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Symmetries for the Semi-Discrete Lattice Potential Korteweg–de Vries Equation

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Abstract: In this paper, we prove that the isospectral flows associated with both the x -part and the n -part of the Lax pair of the semi-discrete lattice potential Korteweg–de Vries equation are symmetries of the equation. Furthermore, we show that these two hierarchies of symmetries are equivalent. Additionally, we construct the non-isospectral flows associated with the x -part of the Lax pair, which can be interpreted as the master symmetries of the semi-discrete lattice potential Korteweg–de Vries equation.

Keywords: semi-discrete lattice potential KdV equation; symmetries; Lax pair; zero-curvature representation

MSC: 39A36

1. Introduction

In the theory of integrable systems, integrable equations usually admit infinitely many symmetries and conservation laws [1–3], which is often regarded as a fundamental integrability criterion [4–8]. These symmetries typically exhibit elegant algebraic structures. Discovering these symmetries and studying their algebraic structures is an interesting and important problem [9–15]. In the study of infinite symmetries, the recent work in [9] presents several groundbreaking results that have a profound impact, providing an important theoretical framework and direction for further research and applications.

For a Lax-integrable equation, infinitely many symmetries can be constructed using zero-curvature representations of isospectral and non-isospectral flows associated with the Lax pair [10–14,16–18], with the non-isospectral flows usually acting as master symmetries of the equation [12,19]. Recently, we constructed symmetries for the lattice potential Korteweg–de Vries (lpKdV) equation using this method [20]. Due to the symmetric property of the Lax pair of lpKdV equation, we have proven that the isospectral flows associated with both the n -part and m -part of the Lax pair are symmetries of this equation, and these two hierarchies of symmetries are equivalent modulo the lpKdV equation.

A natural problem that arises is whether similar results hold for semi-discrete integrable systems. However, the Lax pairs of these systems typically lack clear symmetric properties, so analogous results are not immediately apparent. In this paper, we use the semi-discrete lattice potential Korteweg–de Vries (sdlpKdV) equation as an example to investigate this problem.

The semi-discrete lattice potential Korteweg–de Vries (sdlpKdV) equation is given by

$$\tilde{u}_x + u_x = (\tilde{u} - u)^2 - 2p(\tilde{u} - u), \quad (1)$$



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where $u = u(x, n)$, $\tilde{u} = E_n u = u(x, n + 1)$, with E_n serving as the shift operator in the n direction, while p is the spacing parameter of the direction n . It has the following Lax pair [21]:

$$\tilde{\phi} = M\phi = \begin{pmatrix} v & v^2 - p^2 - r \\ 1 & v \end{pmatrix} \phi, \quad (2a)$$

$$\phi_x = U\phi = \begin{pmatrix} 0 & -r + 2u_x \\ 1 & 0 \end{pmatrix} \phi, \quad (2b)$$

where $\phi = (\phi_1, \phi_2)^T$, $v = \tilde{u} - u - p$, and r stands for a spectral parameter. This implies that

$$M_x = \tilde{U}M - MU \quad (3)$$

if and only if u satisfies Equation (1). This equation is known as the straight limit of the lpKdV equation [22]. The bilinear form and Casoratian solutions for (1) have been investigated in [23].

In this paper, we prove that the isospectral flows associated with both (2a) and (2b) are symmetries of the sdIpKdV equation and that these two hierarchies of symmetries are equivalent modulo the sdIpKdV equation, using a method similar to the one in [20]. Additionally, we construct the non-isospectral flows associated with (2b), which can be interpreted as the master symmetries of the sdIpKdV equation.

This paper is organized as follows. The notations necessary for the following discussions are provided in Section 2. In Section 3, we present the isospectral flows $K^{(l)}$ associated with (2a) and point out that these flows are commuting symmetries of the Equation (1). In Section 4, we start with (2b) to construct the isospectral flows $\{H^{(l)}\}$ and the non-isospectral flows $\{\sigma^{(k)}\}$. These flows are interpreted as symmetries and master symmetries of the sdIpKdV equation, respectively.

2. Notations

We introduce some notations that will be used in the following discussions.

Let $u = u(t, x, n)$ be a real or complex function defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{Z}$ that vanishes rapidly as $|x| \rightarrow \infty$ or $|n| \rightarrow \infty$. We define $\mathcal{V}$ as the linear space of all functions f that depend on u , its partial derivatives of any order with respect to x , and the shifts $u(t, x, n + j)$ for $j \in \mathbb{Z}$, satisfying $f|_{u=0} = 0$. Each f is C^∞ differentiable with respect to t , x and n , and C^∞ -Gâteaux differentiable with respect to u , its partial derivatives and shifts. The Gâteaux (or Fréchet) derivative of $f \in \mathcal{V}$ (or f as an operator living on $\mathcal{V}$) in direction $g \in \mathcal{V}$ is defined as

$$f'[g] = \frac{d}{d\epsilon} f(u + \epsilon g)|_{\epsilon=0}. \quad (4)$$

Using the Gâteaux derivative, we can define the Lie product for any $f, g \in \mathcal{V}$ as

$$[f, g] = f'[g] - g'[f]. \quad (5)$$

We introduce the space of Laurent matrix polynomials, denoted $\mathcal{Q}_2$, which consists of all 2×2 matrices $\mathcal{M} = (q_{ij}(r, u))_{2 \times 2}$, where each $\{q_{ij}\}$ is a Laurent polynomial of r . Additionally, we denote $\mathcal{Q}_2^{(0)} = \{\mathcal{M} \in \mathcal{Q}_2 \mid \mathcal{M}|_{u=0} = 0\}$. We need operators ∂ and ∂^{-1} , which are defined as

$$\partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \int_{-\infty}^x \cdot dx, \quad \partial \partial^{-1} = \partial^{-1} \partial = 1.$$

We also need operators $\Delta_n^{(\pm)} = E_n \pm 1$, whose inverse operators can be denoted as

$$(\Delta_n^{(-)})^{-1}u(n) = -\sum_{k=0}^{\infty} E_n^k u(n) = -\sum_{k=0}^{\infty} u(n+k),$$

$$(\Delta_n^{(+)})^{-1}u(n) = \sum_{k=0}^{\infty} (-1)^k E_n^k u(n) = \sum_{k=0}^{\infty} (-1)^k u(n+k),$$

where u vanishes rapidly as $|n| \rightarrow \infty$.

3. Symmetries $\{K^{(l)}\}$

As the isospectral flows $\{K^{(l)}\}$ have been derived in reference [20], where they are interpreted as symmetries of the lattice potential Korteweg–de Vries equation, we will only briefly summarize them here without detailed derivation.

We assume that ϕ evolves with t as

$$\phi_t = W\phi, \quad W \in \mathcal{Q}_2. \quad (6)$$

Then, we can obtain the isospectral hierarchy associated with (2a) and (6) as follows:

$$u_{t_l} = K^{(l)} = T^l K^{(0)}, l = 0, 1, 2, \dots, \quad (7a)$$

where

$$K^{(0)} = (\Delta_n^{(+)})^{-1}(v^2 - p^2), \quad (7b)$$

$$T = (\Delta_n^{(-)})^{-1} \{ [2(\Delta_n^{(+)})^{-1} - 1] v \Delta_n^{(-)} (\Delta_n^{(+)})^{-1} v \Delta_n^{(-)} + [1 - (\Delta_n^{(+)})^{-1}] (v^2 - p^2) \Delta_n^{(+)} - v^2 + p^2 \}. \quad (7c)$$

(7) have the zero-curvature representations

$$M'[K^{(l)}] = \tilde{W}^{(l)} M - M W^{(l)}, \quad l = 0, 1, 2, \dots, \quad (8)$$

where $W^{(l)}$ is given by (68b) in [20], and

$$W^{(l)}|_{u=0} = \begin{pmatrix} 0 & -r \\ 1 & 0 \end{pmatrix} r^l. \quad (9)$$

The following two propositions are also results from [20].

Proposition 1. *The matrix equation*

$$M'[X] = \tilde{W}M - MW, \quad \text{where } X \in \mathcal{V}, \quad W \in \mathcal{Q}_2^{(0)}, \quad (10)$$

has only zero solutions $X = 0, W = 0$.

Proposition 2. *The isospectral flows $\{K^{(l)}\}$ satisfy*

$$[[K^{(l_1)}, K^{(l_2)}]] = 0. \quad (11)$$

We note that Equation (1) can be rewritten as

$$u_x = K = (E_n + 1)^{-1}[(\tilde{u} - u)^2 - 2p(\tilde{u} - u)]. \quad (12)$$

It is clear that the K in (12) corresponds to the $K^{(0)}$ given by (7b). Therefore, based on Proposition 2 we can deduce the following theorem.

Theorem 1. *The isospectral flows $\{K^{(l)}\}$ are commuting symmetries of the sdIpKdV Equation (1).*

4. Symmetries $\{H^{(l)}\}$ and Master Symmetries $\{\sigma^{(k)}\}$

4.1. The Matrix Equation

We begin by assuming that ϕ evolves with t as

$$\phi_t = V\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \phi, \quad V \in \mathcal{Q}_2. \quad (13)$$

The compatibility condition between (2b) and (13) is

$$U_t = V_x + [V, U], \quad (14)$$

where $[V, U] = VU - UV$. To derive both isospectral and non-isospectral flows, we consider the following matrix equation:

$$U'[X - rY] = V_x + [V, U] - U_r r_t, \quad X, Y \in \mathcal{V}. \quad (15)$$

From (15) we can obtain

$$0 = A_x + B - (-r + 2u_x)C, \quad (16a)$$

$$-r_t + 2(X - rY)_x = (-r + 2u_x)A + B_x - (-r + 2u_x)D, \quad (16b)$$

$$0 = C_x + D - A, \quad (16c)$$

$$0 = (-r + 2u_x)C + D_x - B. \quad (16d)$$

Adding Equations (16a) and (16d) results in

$$(A + D)_x = 0, \quad (17)$$

which implies that

$$D = -A + c, \quad (18)$$

where c is independent of x . Substituting (18) into (16), we obtain

$$A = \frac{1}{2}(C_x + c), \quad (19a)$$

$$B = (-r + 2u_x)C - \frac{1}{2}C_{xx}, \quad (19b)$$

$$-\frac{1}{2}r_t + (X - rY)_x = -rC_x + \left(-\frac{1}{4}\partial^3 + 2u_x\partial + u_{xx}\right)C. \quad (19c)$$

4.2. Isospectral Flows $\{H^{(l)}\}$

To obtain the isospectral flows, we set $r_t = 0$ in (19c). For the first isospectral flow, we set $Y = 0$ in (19c) and require the matrix V to satisfy

$$V|_{u=0} = V_0 = \begin{pmatrix} 0 & -r \\ 1 & 0 \end{pmatrix}. \quad (20)$$

Given the condition (20), we can deduce that $c = 0$ in (19a). By comparing the coefficients of the same powers of r in (19c), we obtain

$$C = 1, \quad (21a)$$

$$X = u_x. \quad (21b)$$

Substituting (21a) into (19a) and (19b) yields

$$A = 0, \quad (22a)$$

$$B = -r + 2u_x. \quad (22b)$$

Thus, there exist a unique $H^{(0)} \in \mathcal{V}$ and a unique $N^{(0)} \in \mathcal{Q}_2$, given by

$$H^{(0)} = u_x, \quad (23)$$

$$N^{(0)} = \begin{pmatrix} 0 & -r + 2u_x \\ 1 & 0 \end{pmatrix}, \quad (24)$$

which satisfy

$$U'[H^{(0)}] = N_x^{(0)} + [N^{(0)}, U]. \quad (25)$$

Next, we derive the second isospectral flow $H^{(1)}$. To do this, we take $Y = H^{(0)}$ in (19c). Then, by comparing the coefficients of the same powers of r in (19c), we obtain

$$C = H^{(0)}, \quad (26)$$

$$X = LH^{(0)} = -\frac{1}{4}u_{xxx} + \frac{3}{2}u_x^2, \quad (27)$$

where

$$L = \partial^{-1}(-\frac{1}{4}\partial^3 + 2u_x\partial + u_{xx}). \quad (28)$$

Substituting (26) into (19a) and (19b), we obtain

$$A = \frac{1}{2}u_{xx}, \quad (29a)$$

$$B = -ru_x + 2u_x^2 - \frac{1}{2}u_{xxx}, \quad (29b)$$

Hence, there exist a unique $H^{(1)} \in \mathcal{V}$ and a unique $N^{(1)} \in \mathcal{Q}_2^{(0)}$, given by

$$H^{(1)} = LH^{(0)} \in \mathcal{V}, \quad (30)$$

$$N^{(1)} = \begin{pmatrix} \frac{1}{2}u_{xx} & -ru_x + 2u_x^2 - \frac{1}{2}u_{xxx} \\ u_x & -\frac{1}{2}u_{xx} \end{pmatrix} \in \mathcal{Q}_2^{(0)}, \quad (31)$$

which satisfy the following equation:

$$U'[H^{(1)} - rH^{(0)}] = N_x^{(1)} + [N^{(1)}, U]. \quad (32)$$

Repeating the above procedure from $H^{(0)}$ to $H^{(1)}$ allows us to obtain higher-order flows $H^{(l)} = L^l H^{(0)} \in \mathcal{V}$ and matrices $N^{(l)} \in \mathcal{Q}_2^{(0)}$, $l = 2, 3, \dots$, which satisfy

$$U'[H^{(l)} - rH^{(l-1)}] = N_x^{(l)} + [N^{(l)}, U]. \quad (33)$$

From Equations (25), (32) and (33), we can derive

$$U'[H^{(l)}] = V_x^{(l)} + [V^{(l)}, U], \quad l = 0, 1, 2, \dots \quad (34a)$$

where

$$V^{(l)} = \sum_{j=0}^{j=l} N^{(j)} r^{l-j}, \quad V^{(l)}|_{u=0} = V_0 r^l. \quad (34b)$$

Up to this point, we have obtained the isospectral hierarchy

$$u_t = H^{(l)} = L^l H^{(0)}, \quad l = 0, 1, 2, \dots, \quad (35)$$

where $H^{(0)}$ and L are given by (23) and (28), with (34) as their zero-curvature representations. The second equation in this hierarchy is the potential Korteweg–de Vries equation:

$$u_t = H^{(1)} = -\frac{1}{4}u_{xxx} + \frac{3}{2}u_x^2. \quad (36)$$

Based on the above derivation, we can obtain the following proposition.

Proposition 3. *The matrix equation*

$$U'[X] = V_x + [V, U], \quad X \in \mathcal{V}, \quad V \in \mathcal{Q}_2^{(0)}, \quad (37)$$

has only zero solutions $X = 0$ and $V = 0$.

4.3. Non-Isospectral Flows $\{\sigma^{(k)}\}$

To obtain the first non-isospectral flow, we take $r_t = 2r$ and $Y = 0$ in (19c) and require the matrix V in (15) to satisfy

$$V|_{u=0} = Z_0 = \begin{pmatrix} \frac{1}{2} & -rx \\ x & -\frac{1}{2} \end{pmatrix}. \quad (38)$$

Then, by comparing the coefficients of the same powers of r in (19c), we can obtain

$$C = x, \quad (39a)$$

$$X = xu_x + u. \quad (39b)$$

Substituting (39a) into (19a) and (19b) yields

$$A = \frac{1}{2}, \quad (40a)$$

$$B = -rx + 2xu_x. \quad (40b)$$

Thus, there exist a unique $\sigma^{(0)} \in \mathcal{V}$ and a unique $Z^{(0)} \in \mathcal{Q}_2$, given by

$$\sigma^{(0)} = xu_x + u, \quad (41)$$

$$Z^{(0)} = \begin{pmatrix} \frac{1}{2} & -rx + 2xu_x \\ x & -\frac{1}{2} \end{pmatrix}, \quad (42)$$

which satisfy

$$U'[\sigma^{(0)}] = Z_x^{(0)} + [Z^{(0)}, U] - U_r(2r). \quad (43)$$

For the second non-isospectral flow, we take $r_t = 0$ and $Y = \sigma^{(0)}$ in (19c). Then, by comparing the coefficients of the same powers of r in (19c), we obtain

$$C = \sigma^{(0)}, \quad (44)$$

$$X = L\sigma^{(0)}. \quad (45)$$

Substituting (44) into (19a) and (19b) yields

$$A = \frac{1}{2}\sigma_x^{(0)}, \quad (46a)$$

$$B = -r\sigma^{(0)} + 2u_x\sigma^{(0)} - \frac{1}{2}\sigma_{xx}^{(0)}. \quad (46b)$$

Thus, there exist a unique $\sigma^{(1)} \in \mathcal{V}$ and a unique $Z^{(1)} \in \mathcal{Q}_2^{(0)}$ given by

$$\sigma^{(1)} = L\sigma^{(0)}, \quad (47)$$

$$Z^{(1)} = \begin{pmatrix} \frac{1}{2}\sigma_x^{(0)} & -r\sigma^{(0)} + 2u_x\sigma^{(0)} - \frac{1}{2}\sigma_{xx}^{(0)} \\ \sigma^{(0)} & -\frac{1}{2}\sigma_x^{(0)} \end{pmatrix}, \quad (48)$$

which satisfy

$$U'[\sigma^{(1)} - r\sigma^{(0)}] = Z_x^{(1)} + [Z^{(1)}, U]. \quad (49)$$

Repeating the above procedure from $\sigma^{(0)}$ to $\sigma^{(1)}$ allows us to obtain higher-order non-isospectral flows $\sigma^{(k)} = L^k\sigma^{(0)}$ and matrices $Z^{(k)} \in \mathcal{Q}_2^{(0)}$, $k = 2, 3, \dots$, which satisfy

$$U'[\sigma^{(k)} - r\sigma^{(k-1)}] = Z_x^{(k)} + [Z^{(k)}, U]. \quad (50)$$

From (43), (49) and (50), we can derive

$$U'[\sigma^{(k)}] = P_x^{(k)} + [P^{(k)}, U] - U_r(2r^{k+1}), \quad k = 0, 1, 2, \dots, \quad (51a)$$

where

$$P^{(k)} = \sum_{j=0}^{j=k} Z^{(j)} r^{k-j}, \quad P^{(k)}|_{u=0} = Z_0 r^k. \quad (51b)$$

So far, we have obtained the non-isospectral hierarchy

$$u_t = \sigma^{(k)} = L^k\sigma^{(0)}, \quad k = 0, 1, 2, \dots, \quad (52)$$

where $\sigma^{(0)}$ and L are given by (41) and (28), with (51) as their zero-curvature representations.

4.4. Algebra of Flows

We present the following proposition without proof, as its proof follows the same steps as the proof of Theorem 2 in [10].

Proposition 4.

$$U'[[H^{(l)}, H^{(k)}]] = \langle V^{(l)}, V^{(k)} \rangle_x + [\langle V^{(l)}, V^{(k)} \rangle, U], \quad (53a)$$

$$U'[[H^{(l)}, \sigma^{(k)}]] = \langle V^{(l)}, P^{(k)} \rangle_x + [\langle V^{(l)}, P^{(k)} \rangle, U], \quad (53b)$$

$$U'[[\sigma^{(l)}, \sigma^{(k)}]] = \langle P^{(l)}, P^{(k)} \rangle_x + [\langle P^{(l)}, P^{(k)} \rangle, U] - 2(l-k)U_r(2r^{l+k+1}), \quad (53c)$$

where

$$\langle V^{(l)}, V^{(k)} \rangle = V^{(l)'}[H^{(k)}] - V^{(k)'}[H^{(l)}] + [V^{(l)}, V^{(k)}], \quad (53d)$$

$$\langle V^{(l)}, P^{(k)} \rangle = V^{(l)'}[\sigma^{(k)}] - P^{(k)'}[H^{(l)}] + [V^{(l)}, P^{(k)}] + V_r^{(l)}(2r^{k+1}), \quad (53e)$$

$$\langle P^{(l)}, P^{(k)} \rangle = P^{(l)'}[\sigma^{(k)}] - P^{(k)'}[\sigma^{(l)}] + [P^{(l)}, P^{(k)}] + P_r^{(l)}(2r^{k+1}) - P_r^{(k)}(2r^{l+1}). \quad (53f)$$

Based on (34b) and (51b), it is straightforward to draw the following conclusions.

$$\langle V^{(l)}, V^{(k)} \rangle|_{u=0} = 0, \quad (54a)$$

$$\langle V^{(l)}, P^{(k)} \rangle|_{u=0} = (2l+1)V^{(l+k)}|_{u=0}, \quad (54b)$$

$$\langle P^{(l)}, P^{(k)} \rangle|_{u=0} = 2(l-k)P^{(l+k)}|_{u=0}. \quad (54c)$$

Then, using Proposition 3 and Proposition 4, we can derive the following proposition.

Proposition 5. The isospectral flows and non-isospectral flows $H^{(l)}$ in (35) and $\sigma^{(k)}$ in (52) compose an infinite-dimensional Lie algebra $\mathcal{F}$ through the Lie product $[\![\cdot, \cdot]\!]$ and possess the following relations:

$$[\![H^{(l)}, H^{(k)}]\!] = 0, \quad (55a)$$

$$[\![H^{(l)}, \sigma^{(k)}]\!] = (2l+1)H^{(l+k)}, \quad (55b)$$

$$[\![\sigma^{(l)}, \sigma^{(k)}]\!] = 2(l-k)\sigma^{(l+k)}. \quad (55c)$$

In light of Proposition 5, we immediately obtain the following theorem.

Theorem 2. Any given member $u_t = H^{(l)}$ in the isospectral hierarchy (35) possesses the following two sets symmetries, i.e.,

$$\{H^{(k)}\}, \quad \{\tau^{(l,k)} = (2l+1)tH^{(l+k)} + \sigma^{(k)}\}, \quad k \in \{0, 1, 2, \dots\}. \quad (56)$$

These symmetries form a centerless Kac–Moody–Virasoro algebra, with the following structure:

$$[\![H^{(s)}, H^{(k)}]\!] = 0, \quad (57a)$$

$$[\![H^{(s)}, \tau^{(l,k)}]\!] = (2s+1)H^{(s+k)}, \quad (57b)$$

$$[\![\tau^{(l,s)}, \tau^{(l,k)}]\!] = 2(s-k)\tau^{(l,s+k)}. \quad (57c)$$

4.5. Symmetries and Master Symmetries

Theorem 3. The isospectral flows $\{H^{(l)}\}$ in (35) are symmetries of the sdIpKdV Equation (1), and they are equivalent to the isospectral flows $\{K^{(l)}\}$ in (7) (modulo Equation (1)). The non-isospectral flows $\{\sigma^{(k)}\}$ in (52) are master symmetries of the sdIpKdV Equation (1).

Proof. The proof method we employed is similar to that in reference [20]. Specifically, to demonstrate that the isospectral flows $\{H^{(l)}\}$ are symmetries of the sdIpKdV Equation (1), it is sufficient to prove that

$$(\tilde{u})_{t_l} = \widetilde{(u_{t_l})}. \quad (58)$$

Firstly, based on (3) and (14), we can derive

$$(\tilde{U})_{t_l} = \tilde{U}'[(\tilde{u})_{t_l}] = [(M_{t_l} + MV^{(l)})M^{-1}]_x + [(M_{t_l} + MV^{(l)})M^{-1}, \tilde{U}], \quad (59)$$

where $\tilde{U}'[(\tilde{u})_{t_l}]$ represents the Gâteaux derivative of $\tilde{U}$ in the direction $(\tilde{u})_{t_l}$ with respect to $\tilde{u}$. From (34), we can obtain

$$\tilde{U}'[\tilde{H}^{(l)}] = \tilde{V}_x^{(l)} + [\tilde{V}^{(l)}, \tilde{U}]. \quad (60)$$

where $\tilde{U}'[\tilde{H}^{(l)}]$ represents the Gâteaux derivative of $\tilde{U}$ in the direction $\tilde{H}^{(l)}$ with respect to $\tilde{u}$. Through straightforward calculations, we can obtain

$$(M_{t_l} + MV^{(l)})M^{-1}|_{u=0} = \tilde{V}^{(l)}|_{u=0} = \begin{pmatrix} 0 & -r \\ 1 & 0 \end{pmatrix} r^l. \quad (61)$$

Comparing (59) and (60), in light of (61) and Proposition 3, we can obtain

$$(\tilde{u})_{t_l} = \tilde{H}^{(l)} = \widetilde{(u_{t_l})}, \quad l = 0, 1, 2, \dots \quad (62)$$

and

$$(M_{t_l} + MV^{(l)})M^{-1} = \tilde{V}^{(l)}, \quad l = 0, 1, 2, \dots \quad (63)$$

From (62) we can conclude that the flows $\{H^{(l)}\}$ in (35) are symmetries of (1). Furthermore, according to Proposition 5, $\{\sigma^{(k)}\}$ are master symmetries of Equation (1). Rearranging (63), we find

$$M_{t_l} = M'[H^{(l)}] = \tilde{V}^{(l)}M - MV^{(l)} \quad l = 0, 1, 2, \dots \quad (64)$$

From (9) and (34b) we can obtain

$$W^{(l)}|_{u=0} = V^{(l)}|_{u=0}. \quad (65)$$

By comparing (8) and (64), and in the light of (65) and Proposition 1, we can conclude that

$$K^{(l)} = H^{(l)}, \quad l = 0, 1, 2, \dots \quad (66)$$

□

5. Conclusions

In this paper, we derive the symmetries and master symmetries for the sdIpKdV equation, along with their algebras, using the zero-curvature representations of flows associated with the Lax pair of this equation. We first derive the isospectral flows associated with (2a) and the isospectral and non-isospectral flows associated with (2b), including their zero-curvature representations. Among these, the isospectral flows associated with (2a) have already been obtained in [20], where they are interpreted as symmetries of the lattice potential Korteweg–de Vries equation. Next, based on the zero-curvature representations of these flows and Propositions 1 and 3, we derive the Lie algebras of flows, i.e., Propositions 2 and 5. It follows from Proposition 2 that the isospectral flows $\{K^{(l)}\}$ are symmetries of the sdIpKdV equation. Furthermore, we demonstrate that the isospectral flows $\{H^{(l)}\}$ are also symmetries of sdIpKdV equation, and that these two hierarchies of symmetries are equivalent modulo the sdIpKdV equation. The proof process mainly relies on the zero-curvature representations of the sdIpKdV equation and the isospectral flows $\{H^{(l)}\}$ and $\{K^{(l)}\}$, as well as Proposition 3, whose proof method is similar to the one used in [20]. Finally, according to Proposition 5, the non-isospectral flows $\{\sigma^{(k)}\}$ are master symmetries of the sdIpKdV equation. The Lie algebra structure of the flows in Proposition 5 differs slightly from the one presented in reference [10]. This difference arises because, in the derivation of the non-isospectral flow $\sigma^{(0)}$, we take $r_t = 2r$ to simplify the expressions

of the non-isospectral flows. If this choice is not made, each flow's expression will be additionally multiplied by one-half. In fact, if $r_t = r$ is taken, the resulting Lie algebra structure is consistent with the one presented in reference [10].

The physical significance of the infinite K -symmetries and τ -symmetries, as well as the completeness of the known symmetries, has been thoroughly explored in recent research presented in reference [9]. The insightful discussions in this work provided valuable guidance for future research in the field. Integrable systems have important applications in both mathematics and physics [24,25]. Investigating the applications of integrable hierarchies in these fields is an important topic [26].

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