

Article

Contact Interaction of a Rigid Stamp and a Porous Elastic Cylinder of Finite Dimensions

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Abstract: This article investigates an axisymmetric contact problem involving the interaction between a rigid cylindrical stamp and a poroelastic cylinder of finite dimensions, based on the Cowin–Nunziato theory of media with voids. The stamp is assumed to have a flat base and to be in frictionless contact with the cylinder. The cylinder, in turn, rests on a rigid base without friction, with no normal displacements or tangential stresses on its lateral surface. Under an applied vertical force, the stamp undergoes displacement, compressing the poroelastic cylinder. The mathematical formulation of this problem involves expressing the unknown displacements within the cylinder and the variation in pore volume fraction as a series of Bessel functions. This representation reduces the problem to an integral equation of the first kind, describing the distribution of contact stresses beneath the stamp. The kernel of the integral equation is explicitly provided in its transformed form. The collocation method is employed to solve the integral equation, enabling the determination of contact stresses and the relationship between the indenter’s displacement and the applied force. A comparative model parameter analysis is performed to examine the effects of different material porosity parameters and model geometrical characteristics on the results.

Keywords: contact problem; Cowin–Nunziato model; analytical solutions; indentation**MSC:** 74M15; 74M25; 74A60; 75A10

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1. Introduction

The Cowin–Nunziato theory of elastic materials with voids represents a significant generalization of classical elasticity, developed to model materials containing distributed voids or unfilled pores. These voids, often small and dispersed throughout the material’s volume, substantially influence mechanical behavior. This theory has been particularly impactful in the study of geological, biological, and synthetic porous materials, where classical elasticity falls short in capturing the effects of voids.

Originally introduced by Cowin and Nunziato in Ref. [1], the theory of porous elastic materials (the micro-dilatation theory or void elasticity) provides a robust framework for analyzing materials with empty pores. This theory is particularly advantageous as it reduces to classical elasticity in the absence of a pore volume fraction, ensuring consistency with traditional models when void effects are negligible. The linearized version of the theory, presented in Ref. [2], facilitates its application in practical scenarios and has been widely utilized in fields such as geomechanics, biomechanics, and advanced material design. Notable studies have expanded its scope, including work on thermoelastic behavior

in porous media [3]. For instance, Carini and Zampoli [4,5] examined porous matrices incorporating three delay times within the context of linear thermoelasticity, highlighting the spatial behavior of solutions in such frameworks. Their contributions emphasize the relevance of delay effects in understanding thermal responses in porous structures and the applicability of the micro-dilatation theory for tackling such problems. The Cowin–Nunziato theory’s adaptability to different material systems is demonstrated in studies incorporating piezoelectric media with voids [6] and anisotropic behavior [7], reflecting its versatility across various disciplines.

Numerical approaches have further enriched the possible application domains for problems evolving materials with voids, the behavior of which is described by employing the Cowin–Nunziato theory. The theoretical aspects of the finite element approach are addressed in, e.g., Refs. [8–10], with the latter also discussing the restrictions on the micro–macro dilatation coupling parameter. Sladek et al. [11] developed a meshless local Petrov–Galerkin (MLPG) model for porous elastic materials based on micro-dilatation theory, offering insights into both the static and dynamic responses of these materials. Scalia and Sumbatyan [12] explored integral equations in contact problems for porous elastic strips, contributing to a deeper understanding of the mathematical formulations arising in such scenarios.

Applications of the Cowin–Nunziato theory also extend to contact mechanics. Among these applications, axisymmetric contact problems hold particular significance. These problems are essential for understanding the interactions between rigid and deformable bodies, especially with porous materials and in situations where the axisymmetric assumption holds. For example, Vasiliev et al. [13] investigated contact problems with applications in nanoindentation testing, providing analytical solutions for a coated infinite substrate under a conical indenter. Their work underscores the utility of the micro-dilatation theory when post-processing data from indentation experiments, a critical step in material characterization, especially when the material properties of these films and coatings have to be assessed. In a series of works listed in the references found in Ref. [14], Chebakov et al. explore various formulations and then present solutions to contact problems involving materials with voids. However, notable gaps remain, particularly in addressing cases where the substrate has finite dimensions. To address this gap, the present work proposes a semi-analytical solution to a contact problem for a finite-sized homogeneous poroelastic cylindrical substrate sample and a rigid cylindrical indenter (flat punch). The behavior of the substrate material is described by the Cowin–Nunziato framework. This contact problem has not previously been investigated, and the proposed solution offers a novel means to analyze the influence of finite sample dimensions on the interpretation of indentation tests. Such considerations are crucial for accurately characterizing material properties, as the mechanical response observed during indentation experiments is significantly affected by the size of the sample. Validation of the proposed solution is achieved through comparisons with established results for contact problems concerning linear elastic cylindrical substrate samples, which serve as benchmarks for specific parameter configurations.

In this study, by addressing contact problems in solids with voids and emphasizing the significance of substrate size, our work is expected to enhance the interpretation of high-accuracy local-scale tests conducted on small volumes. By presenting a semi-analytical solution to the problem, this work is expected to provide a robust tool for the verification of more complex indentation test scenarios involving materials that exhibit micro-dilatation, thereby contributing to the advancement of methodologies for interpreting experimental data with greater accuracy.

2. Problem Formulation

In a cylindrical coordinate system (r, φ, z) , we examine an axisymmetric contact problem involving the interaction between a rigid stamp of radius a and a deformable cylinder of radius R and height h , with a deformation described by the Cowin–Nunziato model [2]. The base of the rigid cylindrical stamp is assumed to be flat. A sketch illustrating the problem geometry is presented in Figure 1.

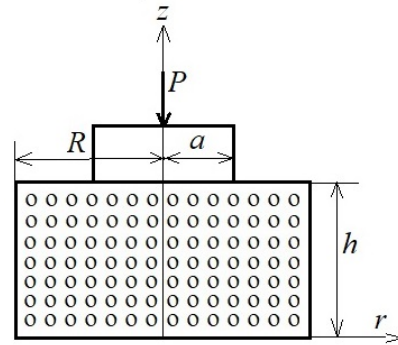


Figure 1. Sketch of the problem geometry, including the notations for the dimensions of the rigid cylindrical stamp and the finite-sized deformable cylinder with voids, along with the applied load configuration (axis of symmetry— Oz).

According to the Cowin–Nunziato theory, the deformation of a homogeneous isotropic material with voids in cylindrical coordinates is described by the following system of partial differential equations [2]:

$$\begin{aligned} (\lambda + \mu) \frac{\partial \theta}{\partial r} + \mu \left(\Delta u - \frac{u}{r^2} \right) + \beta \frac{\partial \Phi}{\partial r} &= 0, \\ (\lambda + \mu) \frac{\partial \theta}{\partial z} + \mu \Delta w + \beta \frac{\partial \Phi}{\partial z} &= 0, \quad \alpha \Delta \Phi - \xi \Phi - \beta \theta = 0 \\ \Delta &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}, \quad \theta = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \end{aligned} \quad (1)$$

Here, λ and μ are the Lamé coefficients, and α is the void diffusion coefficient, β is the parameter for the relationship between micro-dilatation and macro-dilatation properties, and ξ is the void stiffness; the function $\Phi(r, z)$ describes the change in the volume fraction of pores, and u and w are the displacements along the r and z axes, respectively. In a scenario where $\beta = 0$, the deformation of the cylinder is linear elastic.

The components of the stress tensor are determined from the following relationships [2]:

$$\sigma_z = \lambda \theta + 2\mu \frac{\partial w}{\partial z} + \beta \Phi, \quad \tau_{rz} = \mu \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right). \quad (2)$$

The boundary conditions of the problem at $r = R$, $z = h$, and $z = 0$ are given in the following form:

$$\begin{aligned} \tau_{rz}(r, z) &= 0, \quad \frac{\partial \Phi}{\partial z} = 0, \quad w(r, z) = \delta \quad (z = h, \quad r \leq a), \\ \sigma_z(r, z) &= 0, \quad \tau_{rz}(r, z) = 0, \quad \frac{\partial \Phi}{\partial z} = 0 \quad (z = h, \quad a \leq r \leq R), \\ \sigma_z &= \lambda \theta + 2\mu \frac{\partial w}{\partial z} + \beta \Phi, \quad \tau_{rz} = \mu \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right), \\ \tau_{rz}(r, z) &= 0, \quad \frac{\partial \Phi}{\partial z} = 0, \quad w(r, z) = 0 \quad (z = 0), \\ \tau_{rz}(r, z) &= 0, \quad u(r, z) = 0 \quad (0 \leq z \leq h, \quad r = R) \end{aligned} \quad (3)$$

The parameter δ represents the penetration depth of the rigid stamp into the deformable cylindrical substrate, resulting from the applied force P , as shown in Figure 1.

3. Derivation of the Integral Equation

Initially, we assume that the normal stresses within the contact zone are known and depend on r , i.e., $\sigma_z(r, h) = q(r)$. Subsequently, an integral equation will be derived to determine these stresses. We now seek a solution to system (1) subject to the given boundary conditions, as follows:

$$\begin{aligned}\tau_{rz}(r, z) &= 0, \quad \frac{\partial \Phi}{\partial z} = 0, \quad \sigma_z(r, z) = q(r) (z = h, r \leq a), \\ \sigma_z(r, z) &= 0, \quad \tau_{rz}(r, z) = 0, \quad \frac{\partial \Phi}{\partial z} = 0 (z = h, a \leq r \leq R), \\ \tau_{rz}(r, z) &= 0, \quad \frac{\partial \Phi}{\partial z} = 0, \quad w(r, z) = 0 (z = 0), \\ \tau_{rz}(r, z) &= 0, \quad u(r, z) = 0 (0 \leq z \leq h, r = R).\end{aligned}\quad (4)$$

To obtain the solution, the unknown displacements and the function $\Phi(r, z)$ are represented as a series expansion in terms of Bessel functions $J_i(u_k)$ ($i = 0, 1$):

$$\begin{aligned}u(r, z) &= \sum_{k=1}^{\infty} A(u_k, z) J_1(u_k r), \quad w(r, z) = \sum_{k=1}^{\infty} B(u_k, z) J_0(u_k r), \\ \Phi(r, z) &= \sum_{k=1}^{\infty} F(u_k, z) J_0(u_k r).\end{aligned}\quad (5)$$

To satisfy the boundary condition at $r = R$, the roots u_k of the equation $J_1(u_k R) = 0$ have to be determined. Next, the unknown functions $A(u_k, z)$, $B(u_k, z)$, and $F(u_k, z)$ are obtained by solving the following system of second-order ordinary differential equations:

$$\begin{aligned}c^2 \frac{d^2 A}{dz^2} - u_k^2 A - (1 - c^2) u_k \frac{dB}{dz} - u_k H F &= 0, \\ (1 - c^2) u_k \frac{dA}{dz} + \frac{d^2 B}{dz^2} - c^2 u_k^2 B + H \frac{dF}{dz} &= 0, \\ l_1^2 \left(\frac{d^2 F}{dz^2} - u_k^2 F \right) - \frac{l_1^2}{l_2^2} F - u_k A - \frac{dB}{dz} &= 0.\end{aligned}\quad (6)$$

Applying the boundary conditions (4) at $z = h$ and $z = 0$, we obtain the following relationships:

$$\frac{dA}{dz} - u_k B = 0, \quad (1 - 2c^2) u_k A + \frac{dB}{dz} + H F = Q_k c^2 \mu^{-1}, \quad \frac{dF}{dz} = 0 (z = h), \quad (7a)$$

$$B(u_k, 0) = 0, \quad \frac{dA}{dz} - u_k B = 0, \quad \frac{dF}{dz} = 0 (z = 0), \quad (7b)$$

$$q(r) = \sum_{k=1}^{\infty} Q_k J_0(u_k r), \quad Q_k = 2R^{-2} J_0^{-2}(u_k R) \int_0^a q(r) J_0(u_k r) r dr. \quad (8)$$

To determine the expansion coefficients Q_k in the above expressions, the orthogonality condition of cylindrical functions is used [15], i.e.:

$$\int_0^R r J_0(u_k r) J_0(u_n r) dr = \begin{cases} 0 & k \neq n \\ \frac{R^2}{2} J_0^2(u_k R) & k = n \end{cases} \quad (9)$$

provided that, as noted above, u_k are the roots of the equation $J_1(u_k R) = 0$.

It should be noted that the following notations have been used for brevity in the equations above, as detailed in Ref. [16]:

$$c^2 = \frac{\mu}{(\lambda + 2\mu)}, \quad H = \frac{\beta}{\lambda + 2\mu}, \quad l_1^2 = \frac{\alpha}{\beta}, \quad l_2^2 = \frac{\alpha}{\xi}. \quad (10)$$

Following the theory of second-order ordinary differential equations, the solution to system (6) with boundary conditions (7a) and (7b) can be obtained using standard methods.

Furthermore, in order to establish the integral equation for determining the contact stresses, an expression for the function $B(u_k, z)$ is required. Using the boundary conditions (4), and after a series of transformations, we end up with:

$$B(u_k, 0) = \frac{Q_k}{2\mu(1-c^2)} L(u_k), \quad L(u_k) = \frac{L_1(u_k)}{L_2(u_k)}'' \quad (11)$$

where, using the notations $N = Hl_2^2/l_1^2$ and $T = \sqrt{u_k^2 - N + 1}$ for brevity, L_1 and L_2 have the following meaning:

$$L_1(u_k) = 4(N-1)^4(u_k^2 - N + 1) \sinh(Th) [1 - \cosh(2hu_k)],$$

$$L_2(u_k) = S_1 \cosh(Th) [\cosh(2u_k h) - 1] + \sinh(Th) [\sinh(2u_k h) S_2 + S_3],$$

$$\text{with } S_1 = 16NTc^2u_k^3(N-1)^2, \quad S_2 = -8(N-1)^2T^2[(2Nu_k^2 + N-1)c^2 + (N-1)^2],$$

$$\text{and } S_3 = -16u_k h(N-1)^3T^2(c^2 + N-1).$$

It can be proven that:

$$\lim_{u \rightarrow 0} L(u) = Ah \quad (u \rightarrow 0), \quad A = \frac{(N-1)(1-c^2)}{2(N-1+c^2)}, \quad \lim_{u \rightarrow \infty} L(u) = 1 \quad (u \rightarrow \infty). \quad (12)$$

By applying Equations (5) and (11), we obtain:

$$w(r, h) = \frac{1}{2\mu(1-c^2)} \sum_{k=1}^{\infty} Q_k L(u_k) J_0(u_k r) \quad (13)$$

Substituting expression (8) for Q_k into (13) and using the third boundary condition from (3) ($w(r, z) = \delta$, ($z = h$, $r \leq a$)), after straightforward transformations, we derive the following integral equation to determine the contact stress $q(r)$:

$$\int_0^a q(\rho) \rho k(\rho, r) d\rho = \frac{\mu}{1-\nu} \delta \quad (r \leq a) \quad (14)$$

where:

$$k(\rho, r) = \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{L(u_k)}{J_0^2(u_k R)} \rho J_0(u_k r) J_0(u_k \rho), \quad J_1(u_k R) = 0. \quad (15)$$

Note that $2(1-c^2) = (1-\nu)^{-1}$, where ν is the Poisson ratio.

At $N = 0$, the integral Equation (14) corresponds to the equation for the analogous contact problem when the deformable cylinder is linearly elastic, and:

$$L(u_k) = \frac{\cosh(2u_k h) - 1}{u_k (\sinh(2u_k h) + 2u_k h)}. \quad (16)$$

Analyses of similar contact problems employing a leaner elastic constitutive relationship have been carried out in earlier works, as discussed in Refs. [14,17].

4. Solution of the Derived Integral Equation

To solve the integral Equation (14), we apply the direct collocation method [18]. We divide the segment $[0, a]$ into n parts by a set of points $b_j = \varepsilon j$ ($\varepsilon = a/n, j = 0, 1, \dots, n$), and assume that on each segment $[b_{j-1}, b_j]$, the contact stresses have a constant value q_j . Let $r_k = (b_k + b_{k-1})/2$ be the collocation points; then, we discretize the integral equation according to the following rule:

$$\sum_{j=1}^n q_j \int_{b_{j-1}}^{b_j} k(\rho, r_k) d\rho = \frac{\delta \mu}{1 - \nu} \quad (k = 1, 2, \dots, n).$$

As a result, we arrive at the following system for determining q_j :

$$\sum_{j=1}^n q_j a_{kj} = \frac{\delta \mu}{1 - \nu} \quad (k = 1, 2, \dots, n) \quad (17)$$

$$\text{where } a_{kj} = \frac{2}{R^2} \sum_{m=1}^{\infty} \frac{L(u_m) J_0(u_m r_k)}{u_m J_0^2(u_m R)} [b_j J_1(u_m b_j) - b_{j-1} J_1(u_m b_{j-1})]. \quad (18)$$

When deriving relationship (18) according to [19], the following equality is used:

$$\int \rho J_0(u_m \rho) d\rho = \rho u_m^{-1} J_1(u_m \rho).$$

Once the contact stresses are found, the force P acting on the rigid indenter can be calculated as:

$$P = 2\pi \int_0^a q(r) r dr = 2\pi \varepsilon \sum_{k=1}^n q_k r_k. \quad (19)$$

5. Numerical Example and Parameter Study

This section focuses on analyzing the following dimensionless quantities, which represent the contact stresses and forces applied to the rigid cylindrical indenter, as defined in Equation (20):

$$q^*(r) = \frac{1 - \nu}{\mu} \frac{a}{\delta} q(r), \quad P^* = \frac{1 - \nu}{\mu} \frac{1}{\delta a} P. \quad (20)$$

For simplicity and to standardize the analysis, the parameter a (representing the radius of the rigid cylindrical indenter) is set to a value of 1. The numerical results are presented using q^* and P^* , with the model geometry dimensions scaled relative to a .

The numerical results are verified and their accuracy is evaluated by considering a scenario where $N = 0$, which corresponds to a scenario when the substrate is a linear elastic cylinder. These results are then compared with the existing solutions presented in Ref. [17]. Table 1 presents the values of $q^*(0)$ and P^* obtained for $N = 0$, $h = 0.5$, and four different values of R . For comparison, the corresponding results for the same variables from Table 2.2 in Ref. [17] are also included. The comparison reveals a strong agreement between

the values derived in this study and those reported in the literature. This consistency underscores the validity of the employed numerical approach and confirms its accuracy in addressing the problem under consideration. It should be noted that, based on benchmark estimates, the number of equations in system (17) and the number of terms in sum (18) were selected to ensure that the relative error in the calculations did not exceed 0.01%.

Table 1. Contact stress and the applied force in a scenario where the substrate is a linear elastic finite cylinder.

R	P^*	$P^*, [17]$	$q^*(0)$	$q^*(0), [17]$
1.1	14.26	14.26	3.976	3.976
1.3	14.88	14.92	3.934	3.897
1.5	14.74	14.76	3.942	3.857
2	14.68	14.54	3.935	3.835

Table 2 presents the values of the contact stresses and applied force, analogous to those in Table 1, but for a scenario where the substrate is a porous material. These quantities were calculated using the computational scheme proposed in this study. Specifically, the results correspond to the solution of the contact problem for $N = 0.5$, with the substrate geometry remaining the same as in the verification example, i.e., $h = 0.5$, and with the values of R as in Table 1.

Table 2. Contact stress and the applied force in scenarios where $N = 0$ and $N = 0.5$, with a cylindrical substrate height of $h = 0.5$.

R	$P^*, N = 0.5^*$	$P^*, N = 0$	$q^*(0), N = 0.5$	$q^*(0), N = 0$
1.1	8.773	14.26	2.442	3.976
1.3	10.12	14.88	2.656	3.934
1.5	10.87	14.74	2.867	3.942
2	12.01	14.68	3.157	3.935

Figure 2 illustrates the distribution of contact stresses along the contact surface for two different substrate radii and three distinct values of the porosity parameter N , while keeping all other model parameters and the substrate height constant ($h = 0.5$).

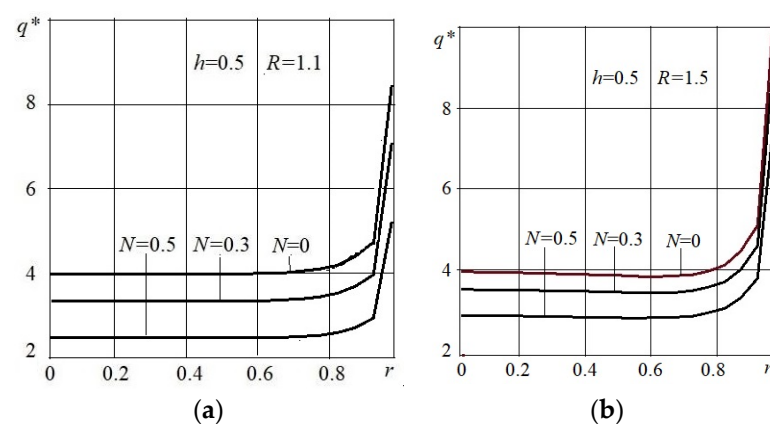


Figure 2. Contact stresses for different values of N and two different values of the substrate radius: (a) $R = 1.1$, (b) $R = 1.5$, with a cylindrical substrate height of $h = 0.5$.

The influence of the cylindrical substrate's height on the contact behavior is illustrated in Table 3. For comparison purposes, the table presents the contact stress at the midpoint of

the contact area and the applied load required for the indenter to achieve a specified fixed penetration depth. This allows for a direct evaluation of how variations in substrate height affect the stress distribution and the force needed for indentation.

Table 3. Contact stress and the applied force in a scenario where the substrate is a porous cylinder with $N = 0.5$, with a cylindrical substrate height of 0.5 and 1.

R	$P^*, h = 0.5$	$P^*, h = 1$	$q^*(0), h = 0.5$	$q^*(0), h = 1$
1.1	8.773	4.471	2.442	1.209
1.3	10.12	5.554	2.656	1.249
1.5	10.87	6.218	2.867	1.307
2	12.01	6.961	3.157	1.417

To further explore the influence of the model's geometry and constitutive parameters, Table 4 provides the contact stresses at equally spaced points along the radius of the contact surface, as well as the corresponding applied force to the rigid indenter for a substrate height that is twice as large ($h = 1$). Table 5 provides similar results for a cylinder height of 3. These results facilitate a detailed comparison and provide insights into how the substrate radius, height, and porosity collectively influence the contact stress distribution and the applied penetration force.

Table 4. Contact stress along the contact surface and the applied force in a scenario where the height of the substrate is $h = 1$, and for two different values of N .

R	1.1		1.3		1.5		2		3	
N	0.5	0.3	0.5	0.3	0.5	0.3	0.5	0.3	0.5	0.3
P^*	4.471	6.128	5.554	7.316	6.218	7.810	6.961	8.050	7.472	8.159
$q^*(0)$	1.209	1.657	1.249	1.651	1.307	1.652	1.417	1.658	1.511	1.677
$q^*(0.2)$	1.211	1.661	1.256	1.659	1.308	1.653	1.420	1.661	1.516	1.681
$q^*(0.4)$	1.217	1.669	1.278	1.688	1.336	1.686	1.454	1.697	1.561	1.726
$q^*(0.6)$	1.234	1.692	1.339	1.766	1.424	1.793	1.555	1.809	1.668	1.836
$q^*(0.8)$	1.301	1.784	1.550	2.042	1.716	2.155	1.905	2.204	2.036	2.224

Table 5. Contact stress along the contact surface and the applied force in a scenario where the height of the substrate is $h = 3$, and for two different values of N .

R	1.1		1.3		1.5		2		3	
N	0.5	0.3	0.5	0.3	0.5	0.3	0.5	0.3	0.5	0.3
P^*	1.51	2.08	2.023	2.745	2.511	3.329	3.467	4.302	4.279	4.866
$q^*(0)$	0.408	0.562	0.445	0.606	0.492	0.656	0.596	0.749	0.68	0.789
$q^*(0.2)$	0.409	0.563	0.448	0.61	0.498	0.663	0.605	0.76	0.696	0.807
$q^*(0.4)$	0.411	0.566	0.459	0.624	0.517	0.688	0.642	0.804	0.748	0.864
$q^*(0.6)$	0.417	0.574	0.485	0.659	0.562	0.748	0.725	0.905	0.864	0.991
$q^*(0.8)$	0.44	0.606	0.566	0.768	0.696	0.923	0.955	1.186	0.173	1.335

6. Discussion

It should be noted that for $N = 0$, when the penetration depth of the rigid indenter is fixed, the dimensionless force P^* exhibits a non-monotonic dependence on the radius $R > a$ of the cylindrical substrate. Specifically, as R increases, P^* initially rises to a maximum value and then begins to decrease as R continues to grow. This behavior is depicted in Figure 2.5 of Ref. [17]. A similar trend can even be observed for relatively small values of N .

However, for N values exceeding a certain threshold (from the results presented in the previous section, this can be 0.5), the force P^* increases monotonically with R , eventually

converging to a value corresponding to the solution for a similar problem involving a flat indenter penetrating a layer with the same value of the parameter N [14]. This highlights the influence of the porosity parameter N on the force–displacement relationship. A comparison of the results presented in Table 2 indicates that the presence of pores leads to a reduction in both the contact stress and the required force for the same indenter displacement. As calculations show, with an increase in the porosity parameter N at fixed values of other parameters, contact stresses decrease.

These observations are further supported by the data presented in Table 3, which illustrate the dependence of the dimensionless force (P^*) and contact stress (q^*) on the substrate radius (R) and height (h) for different values of the parameter N . Additionally, the data in Tables 4 and 5 highlight how these quantities vary along the contact surface, offering a comprehensive view of the influence of substrate dimensions and material properties on the contact response. As the substrate height and radius increase, both the force P^* and the contact stress decrease; the larger volume allows for more micro-dilatation, and the influence of the fixed boundary conditions becomes less pronounced. This can also be seen from the results provided in Figure 2, where the relatively small substrate height is a major factor.

In the context of this study, it is important to highlight the simplifying assumptions made in the application of the Cowin–Nunziato theoretical framework. These assumptions allowed us to derive a semi-analytical solution for the contact problem involving materials with micro-dilatation. While these simplifications were necessary to achieve a tractable solution, they may limit the direct applicability of the results to real-world scenarios involving more complex material behaviors. Key factors such as temperature variations, contact friction, and substrate roughness—which significantly affect material behavior—were not included in this study. For instance, thermal effects can alter stress distributions, frictional forces can influence deformation patterns, and surface roughness impacts stress concentrations and contact areas.

7. Conclusions

This study addressed a contact problem involving a rigid indenter and a finite-sized cylindrical substrate characterized by the micro-dilatation theory. The key conclusions can be summarized as follows:

- Importance of considering the finite size of the porous substrate: unlike infinite or semi-infinite substrate models, the finite dimensions of the substrate impose boundary constraints that influence the overall stress distribution. Ignoring these finite-size effects can lead to inaccurate predictions in practical applications such as materials testing or engineering design involving comparatively small volumes of porous materials.
- Role of porosity characteristics: the porosity parameter N was found to have a marked impact on the contact stresses and penetration force. Substrates with higher porosity exhibit reduced stiffness, resulting in lower contact stresses and applied forces for the same indentation depth. This highlights the sensitivity of the mechanical response to microstructural properties.
- Importance of semi-analytical solutions: the semi-analytical approach proposed in this study facilitates the efficient and accurate analysis of contact problems for poroelastic bodies with finite dimensions. Unlike purely numerical methods, this approach provides closed-form relationships for critical parameters, allowing for better insight into the influence of substrate geometry and material characteristics. Semi-analytical solutions offer a robust framework for understanding the relationship between indentation depth, contact stresses, and material properties, improving the accuracy of property

estimation in experimental settings. Such solutions are particularly advantageous for parametric studies and serve as a foundation for validating numerical models.

- **Applicability to indentation testing:** the findings underscore the importance of accounting for the finite size of the sample when interpreting results from indentation experiments or other applications where the size of the substrate is of the same order as that of the indenter. Neglecting these effects may lead to incorrect estimations of material properties, particularly for substrates governed by the micro-dilatation theory. The proposed solution provides a reliable framework for addressing these limitations. These findings emphasize the importance of considering the finite size of the sample when interpreting the results of indentation tests.

Future studies could explore the effects of multi-material interfaces, contact friction, material inhomogeneity, or dynamic loading conditions to broaden the applicability of the findings. Experimental validation of the proposed approach in assessing the response of porous material properties using indentation tests, as well as comparisons with pure numerical solutions, could further strengthen the conclusions. However, these extensions will likely require numerical solutions for the corresponding contact problem, as closed-form analytical or semi-analytical solutions may not be feasible.

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