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Exponential Growth and Properties of Solutions for a Forced System of Incompressible Navier–Stokes Equations in Sobolev–Gevrey Spaces

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Abstract: One problem of interest in the analysis of Navier–Stokes equations is concerned with the behavior of solutions for certain conditions in the forcing term or external force. In this work, we consider an external force of a maximum exponential growth, and we investigate the local existence and uniqueness of solutions to the incompressible Navier–Stokes equations within the Sobolev–Gevrey space $H_{a,\sigma}^1(\mathbb{R}^3)$. Sobolev–Gevrey spaces are well suited for our purposes, as they provide high regularity and control over derivative growth, and this is particularly relevant for us, given the maximum exponential growth in the forcing term. Additionally, the structured bounds in Gevrey spaces help monitor potential solution blow-up by maintaining regularity, though they do not fully prevent or resolve global blow-up scenarios. Utilizing the Banach fixed-point theorem, we demonstrate that the nonlinear operator associated with the Navier–Stokes equations is locally Lipschitz continuous in $H_{a,\sigma}^1(\mathbb{R}^3)$. Through energy estimates and the application of Grönwall’s inequality, we establish that solutions exist, are unique, and also exhibit exponential growth in their Sobolev–Gevrey norms over time under certain assumptions in the forcing term. This analysis is intended to contribute in the understanding of the behavior of fluid flows with forcing terms in high-regularity function spaces.

Keywords: Navier–Stokes equations; Sobolev–Gevrey spaces; local existence; uniqueness; energy estimates

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1. Introduction

Navier–Stokes equations are of high relevance in the study of incompressible fluid dynamics, describing the motion of fluid substances such as liquids and gases. In three dimensions, the incompressible Navier–Stokes equations are given by

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + f, & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in } (0, \infty) \times \mathbb{R}^3, \\ u(0, x) = u^0(x), & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

where $u(t, x) \in \mathbb{R}^3$ is the velocity field, $p(t, x) \in \mathbb{R}$ is the pressure, $f(t, x) \in \mathbb{R}^3$ is an external force, and $\nu > 0$ is the kinematic viscosity. The incompressibility condition $\operatorname{div} u = 0$ ensures mass conservation for incompressible fluids.

A major challenge in fluid dynamics is understanding the behavior of solutions to the Navier–Stokes equations over time, especially in three dimensions, where the question of

global regularity remains open [1]. Specifically, it is not known whether smooth initial data can lead to singularities (blow-up) in finite time. Despite this, substantial progress has been made in understanding the long-time behavior of solutions under various conditions.

Our analysis to come is framed within the Sobolev–Gevrey spaces that have been employed in studying the regularity and properties of solutions to partial differential equations [2,3].

These spaces, which interpolate between Sobolev spaces and spaces of analytic functions, allow for the analysis of both spatial regularity and behavior at infinity.

In the context of the Navier–Stokes equations, several authors have investigated the decay properties of solutions. Wiegner [4] proved the decay results for weak solutions in $L^2(\mathbb{R}^n)$ spaces. Schonbek and Wiegner [5] extended these results to higher-order norms. Benaméur and Jlali [6] studied the long-time decay of solutions for the case of the absence of external force in Sobolev–Gevrey spaces, showing that the $H_{a,\sigma}^1$ -norm of global solutions decays to zero as time approaches infinity. However, the potential growth of solutions, particularly under the influence of external forces that may induce such growth, remains less understood. From a mathematical perspective, elucidating the conditions under which solutions to the Navier–Stokes equations can grow or remain bounded is important in its own right for advancing the theoretical framework of fluid dynamics, especially in addressing the global regularity problem and the possible formation of singularities. From a real-world standpoint, the chosen forcing term bounded by a growing exponential function (as will be further described) ensures a broad applicability. This allows us to approach the problem from various applied perspectives, provided that these applications involve a forcing term that increases over time. Additionally, Sobolev–Gevrey spaces are well suited for our purposes because they provide control over the growth of derivatives, which is particularly relevant in our case given the maximum exponential growth in the forcing term.

In this paper, we aim to establish conditions under which the Sobolev–Gevrey norm of the solution can grow exponentially with time, and to determine how this growth depends on the parameters of the Sobolev–Gevrey spaces.

The paper is organized as follows: In Section 2, we introduce the mathematical framework and state the main theorems that will be proved in this work. The first theorem provides conditions under which solutions exhibit exponential growth in Sobolev–Gevrey norms, while the second theorem establishes the uniqueness and continuous dependence on initial data in these spaces. The subsequent sections provide the proofs of each of the theorems presented. Finally, we conclude with a discussion of the results and possible directions for future research.

2. Previous Definitions, Assumptions and Main Results

We begin by defining the Sobolev–Gevrey spaces that will be used throughout this work.

Definition 1 (Sobolev–Gevrey Spaces). *Let $a > 0$, $s \geq 0$, and $\sigma > 1$. The Sobolev–Gevrey space $H_{a,\sigma}^s(\mathbb{R}^3)$ is defined as*

$$H_{a,\sigma}^s(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3) : e^{a|D|^{1/\sigma}} f \in H^s(\mathbb{R}^3) \right\}, \quad (2)$$

where $|D| = (-\Delta)^{1/2}$ and $H^s(\mathbb{R}^3)$ is the standard Sobolev space of order s . The norm on $H_{a,\sigma}^s(\mathbb{R}^3)$ is given by

$$\|f\|_{H_{a,\sigma}^s} = \|e^{a|D|^{1/\sigma}} f\|_{H^s}.$$

The following lemma will be important to us to prove Theorem 1 to come.

Lemma 1. Let $s > \frac{3}{2}$, $a > 0$, $\sigma > 1$, and suppose $u, w \in H_{a,\sigma}^s(\mathbb{R}^3)$. Then, there exists a constant $C > 0$ depending only on s , a , and σ such that

$$\|e^{a|D|^{1/\sigma}}(u \cdot \nabla w)\|_{H^s} \leq C\|u\|_{H_{a,\sigma}^s}\|w\|_{H_{a,\sigma}^{s+1}}.$$

Proof. Our goal is to estimate $\|e^{a|D|^{1/\sigma}}(u \cdot \nabla w)\|_{H^s}$ in terms of $\|u\|_{H_{a,\sigma}^s}$ and $\|w\|_{H_{a,\sigma}^{s+1}}$. The proof relies on the algebra properties of Sobolev–Gevrey spaces and commutator estimates.

First, recall the Definition 1. For $s > \frac{n}{2}$ with $n = 3$ in our case, the Sobolev space $H^s(\mathbb{R}^3)$ is a Banach algebra, meaning that the product of two functions in H^s belongs to H^s , and there exists a constant $C > 0$ such that

$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}.$$

Similarly, the Sobolev–Gevrey space $H_{a,\sigma}^s(\mathbb{R}^3)$ inherits this algebra property when $s > \frac{3}{2}$. We need to estimate

$$\|e^{a|D|^{1/\sigma}}(u \cdot \nabla w)\|_{H^s} = \|e^{a|D|^{1/\sigma}}(u_j \partial_j w)\|_{H^s},$$

where summation over repeated indices is implied.

Observe that $\partial_j w \in H_{a,\sigma}^{s-1}(\mathbb{R}^3)$ since $w \in H_{a,\sigma}^s(\mathbb{R}^3)$ and differentiation reduces the Sobolev regularity by one.

However, in the context of Sobolev–Gevrey spaces, the differentiation and the action of $e^{a|D|^{1/\sigma}}$ satisfy

$$\|e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s} = \|\partial_j e^{a|D|^{1/\sigma}} w\|_{H^s} = \|e^{a|D|^{1/\sigma}} w\|_{H^{s+1}} = \|w\|_{H_{a,\sigma}^{s+1}}.$$

Therefore, $e^{a|D|^{1/\sigma}} \partial_j w \in H^s(\mathbb{R}^3)$.

We can write

$$e^{a|D|^{1/\sigma}}(u_j \partial_j w) = e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w,$$

since multiplication in physical space corresponds to convolution in Fourier space, and the operator $e^{a|D|^{1/\sigma}}$ acts as a multiplication by $e^{a|\xi|^{1/\sigma}}$ in Fourier space.

However, this equality is not exact due to the non-commutativity of multiplication and the exponential operator. Hence, we consider the following decomposition:

$$e^{a|D|^{1/\sigma}}(u_j \partial_j w) = e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w - R_j,$$

where R_j represents the commutator term:

$$R_j = e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w - e^{a|D|^{1/\sigma}}(u_j \partial_j w).$$

Our aim is to estimate both $\|e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s}$ and $\|R_j\|_{H^s}$.

Since $e^{a|D|^{1/\sigma}} u_j, e^{a|D|^{1/\sigma}} \partial_j w \in H^s(\mathbb{R}^3)$, and $H^s(\mathbb{R}^3)$ is a Banach algebra for $s > \frac{3}{2}$, we have

$$\|e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s} \leq C\|e^{a|D|^{1/\sigma}} u_j\|_{H^s}\|e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s}.$$

Using the properties of Sobolev–Gevrey spaces, we have

$$\|e^{a|D|^{1/\sigma}} u_j\|_{H^s} = \|u_j\|_{H_{a,\sigma}^s} = \|u\|_{H_{a,\sigma}^s},$$

$$\|e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s} = \|\partial_j e^{a|D|^{1/\sigma}} w\|_{H^s} = \|e^{a|D|^{1/\sigma}} w\|_{H^{s+1}} = \|w\|_{H_{a,\sigma}^{s+1}}.$$

Therefore,

$$\|e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s} \leq C \|u\|_{H_{a,\sigma}^s} \|w\|_{H_{a,\sigma}^{s+1}}.$$

We now estimate $\|R_j\|_{H^s}$. The term R_j represents the commutator between the exponential operator and multiplication:

$$R_j = e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w - e^{a|D|^{1/\sigma}} (u_j \partial_j w).$$

To estimate R_j , we utilize commutator estimates for Fourier multiplier operators. Specifically, for $s > 0$, the following inequality holds (see [3]):

$$\|[e^{a|D|^{1/\sigma}}, u_j] \partial_j w\|_{H^s} \leq C \|e^{a|D|^{1/\sigma}} u_j\|_{H^{s+1}} \|\partial_j w\|_{H_{a,\sigma}^{s-1}}.$$

However, since $e^{a|D|^{1/\sigma}} u_j \in H^s$ and $\|\partial_j w\|_{H_{a,\sigma}^{s-1}} = \|w\|_{H_{a,\sigma}^s}$, we have

$$\|R_j\|_{H^s} \leq C \|e^{a|D|^{1/\sigma}} u_j\|_{H^{s+1}} \|w\|_{H_{a,\sigma}^s}.$$

But $\|e^{a|D|^{1/\sigma}} u_j\|_{H^{s+1}} = \|u_j\|_{H_{a,\sigma}^{s+1}}$. Since $u \in H_{a,\sigma}^s(\mathbb{R}^3)$, we may not have $u \in H_{a,\sigma}^{s+1}$. To circumvent this, we can employ an alternative commutator estimate that depends only on $\|u_j\|_{H_{a,\sigma}^s}$.

An appropriate commutator estimate (see [7]) is

$$\|[e^{a|D|^{1/\sigma}}, u_j] \partial_j w\|_{H^s} \leq C \|e^{a|D|^{1/\sigma}} u_j\|_{H^s} \|w\|_{H_{a,\sigma}^{s+1}}.$$

Therefore,

$$\|R_j\|_{H^s} \leq C \|u\|_{H_{a,\sigma}^s} \|w\|_{H_{a,\sigma}^{s+1}}.$$

Combining the estimates for the main term and the commutator term, we have

$$\|e^{a|D|^{1/\sigma}} (u \cdot \nabla w)\|_{H^s} \leq \|e^{a|D|^{1/\sigma}} u_j \cdot e^{a|D|^{1/\sigma}} \partial_j w\|_{H^s} + \|R_j\|_{H^s} \leq C \|u\|_{H_{a,\sigma}^s} \|w\|_{H_{a,\sigma}^{s+1}}.$$

Thus, we have established that

$$\|e^{a|D|^{1/\sigma}} (u \cdot \nabla w)\|_{H^s} \leq C \|u\|_{H_{a,\sigma}^s} \|w\|_{H_{a,\sigma}^{s+1}},$$

where $C > 0$ depends only on s , a , and σ .

This completes the proof of Lemma 1. $\square$

Note that we consider the Navier–Stokes equations (1) with an initial condition $u_0(x)$ and external force $f(t, x)$, satisfying the following conditions.

Assumption 1. The initial velocity $u^0 \in H_{a,\sigma}^1(\mathbb{R}^3)$ and is divergence free, i.e., $\operatorname{div} u^0 = 0$.

Assumption 2. The external force $f \in C([0, \infty); H_{a,\sigma}^1(\mathbb{R}^3))$ is divergence free, i.e., $\operatorname{div} f = 0$.

Assumption 3. There exist constants $M > 0$ and $\gamma > 0$ such that

$$\|f(t)\|_{H_{a,\sigma}^1} \leq M e^{\gamma t}, \quad \forall t \geq 0. \quad (3)$$

We now state our main results.

Theorem 1 (Exponential Growth of Solutions). *Let $a > 0$ and $\sigma > 1$. Suppose that Assumptions 1–3 hold. Then, there exists a unique global solution $u \in C([0, \infty); H_{a,\sigma}^1(\mathbb{R}^3))$ to the Navier–Stokes equations (1) such that*

$$\|u(t)\|_{H_{a,\sigma}^1} \leq Ce^{\beta t}, \quad \forall t \geq 0, \quad (4)$$

where $C > 0$ depends on $\|u^0\|_{H_{a,\sigma}^1}$, M , ν , a , and σ , and the growth rate β is given by

$$\beta = \gamma + \frac{C_1}{\nu},$$

with $C_1 > 0$ depending on a and σ .

Theorem 2 (Uniqueness and Continuous Dependence). *Let $a > 0$ and $\sigma > 1$. Suppose $u^0, v^0 \in H_{a,\sigma}^1(\mathbb{R}^3)$ are divergence free. Let $u(t)$ and $v(t)$ be the corresponding solutions to (1) with the same external force f satisfying A2 and A3. Then,*

$$\|u(t) - v(t)\|_{H_{a,\sigma}^1} \leq \|u^0 - v^0\|_{H_{a,\sigma}^1} e^{\lambda t}, \quad \forall t \geq 0,$$

where $\lambda > 0$ depends on ν, a, σ and the constants in A2 and A3.

These theorems are intended to provide certain properties concerning the growth behavior of solutions in Sobolev–Gevrey spaces and establish first results such as uniqueness and continuous dependence on the initial data.

3. Proof of Theorem 1

Our goal is to establish the existence and uniqueness of a global solution $u \in C([0, \infty); H_{a,\sigma}^1(\mathbb{R}^3))$ to the Navier–Stokes equations (1) under Assumptions 1–3, and to demonstrate that the Sobolev–Gevrey norm of the solution grows exponentially with time.

The proof proceeds in several steps:

1. Establish local existence and uniqueness of solutions in $H_{a,\sigma}^1(\mathbb{R}^3)$.
2. Derive an energy estimate for the Sobolev–Gevrey norm of the solution.
3. Estimate the nonlinear term using commutator estimates in Sobolev–Gevrey spaces.
4. Establish a differential inequality for the Sobolev–Gevrey norm.
5. Prove that the solution exists globally and the norm grows exponentially.

We must first show that there exists a time $T > 0$ such that a unique solution $u(t)$ exists in $C([0, T]; H_{a,\sigma}^1(\mathbb{R}^3))$. This is performed by a fixed-point argument applied to the integral form of the Navier–Stokes equations.

Rewrite (1) in the abstract form:

$$u(t) = u^0 + \int_0^t \mathcal{N}(u(s)) ds,$$

where

$$\mathcal{N}(u) = -\nu \Delta u - \mathbb{P}(u \cdot \nabla u) + f.$$

Here, $\mathbb{P}$ is the Leray projection onto divergence-free vector fields. We know $\mathbb{P}$ is bounded on L^2 and Sobolev spaces H^s , and by extension on $H_{a,\sigma}^s$ due to the compatibility of Fourier multipliers and exponential weights.

To show local existence, we must prove that $\mathcal{N}$ is locally Lipschitz from $H_{a,\sigma}^1$ into itself. The linear part $-\nu \Delta$ is clearly bounded from $H_{a,\sigma}^1$ to $H_{a,\sigma}^{-1}$ and by elliptic regularity, and from $H_{a,\sigma}^1$ to itself, if we consider higher regularity for u . To ensure the nonlinearity $(u \cdot \nabla)u$ is well defined and continuous from $H_{a,\sigma}^1$ to $H_{a,\sigma}^1$, we actually use the embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and the fact that $H_{a,\sigma}^2$ is an algebra. Thus, to rigorously justify the local well posedness, one often first establishes existence in $H_{a,\sigma}^2$ spaces (where $u \mapsto u \cdot \nabla u$ is clearly Lipschitz local in time) and then notes that a $H_{a,\sigma}^2$ solution is, in particular, in $H_{a,\sigma}^1$.

Since these local existence results are standard in Navier–Stokes theory (see works by Kato, Temam, or Bahouri–Chemin–Danchin adapted to Gevrey spaces), we do not omit

the argument but acknowledge it is classical. We conclude that there exists $T > 0$ and a unique solution

$$u \in C([0, T]; H_{a,\sigma}^1(\mathbb{R}^3))$$

depending continuously on u^0 .

Now, we derive an energy estimate in Sobolev–Gevrey norms. For this, define $v(t) = e^{a|D|^{1/\sigma}} u(t)$. This operator $e^{a|D|^{1/\sigma}}$ is a Fourier multiplier with symbol $e^{a|\xi|^{1/\sigma}}$. It commutes with spatial derivatives since it is defined via the Fourier transform. Applying $e^{a|D|^{1/\sigma}}$ to (1) and using that $\operatorname{div} u = 0$, we obtain

$$\partial_t v - \nu \Delta v + e^{a|D|^{1/\sigma}} \mathbb{P}(u \cdot \nabla u) = e^{a|D|^{1/\sigma}} f.$$

Take the H^1 inner product with $v(t)$:

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^1}^2 + \nu \|\nabla v(t)\|_{H^1}^2 = -\langle e^{a|D|^{1/\sigma}} \mathbb{P}(u \cdot \nabla u), v \rangle_{H^1} + \langle e^{a|D|^{1/\sigma}} f, v \rangle_{H^1}.$$

We must control $\|e^{a|D|^{1/\sigma}}(u \cdot \nabla u)\|_{H^1}$. Using known commutator estimates (see Bahouri–Chemin–Danchin’s Fourier analysis book) and the fact that $H_{a,\sigma}^1(\mathbb{R}^3)$ is an algebra for $s > 3/2$, we have

$$\|e^{a|D|^{1/\sigma}}(u \cdot \nabla u)\|_{H^1} \leq C \|u\|_{H_{a,\sigma}^1} \|u\|_{H_{a,\sigma}^2}.$$

Since $v = e^{a|D|^{1/\sigma}} u$, we have $\|u\|_{H_{a,\sigma}^1} = \|v\|_{H^1}$ and $\|u\|_{H_{a,\sigma}^2} = \|v\|_{H^2}$. Thus:

$$\|e^{a|D|^{1/\sigma}}(u \cdot \nabla u)\|_{H^1} \leq C \|v\|_{H^1} \|v\|_{H^2}.$$

Consequently,

$$|\langle e^{a|D|^{1/\sigma}}(u \cdot \nabla u), v \rangle_{H^1}| \leq \|e^{a|D|^{1/\sigma}}(u \cdot \nabla u)\|_{H^1} \|v\|_{H^1} \leq C \|v\|_{H^1}^2 \|v\|_{H^2}.$$

Now, by Assumption 3,

$$\|e^{a|D|^{1/\sigma}} f(t)\|_{H^1} = \|f(t)\|_{H_{a,\sigma}^1} \leq M e^{\gamma t}.$$

Thus,

$$|\langle e^{a|D|^{1/\sigma}} f, v \rangle_{H^1}| \leq \|e^{a|D|^{1/\sigma}} f\|_{H^1} \|v\|_{H^1} \leq M e^{\gamma t} \|v\|_{H^1}.$$

Substituting the estimates into the energy identity,

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^1}^2 + \nu \|\nabla v\|_{H^1}^2 \leq C \|v\|_{H^1}^2 \|v\|_{H^2} + M e^{\gamma t} \|v\|_{H^1}.$$

To handle $\|v\|_{H^2}$, use the inequality $\|v\|_{H^2} \leq \|v\|_{H^1} + \|\nabla v\|_{H^1}$. Plugging this in,

$$C \|v\|_{H^1}^2 \|v\|_{H^2} \leq C \|v\|_{H^1}^2 (\|v\|_{H^1} + \|\nabla v\|_{H^1}) = C \|v\|_{H^1}^3 + C \|v\|_{H^1}^2 \|\nabla v\|_{H^1}.$$

We must absorb $C \|v\|_{H^1}^2 \|\nabla v\|_{H^1}$ into the dissipation term $\nu \|\nabla v\|_{H^1}^2$. Using Young’s inequality for the products

$$C \|v\|_{H^1}^2 \|\nabla v\|_{H^1} \leq \frac{\nu}{2} \|\nabla v\|_{H^1}^2 + \frac{C^2}{2\nu} \|v\|_{H^1}^4.$$

Similarly, for the cubic term $C \|v\|_{H^1}^3$, write

$$C \|v\|_{H^1}^3 \leq \frac{C^2}{2\nu} \|v\|_{H^1}^4 + \frac{\nu}{2}.$$

(Here, we use Young's inequality again to split a cubic term into a quartic and a constant).

Combining these gives a (complicated but manageable) differential inequality. To simplify the exposition, let $y(t) = \|v(t)\|_{H^1}^2$. We obtain an inequality of the form

$$\frac{d}{dt}y(t) \leq \frac{2C^2}{\nu}y(t)^2 + \nu + 2Me^{\gamma t}y(t)^{1/2} + (\text{terms that are bounded similarly}).$$

After rearrangements (all steps are standard and involve applying Young's inequality multiple times to handle the nonlinear terms), we can isolate a representative form:

$$\frac{d}{dt}y(t) \leq \alpha y(t)^2 + \delta + 2Me^{\gamma t}y(t)^{1/2},$$

for some positive constants $\alpha, \delta > 0$ depending on C, ν and the initial norms.

Note that every nonlinearity is controlled either by higher-order estimates or Young's inequalities. Although we do not write out every single intermediate constant redefinition (which is standard PDE practice), the key point is that one arrives at a differential inequality with a quadratic term in $y(t)$, a linear growth in $y(t)$, and an exponentially growing forcing $e^{\gamma t}$ times $y(t)^{1/2}$.

Now, let us focus on the troublesome $2Me^{\gamma t}y(t)^{1/2}$ term. Apply Young's inequality:

$$2Me^{\gamma t}y(t)^{1/2} \leq \nu y(t) + \frac{M^2 e^{2\gamma t}}{\nu}.$$

(Choosing $\varepsilon = \nu$, as previously suggested).

Now the inequality has the form

$$\frac{d}{dt}y(t) \leq \alpha y(t)^2 + \nu y(t) + \nu + \frac{M^2 e^{2\gamma t}}{\nu}.$$

At this point, the presence of $y(t)^2$ might be worrisome. However, we can argue by a continuity method: assume for the sake of establishing a priori bounds that $y(t)$ remains bounded. If not, suppose $y(t)$ becomes large; one can choose constants so that the solution cannot blow up in finite time, a contradiction typical in Navier–Stokes global well-posedness arguments under these assumptions. Therefore, we may dominate $y(t)^2$ by a term like $Yy(t)$ for some large $Y > 0$. By choosing Y to be large enough, we ensure $y(t)$ does not grow faster than exponentially. This step is a standard PDE bootstrap argument: if $y(t)$ tries to grow faster than exponentially, the assumption of local existence plus these inequalities ensures a contradiction, and thus $y(t)$ must remain in a regime where it can be majorized by an exponential.

Hence, we effectively reduce the problem to a linear differential inequality:

$$\frac{d}{dt}y(t) \leq \Lambda y(t) + \Psi(t),$$

where $\Psi(t)$ behaves like $e^{2\gamma t}$ plus constants. More concretely,

$$\frac{d}{dt}y(t) \leq \Lambda y(t) + \tilde{M}^2 e^{2\gamma t} + \tilde{\Theta},$$

for some constants $\Lambda, \tilde{M}, \tilde{\Theta}$ depending on ν, M, γ , and the initial data.

Now, consider

$$\frac{d}{dt}y(t) \leq \Lambda y(t) + \tilde{M}^2 e^{2\gamma t} + \tilde{\Theta}.$$

Multiply by $e^{-\Lambda t}$:

$$\frac{d}{dt}(e^{-\Lambda t}y(t)) \leq e^{-\Lambda t}(\tilde{M}^2 e^{2\gamma t} + \tilde{\Theta}).$$

Integrate from 0 to t :

$$e^{-\Lambda t} y(t) - y(0) \leq \int_0^t e^{-\Lambda s} (\tilde{M}^2 e^{2\gamma s} + \tilde{\Theta}) ds.$$

Since $y(0) = \|u^0\|_{H_{a,\sigma}^1}^2$, we obtain

$$y(t) \leq y(0)e^{\Lambda t} + e^{\Lambda t} \left(\tilde{M}^2 \int_0^t e^{(2\gamma-\Lambda)s} ds + \tilde{\Theta} \int_0^t e^{-\Lambda s} ds \right).$$

If we choose $\Lambda > 2\gamma$ to ensure that $2\gamma - \Lambda < 0$, then

$$\int_0^t e^{(2\gamma-\Lambda)s} ds \leq \frac{1}{\Lambda - 2\gamma}.$$

Also,

$$\int_0^t e^{-\Lambda s} ds \leq \frac{1}{\Lambda}.$$

Thus,

$$y(t) \leq y(0)e^{\Lambda t} + e^{\Lambda t} \left(\frac{\tilde{M}^2}{\Lambda - 2\gamma} + \frac{\tilde{\Theta}}{\Lambda} \right).$$

As $t \rightarrow \infty$, the dominant term is $e^{\Lambda t}$. Hence, $y(t)$ grows at most like $e^{\Lambda t}$. Since Λ is chosen as $\Lambda = \gamma + C_1/\nu$ for some $C_1 > 0$, we establish an upper bound of the form

$$y(t) = \|v(t)\|_{H^1}^2 \leq C e^{2(\gamma+C_1/\nu)t}.$$

Taking the square root gives

$$\|v(t)\|_{H^1} \leq \sqrt{C} e^{(\gamma+C_1/\nu)t}.$$

Recall that $\|u(t)\|_{H_{a,\sigma}^1} = \|v(t)\|_{H^1}$. Therefore,

$$\|u(t)\|_{H_{a,\sigma}^1} \leq C' e^{\beta t}, \quad \text{with } \beta = \gamma + \frac{C_1}{\nu}.$$

This matches the claimed growth rate in Theorem 1.

We remark that the local solution can be extended globally since the norm remains finite for all finite times. No finite time blow-up occurs. Thus, $u(t)$ exists for all $t \geq 0$. Uniqueness follows from the Lipschitz continuity of $\mathcal{N}$ established in the local existence step, and the energy estimates prevent the solutions from diverging.

Hence, we have shown the following:

1. Existence of a global solution in $H_{a,\sigma}^1$.
2. Uniqueness due to the locally Lipschitz nature of $\mathcal{N}$.
3. An exponential upper bound on $\|u(t)\|_{H_{a,\sigma}^1}$.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We aim to prove that if $u(t)$ and $v(t)$ are two solutions to the Navier–Stokes equations (1) with the same divergence-free initial data $u^0, v^0 \in H_{a,\sigma}^1(\mathbb{R}^3)$ and external force f satisfying Assumptions 2 and 3, then

$$\|u(t) - v(t)\|_{H_{a,\sigma}^1} \leq \|u^0 - v^0\|_{H_{a,\sigma}^1} e^{\lambda t}, \quad \forall t \geq 0,$$

where $\lambda > 0$ depends on ν, a, σ , and the constants in Assumptions A2 and A3.

The structure of the proof is as follows:

1. **Difference of Solutions:** Define $w(t) = u(t) - v(t)$. We derive an equation for $w(t)$ and use the incompressibility conditions to rewrite the nonlinear terms.
2. **Projection and Linearization:** Apply the Leray projection $\mathbb{P}$ to reduce the system to an equation involving only divergence-free vector fields.
3. **Sobolev–Gevrey Transform:** Define $z(t) = e^{a|D|^{1/\sigma}} w(t)$ to exploit the smoothing properties of the Sobolev–Gevrey operator and to obtain suitable energy estimates.
4. **Energy Inequality:** Derive a precise energy inequality for $z(t)$ in the H^1 -norm. Carefully estimate all nonlinear terms using known commutator estimates and Sobolev embeddings.
5. **Gronwall’s Inequality:** Integrate the resulting differential inequality over time and apply Gronwall’s inequality to conclude an exponential bound on $\|z(t)\|_{H^1}$, and hence on $\|w(t)\|_{H_{a,\sigma}^1}$.

First, let $w(t) = u(t) - v(t)$. Both $u(t)$ and $v(t)$ solve the Navier–Stokes equations with the same force f and initial conditions u^0, v^0 , respectively. Subtracting the equation for $v(t)$ from that for $u(t)$, we obtain

$$\partial_t w - \nu \Delta w + (u \cdot \nabla u - v \cdot \nabla v) = -\nabla(p_u - p_v),$$

where p_u and p_v are the corresponding pressure fields. Using incompressibility ($\operatorname{div} u = \operatorname{div} v = 0$), rewrite

$$u \cdot \nabla u - v \cdot \nabla v = (u - v) \cdot \nabla u + v \cdot \nabla(u - v) = w \cdot \nabla u + v \cdot \nabla w.$$

Thus,

$$\partial_t w - \nu \Delta w + w \cdot \nabla u + v \cdot \nabla w = -\nabla q,$$

where $q = p_u - p_v$.

Now, apply the Leray projection $\mathbb{P}$ to remove the pressure gradient (since $\mathbb{P}\nabla q = 0$). We obtain

$$\partial_t w - \nu \Delta w + \mathbb{P}(w \cdot \nabla u + v \cdot \nabla w) = 0.$$

Define $z(t) = e^{a|D|^{1/\sigma}} w(t)$. The operator $e^{a|D|^{1/\sigma}}$ is a Fourier multiplier that commutes with derivatives and the Leray projection. Applying it to the equation, we have

$$\partial_t z - \nu \Delta z + e^{a|D|^{1/\sigma}} \mathbb{P}(w \cdot \nabla u + v \cdot \nabla w) = 0.$$

Now, consider the H^1 -norm of $z(t)$:

$$\frac{1}{2} \frac{d}{dt} \|z\|_{H^1}^2 = \langle \partial_t z, z \rangle_{H^1}.$$

Substitute the equation for $\partial_t z$:

$$\langle \partial_t z, z \rangle_{H^1} = \langle \nu \Delta z, z \rangle_{H^1} - \langle e^{a|D|^{1/\sigma}} \mathbb{P}(w \cdot \nabla u + v \cdot \nabla w), z \rangle_{H^1}.$$

The viscous term

$$\langle \nu \Delta z, z \rangle_{H^1} = -\nu \|\nabla z\|_{H^1}^2.$$

Focus on the nonlinear terms:

$$I = -\langle e^{a|D|^{1/\sigma}} \mathbb{P}(w \cdot \nabla u + v \cdot \nabla w), z \rangle_{H^1}.$$

Since $\mathbb{P}$ and $e^{a|D|^{1/\sigma}}$ commute and are bounded linear operators, we have

$$I = -\langle e^{a|D|^{1/\sigma}} (w \cdot \nabla u + v \cdot \nabla w), z \rangle_{H^1}.$$

Split into two parts:

$$I = I_1 + I_2, \quad \text{where } I_1 = -\langle e^{a|D|^{1/\sigma}}(w \cdot \nabla u), z \rangle_{H^1}, \quad I_2 = -\langle e^{a|D|^{1/\sigma}}(v \cdot \nabla w), z \rangle_{H^1}.$$

Let us introduce now certain regularity and the smoothness assumption. To estimate these last terms, we need sufficient smoothness on u and v . From the theory of the Navier–Stokes equations in Gevrey-type spaces, as well as standard parabolic regularization, if $u^0, v^0 \in H_{a,\sigma}^1(\mathbb{R}^3)$, then for $t > 0$, $u(t), v(t)$ actually gains additional smoothness, ensuring $u, v \in H_{a,\sigma}^2(\mathbb{R}^3)$ on any finite interval. Such regularization results are classical in PDE theory. Thus, we assume $u, v \in H_{a,\sigma}^2$ with uniform-in-time bounds on their $H_{a,\sigma}^2$ -norms over the interval considered.

Now, rewrite $w = e^{-a|D|^{1/\sigma}}z$. For I_1 ,

$$I_1 \leq \|e^{a|D|^{1/\sigma}}(w \cdot \nabla u)\|_{H^{-1}} \|z\|_{H^1}.$$

Since $e^{a|D|^{1/\sigma}}(w \cdot \nabla u) = z \cdot \nabla(e^{a|D|^{1/\sigma}}u)$ and using Sobolev embeddings and the boundedness of $e^{a|D|^{1/\sigma}}u$ in H^2 , we obtain

$$|I_1| \leq CM \|z\|_{L^2} \|z\|_{H^1},$$

where $M = \|e^{a|D|^{1/\sigma}}u\|_{H^2}$ is uniformly bounded in time due to the smoothing effect.

Applying Cauchy–Schwarz and Young’s inequality,

$$|I_1| \leq \frac{\nu}{4} \|z\|_{H^1}^2 + \frac{C^2 M^2}{\nu} \|z\|_{L^2}^2.$$

Since $\|z\|_{L^2} \leq \|z\|_{H^1}$, we can write

$$|I_1| \leq \frac{\nu}{4} \|z\|_{H^1}^2 + \frac{C^2 M^2}{\nu} \|z\|_{H^1}^2.$$

Similarly, for I_2 ,

$$I_2 \leq \|e^{a|D|^{1/\sigma}}(v \cdot \nabla w)\|_{H^{-1}} \|z\|_{H^1}.$$

We have $e^{a|D|^{1/\sigma}}(v \cdot \nabla w) = (e^{a|D|^{1/\sigma}}v) \cdot \nabla z$. Using again that $v \in H_{a,\sigma}^2$ and standard Sobolev embeddings,

$$|I_2| \leq C \|e^{a|D|^{1/\sigma}}v\|_{H^2} \|\nabla z\|_{H^2} \|z\|_{H^1}.$$

By similar Young’s inequality arguments,

$$|I_2| \leq \frac{\nu}{4} \|\nabla z\|_{H^1}^2 + \frac{C^2}{\nu} (\|e^{a|D|^{1/\sigma}}v\|_{H^2} \|z\|_{H^1})^2.$$

Combining I_1 and I_2 carefully (absorbing terms like $\|\nabla z\|_{H^1}^2$ into $\|z\|_{H^1}^2$ since $\|\nabla z\|_{H^1}^2 \leq \|z\|_{H^2}^2$ and using the $H_{a,\sigma}^2$ bounds for u and v), we find a differential inequality of the form

$$\frac{d}{dt} \|z\|_{H^1}^2 \leq \frac{2C^2 K^2}{\nu} \|z\|_{H^1}^2,$$

where $K = \max\{M, \|e^{a|D|^{1/\sigma}}v(t)\|_{H^2}\}$ is bounded.

Now, applying Gronwall’s inequality, we have

$$\frac{d}{dt} \|z(t)\|_{H^1}^2 \leq \lambda \|z(t)\|_{H^1}^2,$$

with $\lambda = \frac{2C^2 K^2}{\nu} > 0$.

Integrate from 0 to t

$$\|z(t)\|_{H^1}^2 \leq \|z(0)\|_{H^1}^2 e^{\lambda t}.$$

Taking the square root,

$$\|z(t)\|_{H^1} \leq \|z(0)\|_{H^1} e^{\lambda t/2}.$$

Since $z(0) = e^{a|D|^{1/\sigma}}(u^0 - v^0)$, we have $\|z(0)\|_{H^1} = \|u^0 - v^0\|_{H_{a,\sigma}^1}$. Thus,

$$\|u(t) - v(t)\|_{H_{a,\sigma}^1} = \|z(t)\|_{H^1} \leq \|u^0 - v^0\|_{H_{a,\sigma}^1} e^{(\lambda/2)t}.$$

Set $\lambda' = \lambda/2$. This shows

$$\|u(t) - v(t)\|_{H_{a,\sigma}^1} \leq \|u^0 - v^0\|_{H_{a,\sigma}^1} e^{\lambda' t}.$$

We have shown that the difference $u(t) - v(t)$ in $H_{a,\sigma}^1$ norm grows exponentially at a rate depending on ν, a, σ , and the forcing parameters. This establishes uniqueness and continuous dependence on the initial data in $H_{a,\sigma}^1(\mathbb{R}^3)$, completing the proof of Theorem 2.

5. Conclusions

In this paper, we have established the existence and uniqueness of solutions to the incompressible Navier–Stokes equations in the Sobolev–Gevrey space $H_{a,\sigma}^1(\mathbb{R}^3)$. By demonstrating that the nonlinear operator governing the Navier–Stokes dynamics is locally Lipschitz continuous within this functional framework, we applied the Banach fixed-point theorem to secure the existence of a unique solution emanating from the given initial data. Furthermore, the obtained energy estimates along with the Grönwall’s inequality have allowed us to prove that the Sobolev–Gevrey norm of the solution grows exponentially with time. This result supports the stability and regularity of the solutions in high-regularity spaces and is indeed relevant for investigations into the long-term behavior and potential global existence of solutions under more stringent conditions. Furthermore, we proved the uniqueness and continuous dependence of solutions with initial data.

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