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Two-Variable q-Hermite-Based Appell Polynomials and Their Applications

Mohammed Fadel [1], Maryam Salem Alatawi [2] and Waseem Ahmad Khan [3,*]

[1] Department of Mathematics, Lahej University, Lahej 73560, Yemen; muhammad.fadl.edu.s@aden-univ.net or mohdfadel80@gmail.com
[2] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; msoalatawi@ut.edu.sa
[3] Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
* Correspondence: wkhan1@pmu.edu.sa

Abstract: A noteworthy advancement within the discipline of q-special function analysis involves the extension of the concept of the monomiality principle to q-special polynomials. This extension helps analyze the quasi-monomiality of many q-special polynomials. This extension is a helpful tool for considering the quasi-monomiality of several q-special polynomials. This study aims to identify and establish the characteristics of the 2-variable q-Hermite–Appell polynomials via an extension of the concept of monomiality. Also, we present some applications that are taken into account.

Keywords: extension of monomility principle; q-Appell polynomials; q-Hermite polynomials; q-dilatation operator

MSC: 33C45; 11B68; 11B83

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1. Introduction

Arguably, the most significant extension of traditional calculus is quantum calculus, often referred to as q-calculus, which is more applicable to the study of quantum physics and other scientific fields like mathematical numerology, combinatorics, orthogonal polynomials, and so forth. The q-calculus structure was first presented by Jackson [1] and later expanded upon through others as well. The introduction of q-calculus allows one to look for an analysis of q-analogs, which represent various fundamental and special functions. Multiple researchers have recently investigated and studied a few strange polynomials connected by q-calculus. (See for example [2–10]).

For $s \in \mathbb{C}$, the q-analog of s is specified in the following manner [11–13]:

$$[s]_q = \frac{1-q^s}{1-q}, \qquad 0 < q < 1.$$

The context regarding the q-factorial is explained as [11–13]:

$$[t]_q! = \begin{cases} \prod_{s=1}^{t} [s]_q, & t \geq 1, \\ 1, & t = 0. \end{cases}$$

The definition of the Gauss q-binomial coefficient is given as [11–13]:

$$\begin{bmatrix} t \\ s \end{bmatrix}_q = \frac{[t]_q\ !}{[t-s]_q![s]_q!}, \qquad s = 0, 1, \ldots, t.$$

The definition of q-exponential function listed through $e_q(\xi)$ is inspired by [11]:

$$e_q(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_q!}, \qquad 0 < q < 1, |\xi| < \frac{1}{1-q}. \tag{1}$$

The function $f(\xi)$ whose q-derivative regarding ξ is shown by the character $\hat{D}_{q,\xi} f(\xi)$, which indicates that [14]:

$$D_{q,\xi} f(\xi) = \frac{f(q\xi) - f(\xi)}{q\xi - \xi}, \qquad \xi \neq 0.$$

In particular,

$$\hat{D}_{q,\xi} \xi^n = [n]_q \xi^{n-1}, \tag{2}$$

$$\hat{D}_{q,\xi} e_q(\alpha\xi) = \alpha e_q(\alpha\xi), \qquad \alpha \in \mathbb{C}, \tag{3}$$

and

$$\hat{D}_{q,\xi}\big(f(\xi)g(\xi)\big) = f(\xi)\hat{D}_{q,\xi} g(\xi) + g(q\xi)\hat{D}_{q,\xi} f(\xi). \tag{4}$$

The q-Appell polynomials can be considered an extension of the Appell polynomials. Sharma and Chak [15] created a family of q-Appell polynomials in 1954. This q-Appell polynomials family was formed, and some of the characteristics were explored by Al-Salaam [16] during 1967. Al-Salaam also contributed to mathematics by presenting the q-Euler and q-Bernoulli polynomials as specific instances of the q-Appell family. In 1982, Srivastava [17] provided multiple descriptions of the renowned Appell polynomials as well as their basic analogs. Furthermore, some new applications of these polynomials are examined and discussed. Srivastava [18] additionally offered q-extensions of the Euler, Genocchi, and Bernoulli polynomials. Ernst [19] has offered a complete analysis for the q-Appell, q-Bernoulli within the structure of q-umbral calculus. By using Equations (5) and (6), we find the generating functions of the appropriate family members of q-Bernoulli, q-Euler and q-Genocchi polynomials that are described in Table 1.

Table 1. Several q-Appell polynomial families.

S. No.	q-Appell Polynomials	Generating Function	$\mathcal{A}_q(t)$
I.	**The q-Bernoulli Polynomials [19,20]**	$\frac{t}{e_q(t)-1} e_q(\xi t) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(\xi) \frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{t}{e_q(t)-1}$
II.	**The q-Euler Polynomials [19,21]**	$\frac{[2]_q}{e_q(t)+1} e_q(\xi t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\xi) \frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{[2]_q}{e_q(t)+1}$
III.	**The q-Genocchi Polynomials [21,22]**	$\frac{[2]_q t}{e_q(t)+1} e_q(\xi t) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(\xi) \frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{[2]_q t}{e_q(t)+1}$

Al Salam provides the generating function of q-Appell polynomials $\mathcal{A}_{n,q}(\xi)$ in the manner of the formula [16]:

$$\mathcal{A}_q(t)\, e_q(\xi t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(\xi) \frac{t^n}{[n]_q!}, \tag{5}$$

where

$$\mathcal{A}_q(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q} \frac{t^n}{[n]_q\ !}, \quad \mathcal{A}_q(t) \neq 0, \qquad \mathcal{A}_{0,q} = 1. \tag{6}$$

The following list shows the suitable option for certain individuals joining the class of q-Appell polynomials:

For $\xi = 0$, the membership of q-Appell polynomials class $\mathcal{A}_{n,q}(\xi)$ give the corresponding q-numbers $\mathcal{A}_{n,q}$. In Table 2, we list the earliest occurrences for the q-Bernoulli numbers $\mathcal{B}_{n,q}$ [19], q-Euler numbers $\mathcal{E}_{n,q}$ [19] and q-Genocchi numbers $\mathcal{G}_{n,q}$ [22].

Table 2. The first five q-numbers $\mathcal{B}_{n,q}$, $\mathcal{E}_{n,q}$ and $\mathcal{G}_{n,q}$.

n	**0**	**1**	**2**	**3**	**4**
$\mathcal{B}_{n,q}$ [19]	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]_q)^{-1}([4]_q)^{-1}$	$q^4(1-q^2-2q^3-q^4+q^6)([2]_q^2![3]_q[5]_q)^{-1}$
$\mathcal{E}_{n,q}$ [19]	1	$-\dfrac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+q^2-q^3)$	$\frac{1}{16}(q-1)[3]_q!(q^2-4q+1)$
$\mathcal{G}_{n,q}$ [22]	0	$\dfrac{2-q}{1+q}$	$\frac{q(q-5)}{(1+q)^2}$	$-\frac{3q^2(q-5)}{(1+q)^3}-\frac{3q(2-q)}{(1+q)^2}-\frac{q}{(1+q)})$	$\frac{-3q}{1+q}\left(\frac{3q^3+10q^2-28q+7}{(1+q)^3}\right)$

Hermite polynomials, deemed to be the most beneficial orthogonal special functions during the classical period, are now frequently used. These were solutions to differential equations combining the quantum mechanical Schrödinger equation with a harmonic oscillator. Furthermore, these polynomials are highly useful for studying classical boundary-value problems on parabolic domains using parabolic coordinates. Regarding extra details on Hermite polynomials as well as applications thereof, readers might turn to academic articles [11,23–33]. Hermite polynomials in two variables have an extensive history in mathematics and have been traced back to Hermite [30]. According to Appel and Kamp de Fériet [34], heat polynomials (2-variable Hermite polynomials) have gained popularity as auxiliary solutions. For further details, see [35]. The q-Hermite polynomials have applications in many areas of mathematics and science, including non-commutative probability, quantum physics, and combinatorics. The polynomials they contained sparked the curiosity of numerous scholars. In certain cases, we additionally reference published findings. (Check, over the instance [6,8,9,36–41]).

Recently, Raza and colleagues [8] identified q-Hermite polynomials with two variables $H_{n,q}(\xi,\nu)$ via a specific generating function:

$$e_q(\xi t)\, e_q(\nu t^2) = \sum_{n=0}^{\infty} H_{n,q}(\xi,\nu) \frac{t^n}{[n]_q\ !} \tag{7}$$

as well as the concept of a series

$$H_{n,q}(\xi,\nu) = [n]_q! \sum_{k=0}^{[n/2]} \frac{\xi^{n-2k}\nu^k}{[n-2k]_q![k]_q!}. \tag{8}$$

Following is the q-derivative for the $e_q(\nu t^2)$ via respect to t [8]:

$$D_{q,t}\, e_q(\nu t^2) = \nu t\, e_q(\nu t^2) + q\nu t\, e_q(q\nu t^2). \tag{9}$$

The operational identity of q-Hermite polynomials having two variables $H_{n,q}(\xi,\nu)$ is given by [8]:

$$H_{n,q}(\xi,\nu) = e_q(\nu \hat{D}^2_{q,\xi})\ \{\xi^n\}. \tag{10}$$

The monomiality concept is an effective tool for investigating the properties of particular special polynomials. The concept of monomiality extends origins in the beginning of the 19th century when J. F. Steffensen [42] coined the term poweroid. Dattoli reconstructed and expanded the concept of quasi-monomiality [43]. Using the monomiality principle, various researchers have recently proposed and investigated hybrid families of special polynomials [44–46]. Recently, the concept of the monomiality principle for q-polynomials has been expanded by Raza et al. [47]. Considering the quasi-monomiality of certain q-special polynomials, this idea is an immensely useful tool. The extension of monomiality treatment is effective for situating q-special polynomials like q-Laguerre polynomials of two variables, see [10].

The two q-operators $\hat{M}_q$ and $\hat{P}_q$, described the q-multiplicative and q-derivative operators within a q-polynomials family $p_{n,q}(\xi)$, are realized by [47]:

$$\hat{M}_q\ \{p_{n,q}(\xi)\} = p_{n+1,q}(\xi) \tag{11}$$

and

$$\hat{P}_q\{p_{n,q}(\xi)\} = [n]_q p_{n-1,q}(\xi), \tag{12}$$

respectively.

The q-operators $\hat{M}_q$ and $\hat{P}_q$ satisfy the following commutation and q-differential relations, which are satisfied by $p_{n,q}(\xi)$ as:

$$[\hat{P}_q, \hat{M}_q] = \hat{P}_q\hat{M}_q - \hat{M}_q\hat{P}_q \tag{13}$$

and

$$\hat{M}_q\hat{P}_q\ \{p_{n,q}(\xi)\} = [n]_q\ p_{n,q}(\xi), \tag{14}$$

or, alternatively

$$\hat{P}_q\hat{M}_q\ \{p_{n,q}(\xi)\} = [n+1]_q\ p_{n,q}(\xi), \tag{15}$$

respectively. Based on formulas (13), (14) as well as (15), we have

$$[\hat{P}_q, \hat{M}_q] = [n+1]_q - [n]_q. \tag{16}$$

Moreover, considering Equation (11), we observed

$$\hat{M}_q^r\ \{p_{n,q}(\xi)\} = p_{n+r,q}(\xi) \tag{17}$$

and

$$p_{n,q}(\xi) = \hat{M}_q^n\{p_{0,q}(\xi)\} = \hat{M}_q^n\{1\}, \tag{18}$$

wherein $p_{0,q}(\xi) = 1$. represents the q-sequel corresponding to the polynomial $p_{n,q}(\xi)$.

Consequently, the generating function of $p_{n,q}(\xi)$ is given as:

$$e_q(\hat{M}_q t)\{1\} = \sum_{n=0}^{\infty} p_{n,q}(\xi)\frac{t^n}{[n]_q!}. \tag{19}$$

The q-dilatation operator is represented by T_z and operates in the following manner for every function associated with the complex variable z [48]:

$$T_z^k f(z) = f(q^k z), \qquad k \in \mathbb{R}. \tag{20}$$

It satisfies the following relation:

$$T_z^{-1}T_z^1 f(z) = f(z).$$

The q-multiplicative and q-derivative operators of 2VqHP $H_{n,q}(\xi,\nu)$ are given as [47]:

$$\hat{M}_{qGH} = \xi T_\nu + \nu\hat{D}_{q,\xi} + q\nu\hat{D}_{q,\nu}T_\nu, \tag{21}$$

or, alternatively

$$\hat{M}_{qGH} = \xi + \nu T_\xi\hat{D}_{q,\xi} + q\nu\hat{D}_{q,\xi}T_\xi T_\nu \tag{22}$$

and

$$\hat{P}_{qH} = \hat{D}_{q,\xi}, \tag{23}$$

respectively.

The following are the q-multiplicative and q-derivative operators for q-Appell polynomials $\mathcal{A}_{n,q}(\nu)$ [47]:

$$\hat{M}_{qA} = \xi + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\xi, \tag{24}$$

or, alternatively

$$\hat{M}_{qA} = \xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}, \tag{25}$$

and

$$\hat{P}_{qA} = \hat{D}_{q,\xi}, \tag{26}$$

respectively.

Inspired by the useful of 2-variable Hermite polynomials and their hybrids in a variety of practical applications, including quantum optics' distribution of coherent or non-coherent radiation fields, multidimensional coupled systems for electromagnetic radiation problems, and associated wave propagation phenomena. Furthermore, inspired by the potential uses of q-special functions in science and mathematics, we present and examine the distinctive features of q-Hermite-Appell polynomials with two variables by employing the q-analog of the concept of the monomiality principle. In addition, we provide applications of these newly certain q-Hermite-Appell polynomials to display their graphical representations. Conclusions are displayed.

2. Characteristics of q-Hermite-Appell Polynomials with Two Variables

For this component, we make the convolution for q-Hermite polynomials of two variables with q-Appell polynomials via the extended concept of monomiality to examine q-Hermite-Appell polynomials. Also, we investigate the monomiality attributes that were newly introduced hybrid q-special polynomials.

The generating function for q-Hermite-Appell polynomials with two variables must be created via selecting 2VqHP $H_{n,q}(\xi,\nu)$ as a base in the generating function of q-Appell polynomials. Thus, substituting ξ in the left hand aspect of Equation (5) by the q-multiplicative operator of $H_{n,q}(\xi,\nu)$ given by Equations (21) or (22) as well as denoting the resultant q-Hermite–Appell polynomials with two variables via ${}_H\mathcal{A}_{n,q}(\xi,\nu)$, we obtain

$$\mathcal{A}_q(t) e_q(\hat{M}_{qGH} t)\{1\} = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu) \frac{t^n}{[n]_q!}. \tag{27}$$

Utilizing Equations (21) and (22), we obtain both analogous types of ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ which occur:

$$\mathcal{A}_q(t) e_q\Big(\Big(\xi T_\nu + \nu\hat{D}_{q,\xi} + q\nu\hat{D}_{q,\xi} T_\xi T_\nu\Big) t\Big)\{1\} = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu) \frac{t^n}{[n]_q!} \tag{28}$$

and

$$\mathcal{A}_q(t) e_q\Big(\Big(\xi + \nu T_x\hat{D}_{q,\xi} + q\nu\hat{D}_{q,\xi} T_\xi T_\nu\Big) t\Big)\{1\} = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu) \frac{t^n}{[n]_q!}, \tag{29}$$

respectively.

In the context of Equations (7), (19) and (21) or (22), we have

$$e_q(\hat{M}_{qH} t)\{1\} = e_q(\xi t)\, e_q(\nu t^2). \tag{30}$$

Utilizing Equation (30) in the left part for the Expression (27), the subsequent generating function of ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ is found:

$$\mathcal{A}_q(t)e_q(\xi t)e_q(\nu t^2) = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}. \tag{31}$$

Extending the left part of Equation (31) with the help of (6) and (7), we receive

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{A}_{k,q}H_{n,q}(\xi,\nu)\frac{t^{n+k}}{[n]_q![k]_q!} = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}, \tag{32}$$

That, when applying the series rearrangement technique then matching the values of the coefficients with the same powers of t from each side of the subsequent equation, produces the subsequent series definition of ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ as follows:

Definition 1.

$$ {}_H\mathcal{A}_{n,q}(\xi,\nu) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{A}_{k,q}H_{n-k,q}(\xi,\nu). \tag{33}$$

Theorem 1. *The 2VqHAP ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ is a solution for the following partial differential equation:*

$$\hat{D}_{q,\nu}\ {}_H\mathcal{A}_{n,q}(\xi,\nu) = \hat{D}^2_{q,\xi}\ {}_H\mathcal{A}_{n,q}(\xi,\nu), \tag{34}$$

under the initial condition

$$ {}_H\mathcal{A}_{n,q}(\xi,0) = \mathcal{A}_{n,q}(\xi).$$

Proof. Partial differentiation of Formula (31) with regard to ξ and ν, by further simplifying the resulting equation, we obtain

$$\hat{D}_{q,\xi}\ {}_H\mathcal{A}_{n,q}(\xi,\nu) = [n]_q\ {}_H\mathcal{A}_{n-1,q}(\xi,\nu) \tag{35}$$

and

$$\hat{D}_{q,\nu}\ {}_H\mathcal{A}_{n,q}(\xi,\nu) = [n]_q[n-1]_q\ {}_H\mathcal{A}_{n-2,q}(\xi,\nu), \tag{36}$$

respectively.

We obtain Statement (34) from Equations (35) and (36). □

Example 1. *Applying (34), we obtain the following q-partial differential equations for the 2-variable q-Hermite-Appell polynomials:*

$$D^2_{2/3,\xi}H_{2,2/3}(\xi,\nu) - D_{2/3,\nu}H_{2,2/3}(\xi,\nu) = 0$$

and

$$\hat{D}^2_{3/4,\xi}\ {}_H\mathcal{A}_{5,3/4}(\xi,\nu) - \hat{D}_{3/4,\nu}\ {}_H\mathcal{A}_{5,3/4}(\xi,\nu) = 0.$$

Following this, we are going to display the subsequent finding:

Theorem 2. *The subsequent operational definition is satisfied by the 2VqHAP ${}_H\mathcal{A}_{n,q}(\xi,\nu)$:*

$$ {}_H\mathcal{A}_{n,q}(\xi,\nu) = e_q(\nu\hat{D}^2_{q,\xi})\{\mathcal{A}_{n,q}(\xi)\}. \tag{37}$$

Proof. After considering Equation (5), Equation (31) gives

$$e_q(\nu t^2)\{\mathcal{A}_{n,q}(\xi)\} = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}. \tag{38}$$

Since, we have

$$\hat{D}_{q,\xi}\{\mathcal{A}_q(t)\, e_q(\xi t)\} = t\mathcal{A}_q(t)\, e_q(\xi t).$$

Similarly, we have

$$\hat{D}^2_{q,\xi}\{\mathcal{A}_q(t)\, e_q(\xi t)\} = t^2\mathcal{A}_q(t)\, e_q(\xi t). \tag{39}$$

Since Equation (1) is taken into consideration, Equation (39) provides

$$e_q(\nu\hat{D}^2_{q,\xi})\mathcal{A}_q(t)\, e_q(\xi t) = e_q(\nu t^2)\mathcal{A}_q(t)\, e_q(\xi t). \tag{40}$$

Utilizing Equation (5) in the left aspect and Equation (31) in the right aspect of the aforementioned equation, then identically obtain the values of the coefficients that possess equal powers of t on each side of the concluding equation, we receive the claim (37). □

The aforementioned theorem is proved to produce the q-multiplicative and q-derivative operators of ${}_H\mathcal{A}_{n,q}(\xi,\nu)$:

Theorem 3. *The q-Hermite-Appell polynomials of two variables ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ are quasi-monomials according to the subsequent q-multiplicative and q-derivative operators:*

$$\hat{M}_{qH\mathcal{A}} = \xi T_\nu + \nu\hat{D}_{q,\xi} + q\nu T_\nu\hat{D}_{q,\xi} + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu, \tag{41}$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi + \nu\hat{D}_{q,\xi}T_\xi + q\nu\hat{D}_{q,\xi}T_\xi T_\nu + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu, \tag{42}$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\nu + \nu\hat{D}_{q,\xi}T_\nu\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + q\nu\hat{D}_{q,\xi}T_\nu\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}, \tag{43}$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \nu\hat{D}_{q,\xi}T_\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + q\nu\hat{D}_{q,\nu}T_\nu T_\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}, \tag{44}$$

and

$$\hat{P}_{qH\mathcal{A}} = \hat{D}_{q,\xi}, \tag{45}$$

respectively.

Proof. Taking q-derivative for each part of Equation (31) partially via regard to t using Equation (4), we have

$$\mathcal{A}_q(t)\hat{D}_{q,t}\left[e_q(\xi t)e_q(\nu t^2)\right] + \hat{D}_{q,t}\mathcal{A}_q(t)(e_q(q\xi t)e_q(q\nu t^2)) = \sum_{n=1}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\hat{D}_{q,t}\frac{t^n}{[n]_q\;!}, \tag{46}$$

which on using Equation (4) by taking $f(t) = e_q(\xi t)$ and $g(t) = e_q(\nu t^2)$, then rearranging the equation that results by utilizing Formulas (3), (9) and (20) in the left aspect, we receive

$$\left(\xi T_\nu + \nu t + q\nu T_\nu t\ + \frac{\mathcal{A}'_q(t)}{\mathcal{A}_q(t)}T_\xi T_\nu\right)\mathcal{A}_q(t)\, e_q(\xi t)e_q(\nu t^2) = \sum_{n=1}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^{n-1}}{[n-1]_q!}. \tag{47}$$

Since

$$\hat{D}_{q,\xi}\mathcal{A}_q(t)\, e_q(\xi t)e_q(\nu t^2) = t\mathcal{A}_q(t)\, e_q(\xi t)e_q(\nu t^2) \tag{48}$$

and $\frac{\mathcal{A}_q'(t)}{\mathcal{A}_q(t)}$ has q-power series expansions in t, as $\mathcal{A}_q(t)$ is invertible series of t.

Therefore, Equation (47) produces

$$\left(\xi T_\nu + \nu\hat{D}_{q,\xi} + q\nu T_\nu \hat{D}_{q,\xi} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu\right)\mathcal{A}_q(t)\, e_q(\xi t)e_q(\nu t^2) = \sum_{n=1}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^{n-1}}{[n-1]_q\ !}, \tag{49}$$

which on using Equation (31), produces

$$\left(\xi T_\nu + \nu\hat{D}_{q,\xi} + q\nu T_\nu \hat{D}_{q,\xi} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu\right)\sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^n}{[n]_q\ !} = \sum_{n=1}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^{n-1}}{[n-1]_q\ !}. \tag{50}$$

Comparing values of the coefficients of t in each aspect of Equation (50) and then from view of Equation (11), the resultant equation, we produce claim (41).

Again, using Equation (46) by taking $f(t) = e_q(\nu t^2)$, $g(t) = e_q(\xi t)$ and following the same paths of verification of statement (41), we produce claim (42).

Taking q-derivative of each side of Formula (31) partially via regard to t by taking $f(t) = e_q(\xi t)e_q(\nu t^2)$ and $g(t) = \mathcal{A}_q(t)$, we produce

$$\hat{D}_{q,t}\mathcal{A}_q(t)\left(e_q(\xi t)e_q(\nu t^2)\right) + \mathcal{A}_q(qt)\hat{D}_{q,t}\left(e_q(\xi t)e_q(\nu t^2)\right) = \sum_{n=1}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^{n-1}}{[n-1]_q\ !}, \tag{51}$$

which on using Equation (4) for $f(t) = e_q(\xi t)$, $g(t) = e_q(\nu t^2)$ and continuing the identical methods for verification of claim (41), we receive a claim (43).

Again, in Equation (51), differentiating $e_q(\xi t)e_q(\nu t^2)$ partially with respect to t by taking $f(t) = e_q(\nu t^2)$ and $g(t) = e_q(\xi t)$ and continuing the identical methods for verification of claim (41), we receive a claim (44).

From Formulas (12) and (35), we receive a claim (45). □

From Formulas (14) and (41)–(45), we receive the subsequent result for q-differential equations satisfied by ${}_H\mathcal{A}_{n,q}(\xi,\nu)$.

Theorem 4. *For q-Hermite-Appell polynomials of two variables, the following q-differential equations hold:*

$$\left(\xi T_\nu \hat{D}_{q,\xi} + \nu\hat{D}^2_{q,\xi} + q\nu T_\nu \hat{D}^2_{q,\xi} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu \hat{D}_{q,\xi} - [n]_q\right) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0, \tag{52}$$

$$\left(\xi \hat{D}_{q,\xi} + \nu\hat{D}^2_{q,\xi}T_\xi + q\nu \hat{D}^2_{q,\xi}T_\xi T_\nu + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi T_\nu \hat{D}_{q,\xi} - [n]_q\right) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0, \tag{53}$$

$$\begin{aligned}\Bigg(\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\nu\hat{D}_{q,\xi} + \nu\hat{D}_{q,\xi}T_\nu\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi} + q\nu\hat{D}_{q,\xi}T_\nu\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi} \\ + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi} - [n]_q\Bigg) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0\end{aligned} \tag{54}$$

and

$$\left(\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi}+\nu\hat{D}_{q,\xi}T_\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi}+q\nu\hat{D}_{q,\xi}T_\nu T_\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi}\right.$$
$$\left.+\frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\hat{D}_{q,\xi}-[n]_q\right)\,{}_H\mathcal{A}_{n,q}(\xi,\nu)=0. \quad (55)$$

Example 2. *Applying Formulas (52)–(55), we obtain the following:*

$$\left(\xi T_\nu\hat{D}_{1/2,\xi}+\nu\hat{D}^2_{1/2,\xi}+1/2\nu T_\nu\hat{D}^2_{1/2,\xi}+\frac{\mathcal{A}'_{1/2}(\hat{D}_{1/2,\xi})}{\mathcal{A}_{1/2}(\hat{D}_{1/2,\xi})}T_\xi T_\nu\hat{D}_{1/2,\xi}\right.$$
$$\left.-[2]_{1/2}\right)\,{}_H\mathcal{A}_{2,1/2}(\xi,\nu)=0,$$

$$\left(\xi\hat{D}_{3/5,\xi}+\nu\hat{D}^2_{3/5,\xi}T_\xi+3/5\nu\hat{D}^2_{3/5,\xi}T_\xi T_\nu+\frac{\mathcal{A}'_{3/5}(\hat{D}_{3/5,\xi})}{\mathcal{A}_{3/5}(\hat{D}_{3/5,\xi})}T_\xi T_\nu\hat{D}_{3/5,\xi}\right.$$
$$\left.-[6]_{3/5}\right)\,{}_H\mathcal{A}_{6,3/5}(\xi,\nu)=0,$$

$$\left(\xi\frac{\mathcal{A}_{2/7}(2/7\hat{D}_{2/7,\xi})}{\mathcal{A}_{2/7}(\hat{D}_{2/7,\xi})}T_\nu\hat{D}_{2/7,\xi}+\nu\hat{D}_{2/7,\xi}T_\nu\frac{\mathcal{A}_{2/7}(2/7\hat{D}_{2/7,\xi})}{\mathcal{A}_{2/7}(\hat{D}_{2/7,\xi})}\hat{D}_{2/7,\xi}\right.$$
$$\left.+2/7\nu\hat{D}_{2/7,\xi}T_\nu\frac{\mathcal{A}_{2/7}(2/7\hat{D}_{2/7,\xi})}{\mathcal{A}_{2/7}(\hat{D}_{2/7,\xi})}\hat{D}_{2/7,\xi}+\frac{\mathcal{A}'_{2/7}(\hat{D}_{2/7,\xi})}{\mathcal{A}_{2/7}(\hat{D}_{2/7,\xi})}\hat{D}_{2/7,\xi}-[9]_{2/7}\right)\,{}_H\mathcal{A}_{9,2/7}(\xi,\nu)=0$$

and

$$\left(\xi\frac{\mathcal{A}_{3/11}(3/11\hat{D}_{3/11,\xi})}{\mathcal{A}_{3/11}(\hat{D}_{3/11,\xi})}\hat{D}_{3/11,\xi}+\nu\hat{D}_{3/11,\xi}T_\xi\frac{\mathcal{A}_{3/11}(3/11\hat{D}_{3/11,\xi})}{\mathcal{A}_{3/11}(\hat{D}_{3/11,\xi})}\hat{D}_{3/11,\xi}+3/11\nu\hat{D}_{3/11,\xi}T_\nu T_\xi\right.$$
$$\left.\frac{\mathcal{A}_{3/11}(3/11\hat{D}_{3/11,\xi})}{\mathcal{A}_{3/11}(\hat{D}_{3/11,\xi})}\hat{D}_{3/11,\xi}+\frac{\mathcal{A}'_{3/11}(\hat{D}_{3/11,\xi})}{\mathcal{A}_{3/11}(\hat{D}_{3/11,\xi})}\hat{D}_{3/11,\xi}-[11]_{3/11}\right)\,{}_H\mathcal{A}_{11,3/11}(\xi,\nu)=0.$$

Remark 1. *From Equations (5) and (19) and (24), we have*

$$e_q(\hat{M}_{q\mathcal{A}}t)\{1\}=\sum_{n=0}^{\infty}\mathcal{A}_{n,q}(\xi)\frac{t^n}{[n]_q\ !}=\mathcal{A}_q(t)\,e_q(\xi t). \quad (56)$$

Replacing ξ with $\hat{M}_{q\mathcal{A}}$ in Equation (7) and indicating the final result of q-Appell-Hermite polynomials in the right-hand side by ${}_{\mathcal{A}}H_{n,q}(\xi,\nu)$, we receive

$$\sum_{n=0}^{\infty}{}_{\mathcal{A}}H_{n,q}(\hat{M}_{q\mathcal{A}},\nu)\{1\}=e_q(\hat{M}_{q\mathcal{A}}t)e_q(\nu t^2)=\sum_{n=0}^{\infty}{}_{\mathcal{A}}H_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}, \quad (57)$$

which in view Equation (56), offers a particular generating function for ${}_{\mathcal{A}}H_{n,q}(\xi,\nu)$:

$$\mathcal{A}_q(t)e_q(\xi t)e_q(\nu t^2)=\sum_{n=0}^{\infty}{}_{\mathcal{A}}H_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}. \quad (58)$$

From Equations (31) and (58), we observe that

$$_{\mathcal{A}}H_{n,q}(\xi,\nu) = \ _{H}\mathcal{A}_{n,q}(\xi,\nu). \tag{59}$$

Remark 2. *According to the q-analog of the concept of quasi-monomiality and Equation (37), we have*

$$_{H}\mathcal{A}_{n,q}(\xi,\nu) = e_q(\nu\hat{D}^2_{q,\xi})(\hat{M}_{q\mathcal{A}})^n\ \{1\}, \tag{60}$$

which on using Equation (19), gives

$$(\hat{M}_{qHA})^n\{1\} = e_q(\nu\hat{D}^2_{q,\xi})(\hat{M}_{q\mathcal{A}})^n\{1\}. \tag{61}$$

Using expressions for $\hat{M}_{q\mathcal{A}}$ given by Equations (24) and (25) in Equation (60), provides the subsequent operational identities for $_{H}\mathcal{A}_{n,q}(\xi,\nu)$:

$$_{H}\mathcal{A}_{n,q}(\xi,\nu) = e_q(\nu\hat{D}^2_{q,\xi})\left(\xi + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}T_\xi\right)^n\Big\{1\Big\} \tag{62}$$

and

$$_{H}\mathcal{A}_{n,q}(\xi,\nu) = e_q(\nu\hat{D}^2_{q,\xi})\left(\xi\frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}'_q(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})}\right)^n\Big\{1\Big\}, \tag{63}$$

respectively.

Also, using different expressions for $\hat{M}_{qHA}$ and $\hat{M}_{q\mathcal{A}}$, given by Equations (24), (25) and (41)–(44) in Equation (61), we can obtain several operational identities for 2VqHAP $_{H}\mathcal{A}_{n,q}(\xi,\nu)$.

The applications of certain members of the q-Hermite-Appell polynomial family are covered in the subsequent section, along with graphical depictions of them.

3. Applications

In this part, we look at how some members of the family of q-Hermite-Appell polynomials can be used. Based on Equation (37), the 2VqHA $_{H}\mathcal{A}_{n,q}(\xi,\nu)$ can be constructed from the 2VqHP $H_{n,q}(\xi,\nu)$ that goes with q-Appell polynomials $\mathcal{A}_{n,q}(\xi)$. For instance, the q-Hermite-Bernoulli polynomials $_{H}\mathcal{B}_{n,q}(\xi,\nu)$, q-Hermite-Euler polynomials $_{H}\mathcal{E}_{n,q}(\xi,\nu)$ and q-Hermite-Genocchi polynomials $_{H}\mathcal{G}_{n,q}(\xi,\nu)$ are specified by the q-operational definitions listed below:

$$_{H}\mathcal{B}_{n,q}(\xi,nu) = e_q(\nu\hat{D}^2_{q,\xi})\{\mathcal{B}_{n,q}(\xi)\}, \tag{64}$$

$$_{H}\mathcal{E}_{n,q}(\xi,nu) = e_q(\nu\hat{D}^2_{q,\xi})\{\mathcal{E}_{n,q}(\xi)\} \tag{65}$$

and

$$_{H}\mathcal{G}_{n,q}(\xi,nu) = e_q(\nu\hat{D}^2_{q,\xi})\{\mathcal{G}_{n,q}(\xi)\}. \tag{66}$$

In view of Equations (31) and (33) and Table 1 ((I)–(III)), we find the generating functions and series definitions of the appropriate members of the family of q-Hermite–Appell polynomials that are described in Table 3.

In order to plot the graphs of $_{H}\mathcal{B}_{n,q}(\xi,\nu)$, $_{H}\mathcal{E}_{n,q}(\xi,\nu)$ and $_{H}\mathcal{G}_{n,q}(\xi,\nu)$, we obtain the following expressions of them for $n = 2,3,4$ and $q = \frac{1}{2}$ by making suitable substitutions from Table 2 and using Equation (8) throughout each of the series definitions provided in Table 3:

Table 3. Members of q-Hermite-Appell polynomials.

S. No.	q-Hermite-Appell Polynomials	Generating Function	Series Definition
I.	**The q-Hermite-Bernoulli Polynomials**	$\frac{t}{e_q(t)-1}e_q(\xi t)e_q(\nu t^2) = \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}(\xi,\nu)\frac{t^n}{[n]q!}$	${}_H\mathcal{B}_{n,q}(\xi,\nu) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{k,q}H_{n-k,q}(\xi,\nu)$
II.	**The q-Hermite-Euler Polynomials**	$\frac{[2]q}{eq(t)+1}e_q(\xi t)e_q(\nu t^2) = \sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}(\xi,\nu)\frac{t^n}{[n]q!}$	${}_H\mathcal{E}_{n,q}(\xi,\nu) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}_{k,q}H_{n-k,q}(\xi,\nu)$
III.	**The q-Hermite-Genocchi Polynomials**	$\frac{[2]qt}{eq(t)+1}e_q(\xi t)e_q(\nu t^2) = \sum_{n=0}^{\infty} {}_H\mathcal{G}_{n,q}(\xi,\nu)\frac{t^n}{[n]q!}$	${}_H\mathcal{G}_{n,q}(\xi,\nu) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}H_{n-k,q}(\xi,\nu)$

Now, using the expressions of ${}_H\mathcal{B}_{n,q}(\xi,\nu)$, ${}_H\mathcal{E}_{n,q}(\xi,\nu)$ and ${}_H\mathcal{G}_{n,q}(\xi,\nu)$ from Table 4 for $n = 2,3,4$ within the program "MATLAB" (version number: ML 2020b), the following surface diagrams have been drawn. Please check Figures 1–8:

Table 4. Expressions for ${}_H\mathcal{B}_{n,\frac{1}{2}}(\xi,\nu)$, ${}_H\mathcal{E}_{n,\frac{1}{2}}(\xi,\nu)$ and ${}_H\mathcal{G}_{n,\frac{1}{2}}(\xi,\nu)$ for $n = 2,3,4$.

n	2	3	4
${}_H\mathcal{B}_{n,\frac{1}{2}}(\xi,\nu)$	$\xi^2 + \frac{3}{2}\nu - \xi + \frac{7}{24}$	$\xi^3 + \frac{21}{8}\xi\nu + \frac{7}{6}\xi^2 - \frac{21}{12}\xi + \frac{1}{45}$	$\xi^4 + \frac{105}{32}\xi^2\nu^2 + \frac{105}{32}\nu^2 - \frac{45}{16}\xi^3 - \frac{945}{128}\xi\nu + \frac{5}{36}\xi + \frac{5}{6}\nu - \frac{1}{24}\xi - \frac{115323157}{262144}$
${}_H\mathcal{E}_{n,\frac{1}{2}}(\xi,\nu)$	$\xi^2 - \frac{3}{2}\nu - \xi + \frac{2}{9}$	$\xi^3 + \frac{21}{8}\xi\nu - \frac{14}{12}\xi^2 - \frac{7}{4}\nu + \frac{28}{72}\xi - \frac{7}{27}$	$\xi^4 + \frac{105}{32}\xi^2\nu + \frac{105}{32}\nu^2 - \frac{30}{24}\xi^3 - \frac{105}{32}\xi\nu - \frac{35}{72}\xi^2 - \frac{105}{288}\nu + \frac{360}{1728}\xi - \frac{126}{324}$
${}_H\mathcal{G}_{n,\frac{1}{2}}(\xi,\nu)$	$\frac{3}{2}\xi - 1$	$\frac{7}{4}\xi^3 + \frac{147}{132}\nu - \frac{7}{2}\xi - \frac{32}{27}$	$\frac{7}{4}\xi^3 + \frac{147}{32}\xi\nu - \frac{105}{16}\xi^2 - \frac{35}{16}\xi^2 - \frac{105}{32}\nu + \frac{1245}{96}\xi - \frac{30}{27}$

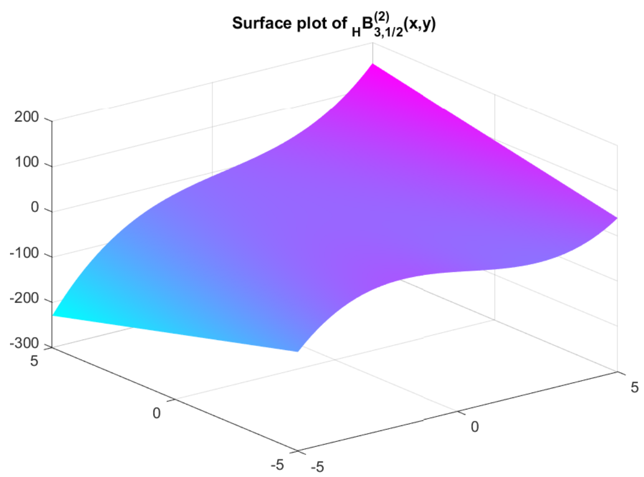

Figure 1. The surface plot of ${}_HB_{3,1/2}(\xi,\nu)$.

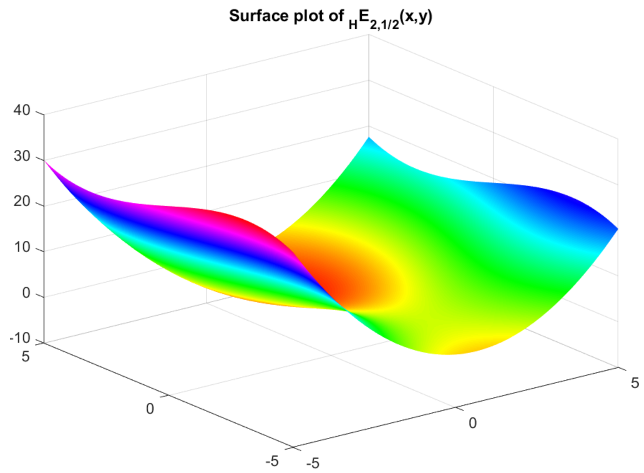

Figure 2. The surface plot of ${}_{H}E_{2,1/2}(\xi, \nu)$.

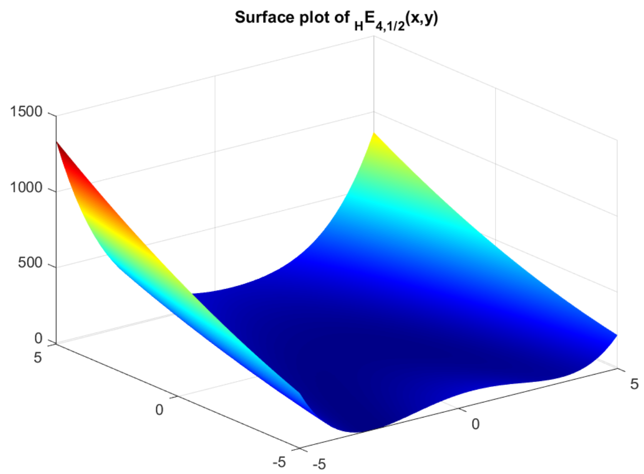

Figure 3. The surface plot of ${}_{H}E_{4,1/2}(\xi, \nu)$.

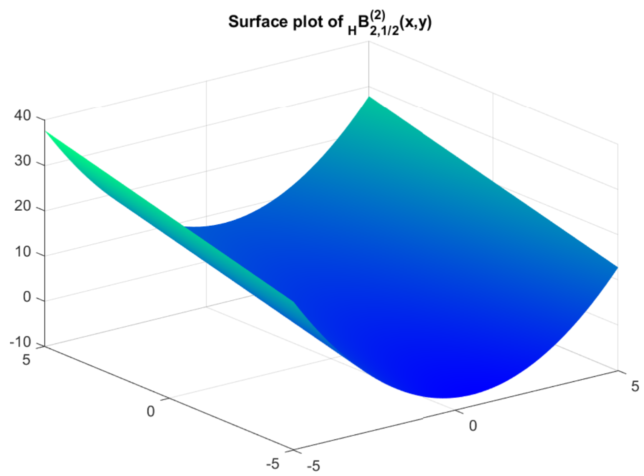

Figure 4. The surface plot of ${}_{H}B_{2,1/2}(\xi, \nu)$.

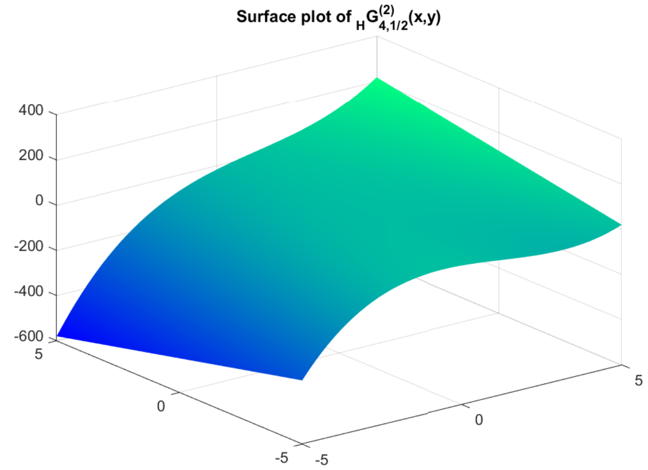

Figure 5. The surface plot of $_HG_{4,1/2}(\xi,\nu)$.

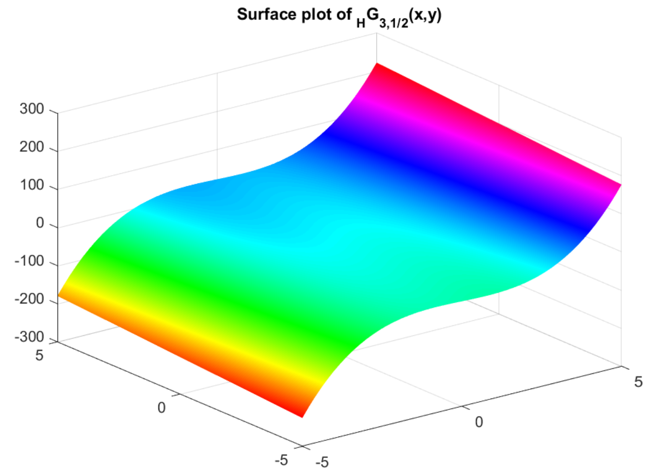

Figure 6. The surface plot of $_HG_{3,1/2}(\xi,\nu)$.

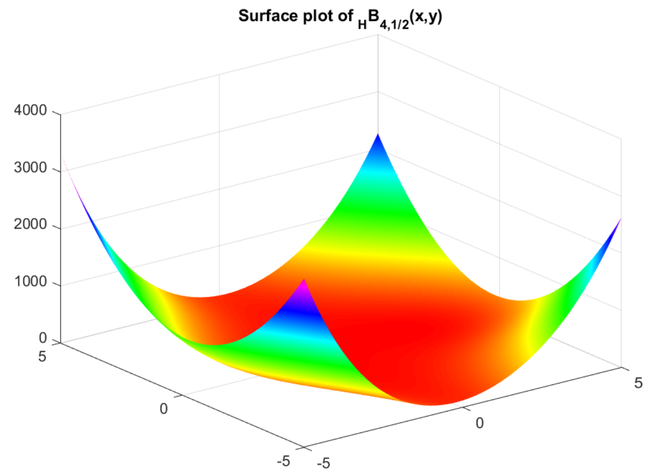

Figure 7. The surface plot of $_HB_{4,1/2}(\xi,\nu)$.

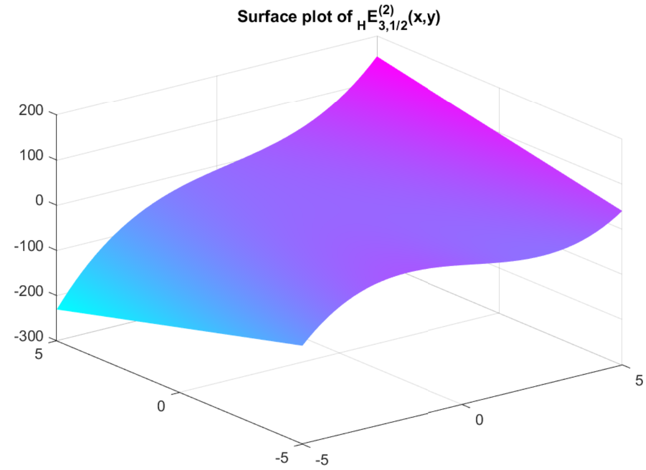

Figure 8. The surface plot of $_HE_{3,1/2}(\xi,\nu)$.

4. Conclusions

Special function experts value q-calculus as a powerful tool for several disciplines of sciences and engineering. Likewise, it has recently been discovered that the q-Hermite polynomials have applications in quantum physics, combination theory, non-commutative probability, and various other areas. Also, the 2-variable Hermite polynomials, along with their hybrids, hold significance in various physical applications. Examples include quantum optics' distribution of coherent or non-coherent radiation fields, multidimensional coupled systems for electromagnetic radiation issues, and associated wave propagation phenomena. In the article, we examined the quasi-monomiality of some q-special polynomials, specifically q-Hermite-Appell polynomials, and established their monomiality properties by extending the monomiality principle to q-special polynomials as follows:

- **Generating function** (See Equation (31)):
  The subsequent generating function for q-Hermite polynomials of 3-variables holds true:
  $$\mathcal{A}_q(t)e_q(\xi t)e_q(\nu t^2) = \sum_{n=0}^{\infty} {}_H\mathcal{A}_{n,q}(\xi,\nu)\frac{t^n}{[n]_q!}.$$
- **Series definition** (See Definition 1):
  The following series definition for the 2VqHAP $_H\mathcal{A}_{n,q}(\xi,\nu)$ holds true:
  $${}_H\mathcal{A}_{n,q}(\xi,\nu) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{A}_{k,q}H_{n-k,q}(\xi,\nu).$$
- **Partial differential equation:** (See Theorem 1):
  The following partial differential equation is a solution for the 2VqHAP $_H\mathcal{A}_{n,q}(\xi,\nu)$:
  $$\hat{D}_{q,\nu}\ {}_H\mathcal{A}_{n,q}(\xi,\nu) = \hat{D}^2_{q,\xi}\ {}_H\mathcal{A}_{n,q}(\xi,\nu),$$
  under the initial condition
  $${}_H\mathcal{A}_{n,q}(\xi,0) = \mathcal{A}_{n,q}(\xi).$$
- **Operational formulas** (See Theorem 2):
  The subsequent operational definition is satisfied by the 2VqHAP $_H\mathcal{A}_{n,q}(\xi,\nu)$:
  $${}_H\mathcal{A}_{n,q}(\xi,\nu) = e_q(\nu\hat{D}^2_{q,\xi})\{\mathcal{A}_{n,q}(\xi)\}.$$
  where $D^2_{q,x}$ is the second and third q-derivative operator.

- ***q*-Multiplicative and *q*-derivative operators** (See Theorem 3):
The q-Hermite-Appell polynomials of two variables ${}_H\mathcal{A}_{n,q}(\xi,\nu)$ are quasi-monomials according to the subsequent q-multiplicative and q-derivative operators:

$$\hat{M}_{qH\mathcal{A}} = \xi T_\nu + \nu \hat{D}_{q,\xi} + q\nu T_\nu \hat{D}_{q,\xi} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\xi T_\nu,$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi + \nu \hat{D}_{q,\xi} T_\xi + q\nu \hat{D}_{q,\xi} T_\xi T_\nu + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\xi T_\nu,$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\nu + \nu \hat{D}_{q,\xi} T_\nu \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + q\nu \hat{D}_{q,\xi} T_\nu \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})},$$

or, equivalently

$$\hat{M}_{qH\mathcal{A}} = \xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \nu \hat{D}_{q,\xi} T_\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + q\nu \hat{D}_{q,\nu} T_\nu T_\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})},$$

and

$$\hat{P}_{qH\mathcal{A}} = \hat{D}_{q,\xi},$$

respectively.
- ***q*-Differential equations** (See Theorem 4):
For q-Hermite-Appell polynomials of two variables, the following q-differential equations hold:

$$\left(\xi T_\nu \hat{D}_{q,\xi} + \nu \hat{D}_{q,\xi}^2 + q\nu T_\nu \hat{D}_{q,\xi}^2 + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\xi T_\nu \hat{D}_{q,\xi} - [n]_q\right) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0,$$

$$\left(\xi \hat{D}_{q,\xi} + \nu \hat{D}_{q,\xi}^2 T_\xi + q\nu \hat{D}_{q,\xi}^2 T_\xi T_\nu + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\xi T_\nu \hat{D}_{q,\xi} - [n]_q\right) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0,$$

$$\begin{aligned}\Bigg(\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} T_\nu \hat{D}_{q,\xi} + \nu \hat{D}_{q,\xi} T_\nu \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} + q\nu \hat{D}_{q,\xi} T_\nu \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} \\ + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} - [n]_q\Bigg) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0\end{aligned}$$

and

$$\begin{aligned}\Bigg(\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} + \nu \hat{D}_{q,\xi} T_\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} + q\nu \hat{D}_{q,\xi} T_\nu T_\xi \frac{\mathcal{A}_q(q\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} \\ + \frac{\mathcal{A}_q'(\hat{D}_{q,\xi})}{\mathcal{A}_q(\hat{D}_{q,\xi})} \hat{D}_{q,\xi} - [n]_q\Bigg) {}_H\mathcal{A}_{n,q}(\xi,\nu) = 0.\end{aligned}$$

As applications, we have taken into consideration such as q-Hermite-Bernoulli polynomials, ${}_H\mathcal{B}_{n,q}(\xi,\nu)$, q-Hermite-Euler polynomials ${}_H\mathcal{E}_{n,q}(\xi,\nu)$ as well as q-Hermite-

Genocchi polynomials ${}_H\mathcal{G}_{n,q}(\xi,\nu)$ in Section 3. Our outcomes will be useful in obtaining novel expression results and their associated hybrid polynomials.

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