities in (47). We obtain

$$\begin{aligned} J_2 \leq\ &c\textstyle\sum_{|\alpha|=1,2}\int\limits_\Omega |E_\alpha|\,|u|^{\lambda(p_\alpha-1)}\frac{(k+s+1)^{|\alpha|}\,[v_0]^k}{(\omega_2-u+R^r)^{q-1}}\,\frac{1}{R^{|\alpha|}}\,\psi^{s-|\alpha|}\,\mathrm{d}x \\ &+\textstyle\sum_{|\alpha|=2}\int\limits_\Omega |E_\alpha|\,|u|^{\lambda(p-1)}\frac{(k+s+1)^2[v_0]^k}{(\omega_2-u+R^r)^{q+1}}\,\sum_{|\beta|=1}|\mathrm{D}^\beta u|^2\,\psi^s\,\mathrm{d}x. \end{aligned} \tag{53}$$

Since $|\alpha|p_\alpha + r(q-p_\alpha) \le q$, by using Young's inequality and (37), the first sum on the right-hand side can be evaluated as follows:

$$\sum_{|\alpha|=1,2} \int_\Omega |E_\alpha|\,|u|^{p_\alpha-1} \frac{(k+s+1)^{|\alpha|}\,[v_0]^k}{(\omega_2-u+R^r)^{q-1}} \frac{1}{R^{|\alpha|}}\,\psi^{s-|\alpha|}\,\mathrm{d}x$$
$$\le c(M+k+s+1)^q \sum_{|\alpha|=1,2} \left\{ \int_\Omega \frac{[v_0]^k\,|E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}}}{(\omega_2-u+R^r)^q}\psi^s\,\mathrm{d}x + \int_\Omega \frac{[v_0]^k\,\psi^{s-|\alpha|p_\alpha}}{R^q}\,\mathrm{d}x \right\}. \tag{54}$$

Using the assumption $r\left(q+\frac{pq}{q-2p}\right) \le q$ and, again, Young's inequality and (37), the last sum on the right-hand side in (53) can be estimated as

$$\begin{aligned}
&\textstyle\sum_{|\alpha|=2} \int_\Omega |E_\alpha|\,|u|^{\lambda(p-1)} \frac{(k+s+1)^2[v_0]^k}{(\omega_2-u+R^r)^{q+1}} \sum_{|\beta|=1} |\mathrm{D}^\beta u|^2\,\psi^s\,\mathrm{d}x \\
&\quad \le \varepsilon \int_\Omega \frac{[v_0]^k}{(\omega_2-u+R^r)^q} \left[\textstyle\sum_{|\alpha|=1} |\mathrm{D}^\alpha u|^q\right]\psi^s\,\mathrm{d}x \\
&+C(c,\varepsilon)(M+k+s+1)^q \left\{ \textstyle\sum_{|\alpha|=2} \int_\Omega \frac{[v_0]^k\,|E_\alpha|^{\frac{p}{p-1}}}{(\omega_2-u+R^r)^q}\,\psi^s\,\mathrm{d}x + \sum_{|\alpha|=2} \int_\Omega \frac{[v_0]^k\,\psi^s}{R^q}\,\mathrm{d}x \right\}.
\end{aligned} \tag{55}$$

From inequalities (53), (54) and (55), we obtain

$$\begin{aligned}
J_2 &\le \varepsilon \int_\Omega \left[\textstyle\sum_{|\alpha|=1} |\mathrm{D}^\alpha u|^q\right] \frac{[v_0]^k}{(\omega_2-u+R^r)^q}\psi^s\,\mathrm{d}x \\
&+C(c,\varepsilon)(M+k+s+1)^q \left\{ \textstyle\sum_{|\alpha|=1,2} \int_\Omega \frac{[v_0]^k\,|E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}}}{(\omega_2-u+R^r)^q}\psi^s\,\mathrm{d}x + \int_\Omega \frac{[v_0]^k\,\psi^{s-q}}{R^q}\,\mathrm{d}x \right\}.
\end{aligned} \tag{56}$$

Finally , let us estimate J_3. By using (6) and taking into account (47), we have

$$\begin{aligned}
J_3 \le c\Bigg\{ &\textstyle\sum_{|\alpha|=1,2} \int_\Omega \left[\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right]^{\frac{p_\alpha-1}{p_\alpha}} \frac{(k+s+1)^{|\alpha|}\,[v_0]^k}{R^{|\alpha|}\,(\omega_2-u+R^r)^{q-1}}\psi^{s-|\alpha|}\,\mathrm{d}x \\
&+ \int_\Omega \left[\textstyle\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right]^{\frac{p-1}{p}} \frac{(k+s+1)^2[v_0]^k}{(\omega_2-u+R^r)^{q+1}} \sum_{|\beta|=1} |\mathrm{D}^\beta u|^2\psi^s\,\mathrm{d}x \Bigg\}.
\end{aligned} \tag{57}$$

Applying Young's inequality with the exponents $\frac{p_\alpha}{p_\alpha-1}$ and p_α, we can evaluate the first integrals in (57) as follows:

$$\begin{aligned}
&\textstyle\sum_{|\alpha|=1,2} \int_\Omega \left[\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right]^{\frac{p_\alpha-1}{p_\alpha}} \frac{(k+s+1)^{|\alpha|}\,[v_0]^k\,\psi^{s-|\alpha|}}{R^{|\alpha|}\,(\omega_2-u+R^r)^{q-1}}\,\mathrm{d}x \\
&\quad \le \varepsilon \int_\Omega \left[\textstyle\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right] \frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x + C(c,\varepsilon)(k+s+1)^q \int_\Omega \frac{[v_0]^k\,\psi^{s-q}}{R^q}\,\mathrm{d}x,
\end{aligned} \tag{58}$$

where we have used condition (39), that is, $|\alpha|p_\alpha + r(q-p_\alpha) \le q$, $|\alpha| = 1,2$.

In the last integrals of (57), we use Young's inequality with the exponents $\frac{p}{p-1}$, $\frac{q}{2}$ and $\frac{pq}{q-2p}$ and condition (39): $r\left(q+\frac{pq}{q-2p}\right) \le q$. We obtain

$$\begin{aligned}
&\int_\Omega \left[\textstyle\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right]^{\frac{p-1}{p}} \frac{(k+s+1)^2[v_0]^k}{(\omega_2-u+R^r)^{q+1}} \sum_{|\beta|=1} |\mathrm{D}^\alpha u|^2\psi^s\,\mathrm{d}x \\
&\quad \le \varepsilon \int_\Omega \left[\textstyle\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right] \frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x + C(c,\varepsilon)(k+s+1)^q \int_\Omega \frac{[v_0]^k\,\psi^s}{R^q}\,\mathrm{d}x.
\end{aligned} \tag{59}$$

From (58) and (59), we deduce

$$J_3 \le \varepsilon \int_\Omega \left[\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha}\right] \frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x + C(c,\varepsilon)(k+s+1)^q \int_\Omega \frac{[v_0]^k\,\psi^s}{R^q}\,\mathrm{d}x. \tag{60}$$

Finally, we estimate the integral involving the datum. Since (37) holds, we have

$$\int_\Omega \frac{|f|\,[v_0]^k}{(\omega_2-u+R^r)^{q-1}}\psi^s\,\mathrm{d}x \le c(2M+1)\int_\Omega \frac{|f|\,[v_0]^k}{(\omega_2-u+R^r)^{q}}\psi^s\,\mathrm{d}x. \tag{61}$$

Now, gathering (48), (52), (56), (60) and (61) and choosing a suitable $\varepsilon>0$, we obtain

$$\begin{aligned}&\int_\Omega \left[\sum_{|\alpha|=1,2}|\mathrm{D}^\alpha u|^{p_\alpha}\right]\frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x\\ &\quad\le c(1+k+s)^q(M+1)^{q+\theta(q-1)}\left\{\int_\Omega F\,\frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x+\int_\Omega\frac{[v_0]^k\,\psi^{s-q}}{R^q}\,\mathrm{d}x\right\}\end{aligned} \tag{62}$$

where we have denoted by F the following function:

$$F=\sum_{|\alpha|=1,2}|E_\alpha|^{\frac{p_\alpha}{p_\alpha-1}}+|f|+1. \tag{63}$$

Thanks to (8) and (9), it follows that there exists $\tau>\frac{N}{q}$ such that $F\in L^\tau(\Omega)$. Therefore, choosing $r<1-\frac{N}{q\tau}$ and applying Hölder's inequality, from (62), we obtain

$$\begin{aligned}&\int_\Omega \left[\sum_{|\alpha|=1,2}|\mathrm{D}^\alpha u|^{p_\alpha}\right]\frac{[v_0]^k\,\psi^s}{(\omega_2-u+R^r)^q}\,\mathrm{d}x\\ &\quad\le c(1+k+s)^q\left\{\frac{1}{R^{rq}}\left(\int_\Omega|F|^\tau\,\mathrm{d}x\right)^{\frac{1}{\tau}}\left(\int_\Omega\left|[v_0]^k\,\psi^{s-q}\right|^{\frac{\tau}{\tau-1}}\,\mathrm{d}x\right)^{\frac{\tau-1}{\tau}}\right.\\ &\qquad\left.+\frac{1}{R^{q-\frac{N}{\tau}}}\left(\int_\Omega\left|[v_0]^k\,\psi^{s-q}\right|^{\frac{\tau}{\tau-1}}\,\mathrm{d}x\right)^{\frac{\tau-1}{\tau}}\right\}\\ &\quad\le c(1+k+s)^q\left\{\frac{1}{R^{q-\frac{N}{\tau}}}\left(\int_\Omega\left|[v_0]^k\,\psi^{s-q}\right|^{\frac{\tau}{\tau-1}}\,\mathrm{d}x\right)^{\frac{\tau-1}{\tau}}\right\}\end{aligned} \tag{64}$$

The above relation is crucial for organizing the iterative Moser's method.
Given $k\ge 0$ and $s>0$, let us define

$$J(k,s)=\frac{1}{R^N}\int_\Omega [v_0]^k\psi^s\,\mathrm{d}x.$$

The following Lemma holds:

Lemma 9. *We assume that conditions (5)–(9) are satisfied. Let $u\in \mathring{W}^{1,q}_{2,p}(\Omega)$ be a bounded solution of problem (10).*

Then, if $k>0$ and $2q^*<s<C_0(k+1)$ (C_0 is an absolute constant that we shall define later), the inequality

$$J(k,s)\le c(k+1)^{q^*}\left[J(kl,sl-2lq^*)\right]^{\frac{1}{l}} \tag{65}$$

holds, where

$$l=\frac{\tau}{\tau-1}\,\frac{N-q}{N}<1.$$

Proof. Let $k>0$ and $2q^*<s<C_0(k+1)$. Applying Sobolev's embedding to the function $[v_0]^{\frac{k}{q^*}}\,\psi^{\frac{s}{q^*}}$ and taking into account that $v_0(x)\ge 1$, we obtain

$$J(k,s) \le c\,\mathcal{S}_1 \frac{(k+s+1)^{q^*}}{R^N} \Bigg(\int\limits_\Omega \sum_{|\beta|=1} |\mathrm{D}^\beta u|^q \frac{[v_0]^{\frac{kq}{q^*}} \psi^{\frac{sq}{q^*}-q}}{(\omega_2 - u + R^r)^q}\,\mathrm{d}x + \frac{1}{R^q} \int\limits_\Omega [v_0]^{\frac{kq}{q^*}} \psi^{\frac{sq}{q^*}-q}\,\mathrm{d}x \Bigg)^{\frac{q^*}{q}}, \tag{66}$$

where $\mathcal{S}_1$ is the Sobolev constant.

Now, estimating the first addend on the right-hand side of the above inequality by means of (64) with $\frac{kq}{q^*}$ and $\frac{sq}{q^*} - q$ instead of k and s and the second one by means of Hölder's inequality, we obtain

$$J(k,s) \le c\,C_0 (1+k)^{q^*} \left(\frac{1}{R^N} \int\limits_\Omega [v_0]^{k\frac{kq}{q^*}\frac{\tau}{\tau-1}} \psi^{(\frac{sq}{q^*}-q)\frac{\tau}{\tau-1}}\,\mathrm{d}x \right)^{\frac{\tau-1}{\tau}\frac{q^*}{q}} \tag{67}$$

and (65) follows. □

For $i = 0, 1, 2, \ldots$, and C_0 such that $C_0 > \frac{2l}{1-l}\frac{q^*}{q}$, we define

$$k_i = q l^{-i}, \quad s_i = \frac{2lq^*}{1-l}(l^{-i} - 1), \quad J_i = J(k_i, s_i)$$

In the next lemma, we will prove the boundedness of $J_0 = J(q, 0)$.

Lemma 10. *We assume that conditions (5)–(9) are satisfied. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ be a bounded solution of problem (10). Then, the inequality*

$$\int\limits_{B_{2R}(x_0)} [v_0]^q\,\mathrm{d}x \le C_1 R^N \tag{68}$$

holds, where C_1 is a positive constant depending only on known parameters and independent of R.

Proof. We note that $1 \le [v_0(x)]^q \le (1 + \log 4)^q$ for any $x \in G_1(R)$, so we can apply Lemma 8 to the function $[v_0]^q \in W^{1,1}\big(B_{2R}(x_0)\big)$ and $G = G_1(R)$.

By means of Young's inequality with $\varepsilon > 0$, we obtain

$$\begin{aligned} \frac{1}{R^N} \int\limits_{B_{2R}(x_0)} [v_0]^q\,\mathrm{d}x &\le 1 + \frac{q}{R^{N-1}} \int\limits_{B_{2R}(x_0)} \frac{[v_0]^{q-1}}{\omega_2 - u + R^r} \sum_{|\alpha|=1} |\mathrm{D}^\alpha u|\,\mathrm{d}x \\ &\le c \Bigg\{ 1 + \frac{\varepsilon}{R^N} \int\limits_{B_{2R}(x_0)} [v_0]^q\,\mathrm{d}x + \frac{C(\varepsilon)}{R^{N-q}} \int\limits_{B_{2R}(x_0)} \frac{1}{(\omega_2 - u + R^r)^q} \sum_{|\alpha|=1} |\mathrm{D}^\alpha u|^q\,\mathrm{d}x \Bigg\}. \end{aligned} \tag{69}$$

Now, we estimate the last integral in the above inequality.

We then consider the following cut-off function:

$$\varphi(x) = \psi_0\Big(\frac{x - x_0}{2R}\Big), \quad x \in \Omega$$

where ψ_0 is the same function defined in (45), and we consider the following test function:

$$v_1(x) = \frac{1}{(\omega_2 - u(x) + R^r)^{q-1}} \varphi^q(x).$$

Substituting v_1 into the integral identity (11) and proceeding in the same way as we obtained (64), we deduce

$$\int\limits_{B_{2R}(x_0)} \Big[\sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u|^{p_\alpha} \Big] \frac{1}{(\omega_2 - u + R^r)^q} \,\mathrm{d}x \le \frac{c}{R^{q-\frac{N}{\tau}}} \Big[\mathrm{meas}\big(B_{4R}(x_0)\big) \Big]^{\frac{\tau-1}{\tau}} \le c\, R^{N-q} \tag{70}$$

Choosing a suitable $0 < \varepsilon < 1$ and taking into account (70), from (69), we obtain

$$\frac{1}{R^N} \int\limits_{B_{2R}(x_0)} [v_0]^q \,\mathrm{d}x \le c$$

and the lemma is proved. □

From Lemmas 9 and 10, we deduce

$$J_i^{\frac{1}{k_i}} \le c J_0^{\frac{1}{q}} \quad \text{for } i = 0, 1, 2, \dots \tag{71}$$

and

$$\|v_0\|_{L^\infty\left(B_R(x_0)\right)} = \lim_{i\to+\infty} \left(\frac{1}{R^N} \int\limits_{B_R(x_0)} [v_0]^{k_i} \,\mathrm{d}x \right)^{\frac{1}{k_i}} \le c \limsup_{i\to+\infty} J_i^{\frac{1}{k_i}} \le c\, J_0^{\frac{1}{q}}.$$

Therefore, (43) holds, and the proof of Theorem 2 is now complete.

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