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Four Different Ulam-Type Stability for Implicit Second-Order Fractional Integro-Differential Equation with M-Point Boundary Conditions

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Abstract: In this paper, we discuss the existence and uniqueness of a solution for the implicit two-order fractional integro-differential equation with m-point boundary conditions by applying the Banach fixed point theorem. Moreover, in the paper we establish the four different varieties of Ulam stability (Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam-Rassias stability, and generalized Hyers–Ulam–Rassias stability) for the given problem.

Keywords: fractional differential equation; Caputo's fractional derivative; Ulam stability; two-order; fixed point theorem

MSC: 34A08; 34B10; 34B15; 26D10

1. Introduction

The derivative and integral of a function are generalized to non-integer order in fractional calculus. Despite this calculus's ancient origins, it has only recently been relevant in the fields of science and engineering (mathematical physics, hydrology, biology, and mechanics) [1–5]. The existence and singularity of solutions to differential equations with fractional derivatives have been discussed by some academics ([6–8] and references therein).

In 1940, Ulam delivered a lecture at Wisconsin University, introducing the concept of functional equations' stability. Within this discussion, Ulam posed a question [9,10] that was initially addressed by Hyers in 1941 [11]. Consequently, this type of stability became known as Ulam–Hyers stability. Rassias significantly developed the concept of Ulam–Hyers stability in 1978 [12]. Since then, a functional equation's stability has been extensively studied in a multitude of articles and through presentations delivered at various conferences [13–15].

The primary goal of this paper is to study the Ulam-type stability for the following fractional integro-differential problem with tow-orders (**FIDP**)

$$ {}^{c}D_{0+}^{\beta}\left(p(t){}^{c}D_{0+}^{\alpha}u(t)\right) = f(t,u(t),\Xi_k u(t),\Xi_h u(t)), \tag{1}$$

$$\Xi_k u(t) = \int_0^t k(t,s){}^{c}D_{0+}^{\beta}\left(p(s){}^{c}D_{0+}^{\alpha}u(s)\right)ds, \quad \Xi_h u(t) = \int_0^t h(t,s){}^{c}D_{0+}^{\beta}\left(p(s){}^{c}D_{0+}^{\alpha}u(s)\right)ds,$$

with the conditions

$$u(0) = u'(0) = u^{(3)}(0) = 0, \quad u''(T) = \sum_{i=1}^{m-2}\eta_i u(\xi_i) \quad and \quad {}^{c}D^{\alpha}u(t)\,|_{t=0} = u_{\alpha}, \tag{2}$$

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for all $t \in [0,T]$ and ${}^{c}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $\alpha \in (3,4)$, $\beta \in (0,1)$ such that $3 < \alpha + \beta < 4$, $p(t)$ on $[0,T]$ and $k(t,s)$, $h(t,s)$ on $[0,T] \times [0,T]$ are real given functions, $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a given function.

2. Preliminaries

In this section, we introduce some definitions and proprieties, which are used in this paper.

Definition 1 ([1,2]). *The Riemann–Liouville fractional (arbitrary) integral of order $\alpha > 0$ of the function $f \in L^1[0,T]$ is formally defined by*

$$I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{3}$$

where Γ is the classical Gamma function.

Definition 2 ([1,2]). *The Caputo fractional derivative of order $\alpha > 0$ for a given function $f(t)$ on $[0,T]$ is defined by*

$${}^{c}D^{\alpha}_{0^+} f(t) = D^{\alpha}_{0^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right], \tag{4}$$

where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α and $D^{\alpha}_{0^+}$ is the Riemann- Liouville fractional derivative operator of order α defined by

$$D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-1-\alpha} f(s) ds = D^n I^{n-\alpha}_{0^+} f(t), \ \text{for } t > 0. \tag{5}$$

Lemma 1 ([16]). *Let $\alpha > 0$, $u \in C^n(\mathbb{R})$, then*

$$I^{\alpha}({}^{c}D^{\alpha} u(t)) = u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

To define Ulam's stability for fractional integro-differential problem (1) and (2).

Definition 3. *If there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $w \in AC([0,T], \mathbb{R})$ of the inequality*

$$\left| {}^{c}D^{\beta}_{0^+} \Big(p(t) {}^{c}D^{\alpha}_{0^+} w(t) \Big) - f\Big(t, w(t), \Xi_k w(t), \Xi_h w(t)\Big) \right| \leq \epsilon, \tag{6}$$

*there exists a solution $u \in AC([0,T], \mathbb{R})$ of the **FIDP** (1) and (2) with*

$$|w(t) - u(t)| \leq c_f \epsilon, \qquad t \in [0,T],$$

*then the **FIDP** (1) and (2) is Hyers–Ulam stable.*

Definition 4. *If there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\theta_f(0) = 0$ such that for each solution $w \in AC([0,T], \mathbb{R})$ of the inequality (6), for each $\epsilon > 0$ there exists a solution $u \in AC([0,T], \mathbb{R})$ of (1) and (2) with*

$$|w(t) - u(t)| \leq \theta_f(\epsilon), \quad t \in [0,T],$$

*thus, the **FIDP** (1) and (2) is generalized Hyers–Ulam stable.*

Definition 5. *Let $\psi : [0,T] \to \mathbb{R}_+$ be continues function, for $t \in [0,T]$. The **FIDP** (1) and (2) is Hyers–Ulam–Rassias stable with respect to ψ if there exists $c_{f,\psi} > 0$ where for every $\epsilon > 0$ and for each solution $w \in AC([0,T],\mathbb{R})$ of the inequality*

$$\left|{}^{c}D^{\beta}_{0^+}\left(p(t){}^{c}D^{\alpha}_{0^+}w(t)\right) - f\left(t, w(t), \Xi_k w(t), \Xi_h w(t)\right)\right| \leq \epsilon\psi(t), \tag{7}$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of (1) and (2) with

$$|w(t) - u(t)| \leq c_{f,\psi}\epsilon\psi(t), \qquad t \in [0,T].$$

Definition 6. *Let $\psi : [0,T] \to \mathbb{R}_+$ be continues function, for $t \in [0,T]$. The **FIDP** (1) and (2) is generalized stability according to the Hyers–Ulam–Rassias with respect to ψ if there exists $c_{f,\psi} > 0$ such that for each solution $w \in AC([0,T],\mathbb{R})$ of the inequality*

$$\left|{}^{c}D^{\beta}_{0^+}\left(p(t){}^{c}D^{\alpha}_{0^+}w(t)\right) - f\left(t, w(t), \Xi_k w(t), \Xi_h w(t)\right)\right| \leq \psi(t), \tag{8}$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of (1) and (2) with

$$|w(t) - u(t)| \leq c_{f,\psi}\psi(t), \qquad t \in [0,T].$$

Remark 1. *It is clear that: (i) Definition 3 $\Longrightarrow$ Definition 4 , (ii) Definition 5 $\Longrightarrow$ Definition 6 (iii) Definition 5 for $\psi(t) = 1 \Longrightarrow$ Definition 3.*

Remark 2. *A function $w \in AC([0,T],\mathbb{R})$ meets the inequality (6) if and only if a function $g \in C([0,T],\mathbb{R})$ exists such that*

(i) $|g(t)| \leq \epsilon$, $t \in [0,T]$.

(ii) ${}^{c}D^{\beta}_{0^+}\left(p(t){}^{c}D^{\alpha}_{0^+}w(t)\right) = f\left(t, w(t), \Xi_k w(t), \Xi_h w(t)\right) + g(t)$, $t \in [0,T]$.

Remark 3. *A function $w \in AC([0,T],\mathbb{R})$ meets the inequality (8) if and only if a function $g \in C([0,T],\mathbb{R})$ exists such that*

(i) $|g(t)| \leq \psi(t)$, $t \in [0,T]$.

(ii) ${}^{c}D^{\beta}_{0^+}\left(p(t){}^{c}D^{\alpha}_{0^+}w(t)\right) = f\left(t, w(t), \Xi_k w(t), \Xi_h w(t)\right) + g(t)$, $t \in [0,T]$.

Remark 4. *A function $w \in AC([0,T],\mathbb{R})$ meets the inequality (7) if and only if a function $g \in C([0,T],\mathbb{R})$ exists such that*

(i) $|g(t)| \leq \epsilon\psi(t)$, $t \in [0,T]$.

(ii) ${}^{c}D^{\beta}_{0^+}\left(p(t){}^{c}D^{\alpha}_{0^+}w(t)\right) = f\left(t, w(t), \Xi_k w(t), \Xi_h w(t)\right) + g(t)$, $t \in [0,T]$.

3. Existence and Uniqueness

In this section, by mentioning basic lemmas, we introduce conditions adequate for the existence and the uniqueness of solution to the **FIDP** (1) and (2) using Banach's fixed point theorem.

Lemma 2. *A function $u \in AC([0,T],\mathbb{R})$ is a solution of the **FIDP** (1) and (2) if and only if u is a solution of the following fractional integral equation for $t \in [0,T]$*

$$\begin{aligned} u(t) &= \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0u_\alpha}{p(s)}\right)ds \\ &+ \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}\eta_i\int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0u_\alpha}{p(s)}\right)ds, \end{aligned} \quad (9)$$

where x is the solution of fractional integral equation

$$x(t) = f\left(t, u(t), \int_0^t k(t,s)x(s)ds, \int_0^t h(t,s)x(s)ds\right),$$

and $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^2(T-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, & 0 < s < t < T, \\ -\dfrac{t^2(T-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, & 0 < t < s < T. \end{cases} \quad (10)$$

Proof. Let ${}^cD_{0^+}^{\beta}\left(p(t){}^cD_{0^+}^{\alpha}u(t)\right) = x(t)$ in Equation (1), then

$$x(t) = f\left(t, u(t), \int_0^t k(t,s)x(s)ds, \int_0^t h(t,s)x(s)ds\right),$$

assume that u a solution of the **FIDP** (1) and (2) then we have $I_{0^+}^{\beta}\left({}^cD_{0^+}^{\beta}\left(p(t){}^cD_{0^+}^{\alpha}u(t)\right)\right) = I_{0^+}^{\beta}x(t)$, $\beta \in (0,1)$, then

$${}^cD_{0^+}^{\alpha}u(t) = \frac{1}{p(t)}I_{0^+}^{\beta}x(t) + \frac{c_0}{p(t)},$$

where c_0 is a real constant, for $t = 0$, ${}^cD_{0^+}^{\alpha}u(0) = \dfrac{c_0}{p_0} = u_\alpha$ implies that $c_0 = p_0u_\alpha$.

Applying $I_{0^+}^{\alpha}$, we get

$$I_{0^+}^{\alpha}\left({}^cD_{0^+}^{\alpha}u(t)\right) = I_{0^+}^{\alpha}\left(\frac{1}{p(t)}I_{0^+}^{\beta}x(t) + \frac{p_0u_\alpha}{p(t)}\right), \qquad \alpha \in (3,4),$$

then

$$u(t) = I_{0^+}^{\alpha}\left(\frac{1}{p(t)}I^{\beta}x(t) + \frac{p_0u_\alpha}{p(t)}\right) + a_0 + a_1t + a_2t^2 + a_3t^3, \quad (11)$$

where a_0, a_1, a_2 and a_3 are some real constants. Using the boundary conditions $u(0) = u'(0) = u^{(3)}(0) = 0$, we find

$$a_0 = a_1 = a_3 = 0.$$

The boundary condition $u''(T) = \sum_{i=1}^{m-2}\eta_iu(\xi_i)$ we get

$$\begin{aligned} a_2 &= \frac{1}{2-\sum_1^{m-2}\eta_i\xi_i^2}\Big\{\frac{-1}{\Gamma(\alpha-2)}\int_0^T \frac{(T-s)^{\alpha-3}}{p(s)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}x(\tau)d\tau + p_0u_\alpha\Big)ds \\ &+ \sum_1^{m-2}\eta_i\frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{p(s)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}x(\tau)d\tau + p_0u_\alpha\Big)ds\Big\}. \end{aligned}$$

Substituting a_0, a_1, a_2 and a_3 into (11) we have (9). □

Now, we present our existence result based on the Banach contraction fixed point theorem.

Theorem 1. *Assume that*

(H_1) $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous,* $k(t,s)$, $h(t,s)$ *on* $[0,T] \times [0,T]$ *are continuous and there exist constants* $L, l_k, l_h > 0$ *such that*

$$|f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| \quad \leq \quad L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for any $u_i, v_i \in \mathbb{R}$, $(i = 1,2,3)$ *and* $t \in [0,T]$,

$$\max_{t,s\in[0,T]} k(t,s) = l_k, \quad \max_{t,s\in[0,T]} h(t,s) = l_h,$$

(H_2) $p \in AC([0,T],\mathbb{R})$ *such that* $p(t) \neq 0 \;\; t \in [0,T]$ *with* $p(0) = p_0$.

(H_3) *For* $\inf_{t\in[0,T]}|p(t)| = q$ *we have*

$$\zeta = L\Delta < 1, \tag{12}$$

where

$$\Delta = \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3} G_0}{\left(2 - \sum_{i=1}^{m-2} \eta_i \xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2} |\eta_i| \right) \frac{1}{1 - LT(l_k + l_h)}, \tag{13}$$

and $\max_{t,s} G(t,s) = G_0$. *If* $(H_1) - (H_3)$ *are satisfied, then the* ***FIDP*** (1) *and* (2) *has a unique solution.*

Proof. First, we denote by $X = AC([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the sup norm $\|u\|_\infty = \sup_{t\in[0,T]}|u(t)|$. Define the operator $A : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$\begin{aligned} Au(t) \quad = \quad & \int_0^T G(t,s) \times \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)} x(\tau) d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \\ & + \frac{t^2}{2 - \sum_{i=1}^{m-2} \eta_i \xi_i^2} \sum_{i=1}^{m-2} \eta_i \int_0^T G(\xi_i, s) \times \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)} x(\tau) d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds, \end{aligned} \tag{14}$$

it is clear that the fixed points of the operator A are solution of **FIDP** (1) and (2).

From hypothesis (H_1), we have

$$|f(t,u_1,u_2,u_3) - f(t,0,0,0)| \leq L(|u_1| + |u_2| + |u_3|),$$

then

$$|f(t,u_1,u_2,u_3)| \leq |f(t,u_1,u_2,u_3) - f(t,0,0,0)| + |f(t,0,0,0)| \leq L(|u_1| + |u_2| + |u_3|) + M,$$

where $M = \sup_{t\in[0,T]}|f(t,0,0,0)|$. Then, we define the nonempty, bounded and convex closed set by

$$B_r = \{u \in C([0,T],R) : \|u\| \leq r\},$$

where

$$r \geq \frac{\Delta M + \frac{G_0 T p_0 u_\alpha}{q} + \frac{T^3 G_0}{2 - \sum_{i=1}^{m-2} \eta_i \xi_i^2} \sum_{i=1}^{m-2} |\eta_i| \left(\frac{p_0 u_\alpha}{q} \right)}{1 - \Delta L}, \tag{15}$$

we demonstrate that the operator A defined by (14) meets the hypothesis of the fixed point theorem of Banach. The proof will be two steps.

Step 1. Maps bounded sets into bounded sets in B_r, i.e., $A(B_r) \subset B_r$

For each $t \in [0,T]$, by condition (H_1), we have

$$\begin{aligned}
|Au(t)| &= \left| \int_0^T G(t,s) \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)} x(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right. \\
&+ \left. \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} \eta_i \int_0^T G(\xi_i,s) \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)} x(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right| \\
&\leq \int_0^T |G(t,s)| \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|} |x(\tau)|d\tau + \frac{p_0 u_\alpha}{|p(s)|} \right) ds \\
&+ \frac{|t^2|}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} |\eta_i| \int_0^T |G(\xi_i,s)| \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|} |x(\tau)|d\tau + \frac{p_0 u_\alpha}{|p(s)|} \right) ds. \quad (16)
\end{aligned}$$

We take $u \in AC([0,T],\mathbb{R})$, then for $t \in [0,T]$, we have

$$\begin{aligned}
|x(t)| &= \left| f\left(t, u(t), \int_0^t k(\tau,s)x(s)ds, \int_0^t h(\tau,s)x(s)ds\right) \right| \\
&\leqslant L(|u(t)| + T(l_k+l_h)|x(t)|) + M,
\end{aligned}$$

then

$$\|x\| \leqslant \frac{L\|u\| + M}{1 - LT(l_k+l_h)}.$$

Equation (16) then reduces to

$$\begin{aligned}
|Au(t)| &\leq G_0 \int_0^T \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|} \left(\frac{L\|u\|+M}{1-LT(l_k+l_h)} \right) d\tau + \frac{p_0 u_\alpha}{|p(s)|} \right) ds \\
&+ \frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} |\eta_i| G_0 \int_0^T \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|} \left(\frac{L\|u\|+M}{1-LT(l_k+l_h)} \right) d\tau + \frac{p_0 u_\alpha}{|p(s)|} \right) ds \\
&\leq G_0 T \left(\frac{T^\beta}{\Gamma(\beta+1)q} \left(\frac{L\|u\|+M}{1-LT(l_k+l_h)} \right) \right) + \frac{G_0 T p_0 u_\alpha}{q} \\
&+ \frac{T^3 G_0}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} |\eta_i| \times \left(\frac{T^\beta}{\Gamma(\beta+1)q} \left(\frac{L\|u\|+M}{1-LT(l_k+l_h)} \right) + \frac{p_0 u_\alpha}{q} \right) \\
&\leq \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3} G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2} |\eta_i| \right) \frac{L}{1-LT(l_k+l_h)} r \\
&+ \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3} G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2} |\eta_i| \right) \frac{M}{1-LT(l_k+l_h)} \\
&+ \frac{G_0 T p_0 u_\alpha}{q} + \frac{T^3 G_0}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} |\eta_i| \left(\frac{p_0 u_\alpha}{q} \right) \\
&\leq \Delta L r + \Delta M + \frac{G_0 T p_0 u_\alpha}{q} + \frac{T^3 G_0}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2} |\eta_i| \left(\frac{p_0 u_\alpha}{q} \right) \leq r.
\end{aligned}$$

Taking for $t \in [0,T]$, we have

$$|Au(t)| \leq r,$$

then

$$A(B_r) \subset B_r.$$

Step 2. A is contraction.

Now for $u, v \in AC([0,T],\mathbb{R})$, then for $t \in [0,T]$ we have

$$Au(t) - Av(t) = \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}(x(\tau)-y(\tau))d\tau\right)ds$$
$$+\frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}\eta_i\int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}(x(\tau)-y(\tau))d\tau\right)ds,$$

where $x, y \in AC([0,T],\mathbb{R})$ be such that

$$\begin{aligned} x(t) &= f\left(t, u(t), \int_0^t k(\tau,s)x(s)ds, \int_0^t h(\tau,s)x(s)ds\right), \\ y(t) &= f\left(t, v(t), \int_0^t k(\tau,s)y(s)ds, \int_0^t h(\tau,s)y(s)ds\right), \end{aligned}$$

then

$$|Au(t) - Av(t)| \leq \int_0^T |G(t,s)|\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|x(\tau)-y(\tau)|d\tau\right)ds$$
$$+\frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}|\eta_i|\int_0^T |G(\xi_i,s)|\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|x(\tau)-y(\tau)|d\tau\right)ds. \quad (17)$$

From hypothesis (H_1) we have

$$\begin{aligned} |x(t)-y(t)| &= \left|f\left(t, u(t), \int_0^t k(\tau,s)x(s)ds, \int_0^t h(\tau,s)x(s)ds\right)\right. \\ &\quad \left. - f\left(t, v(t), \int_0^t k(\tau,s)y(s)ds, \int_0^t h(\tau,s)y(s)ds\right)\right| \\ &\leqslant L\left(|u(t)-v(t)| + \int_0^t k(t,s)|x(s)-y(s)|ds + \int_0^t h(t,s)|x(s)-y(s)|ds\right) \\ &\leqslant L(\|u-v\| + T(l_k+l_h)\|x-y\|), \end{aligned}$$

thus

$$\|x-y\| \leq \frac{L}{1-LT(l_k+l_h)}\|u-v\|.$$

We substitute in (17), we have

$$\begin{aligned} |Au(t) - Av(t)| &\leq \left(\frac{G_0T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q}\sum_{i=1}^{m-2}|\eta_i|\right)\frac{L}{1-LT(l_k+l_h)}\|u-v\| \\ &\leqslant \zeta\|u-v\|, \end{aligned}$$

by $\zeta < 1$, the operator A is a contraction. Hence, by Banach's contraction principle, A has a unique fixed point which is a solution of the **FIDP** (1) and (2) on $[0,T]$. □

4. Ulam Types Stability

In this section, we first establish our Hyres–Ulam stability result and secondly our Hyres–Ulam–Rassias stability of the two-orders fractional integro differential Equation (1).

4.1. Hyres–Ulam Stability

We need the following lemma.

Lemma 3. *If $w \in AC([0,T],\mathbb{R})$ is a solution of the fractional differential inequality (6) for each $\epsilon > 0$ and the initial condition (2), then w is a solution of the following integral inequality*

$$\left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right.$$

$$\left. - \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2}\eta_i \int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right|$$

$$\leq \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2}|\eta_i| \right)\epsilon,$$

where

$$z(t) = f\left(t, w(t), \int_0^t k(\tau,s)z(s)ds, \int_0^t h(\tau,s)z(s)ds\right).$$

Proof. Let $w \in AC([0,T],\mathbb{R})$ be a solution of the inequality (6) for each ϵ then, from Remark 2 and for some continuous function $g(t)$ such that $|g(t)| \leqslant \epsilon, t \in [0,T]$ we have

$$\begin{aligned}
|w(t) - Aw(t)| &\leqslant \left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right. \\
&\quad \left. - \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2}\eta_i \int_0^T G(\xi_i,s) \times \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right| \\
&\leq \int_0^T |G(t,s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|g(\tau)|d\tau ds \\
&\quad + \frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2}|\eta_i| \int_0^T |G(\xi_i,s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|g(\tau)|d\tau ds \\
&\leq \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2}|\eta_i| \right)\epsilon.
\end{aligned}$$

□

The main result of this subsection.

Theorem 2. *Assume that $(H_1)-(H_3)$ hold, then the **FIDP** (1) and (2) is Hyers–Ulam stable.*

Proof. Under $(H_1)-(H_3)$, the **FIDP** (1) and (2) has unique solution in $AC([0,T],\mathbb{R})$. Let $w \in AC([0,T],\mathbb{R})$ be a solution of the inequality (6), then by Lemma 3 we have for each $t \in [0,T]$

$$\begin{aligned}
|w(t) - u(t)| &\leqslant \left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right. \\
&\quad \left. + \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2}\eta_i \int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)} \right) ds \right| \\
&\leqslant \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2}|\eta_i| \right)\epsilon \\
&\quad + \int_0^T |G(t,s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|z(\tau)-x(\tau)|d\tau ds \\
&\quad + \frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2} \sum_{i=1}^{m-2}|\eta_i| \int_0^T |G(\xi_i,s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|z(\tau)-x(\tau)|d\tau ds \\
&\leqslant \left(\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q} \sum_{i=1}^{m-2}|\eta_i| \right)\epsilon + \Delta L|w(t)-u(t)|,
\end{aligned}$$

then, we have

$$|w(t)-u(t)| \leqslant \frac{\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q}\sum_{i=1}^{m-2}|\eta_i|}{1-\Delta L}\epsilon,$$

where Δ is defined by (13).

Then, there exists a real number $c_f = \frac{\frac{G_0 T^{\beta+1}}{\Gamma(\beta+1)q} + \frac{T^{\beta+3}G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta+1)q}\sum_{i=1}^{m-2}|\eta_i|}{1-\Delta L} > 0$ such that

$$|w(t)-u(t)| \leqslant c_f\epsilon.$$

Thus, **FIDP** (1) and (2) has the Hyers–Ulam stability. □

4.2. Hyres–Ulam–Rassias Stability

The following lemma will be used in the subsequent proof of the Hyers–Ulam–Rassias stability result.

Lemma 4. *If $w \in AC([0,T],\mathbb{R})$ is a solution of the fractional differential inequality (7), it follows that w a solution of the following integral inequality*

$$\left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}z(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)}\right)ds \right.$$

$$\left. - \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}\eta_i\int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}z(\tau)d\tau + \frac{p_0 u_\alpha}{p(s)}\right)ds \right|$$

$$\leq \frac{G_0\epsilon}{\Gamma(\beta)q}\int_0^t\int_0^s (s-\tau)^{\beta-1}\psi(\tau)d\tau ds$$

$$+\left(\frac{\epsilon G_0}{\Gamma(\beta)q} + \frac{\epsilon T^2 G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta)q}\sum_{i=1}^{m-2}|\eta_i|\right)\int_0^T\int_0^s (s-\tau)^{\beta-1}\psi(\tau)d\tau ds,$$

where

$$z(t) = f\left(t, w(t), \int_0^t k(\tau,s)z(s)ds, \int_0^t h(\tau,s)z(s)ds\right).$$

Proof. Let $w \in AC([0,T],\mathbb{R})$ be a solution of the inequality (7) for each ϵ then, from Remark 4 and for some continuous function $g(t)$ such that $|g(t)| \leqslant \epsilon, t \in [0,T]$, in each $t \in [0T]$, we have

$$\begin{aligned}
|w(t)-Aw(t)| &\leqslant \left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)}\right)ds \right. \\
&\quad \left. - \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}\eta_i\int_0^T G(\xi_i,s)\times\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}(z(\tau)+g(\tau))d\tau + \frac{p_0 u_\alpha}{p(s)}\right)ds \right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}\psi(\tau)\epsilon d\tau ds + \int_0^T G_0\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}\psi(\tau)\epsilon d\tau ds \\
&\quad + \frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}|\eta_i|\int_0^T |G(\xi_i,s)|\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}\psi(\tau)\epsilon d\tau ds \\
&\leq \frac{G_0\epsilon}{\Gamma(\beta)q}\int_0^t\int_0^s (s-\tau)^{\beta-1}\psi(\tau)d\tau ds \\
&\quad + \left(\frac{\epsilon G_0}{\Gamma(\beta)q} + \frac{\epsilon T^2 G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta)q}\sum_{i=1}^{m-2}|\eta_i|\right)\int_0^T\int_0^s (s-\tau)^{\beta-1}\psi(\tau)d\tau ds.
\end{aligned}$$

□

Now we are ready to announce our Hyres–Ulam–Rassias stability result.

Theorem 3. *Assume that* $(H_1)-(H_3)$ *hold and* (H_4) *The function* $\psi \in AC([0,T],\mathbb{R})$ *is increasing and there exists* $\gamma_\psi > 0$ *such that, for each* $t \in [0,T]$*, we have*

$$\int_0^t \int_0^s (s-\tau)^{\beta-1}\psi(\tau)d\tau ds \leqslant \gamma_\psi \psi(t),$$

then the ***FIDP*** (1) *and* (2) *is Hyers–Ulam–Rassias stable with respect to* Π_ψ*, where* Π_ψ *is defined by:*

$$\Pi_\psi(t) = \psi(t) + \frac{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2 + T^2\sum_{i=1}^{m-2}|\eta_i|\right)}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\psi(T),$$

Proof. Under $(H_1)-(H_3)$, the **FIDP** (1) and (2) has unique solution in $AC([0,T],\mathbb{R})$. Let $w \in AC([0,T],\mathbb{R})$ be a solution of the inequality (7), then for each $t \in [0,T]$

$$\begin{aligned}
|w(t)-u(t)| \leqslant & \left| w(t) - \int_0^T G(t,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0u_\alpha}{p(s)}\right)ds \right. \\
& \left. + \frac{t^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}\eta_i\int_0^T G(\xi_i,s)\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)p(s)}x(\tau)d\tau + \frac{p_0u_\alpha}{p(s)}\right)ds \right| \\
\leq & \frac{G_0\epsilon}{\Gamma(\beta)q}\gamma_\psi\psi(t) + \left(\frac{\epsilon G_0}{\Gamma(\beta)q} + \frac{\epsilon T^2G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta)q}\sum_{i=1}^{m-2}|\eta_i|\right)\gamma_\psi\psi(T) \\
& + \int_0^T |G(t,s)|\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|z(\tau)-x(\tau)|d\tau\right)ds \\
& + \frac{T^2}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\sum_{i=1}^{m-2}|\eta_i|\int_0^T |G(\xi_i,s)|\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)|p(s)|}|z(\tau)-x(\tau)|d\tau\right)ds \\
\leqslant & \frac{G_0\epsilon}{\Gamma(\beta)q}\gamma_\psi\psi(t) + \left(\frac{G_0}{\Gamma(\beta)q} + \frac{T^2G_0}{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2\right)\Gamma(\beta)q}\sum_{i=1}^{m-2}|\eta_i|\right)\gamma_\psi\epsilon\psi(T) \\
& + \Delta L|w(t)-u(t)| \\
\leqslant & \frac{G_0\epsilon}{\Gamma(\beta)q}\gamma_\psi\Pi_\psi(t) + \Delta L|w(t)-u(t)|,
\end{aligned}$$

with

$$\Pi_\psi(t) = \psi(t) + \frac{\left(2-\sum_{i=1}^{m-2}\eta_i\xi_i^2 + T^2\sum_{i=1}^{m-2}|\eta_i|\right)}{2-\sum_{i=1}^{m-2}\eta_i\xi_i^2}\psi(T),$$

then, we have

$$|w(t)-u(t)| \leq \frac{G_0\epsilon}{\Gamma(\beta)q(1-\Delta L)}\gamma_\psi\Pi_\psi(t),$$

where Δ is defined by (13).

Then, there exists a real number

$$c_{f,\psi} = \frac{G_0\gamma_\psi}{\Gamma(\beta)q(1-\Delta L)} > 0,$$

such that

$$|w(t)-u(t)| \leqslant c_{f,\psi}\epsilon\Pi_\psi(t).$$

Thus, **FIDP** (1) and (2) has the Hyers–Ulam–Rassias stability with respect to Π_ψ. □

5. Example

To clarify to the reader the importance of our results, we present the following examble.

Example 1. *Consider the following problem*

$$\begin{cases} {}^{c}D_{0^+}^{\frac{1}{2}}\left(\frac{1}{1+t}\,{}^{c}D_{0^+}^{\frac{7}{2}}u(t)\right) = f\left(t,u(t),\int_0^t k(t,s)x(s)ds,\int_0^t h(t,s)x(s)ds\right), \quad t\in[0,1], \\ u(0)=u'(0)=u^{(3)}(0)=0,\ \ u''(1)=\eta\xi \quad \text{and } {}^{c}D^{\frac{7}{2}}u(t)|_{t=0}=u_{\frac{7}{2}}, \end{cases} \tag{18}$$

where $x(t) = {}^{c}D_{0^+}^{\frac{1}{2}}\left(\frac{1}{1+t}\,{}^{c}D_{0^+}^{\frac{7}{2}}u(t)\right)$, $k(t,s)=e^{-(s-t)}$ *and* $h(t,s)=e^{-\frac{s-t}{2}}$,

with

$$f\left(t,u(t),\int_0^t k(t,s)x(s)ds,\int_0^t h(t,s)x(s)ds\right) = \frac{2+u(t)+\int_0^t e^{-(s-t)}x(s)ds+\int_0^t e^{-\frac{s-t}{2}}x(s)ds}{\pi e^{t+1}\left(1+u(t)+\int_0^t e^{-(s-t)}x(s)ds+\int_0^t e^{-\frac{s-t}{2}}x(s)ds\right)},$$

Clearly, the function f *is jointly continuous.*
For any $u_i, v_i\in\mathbb{R}$, $(i=1,2,3)$ *and* $t\in[0,1]$

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)| \le \frac{1}{\pi e}(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|).$$

Then $L=\frac{1}{\pi e}$, $G_0=\frac{1}{\Gamma(\frac{7}{2})}$, $\max_{t,s\in[0,1]}k(t,s)=e=l_k$, $\max_{t,s\in[0,1]}h(t,s)=\sqrt{e}=l_h$.
Substituting L, l_k, l_h, G_0 *into* Δ *yields*

$$\Delta = \left(\frac{1}{\left(\Gamma\left(\frac{7}{2}\right)\right)^2}+\frac{1}{3\left(2-\frac{1}{3}\left(\frac{3}{4}\right)^2\right)\left(\Gamma\left(\frac{7}{2}\right)\right)^2}\right)\times\frac{1}{1-\frac{1}{\pi e}\left(e+\sqrt{e}\right)} = 3.0850,$$

then

$$\varsigma=\Delta L\simeq\frac{3.0850}{\pi e}=0.36125<1.$$

By Theorem 2, ***FIDP*** *(18) is Hyers–Ulam stable.*

6. Conclusions

This research study has a number of important contributions. First, we prove that the integration Equation (9) and **FIDP** (1) and (2) are equivalent. The focus of our investigation is this significant link. The existence and uniqueness of solutions for boundary value issues involving implicit two-order fractional integro-differential equations are then established. The use of the Banach fixed-point theorem forms the foundation of this accomplishment. Additionally, we learn about and investigate a number of stability characteristics, such as Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and generalized Hyers–Ulam–Rassias stability. If boundary conditions of the fractional integral type are added, implicit two-order fractional integro-differential equations of fractional order are also subject to these stability features. Last but not least, we wrap up the article with a concrete example that effectively illustrates the usefulness and applicability of the obtained results.

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