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Applications of the Kosambi–Cartan–Chern Theory to Hamiltonian Systems on a Cotangent Bundle: Linking Geometric Quantities to the Self-Similar Motions of Three Point Vortices

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Abstract: This study presents a differential geometric framework for Hamiltonian systems expressed in terms of first-order differential equations. For systems governed by second-order ordinary differential equations on tangent bundles, such as Euler–Lagrange systems, the stability of trajectories under perturbations is analyzed based on the eigenvalue of the deviation curvature tensor. Building upon this Jacobi stability analysis approach, four geometric quantities for Hamiltonian systems are derived considering perturbations to trajectories on a cotangent bundle. As a specific Hamiltonian system, a hydrodynamic three-point vortex system is examined, and its four geometric quantities are computed using the Hamiltonian equation. The eigenvalues of these geometric quantities are then used to classify the divergent and collapsing trajectories of point vortices. Specifically, for the divergent trajectories of vortices, the eigenvalues of the geometric quantities converge to zero over time. Conversely, for their collapsing trajectories, the eigenvalues increase with time. This result implies that at the point of vortex collapse, the system becomes geometrically unstable, with diverging trajectory perturbations.



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1. Introduction

In the domain of dynamical systems, the Euler–Lagrange and Hamilton formalisms are used for describing systems [1]. The Euler–Lagrange formalism yields equations of motion considering the difference between the kinetic energy and potential energy of a dynamical system. It is represented by second-order ordinary differential equations for position and velocity. Examples of systems described using this formalism include motion particle systems, such as spring–damper systems and oscillating pendulums. Conversely, the Hamilton formalism yields equations of motion based on the total energy (the sum of the kinetic and potential energies) of a system and is described by a set of first-order differential equations related to position and momentum. This approach is used to analyze point vortex systems that model the interaction of vortices in fluid dynamics [2] and rotational motion of rigid bodies [3].

The relationship between such dynamical systems and geometry has been extensively studied. From a geometric perspective, the Euler–Lagrange formalism can be viewed as a

second-order ordinary differential equation system defined on tangent bundles [4]. In the context of the Kosambi–Cartan–Chern (KCC) theory, proposed by Kosambi [5], Cartan [6], and Chern [7], five geometric invariants can be obtained for the coordinate transformations of second-order ordinary differential equations on tangent bundles [8]. Among these, the eigenvalues of the second invariant, known as the deviation curvature tensor, can be used to estimate system stability under trajectory perturbations [9,10]. This type of stability, referred to as Jacobi stability, can enable the analysis of solution stability in the linear and nonlinear regimes.

The applications of the KCC theory span various domains, including population dynamics in mathematical biology [11–15], variations in Earth’s magnetic field [16,17], heat convection through the Earth’s atmosphere [18,19], oscillatory phenomena in mechanics [20], astrophysical systems [21–24], and chaos models [25,26].

Meanwhile, the Hamilton formalism is regarded as a system of first-order ordinary differential equations on cotangent bundles [27,28]. Applying the KCC theory, originally defined on tangent bundles, to a system of first-order ordinary differential equations, such as those representing the Hamilton formalism, requires transforming these equations into second-order ordinary differential equations. For instance, when applying the KCC theory to a point vortex system represented using the Hamilton formalism, the equation of motion for the point vortex is reformulated as a second-order ordinary differential equation on the tangent bundle by differentiating both sides of the original equation [29]. Based on this approach, a previous study applied the KCC theory to a three-point vortex system represented by the Hamilton formalism. Using this method, the collapse and divergent trajectories of point vortices were classified in terms of the eigenvalues of deviation curvature. However, in this prior study, the Hamilton equations were treated as first-order ordinary differential equations that describe only the position of the point vortex, rather than considering them as equations of motion for position and momentum as defined in the standard Hamiltonian system. Consequently, applying the KCC theory to a dynamical system described using the Hamilton formalism by differentiating the underlying equations helps discuss the Hamiltonian system on the tangent bundle. This approach, however, fails to account for the geometric structure intrinsic to the Hamiltonian system, which is fundamentally defined on a cotangent bundle. To address this limitation, the current study proposes a method to investigate the geometric structure of Hamiltonian systems without transforming the underlying first-order ordinary differential equations into second-order differential equations. Specifically, we focus on a geometric quantity on the cotangent bundle corresponding to the deviation curvature in the context of the KCC theory. Furthermore, using this quantity, we demonstrate the relationship between the geometric quantity and deviation curvature of the three-point vortex system established in a previous study [29].

The remainder of this paper is structured as follows. In Section 2, we review the fundamental concepts of the KCC theory. Section 3 discusses geometric quantities associated with perturbations in the solution trajectories of Hamiltonian systems. In Section 4, we examine the three-point vortex system. In Section 5, we explore the relationship between geometric quantities and the motion of the system represented using the Hamilton formalism. Finally, Section 6 concludes this study.

Notably, the Einstein convention is used throughout this paper.

2. Review of the KCC Theory and Jacobi Stability

In this section, the KCC theory and Jacobi stability are reviewed based on the studies [4,8–11]. Let M denote a smooth n -dimensional manifold, and let (TM, π, M) represent its tangent bundle. The projection $\pi : TM \rightarrow M$ represents a mapping from TM

to M . The local coordinates on M are denoted by x^i (where $i = 1, \dots, n$), while those on TM are represented as $(x^i, \dot{x}^i)$, where t denotes time, and $\dot{x}^i = dx^i/dt$.

We consider the following second-order differential equation system:

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \quad (1)$$

where $G^i(x, \dot{x})$ represents a smooth function. Under the following coordinate transformation

$$\tilde{x}^i = f^i(x^j), \quad \tilde{t} = t, \quad (2)$$

the five invariants associated with the second-order system can be derived. Among these, the first invariant, E^i , is expressed as

$$E^i = 2G^i - N_j^i \dot{x}^j, \quad (3)$$

where N_j^i , denoting the coefficient of the nonlinear connection, is defined as follows:

$$N_j^i = \frac{\partial G^i}{\partial \dot{x}^j}. \quad (4)$$

Meanwhile, the second invariant is related to the stability of solution trajectories for second-order ordinary differential equations. Let us consider that the solution trajectory $x^i(t)$ of Equation (1) is perturbed to $\bar{x}^i = x^i + u^i \delta\tau$, where u^i denotes a vector field and $\delta\tau$ represents an infinitesimal parameter. As $\delta\tau \rightarrow 0$, Equation (1) yields the following perturbation equation:

$$\ddot{u}^i + 2\frac{\partial G^i}{\partial \dot{x}^j} \dot{u}^j + 2\frac{\partial G^i}{\partial x^j} u^j = 0. \quad (5)$$

The KCC covariant derivative is defined as

$$\frac{Du^i}{dt} = \dot{u}^i + N_j^i u^j. \quad (6)$$

Accordingly, using (6), Equation (5) can be reformulated as

$$\frac{D^2 u^i}{dt^2} = B_j^i u^j, \quad (7)$$

where B_j^i , the second invariant, denotes the deviation curvature tensor defined by

$$B_j^i = -2\frac{\partial G^i}{\partial x^j} + \frac{\partial N_j^i}{\partial x^k} \dot{x}^k - 2G_{jk}^i G^k + N_k^i N_j^k. \quad (8)$$

Here, G_{jk}^i , a coefficient of the Berwald connection, is defined as

$$G_{jk}^i = \frac{\partial N_j^i}{\partial \dot{x}^k}. \quad (9)$$

The third invariant, B_{jk}^i , fourth invariant, B_{ljk}^i , and fifth invariant, D_{jkl}^i , are defined as follows:

$$B_{jk}^i = \frac{1}{3} \left(\frac{\partial P_j^i}{\partial \dot{x}^k} - \frac{\partial P_k^i}{\partial \dot{x}^j} \right), \quad B_{ljk}^i = \frac{\partial B_{jk}^i}{\partial \dot{x}^l}, \quad D_{jkl}^i = \frac{\partial^3 G^i}{\partial \dot{x}^j \partial \dot{x}^k \partial \dot{x}^l}. \quad (10)$$

The deviation curvature tensor, B_j^i , gives the stability of whole trajectories via the following definition [9,10]: *The trajectories of the system (1) are Jacobi-stable if and only if the real parts of the eigenvalues of B_j^i are strictly negative everywhere, and Jacobi-unstable otherwise.*

To better understand this stability, we examine the case where $N_j^i = 0$, implying that the covariant derivative (6) reduces to the ordinary time derivative. Furthermore, by assuming a one-dimensional system, B_j^i is treated as a scalar. Under these conditions, (7) can be re-expressed as follows:

$$\frac{d^2u}{dt^2} = Bu, \quad (11)$$

where B represents the scalar form of B_j^i . Given that this is a linear second-order ordinary differential equation, its solution is

$$u(t) = \bar{C}e^{\sqrt{B}t} + \tilde{C}e^{-\sqrt{B}t}, \quad (12)$$

where $\bar{C}$ and $\tilde{C}$ represent constants. Equation (12) indicates that the sign of B determines whether the perturbation u diverges. Thus, the sign of the deviation curvature B serves as a reliable geometric indicator of the stability of the system's solution against perturbation.

3. Application of the KCC Theory to Hamiltonian Systems and Geometric Quantities

In this section, based on previous studies [9,10], we examine the perturbation of solution trajectories in Hamiltonian systems expressed using first-order ordinary differential equations. Geometric quantities corresponding to the deviation curvature tensor in the KCC theory are derived. These geometric quantities are then applied to a one-dimensional free-oscillating system, and the relationship between these geometric quantities and the stability of trajectories under perturbations is derived.

3.1. Geometric Quantities for Perturbations in the Solution Trajectories of Hamiltonian Systems

Let M denote an n -dimensional smooth manifold, and let TM and T^*M represent its tangent and cotangent bundles, respectively. Local coordinates on M are defined by (x^i) ($i = 1, 2, \dots, n$), while those on T^*M are denoted by (x^i, p_i) (p_i represents the generalized momentum). In Hamilton spaces [27,28], a regular Hamiltonian on M is defined as a real function $H : T^*M \rightarrow \mathbf{R}$ that is smooth on $\widehat{T^*M} = T^*M / \{0\}$ and continuous on the null section of the cotangent bundle. Hamilton's equations of motion are then expressed as

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \quad (13)$$

Based on the ordinary KCC theory [9,10], let us consider perturbation to a solution trajectory of (13). For the two variables involved in the Hamilton formalism, x^i and p_i , we introduce the following perturbations:

$$\bar{x}^i = x^i + u^i \delta\tau, \quad \bar{p}_i = p_i + v_i \delta\tau, \quad (14)$$

where u^i and v_i denote vector fields, and $\delta\tau$ represents an infinitesimal parameter. For $\delta\tau \rightarrow 0$, the time evolution equations for the perturbations can be derived using Equation (13):

$$\dot{u}^i = H_{p_i x^j} u^j + H_{p_i p_j} v_j, \quad (15)$$

$$\dot{v}_i = H_{x^i x^j} u^j + H_{x^i p_j} v_j, \quad (16)$$

where the partial derivatives of the Hamiltonian H are defined as

$$H_{p_i x^j} = \frac{\partial^2 H}{\partial x^j \partial p_i}, \quad H_{p_i p_j} = \frac{\partial^2 H}{\partial p_j \partial p_i}, \quad (17)$$

$$H_{x^i x^j} = \frac{\partial^2 H}{\partial x^j \partial x^i}, \quad H_{x^i p_j} = \frac{\partial^2 H}{\partial p_j \partial x^i}. \quad (18)$$

Here, the subscripts x^i and p_i denote partial derivatives with respect to these variables, and this convention will be used consistently hereafter.

The covariant derivatives of the perturbation vectors are defined as

$$\frac{\tilde{D}u^i}{dt} = \dot{u}^i - H_{p_i x^j} u^j, \quad (19)$$

$$\frac{\tilde{D}v_i}{dt} = \dot{v}_i + H_{x^i p_j} v_j. \quad (20)$$

Using (19) and (20), Equations (15) and (16) can be reformulated as

$$\frac{\tilde{D}^2 u^i}{dt^2} = P_j^i u^j + Q^{ij} v_j, \quad (21)$$

$$\frac{\tilde{D}^2 v_i}{dt^2} = R_{ij} u^j + P_i^j v_j, \quad (22)$$

where the coefficients are defined as

$$P_j^i = -H_{p_i p_k} H_{x^k x^j}, \quad (23)$$

$$Q^{ij} = H_{p_i p_j x^k} H_{p_k} - H_{p_i p_j p_k} H_{x^k} - H_{p_i x^k} H_{p_k p_j} - H_{p_i p_k} H_{p_j x^k}, \quad (24)$$

$$R_{ij} = H_{x^i x^j p_k} H_{x^k} - H_{x^i x^j x^k} H_{p_k} - H_{p_k x^i} H_{x^k x^j} - H_{x^i x^k} H_{p_k x^j}. \quad (25)$$

Further, additional partial derivatives of the Hamiltonian H are defined as

$$H_{x^k} = \frac{\partial H}{\partial x^k}, \quad H_{p_k} = \frac{\partial H}{\partial p_k}, \quad (26)$$

$$H_{p_i p_j x^k} = \frac{\partial^3 H}{\partial x^k \partial p_j \partial p_i}, \quad H_{p_i p_j p_k} = \frac{\partial^3 H}{\partial p_k \partial p_j \partial p_i}, \quad (27)$$

$$H_{x^i x^j p_k} = \frac{\partial^3 H}{\partial p_k \partial x^j \partial x^i}, \quad H_{x^i x^j x^k} = \frac{\partial^3 H}{\partial x^k \partial x^j \partial x^i}. \quad (28)$$

Finally, we analyze the geometric structure of Hamiltonian systems by studying the real parts of the eigenvalues of the geometric objects P_j^i , Q^{ij} , and R_{ij} . This approach parallels the analysis of the eigenvalues of the deviation curvature tensor in the KCC theory.

3.2. Relationship Between Geometric Quantities of the Hamiltonian System and Jacobi Stability in the KCC Theory

To establish the correspondence between Jacobi stability in the KCC theory and the geometric quantities derived in the previous subsection, we consider Equations (19) and (20) under the condition $H_{p_i x^j} = 0$. This assumption simplifies the covariant derivative to the ordinary time derivative. Furthermore, for simplicity, we assume a one-dimensional Hamiltonian system and express the geometric quantities P_j^i , Q^{ij} , and R_{ij} as scalars. Under these conditions, Equations (21) and (22) are reformulated as follows:

$$\frac{d^2}{dt^2} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P & Q \\ R & P \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \quad (29)$$

Here, the geometric quantities are defined as

$$P = -H_{pp}H_{xx}, \quad Q = -H_{ppp}H_x, \quad R = -H_{xxx}H_p. \quad (30)$$

Equation (29) can be reformulated as a first-order differential equation:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \\ \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ P & Q & 0 & 0 \\ R & P & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \dot{u} \\ \dot{v} \end{bmatrix}. \quad (31)$$

We define matrix A as

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ P & Q & 0 & 0 \\ R & P & 0 & 0 \end{bmatrix}. \quad (32)$$

The eigenvalues of matrix A are then given as

$$\lambda^2 = P \pm \sqrt{QR}. \quad (33)$$

When the right-hand side of (33) is negative, the solution to the simplified perturbation Equation (31) oscillates, resulting in a bounded perturbation. Conversely, when the right-hand side of (33) is positive, perturbation diverges.

To illustrate this, let us consider the eigenvalues (33) of a simple harmonic system described by a one-dimensional Hamiltonian system. The Hamiltonian for this system is defined as

$$H_s = \frac{p^2}{2m} + \frac{Kx^2}{2}, \quad (34)$$

where K denotes the spring constant and m represents the point mass. Then, from (13), the equation of motion is expressed by $\dot{x} = p/m$ and $\dot{p} = -Kx$. For this system, the eigenvalues in (33) can be simplified:

$$\lambda^2 = -\frac{K}{m}. \quad (35)$$

From Equation (35), we can confirm that the perturbations are oscillatory, bounded perturbations. This result aligns with the notion in the KCC theory that the sign of the deviation curvature B relates to trajectory stability. Thus, the geometric quantities P_j^i , Q^{ij} , and R_{ij} are related to the stability of solution trajectories in Hamiltonian systems.

4. Review of the Two-Dimensional Three-Point Vortex System and Its Self-Similar Motions

In the following, we review the two-dimensional three-point vortex system and its self-similar motions in order to apply the geometric quantities obtained in the previous section to a certain system of first-order ordinary differential equations and investigate the relationship with their solutions. Point vortex systems in fluid mechanics are known as one of the dynamical systems represented as Hamiltonian systems. In particular, it has been shown that three vortices on an infinite plane collapse to a single point in finite time under certain conditions. In this case, when the three point vortices collapse to a single point, the triangle formed by the point vortices maintains its shape. In other words, the three point vortices collapse in a self-similar manner. The self-similar collapse of three

point vortices was first studied by Gröbli [30]. Then, the properties of collapse have been discussed by several researchers [31–35]. Furthermore, conditions have been investigated for the self-similar collapse of a three-point vortex system to occur on a three-dimensional sphere [36,37]. Unlike the collapse on a plane, two types of trajectories toward collapse can be obtained on a sphere by reversing the sign of the circulation. One of them corresponds to self-similar expansion on the plane. Also, on the plane, all three point vortices collapse while rotating at the same angular velocity, but on the sphere, each has a different angular velocity. For these self-similar motions of point vortex systems, we discuss the collapse and divergence of three-point vortex systems on a plane for comparison with a previous study [29].

Based on previous studies [31,33], we review the two-dimensional three-point vortex system and its self-similar motions. The equations of motion for the three-point vortex system are

$$\frac{dx^1}{dt} = -\frac{1}{2\pi} \left(\frac{\Gamma_2(y^1 - y^2)}{(s^3)^2} + \frac{\Gamma_3(y^1 - y^3)}{(s^2)^2} \right), \quad (36)$$

$$\frac{dy^1}{dt} = \frac{1}{2\pi} \left(\frac{\Gamma_2(x^1 - x^2)}{(s^3)^2} + \frac{\Gamma_3(x^1 - x^3)}{(s^2)^2} \right), \quad (37)$$

$$\frac{dx^2}{dt} = -\frac{1}{2\pi} \left(\frac{\Gamma_1(y^2 - y^1)}{(s^3)^2} + \frac{\Gamma_3(y^2 - y^3)}{(s^1)^2} \right), \quad (38)$$

$$\frac{dy^2}{dt} = \frac{1}{2\pi} \left(\frac{\Gamma_1(x^2 - x^1)}{(s^3)^2} + \frac{\Gamma_3(x^2 - x^3)}{(s^1)^2} \right), \quad (39)$$

$$\frac{dx^3}{dt} = -\frac{1}{2\pi} \left(\frac{\Gamma_1(y^3 - y^1)}{(s^2)^2} + \frac{\Gamma_2(y^3 - y^2)}{(s^1)^2} \right), \quad (40)$$

$$\frac{dy^3}{dt} = \frac{1}{2\pi} \left(\frac{\Gamma_1(x^3 - x^1)}{(s^2)^2} + \frac{\Gamma_2(x^3 - x^2)}{(s^1)^2} \right), \quad (41)$$

where Γ_1 , Γ_2 , and Γ_3 represent the circulations of the first, second, and third vortices, respectively. Further, s^1 , s^2 , and s^3 represent the relative distances between these vortices and are defined as

$$s^1 = \sqrt{(x^2 - x^3)^2 + (y^2 - y^3)^2}, \quad (42)$$

$$s^2 = \sqrt{(x^3 - x^1)^2 + (y^3 - y^1)^2}, \quad (43)$$

$$s^3 = \sqrt{(x^1 - x^2)^2 + (y^1 - y^2)^2}. \quad (44)$$

In addition, Equations (36)–(41) are rewritten in terms of the relative distances s^1 , s^2 , and s^3 as follows:

$$\frac{d(s^1)^2}{dt} = \frac{\Gamma_1}{2\pi} \frac{J(s^2 + s^3)(s^2 - s^3)}{(s^2 s^3)^2}, \quad (45)$$

$$\frac{d(s^2)^2}{dt} = \frac{\Gamma_2}{2\pi} \frac{J(s^3 + s^1)(s^3 - s^1)}{(s^1 s^3)^2}, \quad (46)$$

$$\frac{d(s^3)^2}{dt} = \frac{\Gamma_3}{2\pi} \frac{J(s^1 + s^2)(s^1 - s^2)}{(s^1 s^2)^2}, \quad (47)$$

where J is represented by

$$J^2 = 2(s^2 s^3)^2 + 2(s^3 s^1)^2 + 2(s^1 s^2)^2 - (s^1)^4 - (s^2)^4 - (s^3)^4. \quad (48)$$

J means an area of the triangle formed by the three point vortices. Then, according to [38], Equations (45)–(47) can be rewritten as follows:

$$\frac{ds^1}{dt} = \frac{\Gamma_1}{4\pi} \frac{J(s^2 + s^3)(s^2 - s^3)}{s^1(s^2s^3)^2}, \quad (49)$$

$$\frac{ds^2}{dt} = \frac{\Gamma_2}{4\pi} \frac{J(s^3 + s^1)(s^3 - s^1)}{s^2(s^1s^3)^2}, \quad (50)$$

$$\frac{ds^3}{dt} = \frac{\Gamma_3}{4\pi} \frac{J(s^1 + s^2)(s^1 - s^2)}{s^3(s^1s^2)^2}. \quad (51)$$

From Equations (49)–(51), it can be seen that the relative distances are constant when the initial positions of the vortices constitute an equilateral triangle. In the following, we treat the case where the relative distances change.

When the three-point vortex system exhibits self-similar motion where relative distances change, the circulations satisfy the following condition:

$$\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0. \quad (52)$$

The initial position of the third vortex across all self-similar motions satisfies

$$x^3 = \frac{\Gamma_1 x^1 + \Gamma_2 x^2}{\Gamma_1 + \Gamma_2} + \sqrt{\frac{\Gamma_1 + \Gamma_2 + \Gamma_3}{\Gamma_1 + \Gamma_2}} |x^1 - x^2| \cos \theta, \quad (53)$$

$$y^3 = \sqrt{\frac{\Gamma_1 + \Gamma_2 + \Gamma_3}{\Gamma_1 + \Gamma_2}} |x^1 - x^2| \sin \theta, \quad (54)$$

while the first and second vortices lie along the x -axis ($y^1 = 0, y^2 = 0$). Here, θ denotes the angle between the x -axis and the line segment from the center of the circle to the position of the third vortex (Figure 1). The center of this circle is defined by

$$(x_c, y_c) = \left(\frac{\Gamma_1 x^1 + \Gamma_2 x^2}{\Gamma_1 + \Gamma_2}, 0 \right). \quad (55)$$

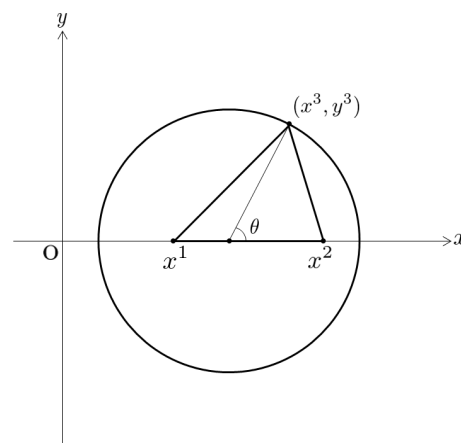


Figure 1. Circle representing the initial positions of the vortices during self-similar expansion or collapse.

Notably, in this study, the trajectories of a three-point vortex system are obtained using the symbolic manipulator “Mathematica”. The explicit Runge–Kutta method is used to obtain numerical solutions of the equations of motions for the point vortex system.

Under self-similar conditions, the trajectory of the divergent motion of the three-point vortex system is computed using the following initial conditions:

$$\theta = \frac{\pi}{3}, \quad (56)$$

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (-10, 20, 20), \quad (57)$$

$$(x^1, x^2, x^3) = (-1, 1, 3 + \sqrt{3}), \quad (58)$$

$$(y^1, y^2, y^3) = (0, 0, 3). \quad (59)$$

Figures 2 and 3 show the calculation results of the self-similar divergence on conditions (56)–(59). In Figure 2, the blue, red, and green lines represent the trajectories of the first, second, and third vortices, respectively. The dashed triangle depicts the initial configuration of the three vortices, while the solid triangle represents the configuration at any given time. In Figure 3, the black, blue, and red lines represent the temporal evolution of s^1 , s^2 , and s^3 , respectively. The self-similar collapse motion of the system is modeled under the following initial conditions:

$$\theta = -\frac{\pi}{6}, \quad (60)$$

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (-1, 2, 2), \quad (61)$$

$$(x^1, x^2, x^3) = (-1, 1, 6), \quad (62)$$

$$(y^1, y^2, y^3) = (0, 0, -\sqrt{3}). \quad (63)$$

Figures 4 and 5 show the calculation results of the self-similar collapse under the conditions represented by Equations (60)–(63). As described in Figure 4, the blue, red, and green lines represent the trajectories of the first, second, and third vortices, respectively, while the dashed triangle depicts the initial configuration of the three vortices. In Figure 5, the black, blue, and red lines represent the temporal evolution of s^1 , s^2 , and s^3 , respectively.

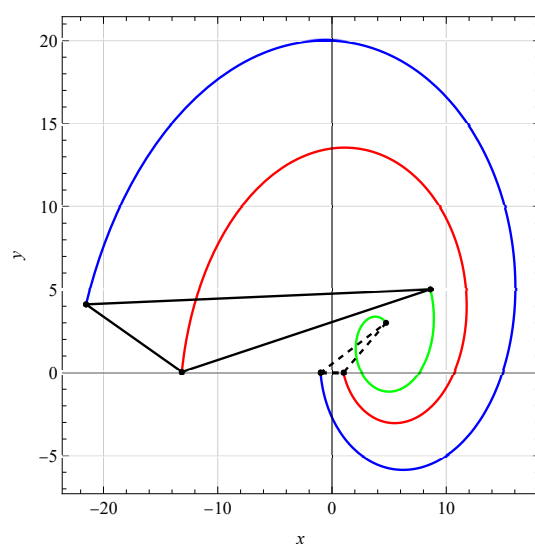


Figure 2. Trajectory of self-similar divergent motion under the conditions (56)–(59) from $t = 0$ to $t = 110$. The blue, red, and green lines represent the trajectories of the first, second, and third vortices, respectively.

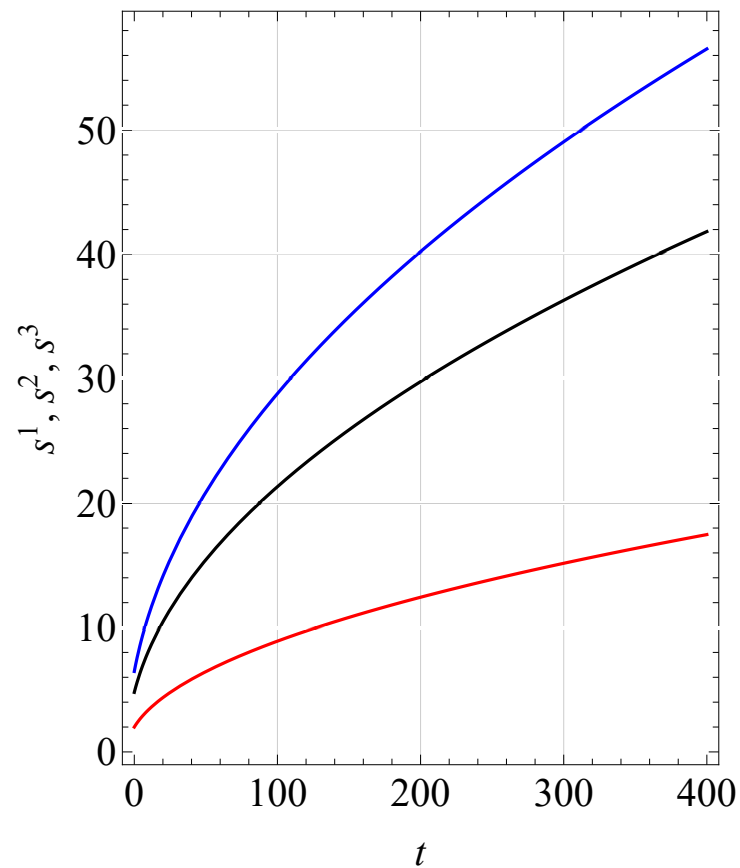


Figure 3. Temporal evolution of relative distances of self-similar divergent motion under the conditions (56)–(59). The black, blue, and red lines represent the temporal evolution of s^1 , s^2 , and s^3 , respectively.

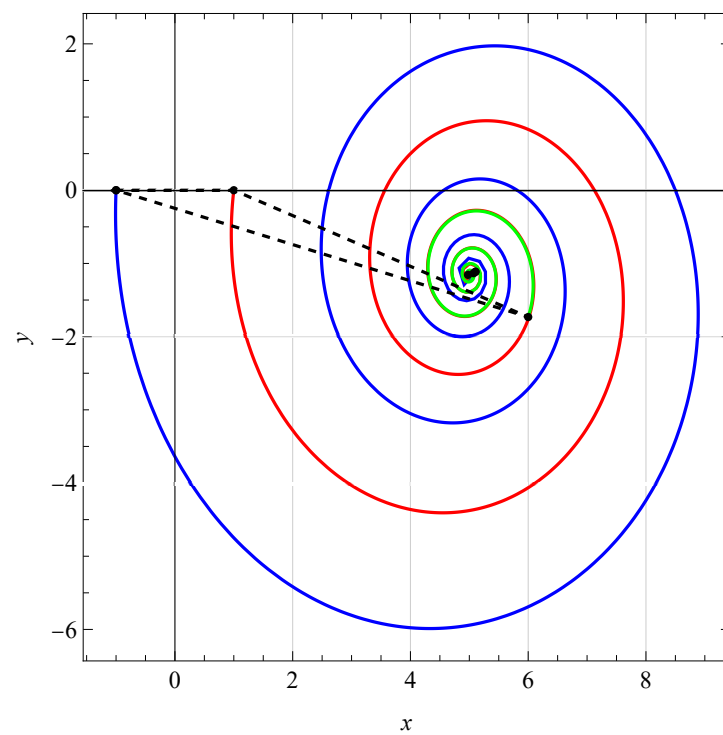


Figure 4. Trajectory of self-similar collapse motion under the conditions (60)–(63) from $t = 0$ to $t = 110$. The blue, red, and green lines represent the trajectories of the first, second, and third vortices, respectively.

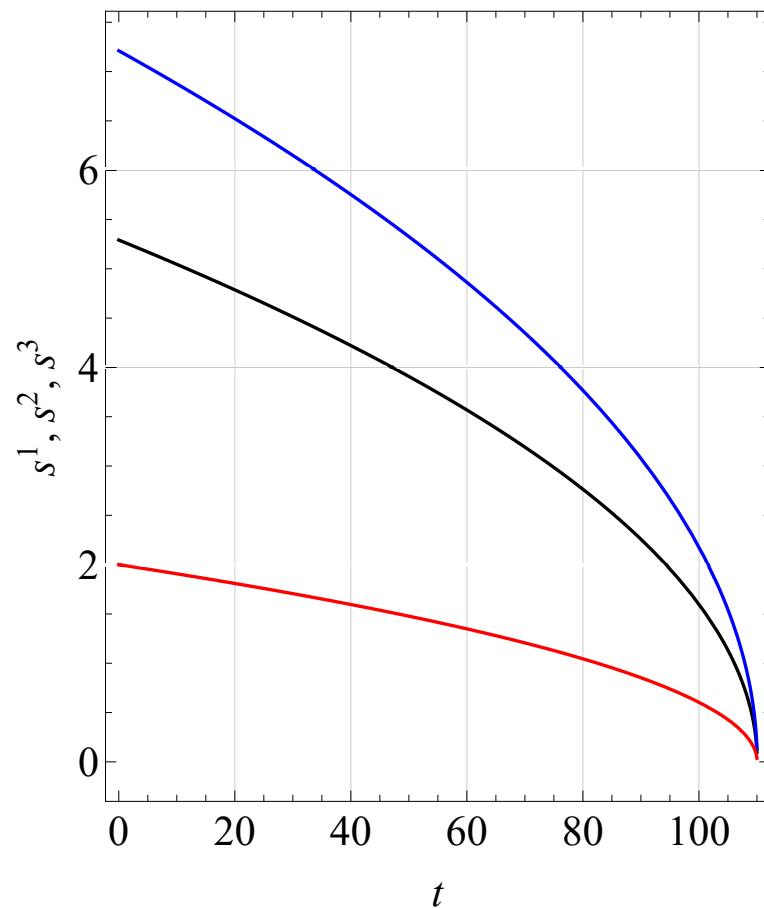


Figure 5. Temporal evolution of relative distances of self-similar collapse motion under the conditions (60)–(63). The black, blue, and red lines represent the temporal evolution of s^1 , s^2 , and s^3 , respectively.

5. Relationship Between Geometric Quantities in Hamiltonian Systems and the Divergent and Collapse Motions of Three-Point Vortex Systems

In this section, the geometric quantities of the Hamiltonian system derived earlier are applied to the equations of motion of the two-dimensional three-point vortex system. Subsequently, its self-similar divergent and collapse motions are analyzed using the eigenvalues of these geometric quantities. Additionally, we discuss these motions of three-point vortex systems on a plane for comparison with a previous study [29].

The two-dimensional three-point vortex system can be expressed using the Hamilton formalism based on generalized momenta $p_1 = \Gamma_1 y^1$, $p_2 = \Gamma_2 y^2$, and $p_3 = \Gamma_3 y^3$. Accordingly, the Hamiltonian for the three-point vortex system is given by

$$H = -\frac{1}{4\pi} \left[\Gamma_1 \Gamma_2 \log \left\{ (x^1 - x^2)^2 + \left(\frac{p_1}{\Gamma_1} - \frac{p_2}{\Gamma_2} \right)^2 \right\} + \Gamma_2 \Gamma_3 \log \left\{ (x^2 - x^3)^2 + \left(\frac{p_2}{\Gamma_2} - \frac{p_3}{\Gamma_3} \right)^2 \right\} + \Gamma_3 \Gamma_1 \log \left\{ (x^3 - x^1)^2 + \left(\frac{p_3}{\Gamma_3} - \frac{p_1}{\Gamma_1} \right)^2 \right\} \right]. \quad (64)$$

The equations of motion for the three-point vortex system in the Hamiltonian canonical form are expressed as

$$\dot{x}^1 = -\frac{1}{2\pi} \left\{ \frac{\Gamma_1 \Gamma_2^2 (\Gamma_2 p_1 - \Gamma_1 p_2)}{r_{12}} - \frac{\Gamma_3^2 \Gamma_1 (\Gamma_1 p_3 - \Gamma_3 p_1)}{r_{31}} \right\}, \quad (65)$$

$$\dot{p}_1 = \frac{1}{2\pi} \left\{ \frac{\Gamma_1^3 \Gamma_2^3 (x^1 - x^2)}{r_{12}} - \frac{\Gamma_3^3 \Gamma_1^3 (x^3 - x^1)}{r_{31}} \right\}, \quad (66)$$

$$\dot{x}^2 = -\frac{1}{2\pi} \left\{ \frac{\Gamma_2 \Gamma_3^2 (\Gamma_3 p_2 - \Gamma_2 p_3)}{r_{23}} - \frac{\Gamma_1^2 \Gamma_2 (\Gamma_2 p_1 - \Gamma_1 p_2)}{r_{12}} \right\}, \quad (67)$$

$$\dot{p}_2 = \frac{1}{2\pi} \left\{ \frac{\Gamma_2^3 \Gamma_3^3 (x^2 - x^3)}{r_{23}} - \frac{\Gamma_1^3 \Gamma_2^3 (x^1 - x^2)}{r_{12}} \right\}, \quad (68)$$

$$\dot{x}^3 = -\frac{1}{2\pi} \left\{ \frac{\Gamma_3 \Gamma_1^2 (\Gamma_1 p_3 - \Gamma_3 p_1)}{r_{31}} - \frac{\Gamma_2^2 \Gamma_3 (\Gamma_3 p_2 - \Gamma_2 p_3)}{r_{23}} \right\}, \quad (69)$$

$$\dot{p}_3 = \frac{1}{2\pi} \left\{ \frac{\Gamma_3^3 \Gamma_1^3 (x^3 - x^1)}{r_{31}} - \frac{\Gamma_2^3 \Gamma_3^3 (x^2 - x^3)}{r_{23}} \right\}, \quad (70)$$

where r_{12} , r_{23} , and r_{31} are, respectively, defined as

$$r_{12} = \Gamma_1^2 \Gamma_2^2 (x^1 - x^2)^2 + (\Gamma_2 p_1 - \Gamma_1 p_2)^2, \quad (71)$$

$$r_{23} = \Gamma_2^2 \Gamma_3^2 (x^2 - x^3)^2 + (\Gamma_3 p_2 - \Gamma_2 p_3)^2, \quad (72)$$

$$r_{31} = \Gamma_3^2 \Gamma_1^2 (x^3 - x^1)^2 + (\Gamma_1 p_3 - \Gamma_3 p_1)^2. \quad (73)$$

Using the initial conditions detailed in Equations (56)–(59), the geometric quantities P_j^i , Q^{ij} , and R_{ij} for self-similar divergence can be derived by reformulating Equation (59) as follows:

$$(p_1, p_2, p_3) = (0, 0, 60). \quad (74)$$

In this case, the trajectories of self-similar divergent motion, the temporal evolution of relative distances, and the Hamiltonian are illustrated in Figures 2, 3 and 6, respectively. The temporal evolutions of the real parts λ of the eigenvalues of P_j^i , Q^{ij} , and R_{ij} are depicted in Figures 7–9, respectively. λ_1^P (blue), λ_2^P (red), and λ_3^P (green) represent the real parts of the eigenvalues of P_j^i . λ_1^Q (blue), λ_2^Q (red), and λ_3^Q (green) represent the real parts of the eigenvalues of Q^{ij} . λ_1^R (blue), λ_2^R (red), and λ_3^R (green) represent the real parts of the eigenvalues of R_{ij} . Figures 7–9 reveal that for all three geometric quantities P_j^i , Q^{ij} , and R_{ij} , at least one eigenvalue is always zero. Furthermore, the real parts of the eigenvalues of all three quantities converge to 0 over time. This behavior is consistent with the findings of a previous study [29], which examined three-point vortex systems in the context of the KCC theory considering tangent bundles.

Similarly, based on the initial conditions outlined in Equations (60)–(63), geometric quantities for self-similar collapse motion can be obtained by formulating Equation (63) as a generalized momentum as follows:

$$(p_1, p_2, p_3) = (0, 0, -2\sqrt{3}). \quad (75)$$

In this case, the trajectories of self-similar collapse motion, the temporal evolution of relative distances, and the Hamiltonian are illustrated in Figures 4, 5 and 10, respectively. The temporal evolution of the real parts λ of the eigenvalues of P_j^i , Q^{ij} , and R_{ij} is depicted in Figures 11–16. λ_1^P (blue), λ_2^P (red), and λ_3^P (green) represent the real parts of the eigenvalues of P_j^i . λ_1^Q (blue), λ_2^Q (red), and λ_3^Q (green) represent the real parts of the eigenvalues of Q^{ij} . λ_1^R (blue), λ_2^R (red), and λ_3^R (green) represent the real parts of the eigenvalues of R_{ij} .

R_{ij} . Figures 12, 14 and 16 depict Figures 11, 13 and 15 from $t = 0$ to $t = 105$, respectively. Figures 11–16 indicate that, as in the divergent case, at least one eigenvalue for each geometric quantity remains zero. However, the real parts of the eigenvalues of the three geometric quantities diverge at the collision point, except for the one that is always zero. The changes in these eigenvalues are consistent with the patterns observed in a previous study [29].

These results demonstrate that the classification of the collapse and divergent motions of a three-point vortex system based on geometric quantities derived from the Hamiltonian system aligns with the results obtained using the deviation curvature tensor on the tangent bundle. This consistency suggests that the stability of perturbations in the solution trajectories of the Hamiltonian system can be analyzed using these geometric quantities.

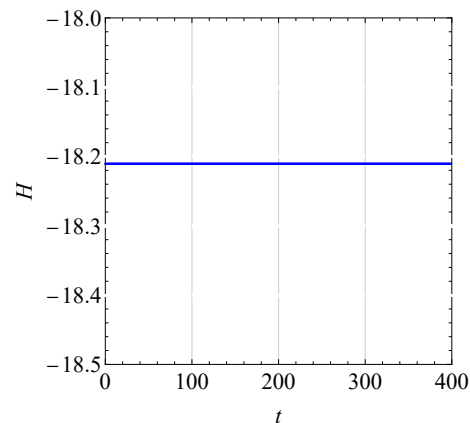


Figure 6. Hamiltonian of self-similar divergent motion under the conditions (56)–(59), and (74).

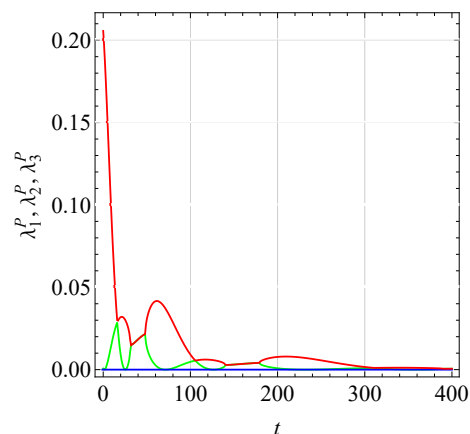


Figure 7. Temporal evolution of the eigenvalues λ_1^P (blue), λ_2^P (red), and λ_3^P (green) of P_j^i during self-similar divergent motion under the conditions (56)–(59), and (74).

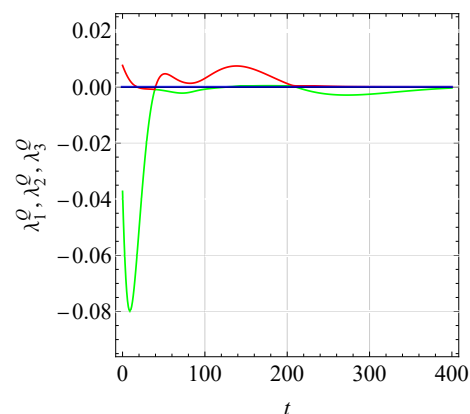


Figure 8. Temporal evolution of the eigenvalues λ_1^Q (blue), λ_2^Q (red), and λ_3^Q (green) of Q_j^i during self-similar divergent motion under the conditions (56)–(59), and (74).

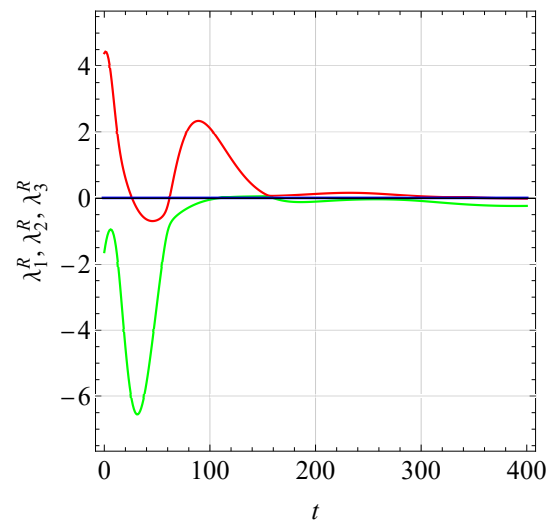


Figure 9. Temporal evolution of the eigenvalues λ_1^R (blue), λ_2^R (red), and λ_3^R (green) of R_{ij} during self-similar divergent motion under the conditions (56)–(59), and (74).

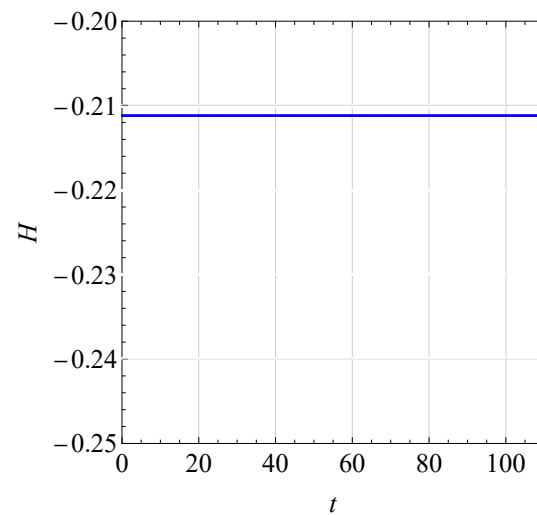


Figure 10. Hamiltonian of self-similar collapse motion under the conditions (60)–(63), and (75).

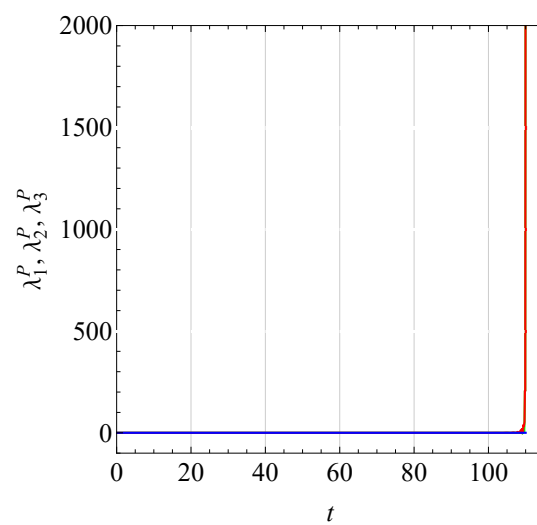


Figure 11. Temporal evolution of the eigenvalues λ_1^P (blue), λ_2^P (red), and λ_3^P (green) of P_j^i during self-similar collapse motion under the conditions (60)–(63), and (75).

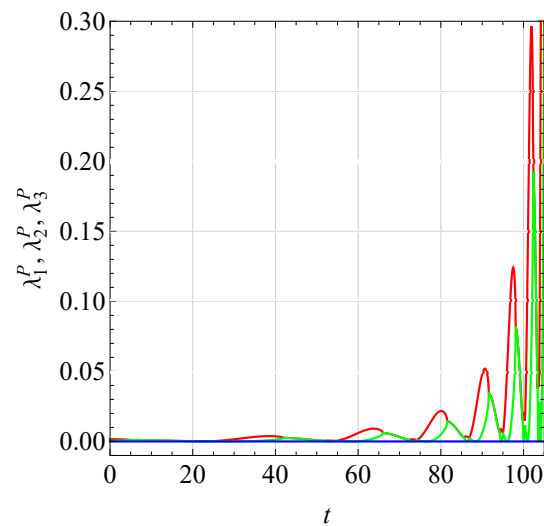


Figure 12. Temporal evolution of the eigenvalues λ_1^P (blue), λ_2^P (red), and λ_3^P (green) of P_j^i during self-similar collapse motion under the conditions (60)–(63), and (75) from $t = 0$ to $t = 105$.

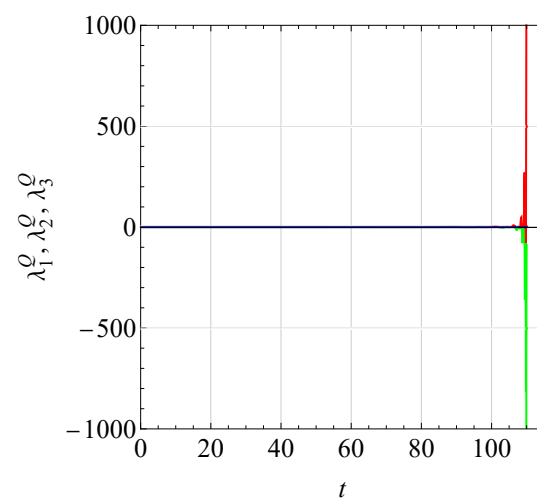


Figure 13. Temporal evolution of the eigenvalues λ_1^Q (blue), λ_2^Q (red), and λ_3^Q (green) of Q^{ij} during self-similar collapse motion under the conditions (60)–(63), and (75).

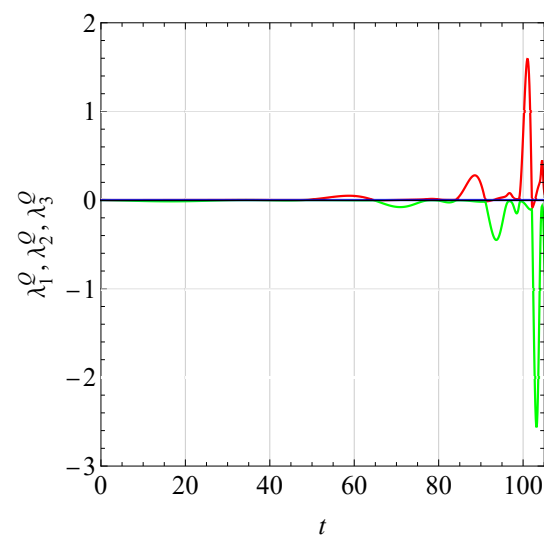


Figure 14. Temporal evolution of the eigenvalues λ_1^Q (blue), λ_2^Q (red), and λ_3^Q (green) of Q^{ij} during self-similar collapse motion under the conditions (60)–(63), and (75) from $t = 0$ to $t = 105$.

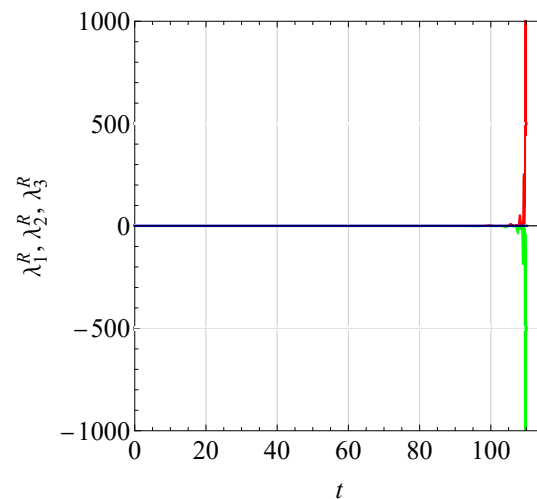


Figure 15. Temporal evolution of the eigenvalues λ_1^R (blue), λ_2^R (red), and λ_3^R (green) of R_{ij} during self-similar collapse motion under the conditions (60)–(63), and (75).

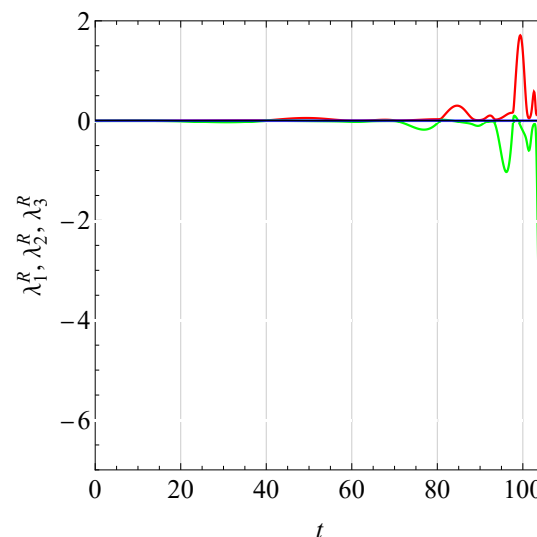


Figure 16. Temporal evolution of the eigenvalues λ_1^R (blue), λ_2^R (red), and λ_3^R (green) of R_{ij} during self-similar collapse motion under the conditions (60)–(63), and (75) from $t = 0$ to $t = 105$.

6. Conclusions

This study examined the trajectories of Hamiltonian systems defined on a cotangent bundle based on geometric quantities. Dynamical systems defined on a tangent bundle are generally represented by a system of second-order ordinary differential equations, where the deviation curvature tensor derived from these equations defines Jacobi stability under trajectory perturbations. Previous studies reported that analyzing the trajectories of Hamiltonian systems using geometric quantities required the transformation of the Hamilton equations into second-order ordinary differential equations. By contrast, this study derived geometric quantities on a cotangent bundle corresponding to the deviation curvature tensor by considering perturbations in the generalized coordinates and momenta of the Hamiltonian system. As an application to a concrete dynamical system, these geometric quantities were derived for a one-dimensional free-oscillating system, revealing that perturbations in this system do not diverge and the system trajectory remains stable. Further, this study applied these geometric quantities to the trajectories of a three-point vortex system in hydrodynamics. During the self-similar evolution of point vortices, the eigenvalues of the geometric quantities were used to classify diverging and collapsing trajectories. It can be seen that the eigenvalues of the geometric quantities converge to zero

with time for the diverging trajectories, while the eigenvalues increase with time for the collapsing trajectories. Specifically, at the point of collapse, the system exhibits a geometric singularity where perturbation diverges to infinity.

As future works, the geometric framework in this paper will be applied to a point vortex system on a sphere. The N -vortex system on sphere has $3N$ equations of motions, and has different conditions of collapse from the system on a plane. Therefore, it is necessary to confirm whether the geometric framework of this study can be applied to the case of the system on a sphere. Furthermore, the KCC theory will be applied to first-order systems that cannot be defined by energy functions. We will develop a framework that allows geometric stability to be discussed for systems of first-order differential equations that does not need to be defined in terms of energy functions without having to be rewritten in terms of second-order differential equations. This is expected to make it possible to investigate more precise geometric properties of various first-order differential equation systems, not just Hamiltonian systems.

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