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A Study of p-Laplacian Nonlocal Boundary Value Problem Involving Generalized Fractional Derivatives in Banach Spaces

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Abstract: The aim of this article is to introduce and study a new class of fractional integro nonlocal boundary value problems involving the p-Laplacian operator and generalized fractional derivatives. The existence of solutions in Banach spaces is investigated with the aid of the properties of Kuratowski's noncompactness measure and Sadovskii's fixed-point theorem. Two illustrative examples are constructed to guarantee the applicability of our results.

Keywords: generalized fractional derivative; p-Laplacian operator; measure of noncompactness; existence; fixed point

MSC: 34A08; 34B10; 34B15; 34G20; 26A33

1. Introduction

Differential equations with the p-Laplacian operator have received considerable attention in recent years. This interest is due to their ability to model physical phenomena occurring in mechanics, electrodynamics, nonlinear elasticity, and many other disciplines. They first appeared in [1], when the author was studying the turbulent flows inside a porous medium. He transformed this physical phenomenon into the following equation:

$$\left(\varphi_p\left(y'(t)\right)\right)' = f(t, y(t)) \tag{1}$$

where $\varphi_p(t) = |t|^{p-2}.t$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$. Since then, this type of equation has attracted many researchers to study with several types of differential operators and equipped by different forms of initial and boundary conditions. Liu et al. in [2] investigated the p-Laplacian differential equation involving the Caputo derivative of order $1 < \alpha \in \mathbb{R}$ and employed the contraction mapping principle for the existence and uniqueness results. The method of upper and lower solutions was utilized to study mixed fractional differential p-Laplacian equations in [3]. Bai [4] studied a p-Laplacian problem involving the Riemann–Liouville fractional derivative and derived the existence and uniqueness criteria of positive solutions by applying Guo–Krasnoselskii's fixed-point theorem and Banach's fixed-point theorem. In 2021, Mater et al. [5] considered a class of p-Laplacian boundary value problems with generalized Caputo derivatives and established the existence results with the aid of the Banach and Schauder theorems. Alsaedi et al. [6] discussed the uniqueness of solutions to a p-Laplacian problem involving the ψ-Hilfer fractional derivative. Recently, Kaihong [7] derived the solvability criteria and UH-stability of a coupled impulsive system of p-Laplacian ABC-fractional differential equations. For more applications of p-Laplacian equations in the scalar case, see [8–14]. Although there are many papers devoted to the study of p-Laplacian fractional differential equations in scalar

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spaces, there are few works concerned with this topic in Banach spaces. To the best of our knowledge, the first article that studied p-Laplacian differential equations in Banach spaces was by Ji and Ge in [15], and then some works followed that. Liu and Lu [16] discussed a class of fractional integro-differential equations with the p-Laplacian operator and nonlocal boundary conditions in Banach spaces. Tan et al. [17] extended Liu et al. [2] by studying their problem in Banach spaces with the aid of the Sadovskii fixed-point theorem, which is one of the most important results in fixed-point theory that relies on the measure of noncompactness. Srivastava et al. [18] investigated the existence and stability results for a class of p-Laplacian differential equations in Banach spaces with the aid of Darbo's fixed-point theorem. Mfadel et al. [19] derived the existence results of a coupled system of differential equations involving ψ-Caputo fractional derivatives with the p-Laplacian operator in Banach spaces by utilizing Mönch's fixed-point theorem. The powerful tool in the study of differential equations in Banach spaces is the measure of noncompactness, which has many applications in infinite-dimensional Banach spaces. For more details and examples of using the measure of noncompactness in arbitrary Banach spaces, see [20–24].

Motivated by the above-mentioned works, we introduce an abstract p-Laplacian fractional boundary value problem involving generalized fractional derivatives and nonlocal boundary conditions. Precisely, we investigate the following problem:

$$\begin{cases} \left(\varphi_p\left({}^{\rho}D^{\eta}_{0^+}y(t)\right)\right)' + \mathcal{J}(t, y(t), {}^{\rho}D^{\zeta}_{0^+}y(t)) = \theta, \ \ t \in [0,1], \\ y(0) = \theta, \quad {}^{\rho}D^{\eta}_{0^+}y(0) = \theta, \\ y(1) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\gamma_i}_{0^+}y(\xi_i), \ 0 < \xi_i < 1, \end{cases} \tag{2}$$

where $\varphi_p(y) = \|y\|^{p-2}.y$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $\eta \in (1,2]$, $\zeta \in (0,1)$, $\rho > 0$, $\gamma_i > 0\,(i = 1,2,\ldots,m)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, ${}^{\rho}D^{\eta}_{0^+}$, ${}^{\rho}D^{\zeta}_{0^+}$ are the generalized fractional derivatives and ${}^{\rho}I^{\gamma_i}_{0^+}$ is the generalized fractional integral, $\mathcal{J} : [0,1] \times B \times B \to B$ is a continuous function, and θ refers to the zero element of the Banach space B.

The crucial issue that should be taken into consideration to deal with p-Laplacian differential equations is that the Lipschitz property of p-Laplacian operators depends on the values of p, as will be explained in Lemma 5. Unfortunately, this issue has not been treated correctly in Banach spaces, especially in demonstrating the continuity and contraction of the operators. As far as we know, the best study of p-Laplacian differential equations in Banach spaces was presented in [15]; however, its results are limited to the case $1 < p \leq 2$ $(q > 2)$. So, based on the nature of the p-Laplacian operators, our goal in this research is to extend the previous arguments in [2,4,6,11] of scalar p-Laplacian problems by establishing two existence results for problem (2) that can be applied for all the values of p in Banach spaces.

The remainder of this article is arranged as follows. Section 2 includes the background material that we need to deduce our main result. In Section 3, we establish our main existence results relying on Sadovskii's theorem for two cases of the value of p. Also, we present two illustrative examples. The conclusion is given in Section 4.

2. Preliminaries

In the present work, we need to define the following spaces:

$\mathcal{BC}([0,1]) = \{y \in C([0,1], B) : \sup_{t\in[0,1]} \frac{\|y(t)\|}{1+t^{\eta-1}} < +\infty\}$, $\mathcal{BC}^1([0,1]) = \{{}^{\rho}D^{\zeta}_{0^+}y \in C([0,1], B) : \sup_{t\in[0,1]} \frac{\|y(t)\|}{1+t^{\eta-1}} < +\infty$ and $\sup_{t\in[0,1]} \|{}^{\rho}D^{\zeta}_{0^+}y(t)\| < +\infty\}$, where $C([0,1], B)$ is the Banach space of all continuous functions $y : [0,1] \to B$ with the norm $\|y\| = \sup_{t\in[0,1]} \|y(t)\|$.

It is obvious that $\mathcal{BC}([0,1])$ and $\mathcal{BC}^1([0,1])$ form Banach spaces with the norm $\|y\|_\eta = \sup_{t\in[0,1]} \frac{\|y(t)\|}{1+t^{\eta-1}}$ and $\|y\|_{\eta'} = \max\{\|y\|_\eta, \|{}^\rho D^\zeta_{0^+} y\|\}$, respectively, where $\|{}^\rho D^\zeta_{0^+} y\| = \sup_{t\in[0,1]} \|{}^\rho D^\zeta_{0^+} y(t)\|$.

Definition 1 ([25]). *For a bounded set Ω in a Banach space B, the Kuratowski measure of noncompactness of Ω is defined by*

$$\mu(\Omega) = \inf\{\epsilon > 0 : \Omega = \bigcup_{i=1}^{n} \Omega_i, diam(\Omega_i) < \epsilon, i = 1, 2, \ldots, n\},$$

where $diam(\Omega_i)$ denotes the diameters of Ω_i.

Definition 2 (δ-set contraction operator, [25]). *For real Banach spaces B_1 and B_2, let $\Omega \subset B_1$, and $F : \Omega \to B_2$ is a bounded and continuous operator. If we can find a constant $\delta \geq 0$ such that $\mu(F(A)) \leq \delta\mu(A)$ for any bounded set A in Ω, then F is called a δ-set contraction operator. F is called a strict set contraction (condensing) operator if $\delta < 1$.*

The notions $\mu_B(.)$, $\mu_C(.)$, $\mu_\eta(.)$, and $\mu_{\eta'}(.)$ denote the Kuratowski measure of noncompactness in B, $C([0,1], B)$, $BC([0,1])$, and $B^1C([0,1])$, respectively.

Lemma 1 ([26]). *If $F \in C([0,1], B)$ is bounded and equicontinuous, then $\mu_B(F(t))$ is continuous on $[0,1]$ and $\mu_C(F) = \max_{t\in[0,1]} \mu_B(F(t))$, $\mu_B(\int_{[0,1]} y(t)dt : y \in F) \leq \int_{[0,1]} \mu_B(F(t))dt$, where $F(t) = \{y(t) : y \in F\}$ for each $t \in [0,1]$.*

Lemma 2 (Sadovskii, [25,27]). *Let a subset S of a Banach space B be convex, bounded, and closed. If the operator $\mathcal{N} : S \to S$ is condensing, then $\mathcal{N}$ has a fixed point in S.*

Definition 3 ([28]). *For $\rho, \varsigma > 0$, the generalized fractional integral of a function $\mathcal{Q} \in X^p_c(c,d)$ for $-\infty < c < t < d < \infty$ is given by*

$$({}^\rho I^\varsigma_{c^+} \mathcal{Q})(t) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_c^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} \mathcal{Q}(s)ds, \tag{3}$$

where $X^p_c(c,d)$ denotes the space of all Lebesgue measurable complex-valued functions on (c,d).

Definition 4 ([29]). *For $\varsigma > 0$, $n = [\varsigma] + 1$ and $\rho > 0$, the generalized fractional derivative is given, for $0 \leq c < x < d < \infty$, by*

$$({}^\rho D^\varsigma_{c+} \mathcal{Q})(t) = \frac{\rho^{\varsigma-n+1}}{\Gamma(n-\varsigma)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_c^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\varsigma-n+1}} \mathcal{Q}(s)ds, \tag{4}$$

if the integral exists.

Lemma 3 ([30]). *Let $1 < \varsigma \leq 2, \rho > 0$, $\mathcal{Q} \in X^p_c(0,T)$, and ${}^\rho I^{2-\varsigma} \mathcal{Q} \in AC^2_\rho(0,T)$. Then, the general solution of the fractional differential equation ${}^\rho D^\varsigma_{0+} \mathcal{Q}(t) = 0$ is*

$$\mathcal{Q}(t) = d_1 t^{\rho(\varsigma-1)} + d_2 t^{\rho(\varsigma-2)},$$

where $d_i \in \mathbb{R}, i = 1, 2$. Furthermore,

$$({}^\rho I^\varsigma_{0+} \, {}^\rho D^\varsigma_{0+} \mathcal{Q})(t) = \mathcal{Q}(t) + c_1 t^{\rho(\varsigma-1)} + c_2 t^{\rho(\varsigma-2)},$$

where the space $AC^2_\rho([c,d])$ is defined by:

$$AC^2_\rho[c,d] = \Big\{h : [c,d] \to B : (t^{1-\rho}\frac{d}{dt}h) \in AC[c,d]\Big\}.$$

Lemma 4. *Let $\psi \in C([0,1],B)$. Then, the solution to the following generalized fractional p-Laplacian problem:*

$$\begin{cases} \big(\varphi_p\big({}^\rho D^\eta_{0^+} y(t)\big)\big)' + \psi(t) = \theta, \ \ t \in [0,1], \\ y(0) = \theta, \quad {}^\rho D^\eta_{0^+} y(0) = \theta, \\ y(1) = \sum_{i=1}^{m} \sigma_i {}^\rho I^{\gamma_i}_{0^+} y(\xi_i), \ 0 < \xi_i < 1, \end{cases} \tag{5}$$

can be represented by

$$\begin{aligned} y(t) \ &= \ -\frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \varphi_q\Big(\int_0^s \psi(v)dv\Big)ds \\ &\quad + \frac{t^{\rho(\eta-1)}}{\Lambda}\Bigg\{\frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1} \varphi_q\Big(\int_0^s \psi(v)dv\Big)ds \\ &\quad - \sum_{i=1}^{m} \frac{\sigma_i \rho^{1-(\eta+\gamma_i)}}{\Gamma(\eta+\gamma_i)} \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\eta+\gamma_i-1} \varphi_q\Big(\int_0^s \psi(v)dv\Big)ds\Bigg\} \end{aligned} \tag{6}$$

where $\Lambda = 1 - \sum_{i=1}^{m} \frac{\sigma_i \Gamma(\eta)}{\rho^{\gamma_i}\Gamma(\eta+\gamma_i)} \xi_i^{\rho(\eta+\gamma_i-1)} \neq 0.$

Proof. Let $\varphi_p\big({}^\rho D^\eta_{0^+} y(t)\big) = K(t)$; then, problem (5) can be decomposed as

$$\begin{cases} -K'(t) = \psi(t), \\ K(0) = \theta, \end{cases} \tag{7}$$

and

$$\begin{cases} {}^\rho D^\eta_{0^+} y(t) = \varphi_q\big(K(t)\big), \\ y(0) = \theta, \quad y(1) = \sum_{i=1}^{m} \sigma_i {}^\rho I^{\gamma_i}_{0^+} y(\xi_i), \end{cases} \tag{8}$$

The equation $-K'(t) = \psi(t)$ is reduced to its equivalent equation

$$K(t) \ = \ -\int_0^t \psi(s)ds + c_0, \tag{9}$$

where c_0 is arbitrary constant. The condition $K(0) = \theta$ implies $c_0 = \theta$. Hence,

$$K(t) \ = \ -\int_0^t \psi(s)ds \tag{10}$$

Now, for the solution to problem (8), applying ${}^\rho I^\alpha_{0+}$ to both sides of the equation in (8), we have

$$y(t) \ = \ \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \varphi_q(K(s))ds + b_0 t^{\rho(\eta-2)} + b_1 t^{\rho(\eta-1)}, \tag{11}$$

where b_0 and b_1 are arbitrary constants. The condition $y(0) = \theta$ leads to $b_0 = \theta$, and since $y(1) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I_{0^+}^{\gamma_i} y(\xi_i)$, we have

$$\begin{aligned} & \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^{\rho})^{\eta-1} \varphi_q(K(s))ds + b_1 \\ = & \sum_{i=1}^{m} \sigma_i \frac{\rho^{1-(\eta+\gamma_i)}}{\Gamma(\eta+\gamma_i)} \int_0^{\xi_i} s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{(\eta+\gamma_i)-1} \varphi_q(K(s))ds + \sum_{i=1}^{m} \sigma_i \frac{\Gamma(\eta)}{\rho^{\gamma_i}\Gamma(\eta+\gamma_i)} b_1 \xi_i^{\rho(\eta+\gamma_i-1)}, \end{aligned}$$

which yields

$$\begin{aligned} b_1 = & \left[1 - \sum_{i=1}^{m} \frac{\sigma_i \Gamma(\eta)}{\rho^{\gamma_i}\Gamma(\eta+\gamma_i)} \xi_i^{\rho(\eta+\gamma_i-1)}\right]^{-1} \left\{ -\frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^{\rho})^{\eta+\gamma_i-1} \varphi_q(K(s))ds \right. \\ & \left. + \sum_{i=1}^{m} \sigma_i \int_0^{\xi_i} s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\eta+\gamma_i-1} \varphi_q(K(s))ds \right\}, \end{aligned} \tag{12}$$

Substituting the values of b_1, b_0, and $K(t)$ in (11), we obtain the solution (6). By direct computation, the converse can be deduced. □

The following lemma is necessary to deal with the p-Laplacian operator, which can be easily generalized to the Banach space B by replacing $|\varphi_p(f)|$ by $\|\varphi_p(f)\| = \sup_n |\|f\|^{p-1} f_n|$.

Lemma 5 ([2], (2.1), and (2.2) on page 3268)**.**

(*i*) *For* $1 < p \leq 2$, $|f_1|, |f_2| \geq \Psi_1 > 0$ *and* $f_1 f_2 > 0$, *we have*

$$|\varphi_p(f_2) - \varphi_p(f_1)| \leq (p-1)\Psi_1^{p-2}|f_2 - f_1|;$$

(*ii*) *For* $p > 2$, $|f_1|, |f_2| \leq \Psi_2$ *and* $f_1 f_2 > 0$, *we have*

$$|\varphi_p(f_2) - \varphi_p(f_1)| \leq (p-1)\Psi_2^{p-2}|f_2 - f_1|.$$

3. Existence Results

In view of Lemma 4, problem (2) can be transformed into a fixed-point problem by defining an operator $\mathcal{P} : \mathcal{BC}^1([0,1]) \to \mathcal{BC}^1([0,1])$ as the following:

$$\begin{aligned} (\mathcal{P}y)(t) = & -\frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\eta-1} \phi_q\Big(\int_0^s \mathcal{J}(v, y(v), {}^{\rho}D_{0^+}^{\zeta} y(v))dv \Big) ds \\ & + \frac{t^{\rho(\eta-1)}}{\Lambda} \left\{ \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^{\rho})^{\eta-1} \varphi_q\Big(\int_0^s \mathcal{J}(v, y(v), {}^{\rho}D_{0^+}^{\zeta} y(v))dv \Big) ds \right. \\ & \left. - \sum_{i=1}^{m} \frac{\sigma_i \rho^{1-(\eta+\gamma_i)}}{\Gamma(\eta+\gamma_i)} \int_0^{\xi_i} s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\eta+\gamma_i-1} \varphi_q\Big(\int_0^s \mathcal{J}(v, y(v), {}^{\rho}D_{0^+}^{\zeta} y(v))dv \Big) ds \right\} \end{aligned} \tag{13}$$

For convenience, we use the following notations:

$$\Delta_1 = \frac{|\Lambda| + 1}{\rho^{\eta}|\Lambda|\Gamma(\eta+1)} + \sum_{i=1}^{m} \frac{|\sigma_i| \xi_i^{\rho(\eta+\gamma_i)}}{\rho^{\eta+\gamma_i}|\Lambda|\Gamma(\eta+\gamma_i+1)}, \tag{14}$$

$$\Delta_2 = \frac{\eta(1+|\Lambda|) - \zeta}{\rho^{\eta-\zeta}\Gamma(\eta-\zeta+1)} + \sum_{i=1}^{m} \frac{|\sigma_i| \xi_i^{\rho(\eta+\gamma_i-\zeta)} \Gamma(\eta)}{\rho^{\eta+\gamma_i-\zeta}|\Lambda|\Gamma(\eta-\zeta)\Gamma(\alpha+\gamma_i+1)}, \tag{15}$$

and

$$\Delta = \max\{\Delta_1, \Delta_2\}. \tag{16}$$

Further, we introduce the following assumptions to establish our results.

(C_1) There are continuous nonnegative functions $\chi, \kappa, \omega : [0,1] \to [0,\infty)$, such that

$$\int_0^t \|\mathcal{J}(s,y,z)\|ds \leq \chi(t)\|y\| + \kappa(t)\|z\| + \omega(t) \quad \forall (t,y,z) \in [0,1] \times B \times B,$$

and

$$H^* = \max_{t\in[0,1]}\{(1+t^{\eta-1})\chi(t) + \kappa(t)\}, \quad \omega^* = \max_{t\in[0,1]} \omega(t).$$

(C_2) For any $\varrho > 0$ and $[a,b] \subset [0,1]$, the function $\mathcal{J}(t,y,{}^{\rho}D^{\zeta}_{0^+}y)$ is uniformly continuous on $[a,b] \times \mathrm{Y}_{\varrho} \times \mathrm{Y}_{\varrho}$, where $\mathrm{Y}_{\varrho} = \{y \in B : \|y\| \leq \varrho\}$, and $\mathcal{J}(t,y,{}^{\rho}D^{\zeta}_{0^+}y) > \theta$.

(C_3) There are two constants $\alpha_1, \alpha_2 > 0$, for all bounded subsets $\Omega_i \subset B, i = 1,2$, such that

$$\mu_B\Big(\mathcal{J}(t,\Omega_1(t),\Omega_2(t))\Big) \leq \alpha_1\mu_B(\Omega_1(t)) + \alpha_1\mu_B(\Omega_2(t)), \ \forall\ t \in [0,1],$$

and $\alpha := (\max_{t\in I}(1+t^{\eta-1})\alpha_1 + \alpha_2)$.

(C_4) There are two constants $\mathcal{A} > 0$ and $0 < \sigma \leq \frac{1}{2-q}$, such that

$$\|\mathcal{J}(t,y,z)\| \geq \mathcal{A}\sigma t^{\sigma-1}, \quad \forall (t,y,z) \in [0,1] \times B \times B.$$

The following lemma is a modified version of ([20], [Lemma 2.6]), and we can easily prove it by using similar arguments, so we omit its proof.

Lemma 6. *For a bounded subset Ω of $\mathcal{BC}^1([0,1])$, if (C_1) is satisfied, then*

$$\mu_{\eta'}(\mathcal{P}\Omega) = \max\Big\{\sup_{t\in[0,1]} \mu_B\Big(\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}\Big), \sup_{t\in[0,1]} \mu_B\Big(({}^{\rho}D^{\zeta}_{0^+}\mathcal{P}\Omega)(t)\Big)\Big\}.$$

Now, let us define, for any $\lambda > 0$, the bounded set

$$D_\lambda = \{y \in \mathcal{BC}^1([0,1]) : \|y\|_{\eta'} \leq \lambda\}. \tag{17}$$

Then, we establish our first existence result for $1 < p \leq 2$ $(q > 2)$.

Theorem 1. *Assume that $1 < p \leq 2$ and the conditions (C_1)–(C_3) hold true. Then, problem (2) has at least one solution on $D_\lambda \subset \mathcal{BC}^1([0,1])$, if the following inequalities are satisfied:*

$$\Delta(q-1)|H^*\lambda + \omega * |^{q-2}\alpha < 1, \ \text{and} \ \lambda \geq \max\{1, \omega^*[\Delta^{1/(1-q)} - H^*]^{-1}\}, \tag{18}$$

where Δ is given by (16).

Proof. In order to apply the conclusion of Lemma 2, we need to satisfy its hypotheses in several steps.

Step 1: *The operator $\mathcal{P} : \mathcal{BC}^1([0,1]) \to \mathcal{BC}^1([0,1])$ is bounded and continuous.*

Firstly, we have to prove that for any $y \in \mathcal{BC}^1([0,1])$, $(\mathcal{P}y)(t) \in \mathcal{BC}^1([0,1])$. So, combining the definitions of the operators $\mathcal{P}$ and φ_q with the condition (C_1), we find

$$\begin{aligned}
&\Big\|\frac{(\mathcal{P}y)(t)}{1+t^{\eta-1}}\Big\| \\
\leq\ & \frac{\rho^{1-\alpha}}{\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s \big\|\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\big\|dv\Big)ds \\
+\ & \frac{\rho^{1-\eta}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s \big\|\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\big\|dv\Big)ds \\
+\ & \frac{\rho^{1-(\eta+\gamma)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m|\sigma_i|\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s \big\|\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\big\|dv\Big)ds \\
\leq\ & \frac{\rho^{1-\alpha}}{\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\Big|\chi(s)\|y(s)\|+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
+\ & \frac{\rho^{1-\eta}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\Big|\chi(s)\|y(s)\|+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
+\ & \frac{\rho^{1-(\eta+\gamma_i)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m|\sigma_i|\int_0^{\xi_i} s^{\rho-1}(\xi_i-s^\rho)^{\eta+\gamma_i-1}\Big|\chi(s)\|y(s)\|+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
\leq\ & \frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\Big|(1+s^{\eta-1})\chi(s)\frac{\|y(s)\|}{1+s^{\eta-1}}+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
+\ & \frac{\rho^{1-\eta}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\Big|(1+s^{\eta-1})\chi(s)\frac{\|y(s)\|}{1+s^{\eta-1}}+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
+\ & \frac{\rho^{1-(\eta+\gamma_i)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m|\sigma_i|\int_0^{\xi_i} s^{\rho-1}(\xi_i-s^\rho)^{\eta+\gamma_i-1}\times \\
& \Big|(1+s^{\eta-1})\chi(s)\frac{\|y(s)\|}{1+s^{\eta-1}}+\kappa(s)\|{}^\rho D^\zeta_{0^+}y(s)\|+\omega(s)\Big|^{q-1}ds \\
\leq\ & |H^*\|y\|_{\eta'}+\omega*|^{q-1}\Big[\frac{|\Lambda|+1}{\rho^\eta|\Lambda|\Gamma(\eta+1)}+\sum_{i=1}^m\frac{|\sigma_i|\xi_i^{\eta+\gamma_i}}{\rho^{\eta+\gamma_i}|\Lambda|\Gamma(\eta+\gamma_i+1)}\Big] \\
=\ & \Delta_1|H^*\|y\|_{\eta'}+\omega*|^{q-1}<\infty.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Big\|{}^\rho D^\zeta_{0^+}(\mathcal{P}y)(t)\Big\| &\leq [H^*\|y\|_{\eta'}+\omega*]\Big[\frac{\eta(1+|\Lambda|)-\zeta}{\rho^{\eta-\zeta}\Gamma(\eta-\zeta+1)}+\sum_{i=1}^m\frac{|\sigma_i|\xi_i^{\eta+\gamma_i-\zeta}\Gamma(\eta)}{\rho^{\eta+\gamma_i-\zeta}|\Lambda|\Gamma(\eta-\zeta)\Gamma(\eta+\gamma_i+1)}\Big] \\
&= \Delta_2|H^*\|y\|_{\eta'}+\omega*|^{q-1}<\infty.
\end{aligned}$$

where

$$\begin{aligned}
{}^\rho D^\zeta_{0^+}(\mathcal{P}y)(t) =\ & -\frac{\rho^{1-\eta+\zeta}}{\Gamma(\eta-\zeta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-\zeta-1}\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big)ds \\
& +\frac{\rho^\zeta\Gamma(\eta)t^{\rho(\eta-\zeta-1)}}{\Gamma(\eta-\zeta)\Lambda}\times \\
& \Big\{\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-t^\rho)^{\eta+\gamma_i-1}\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big)ds \\
& -\sum_{i=1}^m\frac{\sigma_i\rho^{1-(\eta+\gamma_i)}}{\Gamma(\eta+\gamma_i)}\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma_i-1}\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big)ds\Big\}.
\end{aligned}$$

Hence, $(\mathcal{P}y)(t)$ is well defined for all $y \in \mathcal{BC}^1([0,1])$. Also, $\mathcal{P}$ is a bounded operator, that is, for any bounded subset of $\mathcal{BC}^1([0,1])$, say, $D_\nu = \{y \in \mathcal{BC}^1([0,1]) : \|y\|_{\eta'} \leq \nu\}$ and any $y \in D_\nu$, we have

$$\|\mathcal{P}y\|_\eta = \sup_{t\in[0,1]}\frac{\|(\mathcal{P}y)(t)\|}{1+t^{\eta-1}} \leq \Delta_1|H^*\nu+\omega*|^{q-1},$$

and

$$\left\|{}^{\rho}D_{0^+}^{\zeta}\mathcal{P}y\right\| = \sup_{t\in[0,1]}\left\|{}^{\rho}D_{0^+}^{\zeta}(\mathcal{P}y)(t)\right\| \leq \Delta_2|H^*\nu+\omega*|^{q-1},$$

which implies

$$\|\mathcal{P}y\|_{\eta'} \leq \Delta|H^*\nu+\omega*|^{q-1}.$$

Thus, $\mathcal{P}$ maps any bounded set into a bounded set in $\mathcal{BC}^1([0,1])$.

Next, to prove the continuity of $\mathcal{P}$, let us take a sequence $\{y_n\}$ in $\mathcal{BC}^1([0,1])$, such that $\|y_n-y\|_{\eta'}\to 0$ as $n\to\infty$ in $\mathcal{BC}^1([0,1])$. Thus, $\{y_n\}$ is a bounded subset of $\mathcal{BC}^1([0,1])$. Consequently, we can find a constant δ such that $\|y_n\|_{\eta'}\leq\delta$ for $n\geq 1$. So, by taking the limit, we find that $\|y\|_{\eta'}\leq\delta$. Now, from the condition (C_2), there is, for any $\varepsilon>0$, a constant $N>0$ such that

$$\left\|\mathcal{J}(t,y_n(t),{}^{\rho}D_{0^+}^{\zeta}y_n(t))-\mathcal{J}(t,y(t),{}^{\rho}D_{0^+}^{\zeta}y(t))\right\| \leq \left[\Delta(q-1)[H^*\delta+\omega*]^{q-2}\right]^{-1}\varepsilon,$$

for all $n\geq N, t\in[0,1]$. In addition, by the condition (C_1), we have

$$\left\|\int_0^t \mathcal{J}(t,y(t),{}^{\rho}D_{0^+}^{\zeta}y(t))\right\| \leq [H^*\delta+\omega*].$$

In view of property (ii) of Lemma 5, where $q>2$, we find

$$\begin{aligned}
&\left\|\frac{(\mathcal{P}y_n)(t)}{1+t^{\eta-1}}-\frac{(\mathcal{P}y)(t)}{1+t^{\eta-1}}\right\|\\
\leq\ &\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\eta-1}\times\\
&\left\|\varphi_q\Big(\int_0^s \mathcal{J}(v,y_n(v),{}^{\rho}D_{0^+}^{\zeta}y_n(v))dv\Big)-\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^{\rho}D_{0^+}^{\zeta}y(v))dv\Big)\right\|ds\\
&+\frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^{\rho})^{\eta-1}\times\\
&\left\|\varphi_q\Big(\int_0^s \mathcal{J}(v,y_n(v),{}^{\rho}D_{0^+}^{\zeta}y_n(v))dv\Big)-\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^{\rho}D_{0^+}^{\zeta}y(v))dv\Big)\right\|ds\\
&+\frac{\rho^{1-(\eta+\gamma_i)}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^{m}|\sigma_i|\int_0^{\xi_i} s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\eta+\gamma_i-1}\times\\
&\left\|\varphi_q\Big(\int_0^s \mathcal{J}(v,y_n(v),{}^{\rho}D_{0^+}^{\zeta}y_n(v))dv\Big)-\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^{\rho}D_{0^+}^{\zeta}y(v))dv\Big)\right\|ds\\
\leq\ &(q-1)[H^*\delta+\omega*]^{q-2}\Delta_1\left\|\mathcal{J}(t,y_n(t),{}^{\rho}D_{0^+}^{\zeta}y_n(t))-\mathcal{J}(t,y(t),{}^{\rho}D_{0^+}^{\zeta}y(t))\right\|\\
\leq\ &(q-1)[H^*\delta+\omega*]^{q-2}\Delta_1\left[\Delta(q-1)[H^*\delta+\omega*]^{q-2}\right]^{-1}\varepsilon\leq\varepsilon.
\end{aligned}$$

Similarly, we have

$$\left\|{}^{\rho}D_{0^+}^{\zeta}(\mathcal{P}y_n)(t)-{}^{\rho}D_{0^+}^{\zeta}(\mathcal{P}y)(t)\right\| \leq (q-1)[H^*\delta+\omega*]^{q-2}\Delta_2\left[\Delta(q-1)[H^*\delta+\omega*]^{q-2}\right]^{-1}\varepsilon\leq\varepsilon.$$

Thus, $\|(\mathcal{P}y_n)-(\mathcal{P}y)\|_{\eta'}\leq\varepsilon$, which proves that $\mathcal{P}:\mathcal{BC}^1([0,1])\to\mathcal{BC}^1([0,1])$ is a continuous operator.

Step 2: *For a bounded subset Ω of $\mathcal{BC}^1([0,1])$, $\dfrac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}$ and $({}^{\rho}D_{0^+}^{\zeta}\mathcal{P}\Omega)(t)$ are equicontinuous on $[0,1]$.*

The boundeness of Ω means that there is a constant, say r, such that for any $y\in\Omega$, $\|y\|_{\eta'}<r$. Using the condition (C_1), for $0<t_1<t_2<1$, we have

$$
\begin{aligned}
&\Big\|\frac{(\mathcal{P}y)(t_2)}{1+t_2^{\eta-1}}-\frac{(\mathcal{P}y)(t_1)}{1+t_1^{\eta-1}}\Big\|\\
\leq\ &\Big\|\frac{\rho^{1-\eta}}{(1+t_2^{\eta-1})\Gamma(\eta)}\int_0^{t_2}s^{\rho-1}(t_2^\rho-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v))dv\Big)ds\\
&-\frac{\rho^{1-\eta}}{(1+t_1^{\eta-1})\Gamma(\eta)}\int_0^{t_1}s^{\rho-1}(t_1^\rho-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v))dv\Big)ds\Big\|\\
&+\Big|\frac{t_2^{\rho(\eta-1)}}{1+t_2^{\eta-1}}-\frac{t_1^{\rho(\eta-1)}}{1+t_1^{\eta-1}}\Big|\Big\{\frac{\rho^{1-\eta}}{|\Lambda|\Gamma(\eta)}\Big\|\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\varphi_q\Big(\int_0^s\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v))dv\Big)ds\Big\|\\
&+\sum_{i=1}^m\frac{|\sigma_i|\rho^{1-(\eta+\gamma_i)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\Big\|\int_0^{\xi_i}s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma_i-1}\varphi_q\Big(\int_0^s\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v))dv\Big)ds\Big\|\Big\}\\
\leq\ &|H^*\|y\|_{\eta'}+\omega*|^{q-1}\Big\{\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^{t_1}s^{\rho-1}\Big|\frac{(t_2^\rho-s^\rho)^{\eta-1}}{1+t_2^{\eta-1}}-\frac{(t_1^\rho-s^\rho)^{\eta-1}}{1+t_1^{\eta-1}}\Big|ds\\
&+\int_{t_1}^{t_2}s^{\rho-1}\Big|\frac{(t_2^\rho-s^\rho)^{\eta-1}}{1+t_2^{\eta-1}}\Big|ds+\frac{1}{|\Lambda|}\Big|\frac{t_2^{\rho(\eta-1)}}{1+t_2^{\eta-1}}-\frac{t_1^{\rho(\eta-1)}}{1+t_1^{\eta-1}}\Big|\times\\
&\Big[\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}ds+\sum_{i=1}^m\frac{|\sigma_i|\rho^{1-(\eta+\gamma_i)}}{\Gamma(\eta+\gamma_i)}\int_0^{\xi_i}s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta-1}ds\Big]\Big\}\\
\leq\ &|H^*r+\omega*|^{q-1}\Big\{\frac{1}{\rho^\eta\Gamma(\eta+1)}\Big[\frac{2(t_2^\rho-t_1^\rho)^\eta}{1+t_2^{\eta-1}}+\frac{|t_2^{\rho\eta}(1+t_1^{\eta-1})-t_1^{\rho\eta}(1+t_2^{\eta-1})|}{(1+t_1^{\eta-1})(1+t_2^{\eta-1})}\Big]\\
&+\frac{|t_2^{\rho(\eta-1)}(1+t_1^{\eta-1})-t_1^{\rho(\eta-1)}(1+t_2^{\eta-1})|}{|\Lambda|(1+t_1^{\eta-1})(1+t_2^{\eta-1})}\times\\
&\Big[\frac{1}{\rho^\eta\Gamma(\eta+1)}+\sum_{i=1}^m\sigma_i\frac{\xi_i^{\rho(\eta+\gamma_i+1)}}{\rho^{\eta+\gamma_i}\Gamma(\eta+\gamma_i+1)}\Big]\Big\}\to 0\ as\ t_1\to t_2,
\end{aligned}
$$

independently of $y\in\Omega$, which implies that $\dfrac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}$ is equicontinuous on $[0,1]$.

In a similar argument, we can show that $({}^\rho D_{0^+}^\zeta\mathcal{P}\Omega)(t)$ is equicontinuous on $[0,1]$.

Step 3: *The operator $\mathcal{P}$ is a strict set contraction operator.*

At first, we need to show that $\mathcal{P}D_\lambda\subset D_\lambda$, where D_λ is defined by (17) with $\lambda\geq\max\{1,\omega^*[\Delta^{1/(1-q)}-H^*]^{-1}\}$. So, for any $y\in D_\lambda$ and $t\in[0,1]$, we have

$$
\begin{aligned}
&\Big\|\frac{(\mathcal{P}y)(t)}{1+t^{\eta-1}}\Big\|\leq\Delta_1|H^*\|y\|_{\eta'}+\omega*|^{q-1}<\lambda,\ \text{and}\\
&\|{}^\rho D_{0+}^\zeta(\mathcal{P}y)(t)\|\leq\Delta_2|H^*\|y\|_{\eta'}+\omega*|^{q-1}<\lambda,
\end{aligned}
$$

which implies $\|\mathcal{P}y\|_{\eta'}\leq\Delta|H^*\|y\|_{\eta'}+\omega*|^{q-1}\leq\lambda$, and consequently $\mathcal{P}D_\lambda\subset D_\lambda$.

Next, let $\Phi=\overline{co}_{\mathcal{BC}^1}(\mathcal{P}D_\lambda)$, i.e., Φ is the convex closure of $\mathcal{P}D_\lambda$ in $\mathcal{BC}^1([0,1])$. Clearly, Φ is a closed, convex, and nonempty bounded subset of D_λ. $\Phi\subset D_\lambda$ and $\mathcal{P}D_\lambda\subset\Phi$, so $\mathcal{P}:\Phi\to\Phi$. By ***Step 1***, we have that $\mathcal{P}:\Phi\to\Phi$ is bounded and continuous. Moreover, it is obvious from (C_2) that $\{\mathcal{J}(t,y(t),{}^\rho D_{0^+}^\zeta y(t))),y\in\Phi\}$ is equicontinuous on $[0,1]$. Now, let Ω be an arbitrary bounded set in Φ with a partition $\Omega=\bigcup_{j=1}^n\Omega_i$. Then, for any $t\in[0,1]$,

$$
\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}=\left\{\frac{(\mathcal{P}y)(t)}{1+t^{\eta-1}}\Big|y\in\Omega,\ \text{t is fixed}\right\}\subset\Phi,
$$

and

$$
({}^\rho D_{0^+}^\zeta\mathcal{P}\Omega)(t)=\left\{({}^\rho D_{0^+}^\zeta\mathcal{P}y)(t)\big|y\in\Omega,\ \text{t is fixed}\right\}\subset\Phi.
$$

Also, for $y_1, y_2 \in \Omega_i$, we have

$$
\begin{aligned}
& \Big\| \frac{(\mathcal{P}y_1)(t)}{1+t^{\eta-1}} - \frac{(\mathcal{P}y_2)(t)}{1+t^{\eta-1}} \Big\| \\
\leq\ & \frac{\rho^{1-\eta}}{(1+t^{\eta-1})\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \times \\
& \Big\| \varphi_q\Big(\int_0^s \mathcal{J}(v, y_1(t), {}^\rho D^\zeta_{0^+} y_1(v))dv \Big) - \varphi_q\Big(\int_0^s \mathcal{J}(v, y_2(t), {}^\rho D^\zeta_{0^+} y_2(v))dv \Big) \Big\| ds \\
+\ & \frac{\rho^{1-\eta} t^{\rho(\eta-1)}}{(1+t^{\eta-1})\Lambda\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1} \\
& \| \varphi_q\Big(\int_0^s \mathcal{J}(v, y_1(t), {}^\rho D^\zeta_{0^+} y_1(v))dv \Big) - \varphi_q\Big(\int_0^s \mathcal{J}(v, y_2(t), {}^\rho D^\zeta_{0^+} y_2(v))dv \Big) \| ds \\
+\ & \frac{\rho^{1-(\eta+\gamma)} t^{\rho(\eta-1)}}{(1+t^{\eta-1})\Lambda\Gamma(\eta+\gamma)} \sum_{i=1}^{m-2} \sigma_i \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\eta+\gamma-1} \times \\
& \| \varphi_q\Big(\int_0^s \mathcal{J}(v, y_1(t), {}^\rho D^\zeta_{0^+} y_1(v))dv \Big) - \varphi_q\Big(\int_0^s \mathcal{J}(v, y_2(t), {}^\rho D^\zeta_{0^+} y_2(v))dv \Big) \| ds \\
\leq\ & (q-1)|H^*\lambda + \omega *|^{q-2} \Bigg\{ \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \\
& \int_0^s \| \mathcal{J}(v, y_1(t), {}^\rho D^\zeta_{0^+} y_1(v)) - \mathcal{J}(v, y_2(t), {}^\rho D^\zeta_{0^+} y_2(v)) \| dv ds \\
+\ & \frac{\rho^{1-\eta} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1} \int_0^s \| \mathcal{J}(v, y_1(t), {}^\rho D^\zeta_{0^+} y_1(v)) - \mathcal{J}(v, y_2(t), {}^\rho D^\zeta_{0^+} y_2(v)) \| dv ds \\
+\ & \frac{\rho^{1-(\eta+\gamma)} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)} \sum_{i=1}^{m} |\sigma_i| \times \\
& \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\eta+\gamma-1} \int_0^s \| \mathcal{J}(v, y_1(v), {}^\rho D^\zeta_{0^+} y_1(v)) - \mathcal{J}(v, y_2(v), {}^\rho D^\zeta_{0^+} y_2(v)) \| dv ds \Bigg\},
\end{aligned}
$$

which, with the aid of the condition (C_3) and Lemma 1, implies

$$
\begin{aligned}
& \mu_B\Big(\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}} \Big) \\
\leq\ & (q-1)|H^*\lambda + \omega *|^{q-2} \times \\
& \Bigg\{ \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \int_0^s \mu_B(\{\mathcal{J}(v, y(v), {}^\rho D^\zeta_{0^+} y(v)) : y \in \Omega\}) dv ds \\
& + \frac{\rho^{1-\eta} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1} \int_0^s \mu_B(\{\mathcal{J}(v, y(v), {}^\rho D^\zeta_{0^+} y(v)) : y \in \Omega\}) dv ds \\
& + \frac{\rho^{1-(\eta+\gamma)} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)} \sum_{i=1}^{m} \sigma_i \times \\
& \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\eta+\gamma-1} \int_0^s \mu_B(\{\mathcal{J}(v, y(v), {}^\rho D^\zeta_{0^+} y(v)) : y \in \Omega\}) dv ds \\
\leq\ & (q-1)|H^*\lambda + \omega *|^{q-2} \times \\
& \Bigg\{ \frac{\rho^{1-\eta}}{\Gamma(\eta)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \int_0^s \Big[(1+v^{\eta-1})\alpha_1 \mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}} \Big) + \alpha_2 \mu_B\Big({}^\rho D^\zeta_{0^+} \Omega(v) \Big) \Big] dv ds \\
& + \frac{\rho^{1-\eta} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)} \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1} \int_0^s \Big[(1+v^{\eta-1})\alpha_1 \mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}} \Big) + \alpha_2 \mu_B\Big({}^\rho D^\zeta_{0^+} \Omega(v) \Big) \Big] dv ds \\
& + \frac{\rho^{1-(\eta+\gamma)} t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)} \times \\
& \sum_{i=1}^{m} \sigma_i \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\eta+\gamma-1} \int_0^s \Big[(1+v^{\eta-1})\alpha_1 \mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}} \Big) + \alpha_2 \mu_B\Big({}^\rho D^\zeta_{0^+} \Omega(v) \Big) \Big] dv ds \\
\leq\ & \Delta_1 (q-1)|H^*\lambda + \omega *|^{q-2} \Big[\max_{t\in[0,1]} (1+t^{\eta-1})\alpha_1 \sup_{t\in[0,1]} \mu_B\Big(\frac{\Omega(t)}{1+t^{\eta-1}} \Big) + \alpha_2 \sup_{t\in[0,1]} \mu_B\Big({}^\rho D^\zeta_{0^+} \Omega(t) \Big) \Big] \\
\leq\ & \Delta (q-1)|H^*\lambda + \omega *|^{q-2} \alpha\ \mu_{\eta'}(\Omega).
\end{aligned}
$$

Since $t \in [0,1]$ is arbitrary, we have

$$\sup_{t\in[0,1]} \mu_B\Big(\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}\Big) \leq \Delta(q-1)|H^*\lambda+\omega*|^{q-2}\alpha\ \mu_{\eta'}(\Omega).$$

Also,

$$\sup_{t\in[0,1]} \mu_B\Big({}^{\rho}D^{\zeta}_{0+}(P\Omega)(t)\Big) \leq \Delta(q-1)|H^*\lambda+\omega*|^{q-2}\alpha\ \mu_{\eta'}(\Omega).$$

Thus, by using Lemma 6, we find

$$\mu_{\eta'}(\mathcal{P}\Omega) \leq \Delta(q-1)|H^*\lambda+\omega*|^{q-2}\alpha\ \mu_{\eta'}(\Omega)$$

which implies, in view of (18), that $\mathcal{P}$ is a strict set contraction operator from Φ to Φ. Consequently, $\mathcal{P}$ is a condensing operator. Thus, by Lemma 2, we conclude that $\mathcal{P}$ has at least one fixed point in Φ, which is indeed a solution to problem (2) in $D_\lambda \subset \mathcal{BC}^1([0,1])$. □

The following existence result is related to the case when $p > 2\,(1 < q \leq 2)$, for $D_\lambda \subset \mathcal{BC}^1([0,1])$ defined by (17).

Theorem 2. *Assume that $p > 2$, and let the conditions (C_1)–(C_4) hold true. Then, problem (2) has at least one solution in $D_\lambda \subset \mathcal{BC}^1([0,1])$, provided*

$$\Delta(q-1)\mathcal{A}^{q-2}\alpha < 1, \text{ and } \lambda \geq \max\{1, \omega^*[\Delta^{1/(1-q)} - H^*]^{-1}\}, \tag{19}$$

where D_λ, Δ are given by (17), (16), respectively.

Proof. In a similar argument to the last proof, we can show that the operator $\mathcal{P}$ defined by (13) is bounded, and $\dfrac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}$ and $({}^{\rho}D^{\zeta}_{0+}\mathcal{P}\Omega)(t)$ are equicontinuous on $[0,1]$ for any bounded set $\Omega \subset \mathcal{BC}^1([0,1])$.

Now, for the continuity of $\mathcal{P}$, since $p > 2\,(1 < q \leq 2)$, we have to apply property (i) of Lemma 5, where condition (C_4) implies

$$\int_0^t \big\|\mathcal{J}(v, y(v), {}^{\rho}D^{\zeta}_{0+}y(v))\big\|dv \geq \mathcal{A}t^{\sigma}.$$

Thus, as we have done to prove the continuity in the last proof, we can take a bounded sequence $\{y_n\}$ in $\mathcal{BC}^1([0,1])$, with $\|y_n - y\|_{\eta'} \to 0$ as $n \to \infty$ in $\mathcal{BC}^1([0,1])$. Then, from (C_2), there is, for any $\varepsilon > 0$, a constant $N > 0$ such that

$$\big\|\mathcal{J}(t, y_n(t), {}^{\rho}D^{\zeta}_{0+}y_n(t)) - \mathcal{J}(t, y(t), {}^{\rho}D^{\zeta}_{0+}y(t))\big\| \leq \big[\Delta(q-1)\mathcal{A}^{q-2}\big]^{-1}\varepsilon,$$

for all $n \geq N$, $t \in [0,1]$. Consequently, we have

$$\begin{aligned}
&\Big\|\frac{(\mathcal{P}y_n)(t)}{1+t^{\eta-1}} - \frac{(\mathcal{P}y)(t)}{1+t^{\eta-1}}\Big\| \\
\leq\ & \frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1} \times \\
&\Big\|\varphi_q\Big(\int_0^s \mathcal{J}(v, y_n(v), {}^{\rho}D^{\zeta}_{0+}y_n(v))dv\Big) - \varphi_q\Big(\int_0^s \mathcal{J}(v, y(v), {}^{\rho}D^{\zeta}_{0+}y(v))dv\Big)\Big\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\\
&\Big\|\varphi_q\Big(\int_0^s \mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))dv\Big)-\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big)\Big\|ds\\
&+\frac{\rho^{1-(\eta+\gamma_i)}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m\sigma_i\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma_i-1}\times\\
&\Big\|\varphi_q\Big(\int_0^s \mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))dv\Big)-\varphi_q\Big(\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big)\Big\|ds\\
\leq\ &\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-1}\times\\
&(q-1)(\mathcal{A}s^\sigma)^{q-2}\Big\|\int_0^s \mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))dv-\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big\|ds\\
&+\frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\\
&(q-1)(\mathcal{A}s^\sigma)^{q-2}\Big\|\int_0^s \mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))dv-\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big\|ds\\
&+\frac{\rho^{1-(\eta+\gamma_i)}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m\sigma_i\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma_i-1}\times\\
&(q-1)(\mathcal{A}s^\sigma)^{q-2}\Big\|\int_0^s \mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))dv-\int_0^s \mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))dv\Big\|ds\\
\leq\ &\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-1}\times\\
&(q-1)\mathcal{A}^{q-2}s^{\sigma(q-2)+1}\Big\|\mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))-\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\Big\|ds\\
&+\frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\\
&(q-1)\mathcal{A}^{q-2}s^{\sigma(q-2)+1}\Big\|\mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))-\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\Big\|ds\\
&+\frac{\rho^{1-(\eta+\gamma_i)}t^{\rho(\eta-1)}}{|\Lambda|\Gamma(\eta+\gamma_i)}\sum_{i=1}^m\sigma_i\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma_i-1}\times\\
&(q-1)\mathcal{A}^{q-2}s^{\sigma(q-2)+1}\Big\|\mathcal{J}(v,y_n(v),{}^\rho D^\zeta_{0^+}y_n(v))-\mathcal{J}(v,y(v),{}^\rho D^\zeta_{0^+}y(v))\Big\|ds\\
\leq\ &(q-1)\mathcal{A}^{q-2}\Delta_1\big\|\mathcal{J}(t,y_n(t),{}^\rho D^\zeta_{0^+}y_n(t))-\mathcal{J}(t,y(t),{}^\rho D^\zeta_{0^+}y(t))\big\|\\
\leq\ &(q-1)\mathcal{A}^{q-2}\Delta_1\big[\Delta(q-1)\mathcal{A}^{q-2}\big]^{-1}\varepsilon\leq\varepsilon,
\end{aligned}
$$

Similarly, we find

$$\Big\|{}^\rho D^\zeta_{0^+}(\mathcal{P}y_n)(t)-{}^\rho D^\zeta_{0^+}(\mathcal{P}y)(t)\Big\|\leq(q-1)\mathcal{A}^{q-2}\Delta_2\big[\Delta(q-1)\mathcal{A}^{q-2}\big]^{-1}\varepsilon\leq\varepsilon$$

Thus, $\|(\mathcal{P}y_n)-(\mathcal{P}y)\|_{\eta'}\leq\varepsilon$, which means $\mathcal{P}:\mathcal{BC}^1([0,1])\to\mathcal{BC}^1([0,1])$ is continuous.

Finally, to apply the conclusion of Lemma 2, it remains to show that the operator $\mathcal{P}$ is a strict set contraction operator. First, let us take the bounded set D_λ defined by (17); then, similar to the last proof, we can show that $\mathcal{P}D_\lambda\subset D_\lambda$.

Next, assume that the set Φ, which is defined as in ***Step 3*** in the last proof, is a convex closure of $\mathcal{P}D_\lambda$ in $\mathcal{BC}^1([0,1])$, and hence the operator $\mathcal{P}:\Phi\to\Phi$ is bounded and continuous. Also, from (C_2), $\{\mathcal{J}(t,y(t),{}^\rho D^\zeta_{0^+}y(t))),y\in\Phi\}$ is equicontinuous on $[0,1]$. Now, using (C_4), property (i) of Lemma 5, and Lemma 1, we have

$$
\begin{aligned}
& \mu_B\Big(\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}\Big) \\
\leq\ & \frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-1}(q-1)(\mathcal{A}s^\sigma)^{q-2}\int_0^s \mu_B(\{\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v)):y\in\Omega)\})dvds \\
+\ & \frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}(q-1)(\mathcal{A}s^\sigma)^{q-2}\int_0^s \mu_B(\{\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v)):y\in\Omega)\})dvds \\
+\ & \frac{\rho^{1-(\eta+\gamma)}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)}\sum_{i=1}^m \sigma_i\times \\
& \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma-1}(q-1)(\mathcal{A}s^\sigma)^{q-2}\int_0^s \mu_B(\{\mathcal{J}(v,y(v),{}^\rho D_{0^+}^\zeta y(v)):y\in\Omega)\})dvds \\
\leq\ & \frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-1}(\mathcal{A}s^\sigma)^{q-2}\int_0^s \Big[(1+v^{\eta-1})\alpha_1\mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}}\Big)+\alpha_2\mu_B\Big({}^\rho D_{0^+}^\zeta\Omega(v)\Big)\Big]dvds \\
+\ & \frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)}\times \\
& \int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}(\mathcal{A}s^\sigma)^{q-2}\int_0^s \Big[(1+v^{\eta-1})\alpha_1\mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}}\Big)+\alpha_2\mu_B\Big({}^\rho D_{0^+}^\zeta\Omega(v)\Big)\Big]dvds \\
+\ & \frac{\rho^{1-(\eta+\gamma)}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)}\sum_{i=1}^m \sigma_i\times \\
& \int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma-1}(\mathcal{A}s^\sigma)^{q-2}\int_0^s \Big[(1+v^{\eta-1})\alpha_1\mu_B\Big(\frac{\Omega(v)}{1+v^{\eta-1}}\Big)+\alpha_2\mu_B\Big({}^\rho D_{0^+}^\zeta\Omega(v)\Big)\Big]dvds \\
\leq\ & \Bigg\{\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\eta-1}\mathcal{A}^{q-2}s^{\sigma(q-2)+1}ds+\frac{\rho^{1-\eta}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta)}\int_0^1 s^{\rho-1}(1-s^\rho)^{\eta-1}\mathcal{A}^{q-2}s^{\sigma(q-2)+1}ds \\
+\ & \frac{\rho^{1-(\eta+\gamma)}t^{\rho(\eta-1)}}{\Lambda\Gamma(\eta+\gamma)}\sum_{i=1}^{m-2}\sigma_i\int_0^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\eta+\gamma-1}\mathcal{A}^{q-2}s^{\sigma(q-2)+1}ds\Bigg\}\times \\
& \Big[\max_{t\in[0,1]}(1+t^{\eta-1})\alpha_1\sup_{t\in[0,1]}\mu_B\Big(\frac{\Omega(t)}{1+t^{\eta-1}}\Big)+\alpha_2\sup_{t\in[0,1]}\mu_B\Big({}^\rho D_{0^+}^\zeta\Omega(t)\Big)\Big] \\
\leq\ & \Delta_1(q-1)\mathcal{A}^{q-2}\Big[\max_{t\in[0,1]}(1+t^{\eta-1})\alpha_1\sup_{t\in[0,1]}\mu_B\Big(\frac{\Omega(t)}{1+t^{\eta-1}}\Big)+\alpha_2\sup_{t\in[0,1]}\mu_B\Big({}^\rho D_{0^+}^\zeta\Omega(t)\Big)\Big] \\
\leq\ & (q-1)\mathcal{A}^{q-2}\Delta\alpha\mu_{\eta'}(\Omega).
\end{aligned}
$$

Since $t\in[0,1]$ is arbitrary, we have

$$\sup_{t\in[0,1]}\mu_B\Big(\frac{(\mathcal{P}\Omega)(t)}{1+t^{\eta-1}}\Big)\leq(q-1)\mathcal{A}^{q-2}\Delta\alpha\mu_{\eta'}(\Omega).$$

Also,

$$\sup_{t\in[0,1]}\mu_B\Big({}^\rho D_{0^+}^\zeta(P\Omega)(t)\Big)\leq(q-1)\mathcal{A}^{q-2}\Delta\alpha\mu_{\eta'}(\Omega).$$

Thus, by using Lemma 6, we find

$$\mu_{\eta'}(\mathcal{P}\Omega)\leq(q-1)\mathcal{A}^{q-2}\Delta\alpha\mu_{\eta'}(\Omega)$$

which implies, in view of (19), that $\mathcal{P}$ is a strict set contraction operator from Φ to Φ. Thus, by Lemma 2, we conclude that $\mathcal{P}$ has at least one fixed point in Φ, which corresponds to the solution to problem (2) in $D_\lambda\subset\mathcal{BC}^1([0,1])$. □

Remark 1. *As a spacial case, let $\rho \to 1$; then, the result obtained in this paper can be specialized for the following new problem:*

$$\begin{cases} \left(\varphi_p\left(D^{\eta}_{0^+}y(t)\right)\right)' + \mathcal{J}(t, y(t), D^{\zeta}_{0^+}y(t)) = \theta, \ \ t \in I := [0,1], \\ y(0) = \theta, \quad D^{\eta}_{0^+}y(0) = \theta, \\ y(1) = \sum_{i=1}^{m} \sigma_i I^{\gamma_i}_{0^+} y(\xi_i), \ 0 < \xi_i < 1, \end{cases} \tag{20}$$

where $D^{\eta}_{0^+}$, $D^{\zeta}_{0^+}$ refer to the Riemann–Liouville fractional derivatives and $I^{\gamma_i}_{0^+}$ is Riemann–Liouville fractional integral.

Example 1. *Consider the following problem:*

$$\begin{cases} \left(\phi_p\left(\frac{1}{2}D^{\frac{3}{2}}_{0^+}y(t)\right)\right)' + \mathcal{J}(t, y, {}^{1/2}D^{3/4}_{0^+}y) = 0, \\ y_n(0) = \theta, \ {}^{\rho}D^{\eta}_{0^+}y_n(0) = \theta, \\ y_n(1) = \frac{1}{5}\frac{1}{2}I^{\frac{5}{2}}_{0^+}y_n(\frac{1}{3}) + \frac{1}{2}\frac{1}{2}I^{\frac{1}{4}}_{0^+}y_n(\frac{2}{3}), \end{cases} \tag{21}$$

on a Banach space $B = l_\infty = \{y = (y_1, y_2, \dots, y_n, \dots) : \sup_n |y| < \infty, t \in [0,1]\}$, equipped with the norm $\|y\| = \sup_n \|y_n\|$, where $y_n \in \mathbb{R}$

Comparing this example with problem (2), we have $\eta = \frac{3}{2}$, $\zeta = \frac{3}{4}$, $\rho = \frac{1}{2}$, $\sigma_1 = \frac{1}{5}$, $\sigma_2 = \frac{1}{2}$, $\gamma_1 = \frac{5}{2}$, $\gamma_2 = \frac{1}{4}$, $\xi_1 = \frac{1}{3}$, $\xi_1 = \frac{2}{3}$. $\mathcal{J}(t, y(t), {}^{\rho}D^{\zeta}_{0^+}y(t))$ and p will be fixed later.

Using the given data, we find $\Delta_1 \approx 8.177603095$, $\Delta_2 \approx 5.673730752$, which are defined by (14), (15), respectively, and hence $\Delta = \Delta_1 \approx 8.177603095$.

For illustrating Theorem 1, let us assume $p = \frac{4}{3} \Rightarrow q = 4$, and

$$\begin{aligned} &\mathcal{J}(t, y, {}^{1/2}D^{3/4}_{0^+}y) = (\mathcal{J}_1(t, y, {}^{1/2}D^{3/4}_{0^+}y), \mathcal{J}_2(t, y, {}^{1/2}D^{3/4}_{0^+}y), \dots, \mathcal{J}_n(t, y, {}^{1/2}D^{3/4}_{0^+}y), \dots), \\ &\text{with } \mathcal{J}_n(t, y, {}^{1/2}D^{3/4}_{0^+}y) = \frac{|y_n|}{2\sqrt{t^2+900}} + \frac{|\sin\left({}^{1/2}D^{3/4}_{0^+}y_{n+1}\right)|}{25\, e^t} + \frac{t}{25\, e^t}. \end{aligned} \tag{22}$$

Obviously, $\mathcal{J} : [0,1] \times B \times B \to B$ is continuous and uniformly continuous on a bounded domain. Also, since

$$\int_0^t \|\mathcal{J}(s, y, {}^{\rho}D^{\beta}_{0^+}y)\| ds = \int_0^t \sup_n |\mathcal{J}_n(s, y, {}^{\rho}D^{\zeta}_{0^+}y) ds \le \frac{\|y\|}{2\sqrt{t^2+900}} + \frac{\|{}^{\rho}D^{\zeta}_{0^+}y\|}{25e^t} + \frac{t}{25e^t},$$

(C_1) is satisfied with $\chi(t) = \frac{1}{2\sqrt{t^2+900}}, \kappa(t) = \omega(t) = \frac{1}{25\, e^t}$, such that $H^* \approx 0.07331483$. Furthermore, it can easily be verified that (C_3) is satisfied with $\alpha_1 = \frac{1}{60}$, $\alpha_2 = \frac{1}{25}$, where

$$\mu_B(f(t, \Omega_1, \Omega_2)) \le \frac{\mu_B(\Omega_1)}{60} + \frac{\mu_B(\Omega_2)}{25}$$

and $\alpha = (\max_{t\in[0,1]}(1 + t^{\eta-1})\alpha_1 + \alpha_2) = 0.0733333$.

Now, since $\omega^*[\Delta^{1/(1-q)} - H^*]^{-1} \approx 0.09455394364$, we can take $\lambda = 8 > \max\{1, \omega^*[\Delta^{1/(1-q)} - H^*]^{-1}\}$ such that

$$\Delta(q-1)[H^*\lambda + \omega^*]^{q-2}\alpha \approx 0.7061820946 < 1.$$

Thus, all the assumptions of Theorem 1 hold true, and consequently, problem (21) has at least one solution on $D_\lambda \subset BC^1([0,1])$ with $\mathcal{J}(t, y(t), {}^{\rho}D^{\zeta}_{0^+} y)$ given by (22).

Next, to demonstrate the applicability of Theorem 2, let us choose $p = 9/2 \Rightarrow q = \frac{9}{7}$, and

$$\mathcal{J}(t, y, {}^{1/2}D^{3/4}_{0^+} y) = (\mathcal{J}_1(t, y, {}^{1/2}D^{3/4}_{0^+} y), \mathcal{J}_2(t, y, {}^{1/2}D^{3/4}_{0^+} y), \ldots, \mathcal{J}_n(t, y, {}^{1/2}D^{3/4}_{0^+} y), \ldots),$$
$$\text{with } \mathcal{J}_n(t, y, {}^{1/2}D^{3/4}_{0^+} y) = \frac{e^t}{5}\Big(\frac{|y_n|}{|y_n|+1} + |\sin({}^{1/2}D^{3/4}_{0^+} y_{n+1})| + 55\Big). \tag{23}$$

Then, we can take $\sigma = 6/5 < 1/(2-q)$ and $\mathcal{A} = 9$. The condition (C_1) is satisfied with $\chi(t) = \kappa(t) = \frac{e^t}{5}$, $\omega(t) = \frac{55e^t}{5}$ such that $H^* \approx 1.630969097$. Also, condition (C_3) holds with $\alpha_1 = \alpha_2 = \frac{e}{5}$ and $\alpha = (\max_{t\in[0,1]}(1 + t^{\eta-1})\alpha_1 + \alpha_2) = 1.630969097$. In addition, condition (C_4) is satisfied where $\|\mathcal{J}_n(t, y, {}^{1/2}D^{3/4}_{0^+} y)\| = \sup_n |\mathcal{J}_n(s, y, {}^{\rho}D^{\zeta}_{0^+} y)| > \mathcal{A}\sigma t^{1/5}$. Consequently, $\Delta(q-1)\mathcal{A}^{q-2}\alpha \approx 0.7932351008 < 1$ and $\lambda = 30 > \max\{1, \omega^*[\Delta^{1/(1-q)} - H^*]^{-1}\}$, where $\omega^*[\Delta^{1/(1-q)} - H^*]^{-1} \approx 29.90110011$.

As a result of satisfying all the hypotheses of Theorem 2, we conclude that problem (21) has at least one solution on $D_\lambda \subset BC^1([0,1])$ with $\mathcal{J}(t, y(t), {}^{\rho}D^{\zeta}_{0^+} y)$ given by (23).

4. Conclusions

In this article, we have established the existence's criteria for the solutions to a class of p-Laplacian generalized fractional differential equations with nonlocal integral boundary conditions in arbitrary Banach spaces. Using the properties of the measure of noncompactness and Sadovskii's theorem, we have proved under some specific hypotheses the existence of solutions to the obtained problem. We have illustrated our results by constructing two examples in infinite dimensional Banach space. It is worthwhile to observe that the results presented in this study are a novel contribution, providing a corrected extension of the literature on fractional p-Laplacian boundary value problems in abstract spaces.

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