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Extremal Results on ℓ-Connected Graphs or Pancyclic Graphs Based on Wiener-Type Indices

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Abstract: A graph of order n is called pancyclic if it contains a cycle of length y for every $3 \leq y \leq n$. The connectivity of an incomplete graph G, denoted by $\kappa(G)$, is $\min\{|W||W \text{ is a vertex cut of } G\}$. A graph G is said to be ℓ-connected if the connectivity $\kappa(G) \geq \ell$. The Wiener-type indices of a connected graph G are $W_g(G) = \sum_{\{s,t\}\subseteq V(G)} g(d_G(s,t))$, where $g(x)$ is a function and $d_G(s,t)$ is the distance in G between s and t. In this note, we first determine the minimum and maximum values of $W_g(G)$ for ℓ-connected graphs. Then, we use the Wiener-type indices of graph G, the Wiener-type indices of complement graph $\overline{G}$ with minimum degree $\delta(G) \geq 2$ or $\delta(G) \geq 3$ to give some sufficient conditions for connected graphs to be pancyclic. Our results generalize some existing results of several papers.

Keywords: ℓ-connected graph; pancyclic graph; Wiener-type index; sufficient condition

MSC: 05C45; 05C09; 05C92

1. Introduction

The graphs considered here are undirected, finite and simple. Following the convention, we use G to denote a graph (undirected, finite, simple), use $V(G)$ to represent the vertex set (it must be a finite set) of the graph, use $E(G)$ to represent the edge set of the graph and use $uv \in E(G)$ to denote that there exists an edge between u and v. Furthermore, we use C_n to denote an n-cycle (a graph in which all its vertices form a single closed loop, with each vertex connected to exactly two other vertices), $K_{a,n-a}$ to denote a complete bipartite graph of order n (a graph with a partition Q_1, Q_2 of its vertex set such that each vertex of Q_1 is connected to each vertex of Q_2, and the subgraphs induced by Q_1 and Q_2 contain no edges) and K_n to denote a complete graph of order n. Clearly, in a simple graph, a complete graph possesses the highest possible number of edges.

Let $u \in V(G)$. The neighbor of u is a vertex subset $\{v \in V(G)|uv \in E(G)\}$, denoted by $N_G(u)$, and the size of the vertex subset $\{v \in V(G)|uv \in E(G)\}$ is called the degree of vertex u, denoted by $d_G(u)$. Examining the degrees of vertices in graph theory has considerable significance. The degree of vertices is closely linked to the connectivity, Hamiltonian properties and sparsity of a graph. Under certain given conditions, by calculating the degree of the graph, we can determine whether it belongs to a certain class of graphs and whether there exists a cycle of a certain size, and so on. Moreover, in graph algorithms, the degree of vertices is often used as the basis for the algorithm design, as a range of graph parameters are associated with vertex degrees.

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The complement graph of G is the graph with the same vertex set of G but its edge set is $\{v_iv_j|v_iv_j \notin E(G)\}$, denoted by $\overline{G}$. Therefore, for $u, v \in V(G)$, there either exists an edge uv in the graph or in its complement graph, but not both. Regarding complement graphs, a famous result is that either a graph or its complement graph must be connected. Let R_1 and R_2 be two graphs that are disjoint from each other. Now, we introduce the union and the joint of two distinct graphs. The union of R_1 and R_2, denoted by $R_1 + R_2$, is the graph with $V(R_1 + R_2) = V(R_1) \cup V(R_2)$ and $E(R_1 + R_2) = E(R_1) \cup E(R_2)$. It is easy to see that the union is not connected if R_1 and R_2 are not empty graphs. The joint of R_1 and R_2, denoted by $R_1 \vee R_2$, is the graph with $V(R_1 \vee R_2) = V(R_1) \cup V(R_2)$ and $E(R_1 \vee R_2) = E(R_1) \cup E(R_2) \cup \{u_1u_2|u_1 \in V(R_1), u_2 \in V(R_2)\}$. It is easy to see that the joint is connected if R_1 and R_2 are not empty graphs (by the famous result of complement graphs mentioned above).

Let Y be a vertex subset of G. Then, we call Y a vertex cut if $G - Y$ is not connected. Let G be a graph such that $|V(G)| = n$ and it is not isomorphic to K_n, and W be the set consisting of all vertex cuts of the graph G. The connectivity of G, denoted by $\kappa(G)$, is $\min\{|Y||Y \in W\}$. The minimum cut set is widely applied in network flow problems. A graph G is said to be ℓ-connected if $\kappa(G) \geq \ell$, where $\ell > 0$ is an integer. In other words, there does not exist a vertex subset of size $(\ell - 1)$ such that the graph G becomes disconnected after removing this subset. Clearly, if G is $(m + 1)$-connected, then it is m-connected. For convenience, we introduce some notations here. We use $\mathcal{G}^*(n)$ to denote the set consisting of all the graphs which are connected and have n vertices, use $\mathcal{G}^*(n, \ell)$ to denote the set consisting of all the graphs which are connected, have connectivity $\kappa(G) \geq \ell$ and n vertices and use $\mathcal{BG}^*(n)$ to denote the set consisting of all connected bipartite graphs with n vertices.

The distance between s and t is the value of the length of a shortest path in a graph G for vertex s and vertex t, denoted by $d_G(s, t)$ ($d(s, t)$ for short). The distance is a core concept in this paper. Clearly, the distances are all equal to 1 in a complete graph. The eccentricity of vertex u in a graph G, denoted by $ecc(u)$, is $\max\{d(u, v)|v \in V(G)\}$, and the diameter of a graph G, denoted by $D(G)$, is defined as $\max\{d(u, v)|u, v \in V(G)\}$, or equivalently, is $\max\{ecc(u)|u \in V(G)\}$. Typically, a graph with a smaller diameter implies that the vertices in the graph are more easily accessible to each other.

A graph G is called pancyclic if the graph G contains a cycle of length y for every $3 \leq y \leq n$, where $|V(G)| = n$. Clearly, if a graph has the pancyclic property, then this graph is connected (since it includes a cycle whose length is equal to its number of vertices). For other undefined terminologies and notations, we refer readers to [1].

Topological indices are mathematical invariants derived from the structure of a graph. These indices can be categorized into several types based on their computational principles and applications, including the distance-based indices, the counting-based indices, the shape-based indices, the degree-based indices, the spectral indices and the information-theoretic indices. The distance-based indices are calculated based on the distances between vertices, the counting-based indices involve counting specific subgraphs, the shape-based indices capture the topological shape and measure the structural characteristics, the degree-based indices are calculated using the degrees of the vertices, the spectral indices are based on the spectrum of matrices (adjacency matrix, normalized Laplacian matrix) associated with the graph and the information-theoretic indices utilize concepts from information theory in the context of graphs, frequently involving the measurement of entropy or mutual information. Next, we introduce several distance-based indices: *Wiener-type indices*, which are a widely studied class of distance-based indices.

Let $G \in \mathcal{G}^*(n)$. The Wiener index of G, denoted by $W(G)$, was defined as [2]

$$W(G) = \sum_{\{s,t\}\subseteq V(G)} d_G(s,t). \tag{1}$$

Clearly, the value of the Wiener index depends only on the distances. Consequently, this characteristic of the calculation makes the Wiener index highly sensitive to the shape and size of the graph.

The Wiener-type indices of a connected graph G with respect to a function $g(x)$, denoted by $W_g(G)$, was defined as

$$W_g(G) = \sum_{\{s,t\}\subseteq V(G)} g(d_G(s,t)). \tag{2}$$

Wiener-type indices are generalizations of the Wiener index. By introducing different functions $g(x)$, Wiener-type indices provide a broader range of tools and methods for studying and describing the structural characteristics of graphs. Now, we introduce some Wiener-type indices.

- $W_g(G)$ is the Wiener index [2] when $g(x) = x$, denoted by $W(G)$.
- $W_g(G)$ is the modified Wiener index [3] when $g(x) = x^\lambda$ for $\lambda \neq 0$, denoted by $W_\lambda(G)$.
- $W_g(G)$ is the hyper-Wiener index [4] when $g(x) = \frac{1}{2}(x + x^2)$, denoted by $WW(G)$.
- $W_g(G)$ is the Harary index [5] when $g(x) = \frac{1}{x}$, denoted by $H(G)$.

The above Wiener-type indices are important distance-based indices with significant applications in various scientific disciplines, particularly in chemistry. They are widely used topological indices in chemical graph theory. The Wiener index is closely linked to the molecular branching and structural properties of chemical compounds. The modified Wiener index refines the original Wiener index by assigning weights or emphasizing specific graph characteristics, making it more suitable for analyzing complex molecular systems. The hyper-Wiener index incorporates both the distances and their squared values, offering a more sensitive measure for highly branched or cyclic structures. The Harary index focuses on the reciprocal of the shortest distances, capturing the overall connectivity and robustness of a molecular structure. These indices play a crucial role in modeling and predicting physical, chemical and biological properties of molecules. For more advances on Wiener-type indices and the latest work on related topics, we refer readers to [6–10].

The study of finding some sufficient condition for a graph to have a particular property has attracted the attention of many researchers. The Zagreb index is a graph-theoretical index used in mathematical chemistry to study molecular structures. It quantifies the connectivity of atoms in chemical molecules, which provides insights into molecular stability and reactivity. In [11], Liu et al. gave some sufficient conditions on the connective eccentricity index, difference of Zagreb indices and some other indices for a graph $G \in \mathcal{G}^*(n)$ to be k-leaf-connected. Liu et al. [12] obtained some sufficient conditions, in terms of the difference of Zagreb indices, for graphs to be traceable, Hamilton-connected, k-edge-Hamiltonion, k-Hamiltonion, k-path-coverable and k-connected. Others see [13–17] and the references within.

Determining whether a graph has the Hamiltonian property is a famous problem. In [18], Kuang et al. used the Wiener-type invariants to show some sufficient conditions for Hamiltonian property of graphs. It is worth considering the use of the Wiener-type indices to derive other properties. Motivated by this, we consider the pancyclic property of graphs. The pancyclic property of a graph has a wide range of applications in combinatorial optimization, network design, theoretical research and algorithm design. It provides a

useful tool for understanding the connectivity of graphs and solving problems related to cycle structures.

In this paper, we first determine the extreme values of $W_g(G)$ for ℓ-connected graphs. Then, we use the Wiener-type indices of graph G, the Wiener-type indices of complement graph $\overline{G}$ with $\delta(G) \geq 2$ or $\delta(G) \geq 3$ to give some sufficient conditions for a graph $G \in \mathcal{G}^*(n)$ to have the pancyclic property. Our results generalize some existing results of several papers.

The outline of this paper is given below.

- In Section 2, the necessary preliminaries are introduced.
- In Section 3, we show the main results, including some extremal conclusions on ℓ-connected graphs with respect to Wiener-type indices and some conditions that ensure graphs to have the pancyclic property with respect to Wiener-type indices.
- In Section 4, we summarize the main findings of this paper, and pose a problem for future work.

2. Preliminaries

For convenience, we use $e(G)$ to represent $|E(G)|$. Clearly, if $st \in E(G)$, then $d_G(s,t) = 1$. Moreover, if $1 \leq d_G(s,t) \leq 2$ for any $s,t \in V(G)$, then the count of vertex pairs that are separated by a distance of 2 is equivalent to $\frac{n(n-1)}{2}$ minus the count of vertex pairs that are directly connected (i.e., have a distance of 1), where $n = |V(G)|$.

Lemma 1. *Let $G \in \mathcal{G}^*(n)$ with diameter $D(G)$. Then, $W_g(G) = \binom{n}{2}g(2) - (g(2) - g(1))e(G)$ iff $D(G) \leq 2$.*

Proof. Let $y = \binom{n}{2}g(2) - (g(2) - g(1))e(G)$. If $W_g(G) = y$, then $d_G(s,t) \in \{1,2\}$ for two vertices $s,t \in V(G)$. Thus, $D(G) \leq 2$. Conversely, if $D(G) \leq 2$, then $d_G(s,t) \in \{1,2\}$ for $s,t \in V(G)$. Thus, $W_g(G) = (\binom{n}{2} - e(G))g(2) + e(G)g(1) = y$. □

Lemma 2 ([19]). *If $G \in \mathcal{G}^*(n)$ with $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$, then the diameter $D(G) \leq 2$.*

Combining Lemmas 1 and 2, the Wiener-type index of a graph $G \in \mathcal{G}^*(n)$ is determined by $g(1)$ and $g(2)$ if $\delta(G)$ is greater than $\lfloor \frac{n}{2} \rfloor$.

Lemma 3 ([19]). *Let $\ell, n \in \mathbb{Z}^+$, $n - 1 \geq \ell \geq 2$, the graph $G \in \mathcal{G}^*(n,\ell)$. Then, $e(G) \geq \varphi(n,\ell)$, where*

$$\varphi(n,\ell) = \begin{cases} \frac{n\ell+1}{2}, & \text{if both } n \text{ and } \ell \text{ are odd;} \\ \frac{n\ell}{2}, & \text{otherwise.} \end{cases}$$

Let $\ell \in \mathbb{Z}^+$, G^ℓ be the ℓ-th power of G, which is the graph with $V(G^\ell) = V(G)$ and $uv \in E(G^\ell)$ iff $d_G(u,v) \leq \ell$. It is clear that $d_{G^\ell}(u,v) = \lceil \frac{d_G(u,v)}{\ell} \rceil$. The following lemma represents this fact.

Lemma 4 ([19]). *Let $G \in \mathcal{G}^*(n)$ and $\ell \in \mathbb{Z}^+$. Then, $D(G^\ell) = \lceil \frac{D(G)}{\ell} \rceil$.*

For these graphs in $\mathcal{G}^*(n,\ell)$, the following lemma provides an upper bound for their diameter:

Lemma 5 ([19]). *Let $G \in \mathcal{G}^*(n,\ell)$. Then, the diameter $D(G) \leq \lceil \frac{n-1}{\ell} \rceil$.*

For connected graphs with minimum degree greater than 2 or 3, the following two lemmas show that, as long as their number of edges is sufficiently large, these graphs have

pancyclic property, except for bipartite graphs and those in the set $\mathbb{NP}$ or the set $\mathbb{NP}_1$ (the sets $\mathbb{NP}$ and $\mathbb{NP}_1$ will be introduced in next section).

Lemma 6 ([20]). *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ with $e(G)$ edges and $\delta(G) \geq 2$. If $e(G) \geq \binom{n-2}{2} + 4$, then G has pancyclic property unless $G \in \mathcal{BG}^*(n)$ or $G \in \mathbb{NP}$.*

Lemma 7 ([21]). *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ with $e(G)$ edges and $\delta(G) \geq 3$. If $e(G) \geq \binom{n-3}{2} + 9$, then G has pancyclic property unless $G \in \mathcal{BG}^*(n)$ or $G \in \mathbb{NP}_1$.*

Let $D(s) = \sum_{t \in V(G)\setminus\{s\}} g(d(s,t))$, where $g(x)$ is a function. Double counting is a combinatorial technique used to count the same set of objects in two different ways for deriving certain results. This technique is widely used in graph theory. The following lemma can be directly obtained by using double counting:

Lemma 8. *Let $G \in \mathcal{G}^*(n,\ell)$, $s \in V(G)$, $D(s) = \sum_{t \in V(G)\setminus\{s\}} g(d(s,t))$. Then, $W_g(G) = \frac{1}{2}\sum_{j=1}^{n} D(s_j)$, where $V(G) = \{s_1, s_2, \ldots, s_n\}$.*

Proof. By the fact that $\sum_{\{s,t\}\subseteq V(G)} g(d_G(s,t)) = \frac{1}{2}\sum_{j=1}^{n} D(s_j)$, we have $W_g(G) = \frac{1}{2}\sum_{j=1}^{n} D(s_j)$. □

3. Main Results

3.1. Results on ℓ-Connected Graphs

In this subsection, we show the results on ℓ-connected graphs.

For these graphs in $\mathcal{G}^*(n)$ with the connectivity ℓ, the following theorem provides a lower bound for their Wiener-type indices with a monotone increasing function and identifies the extremal graph where the equality holds.

Theorem 1. *Let $\ell \in \mathbb{Z}^+$, $1 \leq \ell \leq n-2$ and $G \in \mathcal{G}^*(n)$ with connectivity ℓ. If $g(x)$ has monotonic increasing property, then*

$$W_g(G) \geq (n-\ell-1)g(2) + \left(\ell(n-\ell) + \binom{\ell}{2} + \binom{n-\ell-1}{2}\right)g(1),$$

with equality iff $G \cong K_\ell \vee (K_1 + K_{n-\ell-1})$.

Proof. Let $G^* \in \mathcal{G}^*(n)$ with connectivity ℓ and minimum Wiener-type indices. Then, there is a vertex cut Y of G^* with $|Y| = \ell$. Let $G_1, G_2, \cdots, G_t$ be the components of the graph $G^* - Y$.

Claim 1. For $1 \leq i \leq t$, G_i is a complete graph.

Suppose that there exists i_0 such that G_{i_0} is not a complete graph and $uv \notin E(G_{i_0})$. Let $G' = G^* + uv$. Then $G' \in \mathcal{G}^*(n)$ with connectivity ℓ. Since $g(x)$ is strictly increasing, then $W_g(G') < W_g(G^*)$, which is a contradiction.

By a similar analysis, we have

Claim 2. $G[Y]$ is a complete graph.

Claim 3. $t = 2$.

Suppose that $t \geq 3$. Let $G'' = G^* + uv$, where $u \in V(G_i)$, $v \in V(G_j)$ and $1 \leq i < j \leq t$. Then, $G'' \in \mathcal{G}^*(n)$ with connectivity ℓ. Since $g(x)$ is strictly increasing, then $W_g(G'') < W_g(G^*)$, which is a contradiction.

Combining Claims 1–3, we have that $G^* \cong K_\ell \vee (K_{n_1} + K_{n_2})$, where $n_1 + n_2 + \ell = n$.

$$\begin{aligned} W_g(G^*) &= \binom{n_1}{2}g(1) + \binom{n_2}{2}g(1) + \binom{\ell}{2}g(1) + \ell(n_1+n_2)g(1) + n_1n_2g(2) \\ &= \frac{1}{2}n^2g(1) - \frac{1}{2}ng(1) + n_1n_2(g(2)-g(1)). \end{aligned}$$

When $n_1 = 1$ or $n_2 = 1$, we obtain the minimum value of $W_g(G^*)$, i.e., $G^* \cong K_\ell \vee (K_1 + K_{n-\ell-1})$ and $W_g(G^*) = \frac{1}{2}n^2g(1) - \frac{1}{2}ng(1) + (n-\ell-1)(g(2)-g(1)) = (n-\ell-1)g(2) + \Big(\ell(n-\ell) + \binom{\ell}{2} + \binom{n-\ell-1}{2}\Big)g(1)$.

By the above arguments, the conclusion holds. □

In Theorem 1, let $g(x) \in \{x, x^\lambda, \frac{1}{2}(x^2+x)\}$ (where $\lambda > 0$). Then, we can derive the minimal values of these Wiener-type indices for ℓ-connected graphs (it should be noted that $g(x)$ is a monotone increasing function when $x > 0$). The results are shown below.

When $g(x) = x$, we have $W_g(G) = W(G)$. The corresponding result on the Wiener index is as follows:

Corollary 1 ([22]). *Let $\ell \in \mathbb{Z}^+$, $1 \le \ell \le n-2$, $y = n-\ell-1$ and $G \in \mathcal{G}^*(n)$ with connectivity ℓ. Then,*

$$W(G) \ge 2y + \ell(y+1) + \binom{\ell}{2} + \binom{y}{2},$$

the equality holds iff $G \cong K_\ell \vee (K_1 + K_{n-\ell-1})$.

When $g(x) = x^\lambda$, we have $W_g(G) = W_\lambda(G)$. The corresponding result on the modified Wiener index is as follows:

Corollary 2. *Let $\lambda > 0$, $\ell \in \mathbb{Z}^+$, $1 \le \ell \le n-2$, $y = n-\ell-1$ and $G \in \mathcal{G}^*(n)$ with connectivity ℓ. Then,*

$$W_\lambda(G) \ge 2^\lambda y + \ell(y+1) + \binom{\ell}{2} + \binom{y}{2},$$

the equality holds iff $G \cong K_\ell \vee (K_1 + K_{n-\ell-1})$.

When $g(x) = \frac{1}{2}(x^2+x)$, we have $W_g(G) = WW(G)$. The corresponding result on the hyper-Wiener index is as follows:

Corollary 3. *Let $\ell \in \mathbb{Z}^+$, $1 \le \ell \le n-2$, $y = n-\ell-1$ and $G \in \mathcal{G}^*(n)$ with connectivity ℓ. Then,*

$$WW(G) \ge 3y + \ell(y+1) + \binom{\ell}{2} + \binom{y}{2},$$

the equality holds iff $G \cong K_\ell \vee (K_1 + K_{n-\ell-1})$.

For graphs in $\mathcal{G}^*(n,\ell)$, the following theorem provides an upper bound for their Wiener-type indices with a monotone increasing function and gives some graphs that make the equality hold.

Theorem 2. *Let $\ell \in \mathbb{Z}^+$, $1 \le \ell \le n-1$ and $G \in \mathcal{G}^*(n,\ell)$. When $g(x)$ is a monotone increasing function, then*

$$W_g(G) \le \frac{n}{2}\left(\sum_{i=1}^{\lfloor \frac{n-1}{\ell} \rfloor} \ell g(i) + (n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor)g(\lceil\frac{n-1}{\ell}\rceil)\right).$$

The bound is the best possible when $\ell \equiv 0(mod\ 2)$, *and*

$$W_g((C_n)^{\frac{\ell}{2}}) = \frac{n}{2}\left(\sum_{i=1}^{\lfloor\frac{n-1}{\ell}\rfloor} \ell g(i) + (n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor)g(\lceil\frac{n-1}{\ell}\rceil)\right).$$

Proof. Let $u \in V(G)$ and $V_i(u) = \{v|v \in V(G), d(u,v) = i\}$. Since $G \in \mathcal{G}^*(n,\ell)$, by Lemma 5, diameter $D(G) \leq \lceil\frac{n-1}{\ell}\rceil$. Since $G \in \mathcal{G}^*(n,\ell)$, then $|V_i(u)| \geq \ell$ for $1 \leq i \leq ecc(v) - 1$. Since $g(x)$ is strictly increasing, we have $D(u) = \sum_{i=1}^{ecc(u)} |V_i(u)|g(i) \leq \sum_{i=1}^{\lfloor\frac{n-1}{\ell}\rfloor} \ell g(i) + (n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor)g(\lceil\frac{n-1}{\ell}\rceil)$. Therefore, by Lemma 8, $W_g(G) \leq \frac{n}{2}\left(\sum_{i=1}^{\lfloor\frac{n-1}{\ell}\rfloor} \ell g(i) + (n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor)g(\lceil\frac{n-1}{\ell}\rceil)\right)$.

Now, we show that, when $\ell \equiv 0(mod\ 2)$, the bound can be obtained by giving an example. It is obvious that

$$D(C_n) = \begin{cases} \frac{n-1}{2}, & if\ n\ is\ odd; \\ \frac{n}{2}, & if\ n\ is\ even. \end{cases}$$

Then, by Lemma 4, when $n \equiv 1(mod\ 2)$, we have $D((C_n)^{\frac{\ell}{2}}) = \lceil\frac{n-1}{\ell}\rceil$; when $n \equiv 0(mod\ 2)$, since ℓ is even, we have $D((C_n)^{\frac{\ell}{2}}) = \lceil\frac{n}{\ell}\rceil = \lceil\frac{n-1}{\ell}\rceil$.

Let $V(C_n) = V = \{u_0, u_1, u_2, \cdots, u_{n-1}\}$ and $E(C_n) = \{u_0u_1, u_1u_2, \cdots, u_{n-2}u_{n-1}, u_{n-1}u_0\}$. Let $V_i^*(u_j) = \{v|v \in V, d(u_j,v) = i\ in\ (C_n)^{\frac{\ell}{2}}\}$ for $1 \leq i \leq \lceil\frac{n-1}{\ell}\rceil$. Then, we have $V_i^*(u_0) = \{u_{\frac{\ell}{2}(i-1)+1}, u_{\frac{\ell}{2}(i-1)+2}, \cdots, u_{\frac{\ell}{2}i}, u_{n-(\frac{\ell}{2}(i-1)+1)}, u_{n-(\frac{\ell}{2}(i-1)+2)}, \cdots, u_{n-\frac{\ell}{2}i}\}$ for $1 \leq i \leq \lfloor\frac{n-1}{\ell}\rfloor$. Therefore, $|V_i^*(u_j)| = |V_i^*(u_0)| = \ell$ for $1 \leq i \leq \lfloor\frac{n-1}{\ell}\rfloor$, $(C_n)^{\frac{\ell}{2}} \in \mathcal{G}^*(n,\ell)$, and $|V^*_{\lceil\frac{n-1}{\ell}\rceil}(u_j)| = n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor$ if $n-1 \not\equiv 0(mod\ \ell)$, $|V^*_{\lceil\frac{n-1}{\ell}\rceil}(u_j)| = \ell$ if $n-1 \equiv 0(mod\ \ell)$. Then, $W_g((C_n)^{\frac{\ell}{2}}) = \frac{n}{2}\left(\sum_{i=1}^{\lfloor\frac{n-1}{\ell}\rfloor} \ell g(i) + (n-1-\ell\lfloor\frac{n-1}{\ell}\rfloor)g(\lceil\frac{n-1}{\ell}\rceil)\right)$.

This concludes the proof. □

In Theorem 2, let $g(x) \in \{x, x^\lambda, \frac{1}{2}(x^2+x)\}$ (where $\lambda > 0$). Then, we can derive the maximal values of these Wiener-type indices for ℓ-connected graphs. The results are shown below.

When $g(x) = x$, we have $W_g(G) = W(G)$. The corresponding result on the Wiener index is as follows:

Corollary 4 ([19]). *Let* $\ell \in \mathbb{Z}^+$, $1 \leq \ell \leq n-1$, $y = \frac{n-1}{\ell}$ *and* $G \in \mathcal{G}^*(n,\ell)$. *Then,*

$$W(G) \leq \frac{n}{2}\left(\sum_{i=1}^{\lfloor y\rfloor} i\ell + (n-1-\ell\lfloor y\rfloor)\lceil y\rceil\right).$$

The bound is the best possible when $\ell \equiv 0(mod\ 2)$, *and*

$$W((C_n)^{\frac{\ell}{2}}) = \frac{n}{2}\left(\sum_{i=1}^{\lfloor y\rfloor} i\ell + (n-1-\ell\lfloor y\rfloor)\lceil y\rceil\right).$$

When $g(x) = x^\lambda$, we have $W_g(G) = W_\lambda(G)$. The corresponding result on the modified Wiener index is as follows:

Corollary 5. *Let $\lambda > 0$, $\ell \in \mathbb{Z}^+$, $1 \leq \ell \leq n-1$, $y = \frac{n-1}{\ell}$ and $G \in \mathcal{G}^*(n,\ell)$. Then,*

$$W_\lambda(G) \leq \frac{n}{2}\left(\sum_{i=1}^{\lfloor y \rfloor} \ell i^\lambda + (n-1-\ell\lfloor y \rfloor)(\lceil y \rceil)^\lambda\right).$$

The bound is the best possible when $\ell \equiv 0 (mod\ 2)$, and

$$W_\lambda((C_n)^{\frac{\ell}{2}}) = \frac{n}{2}\left(\sum_{i=1}^{\lfloor y \rfloor} \ell i^\lambda + (n-1-\ell\lfloor y \rfloor)(\lceil y \rceil)^\lambda\right).$$

When $g(x) = \frac{1}{2}(x^2+x)$, we have $W_g(G) = WW(G)$. The corresponding result on the hyper-Wiener index is as follows:

Corollary 6. *Let $\ell \in \mathbb{Z}^+$, $1 \leq \ell \leq n-1$, $y = \frac{n-1}{\ell}$ and $G \in \mathcal{G}^*(n,\ell)$. Then,*

$$WW(G) \leq \frac{n}{4}\left(\sum_{i=1}^{\lfloor y \rfloor} \ell(i^2+i) + (n-1-\ell\lfloor y \rfloor)\Big((\lceil y \rceil)^2 + \lceil y \rceil\Big)\right).$$

The bound is the best possible when $\ell \equiv 0 (mod\ 2)$, and

$$WW((C_n)^{\frac{\ell}{2}}) = \frac{n}{4}\left(\sum_{i=1}^{\lfloor y \rfloor} \ell(i^2+i) + (n-1-\ell\lfloor y \rfloor)\Big((\lceil y \rceil)^2 + \lceil y \rceil\Big)\right).$$

Theorem 3. *Let $\ell, n \in \mathbb{Z}^+$ such that $\ell \geq \lfloor \frac{n}{2} \rfloor$, $G \in \mathcal{G}^*(n,\ell)$. When $g(x)$ is a monotone increasing function, then*

$$W_g(G) \leq \binom{n}{2} g(2) - (g(2)-g(1))\varphi(n,\ell),$$

where

$$\varphi(n,\ell) = \begin{cases} \frac{n\ell+1}{2}, & if\ both\ n\ and\ \ell\ are\ odd; \\ \frac{n\ell}{2}, & otherwise. \end{cases}$$

Proof. Let $G \in \mathcal{G}^*(n,\ell)$. Then, it is well known that $\delta(G) \geq \ell$. Since $\ell \geq \lfloor \frac{n}{2} \rfloor$, then $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$. By Lemma 2, we have $D(G) \leq 2$. By Lemma 1, $W_g(G) = \binom{n}{2} g(2) - (g(2) - g(1))e(G)$. By Lemma 3 and the monotonicity of $g(x)$, we have $W_g(G) \leq \binom{n}{2} g(2) - (g(2) - g(1))\varphi(n,\ell)$, where

$$\varphi(n,\ell) = \begin{cases} \frac{n\ell+1}{2}, & if\ both\ n\ and\ \ell\ are\ odd; \\ \frac{n\ell}{2}, & otherwise. \end{cases}$$

Based on the above arguments, the conclusion holds. □

For graphs in $\mathcal{G}^*(n,\ell)$ where ℓ and n satisfy that n is even, $n \geq 6$ and $\ell = \frac{n}{2} - 1$, the following theorem provides an upper bound for their Wiener-type indices with a monotone increasing function and gives some graphs that make the equality hold.

Theorem 4. *Let $\ell, n \in \mathbb{Z}^+$, $n \equiv 0 (mod\ 2)$, $n \geq 6$, $\ell = \frac{n}{2} - 1$ and $G \in \mathcal{G}^*(n,\ell)$. If $g(x)$ has monotonic increasing property, then*

$$W_g(G) \leq \frac{n}{2}\Big((\frac{n}{2}-1)(g(1)+g(2)) + g(3)\Big),$$

the bound is obtained if $G \cong K_{\frac{n}{2},\frac{n}{2}} \setminus H^*$, where H^* is a perfect matching in the graph $K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Let $v \in V(G)$ and $V_i(v) = \{u | u \in V(G), d(v,u) = i\}$. Since $\ell = \frac{n}{2} - 1$, then $D(G) \leq \lceil \frac{n-1}{\frac{n}{2}-1} \rceil \leq 3$ by Lemma 5. By $G \in \mathcal{G}^*(n,\ell)$ and the monotonicity of $g(x)$, we have $D(v) = \sum_{i=1}^{ecc(v)} |V_i(v)| g(i) \leq (\frac{n}{2} - 1)(g(1) + g(2)) + 1 \times g(3)$. Thus, we have $W_g(G) \leq \frac{n}{2}\big((\frac{n}{2} - 1)(g(1) + g(2)) + g(3)\big)$. If $G \cong K_{\frac{n}{2},\frac{n}{2}} \setminus H^*$, then the equality holds. □

3.2. Results on Pancyclic Graphs

In this subsection, we show the results on pancyclic graphs.

For graphs in $\mathcal{BG}^*(n)$, the following theorem provides a lower bound (or an upper bound) for their Wiener-type indices with a monotone increasing (or decreasing) function and identifies the extremal graph where the equality holds.

Lemma 9. *Let $G \in \mathcal{BG}^*(n)$.*

(i) *When $g(x)$ has monotonic increasing property, then*

$$W_g(G) \geq \frac{1}{4} n((g(1) + g(2))n - 2g(2)),$$

with equality iff $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

(ii) *When $g(x)$ has monotonic decreasing property, then*

$$W_g(G) \leq \frac{1}{4} n((g(1) + g(2))n - 2g(2)),$$

with equality iff $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Let $G \in \mathcal{BG}^*(n)$ have two disjoint independent sets, denoted by S_1 and S_2 such that $S_1, S_2 \subseteq V(G)$, $|S_1| = a$ and $|S_2| = n - a$ $(a \leq \lfloor \frac{n}{2} \rfloor)$. If $g(x)$ is strictly increasing, then we have

$$\begin{aligned}
W_g(G) &\geq \frac{1}{2}\left\{\sum_{v_i \in V_1} (d_i g(1) + (n - a - d_i)g(3) + (a-1)g(2))\right\} \\
&\quad + \frac{1}{2}\left\{\sum_{v_i \in V_2} (d_i g(1) + (a - d_i)g(3) + (n-a-1)g(2))\right\} \\
&= \frac{1}{2}\left\{\sum_{v_i \in V_1} ((n-a)g(3) + (a-1)g(2) + (g(1) - g(3))d_i)\right\} \\
&\quad + \frac{1}{2}\left\{\sum_{v_i \in V_2} (ag(3) + (n-a-1)g(2) + (g(1) - g(3))d_i)\right\} \\
&\geq \frac{1}{2}\{[(n-a)g(3) + (a-1)g(2)]a + [ag(3) + (n-a-1)g(2)](n-a)\} \\
&\quad + (g(1) - g(3))a(n-a) \\
&= (g(2) - g(1))a^2 + (g(1) - g(2))na + \frac{1}{2}g(2)n^2 - \frac{1}{2}g(2)n \\
&\geq \frac{1}{4}(g(2) - g(1))n^2 + \frac{1}{2}(g(1) - g(2))n^2 + \frac{1}{2}g(2)n^2 - \frac{1}{2}g(2)n \\
&= \frac{1}{4} n((g(1) + g(2))n - 2g(2)),
\end{aligned}$$

with equality iff $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

The proof for the case where $g(x)$ is strictly decreasing is similar, and thus it is omitted here. This completes the proof. □

Let $\mathbb{NP} = \{K_3 \vee 4K_1, K_2 \vee (K_{1,3} + K_1), K_{1,2} \vee 4K_1, K_4 \vee 5K_1, (K_2 \vee 2K_1) \vee 5K_1, K_3 \vee (K_{1,3} + K_2), K_3 \vee (K_{1,4} + K_1), K_3 \vee (K_2 + 3K_1), K_5 \vee 6K_1, K_2 \vee (K_{n-4} + 2K_1)\}$ [20].

Yu et al. showed Lemma 6 in [20]. Based on Lemmas 6 and 9, we show the following results.

Theorem 5. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$.*

(i) *If $g(x)$ has monotonic increasing property, and*

$$W_g(G) \leq \binom{n}{2} g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1)),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

(ii) *If $g(x)$ has monotonic decreasing property, and*

$$W_g(G) \geq \binom{n}{2} g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1)),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

Proof. Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If $g(x)$ is strictly increasing, then

$$\begin{aligned} W_g(G) &\geq \frac{1}{2} \sum_{v_i \in V(G)} (d_i g(1) + (n-1-d_i) g(2)) \\ &= \frac{1}{2} n(n-1) g(2) - \frac{1}{2}(g(2) - g(1)) \sum_{v_i \in V(G)} d_i \\ &= \binom{n}{2} g(2) - (g(2) - g(1)) e(G). \end{aligned}$$

Since $W_g(G) \leq \binom{n}{2} g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$, then $e(G) \geq \binom{n-2}{2} + 4$. By Lemma 6, G is pancyclic unless $G \in \mathcal{BG}^*(n)$ or G is a graph in the set $\mathbb{NP}$.

If G is a pancyclic graph, the conclusion holds. Otherwise, we have $G \in \mathbb{NP}$ or $G \in \mathcal{BG}^*(n)$. By Lemma 9, we have $W_g(G) \geq \frac{1}{4} n((g(1) + g(2))n - 2g(2))$ if G is a bipartite graph. Since $\frac{1}{4} n((g(1) + g(2))n - 2g(2)) > \binom{n}{2} g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$, then G is not a bipartite graph. For all $G \in \mathbb{NP}$, we can easily verify that $W_g(G) \leq \binom{n}{2} g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$.

The proof for the case where $g(x)$ is strictly decreasing is similar, and thus it is omitted here.

By the above arguments, we have completed the proof. □

In Theorem 5, let $g(x) \in \{x, x^\lambda, \frac{1}{2}(x^2 + x), \frac{1}{x}\}$ (where $\lambda \neq 0$). Then, we can derive some conditions that ensure graphs to have the pancyclic property with respect to these Wiener-type indices. The results are shown below.

When $g(x) = x$, we have $W_g(G) = W(G)$. The corresponding result on the Wiener index is as follows.

Corollary 7 ([23]). *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If*

$$W(G) \leq 2\binom{n}{2} - \binom{n-2}{2} - 4,$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

When $g(x) = x^\lambda$, we have $W_g(G) = W_\lambda(G)$. The corresponding result on the modified Wiener index is as follows.

Corollary 8. *Let $\lambda \neq 0$, $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$.*
(i) *If $\lambda > 0$ and*

$$W_\lambda(G) \leq 2^\lambda \binom{n}{2} - (2^\lambda - 1)\left(\binom{n-2}{2} + 4\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.
(ii) *If $\lambda < 0$ and*

$$W_\lambda(G) \geq 2^\lambda \binom{n}{2} - (2^\lambda - 1)\left(\binom{n-2}{2} + 4\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

When $g(x) = \frac{1}{2}(x^2 + x)$, we have $W_g(G) = WW(G)$. The corresponding result on the hyper-Wiener index is as follows.

Corollary 9. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If*

$$WW(G) \leq 3\binom{n}{2} - 2\binom{n-2}{2} - 8,$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

When $g(x) = \frac{1}{x}$, we have $W_g(G) = H(G)$. The corresponding result on the Harary index is as follows.

Corollary 10 ([23])**.** *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If*

$$H(G) \geq \frac{1}{2}\binom{n}{2} + \frac{1}{2}\binom{n-2}{2} + 2,$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}$.

For convenience, we say G is an NCB-graph if G is a non-complete bipartite graph.

Theorem 6. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$.*
(i) *If $g(x)$ has monotonic increasing property, $\overline{G} \in \mathcal{G}^*(n)$ and*

$$W_g(\overline{G}) \geq \binom{n}{2} g(1) + \left(\binom{n-2}{2} + 4\right)(g(n-1) - g(1)),$$

then the graph G has pancyclic property if it is not an NCB-graph.
(ii) *If $g(x)$ has monotonic decreasing property, $\overline{G} \in \mathcal{G}^*(n)$ and*

$$W_g(\overline{G}) \leq \binom{n}{2} g(1) + \left(\binom{n-2}{2} + 4\right)(g(n-1) - g(1)),$$

then the graph G has pancyclic property if it is not an NCB-graph.

Proof. For convenience, let $d_i = d_G(v_i)$ and $\overline{d_i} = d_{\overline{G}}(v_i)$. Then, $\overline{d_i} = n - 1 - d_i$.

Let $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If $g(x)$ has monotonic increasing property, then

$$\begin{aligned} W_g(\overline{G}) &\leq \frac{1}{2}\sum_{v_i \in V(G)} (\overline{d_i}g(1) + (n-1-\overline{d_i})g(n-1)) \\ &= \frac{1}{2}n(n-1)g(n-1) - \frac{1}{2}(g(n-1)-g(1))\sum_{v_i \in V(G)} \overline{d_i} \\ &= \frac{1}{2}n(n-1)g(n-1) - \frac{1}{2}(g(n-1)-g(1))\sum_{v_i \in V(G)} (n-1-d_i) \\ &= \binom{n}{2}g(1) + (g(n-1)-g(1))e(G). \end{aligned}$$

Since $W_g(\overline{G}) \geq \binom{n}{2}g(1) + \left(\binom{n-2}{2}+4\right)(g(n-1)-g(1))$, then $e(G) \geq \binom{n-2}{2}+4$. By Lemma 6, G has the pancyclic property unless it is in the set $\mathbb{NP}$ or the set $\mathcal{BG}^*(n)$.

If G is a pancyclic graph, the conclusion holds. Otherwise, we have $G \in \mathbb{NP}$ or $G \in \mathcal{BG}^*(n)$. If $G \in \mathbb{NP}$ or G is a complete bipartite graph, then $\overline{G} \notin \mathcal{G}^*(n)$. Hence, G is an NCB-graph.

The proof for the case where $g(x)$ is strictly decreasing is similar, and thus it is omitted here.

The proof is complete by the above arguments. □

In Theorem 6, let $g(x) \in \{x, x^\lambda, \frac{1}{2}(x^2+x), \frac{1}{x}\}$ (where $\lambda \neq 0$). Then, we can derive some conditions that ensure graphs to have the pancyclic property with respect to these Wiener-type indices of complement graphs. The results are shown below.

When $g(x) = x$, we have $W_g(\overline{G}) = W(\overline{G})$. The corresponding result on the Wiener index is as follows.

Corollary 11 ([23]). *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If $\overline{G} \in \mathcal{G}^*(n)$ and*

$$W(\overline{G}) \geq \binom{n}{2} + \left(\binom{n-2}{2}+4\right)(n-2),$$

then the graph G has pancyclic property if it is not an NCB-graph.

When $g(x) = x^\lambda$, we have $W_g(\overline{G}) = W_\lambda(\overline{G})$. The corresponding result on the modified Wiener index is as follows.

Corollary 12. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$.*

(i) *If $\lambda > 0$, $\overline{G} \in \mathcal{G}^*(n)$ and*

$$W_\lambda(\overline{G}) \geq \binom{n}{2} + ((n-1)^\lambda - 1)\left(\binom{n-2}{2}+4\right),$$

then the graph G has pancyclic property if it is not an NCB-graph.

(ii) *If $\lambda < 0$, $\overline{G} \in \mathcal{G}^*(n)$ and*

$$W_\lambda(\overline{G}) \leq \binom{n}{2} + ((n-1)^\lambda - 1)\left(\binom{n-2}{2}+4\right),$$

then the graph G has pancyclic property if it is not an NCB-graph.

When $g(x) = \frac{1}{2}(x^2+x)$, we have $W_g(\overline{G}) = WW(\overline{G})$. The corresponding result on the hyper-Wiener index is as follows.

Corollary 13. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If $\overline{G} \in \mathcal{G}^*(n)$ and*

$$WW(\overline{G}) \geq \binom{n}{2} + \left(\binom{n}{2} - 1\right)\left(\binom{n-2}{2} + 4\right),$$

then the graph G has pancyclic property if it is not an NCB-graph.

When $g(x) = \frac{1}{x}$, we have $W_g(\overline{G}) = H(\overline{G})$. The corresponding result on the Harary index is as follows.

Corollary 14 ([23])**.** *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 2$. If $\overline{G} \in \mathcal{G}^*(n)$ and*

$$H(\overline{G}) \leq \binom{n}{2} - \frac{n-2}{n-1}\left(\binom{n-2}{2} + 4\right),$$

then the graph G has pancyclic property if it is not an NCB-graph.

Let $\mathbb{NP}_1 = \{(K_2 + K_{1,5}) \vee K_5, (2K_1 \vee K_4) \vee 7K_1, (K_1 + K_2 + K_{1,4}) \vee K_5, (2K_1 + K_{1,5}) \vee K_5, (K_1 + K_2 \vee 6K_1) \vee K_4, (K_{1,5} + K_2) \vee 2K_1 \vee K_3, 7K_1 \vee K_6, K_5 \vee (2K_1 + K_{1,3} + K_2), K_4 \vee [K_1 + K_1 \vee (K_{1,4} + K_2)], K_4 \vee (K_1 + 2K_1 \vee 6K_1), K_3 \vee K_{3,7}, K_5 \vee (3K_1 + K_{1,4}), K_3 \vee 2K_1 \vee (K_1 + K_{1,4} + K_2), K_4 \vee [K_1 \vee (K_1 + K_{1,5}) + K_1], K_3 \vee 2K_1 \vee (2K_1 + K_{1,5}), (5K_1 + K_2) \vee K_5, K_4 \vee [K_1 \vee (4K_1 + K_2) + K_1], 7K_1 \vee K_5, (5K_1 + K_2) \vee 2K_1 \vee K_3, 6K_1 \vee K_5, K_4 \vee (K_{1,4} + K_2), K_4 \vee (K_1 + K_{1,5}), K_3 \vee K_{2,6}, K_4 \vee (K_1 + K_{1,3} + K_2), K_4 \vee (2K_1 + K_{1,4}), K_3 \vee [(K_2 \vee 5K_1) + K_1], K_2 \vee 2K_1 \vee (K_{1,4} + K_2), K_4 \vee (K_2 + K_{1,2} + 2K_1), K_3 \vee (K_1 + K_{2,5}), K_2 \vee K_{3,6}, K_4 \vee (K_{1,3} + 3K_1), K_2 \vee 2K_1 \vee (K_1 + K_{1,3} + K_2), [K_1 \vee (K_{1,4} + K_1) + K_1] \vee K_3, K_2 \vee 2K_1 \vee (2K_1 + K_{1,4}), (4K_1 + K_2) \vee K_4, K_3 \vee [K_1 + K_1 \vee (3K_1 + K_2)], 6K_1 \vee K_4, (4K_1 + K_2) \vee 2K_1 \vee K_2, 5K_1 \vee K_4, (K_{1,3} + K_2) \vee K_3, (K_1 + K_{1,4}) \vee K_3, 5K_1 \vee 2K_1 \vee K_2, (K_1 + K_{1,2} + K_2) \vee K_3, (2K_1 + K_{1,3}) \vee K_3, (K_{1,3} + K_2) \vee 2K_1 \vee K_1, (3K_1 + K_{n-6}) \vee K_3, (4K_1 + K_3) \vee K_4, (6K_1 + K_2) \vee K_6, 9K_1 \vee K_8, 8K_1 \vee K_7, (K_1 + K_{1,7}) \vee K_6, K_{2,8} \vee K_5, K_6 \vee (2K_1 + K_{1,6}), K_4 \vee 2K_1 \vee (K_2 + K_{1,6}), K_5 \vee (K_1 + K_{1,6})\}$ [21].

Xu et al. showed Lemma 7 in [21]. Based on Lemmas 7 and 9, we show the following results.

Theorem 7. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$.*

(i) *If $g(x)$ has monotonic increasing property and*

$$W_g(G) \leq \binom{n}{2} g(2) - \left(\binom{n-3}{2} + 9\right)(g(2) - g(1)),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

(ii) *If $g(x)$ has monotonic decreasing property and*

$$W_g(G) \geq \binom{n}{2} g(2) - \left(\binom{n-3}{2} + 9\right)(g(2) - g(1)),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

Proof. Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$. If $g(x)$ is strictly increasing, then

$$\begin{aligned} W_g(G) &\geq \frac{1}{2} \sum_{v_i \in V(G)} (d_i g(1) + (n-1-d_i) g(2)) \\ &= \frac{1}{2} n(n-1) g(2) - \frac{1}{2}(g(2) - g(1)) \sum_{v_i \in V(G)} d_i \\ &= \binom{n}{2} g(2) - (g(2) - g(1)) e(G). \end{aligned}$$

Since $W_g(G) \leq \binom{n}{2}g(2) - \left(\binom{n-3}{2}+9\right)(g(2)-g(1))$, then $e(G) \geq \binom{n-3}{2}+9$. By Lemma 7, G is pancyclic unless G is a graph in the set $\mathbb{NP}_1$ or $G \in \mathcal{BG}^*(n)$.

If G is a pancyclic graph, the conclusion holds. Otherwise, we have $G \in \mathbb{NP}_1$ or G is a bipartite graph. By Lemma 9, we have $W_g(G) \geq \frac{1}{4}n((g(1)+g(2))n - 2g(2))$ if G is a bipartite graph. Since $\frac{1}{4}n((g(1)+g(2))n-2g(2)) > \binom{n}{2}g(2) - \left(\binom{n-3}{2}+9\right)(g(2)-g(1))$, then G is not a bipartite graph. For all $G \in \mathbb{NP}_1$, we can easily verify that $W_g(G) \leq \binom{n}{2}g(2) - \left(\binom{n-3}{2}+9\right)(g(2)-g(1))$.

The proof for the case where $g(x)$ is strictly decreasing is similar, and thus it is omitted here.

Based on the above arguments, the proof is complete. □

In Theorem 7, let $g(x) \in \{x, x^\lambda, \frac{1}{2}(x^2+x), \frac{1}{x}\}$ (where $\lambda \neq 0$). Then, we can derive some conditions that ensure graphs to have the pancyclic property with respect to these Wiener-type indices. The results are shown below.

When $g(x) = x$, we have $W_g(G) = W(G)$. The corresponding result on the Wiener index is as follows.

Corollary 15. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$. If*

$$W(G) \leq 2\binom{n}{2} - \left(\binom{n-3}{2}+9\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

When $g(x) = x^\lambda$, we have $W_g(G) = W_\lambda(G)$. The corresponding result on the modified Wiener index is as follows.

Corollary 16. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$.*

(i) *If $\lambda > 0$ and*

$$W_\lambda(G) \leq 2^\lambda\binom{n}{2} - (2^\lambda - 1)\left(\binom{n-3}{2}+9\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

(ii) *If $\lambda < 0$ and*

$$W_\lambda(G) \geq 2^\lambda\binom{n}{2} - (2^\lambda - 1)\left(\binom{n-3}{2}+9\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

When $g(x) = \frac{1}{2}(x^2+x)$, we have $W_g(G) = WW(G)$. The corresponding result on the hyper-Wiener index is as follows.

Corollary 17. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$. If*

$$WW(G) \leq 3\binom{n}{2} - 2\left(\binom{n-3}{2}+9\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

When $g(x) = \frac{1}{x}$, we have $W_g(G) = H(G)$. The corresponding result on the Harary index is as follows.

Corollary 18. *Let $n \geq 5$, $G \in \mathcal{G}^*(n)$ and $\delta(G) \geq 3$. If*

$$H(G) \geq \frac{1}{2}\binom{n}{2} + \frac{1}{2}\left(\binom{n-3}{2} + 9\right),$$

then the graph G has pancyclic property if it is not an element of the set $\mathbb{NP}_1$.

Example 1. *Let $t \geq 2$, $n_i \geq 3$ $(i = 1, 2, \ldots, t)$, $G_i = K_{n_i}$ $(i = 1, 2, \ldots, t)$, $v_i \in V(G_i)$ $(i = 1, 2, \ldots, t)$, $G = (V(G), E(G))$ where $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_t)$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t) \cup \{v_1v_2, v_2v_3, \ldots, v_{t-1}v_t\}$. Then, $W_g(G) > \binom{n}{2}g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$ if $g(x)$ is a monotone increasing function, and $W_g(G) < \binom{n}{2}g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$ if $g(x)$ is a monotone decreasing function.*

Proof. Clearly, G is connected, $|V(G)| > 5$ and $\delta(G) \geq 2$.

Let $n = n_1 + n_2 + \cdots + n_t$. Clearly, $G \notin \mathbb{NP}$. Since there exists exactly one edge between $V(G_1)$ and $V(G_2)$, G contains no C_n. Let $g(x)$ be a monotone increasing function. If $W_g(G) \leq \binom{n}{2}g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$, then by Theorem 5, G is pancyclic, a contradiction. Hence, $W_g(G) > \binom{n}{2}g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$.

Similarly, if $g(x)$ is strictly decreasing, we have $W_g(G) < \binom{n}{2}g(2) - \left(\binom{n-2}{2} + 4\right)(g(2) - g(1))$. □

Example 2. *Let $n \geq 6$, $G_a = K_{n-1}$, $G_b = K_1$, $X \subseteq V(G_a)$, $G = (V(G), E(G))$ where $V(G) = V(G_a) \cup V(G_b)$ and $E(G) = E(G_a) \cup \{uv | u \in X, v \in V(G_b)\}$. Then, G has the pancyclic property if $|X| \geq 2$.*

Proof. Clearly, $G \notin \mathbb{NP}$ and $G \notin \mathbb{NP}_1$. Let $|X| = m$. Then, $W(G) = 2(n-1-m) + \binom{n-1}{2} + m = \frac{n^2+n-2m-2}{2}$. If $m = 2$, then $2(n-1-m) + \binom{n-1}{2} + m = \frac{n^2+n-6}{2} \leq 2\binom{n}{2} - \binom{n-2}{2} - 4 = \frac{n^2+3n-14}{2}$. If $m \geq 3$, then $2(n-1-m) + \binom{n-1}{2} + m = \frac{n^2+n-2m-2}{2} \leq 2\binom{n}{2} - \binom{n-3}{2} - 9 = \frac{n^2+5n-30}{2}$. By Corollaries 7 and 15, G has the pancyclic property if $|X| \geq 2$. □

4. Conclusions and Future Work

In this paper, we introduce several Wiener-type indices, whose corresponding functions $g(x)$ are monotonically increasing or monotonically decreasing. Based on these Wiener-type indices and the monotonicity of their corresponding functions $g(x)$, we show some extremal conclusions on ℓ-connected graphs and some conditions that ensure graphs to have the pancyclic property. Moreover, we give some examples (see Examples 1 and 2) to support our findings.

Wiener-type indices are important and widely studied in chemical graph theory. They can be used to describe molecular structures and determine certain properties of graphs. The pancyclic property of a graph can be used to analyze complex networks. Our findings provide some upper and lower bounds for Wiener-type indices and offer a numerical computation method to determine whether a graph is pancyclic, which will assist in deriving the structure and properties of certain graphs by utilizing Wiener-type indices in practical applications. However, what we explore in this paper is the sufficient condition for a graph to be pancyclic based on the Wiener-type indices. Naturally, one may ask about the necessary conditions, which we believe are worth considering for future work.

Problem 1. *Find some necessary conditions for a graph G to be pancyclic with respect to $W_g(G)$, where $g(x)$ is a monotonic function.*

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